



Existence results for a class of generalized complementarity problems

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Abstract. In this paper, we establish some existence results for the extended generalized complementarity problems in topological vector spaces, using the concept of minimax inequality and copositivity. We then proceed to find some existence of solutions for the generalized complementarity problems in reflexive Banach spaces as well as finite dimensional normed spaces.

1. Introduction

Complementarity problem unifies many mathematical problems, like optimization, economics, finance, game theory, and mathematical programming. It has various applications in science, engineering, mechanics, variational inequalities and equilibrium problems [2, 5], etc. The complementarity problem is closely related to the variational inequality problem. It was Karamardian [14] who first built up an equivalence between these two problems. Afterwards, various authors [6, 17, 20] established equivalences with their various generalized forms. The concept of monotonicity takes a key role in establishing the existence results of complementarity problems. The existence results of the complementarity problems as well as the variational inequalities under different kinds of generalized monotonicities, such as pseudomonotonicity and quasimonotonicity were established by many authors, like [9, 14, 20]. The general form of the complementarity problem is: given a mapping F from the n -dimensional Euclidean space $\mathbb{R}^n$ into itself, find a vector $x \in \mathbb{R}^n$ satisfying

$$x \geq 0, \quad F(x) \geq 0, \quad x^T F(x) = 0. \quad (1)$$

Problem (1) is called the linear complementarity problem (LCP) if F is of the form $F(x) = Mx + b$, where M is a $n \times n$ matrix and b is an n -vector, otherwise it is called the nonlinear complementarity problem (NCP).

Many investigators have been concerned with both the computational as well as the theoretical aspects of the problem (1). There are a large number of results on the existence and uniqueness of solutions for the problem (1), see [13, 16, 19] for instance. In particular, Habetler and Price [11] generalized the NCP, by

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replacing the usual nonnegative partial ordering on $\mathbb{R}^n$ by partial orderings generated by a given cone and its polar. Karamardian [14] extended this work to a locally convex Hausdorff topological vector space X , that is given a mapping F from a closed convex cone K of a locally convex Hausdorff, topological vector space X into a vector space Y , then the generalized complementarity problem (GCP) given by Karamardian [14] is to find a vector $x \in X$ such that

$$x \in K, F(x) \in K^* \text{ and } \langle x, F(x) \rangle = 0, \quad (2)$$

where $\langle \cdot, \cdot \rangle$ is a bilinear form from $X \times Y$ into $\mathbb{R}$ and K^* be the polar of K in Y . He used the finite intersection property and proved that the complementarity problem (2) has a solution if the function $(u, v) \mapsto \langle u, F(v) \rangle$ is continuous on $K \times K$ and there exists a nonempty compact and convex subset D in K with the property that for every $x \in K \setminus D$, there exists $z \in D$ such that $\langle x - z, F(x) \rangle > 0$. Saigal [19] extended the problem considered by Habetler and Price by assuming the function F to be multivalued, that is given a closed, convex cone K , its polar K^* and a multivalued mapping $F : K \rightarrow \mathbb{R}^n$, the complementarity problem considered by Saigal is to find a vector $x \in \mathbb{R}^n$ such that

$$x \in K, y \in F(x) \cap K^* \text{ and } x^T y = 0. \quad (3)$$

Park [17] generalized the complementarity problem (1) to a topological vector space in the form of (2) and proved an existence result by first proving a Karamardian-type equilibrium theorem and using it he established the existence of solutions. On the other hand, Fan [7] extended fixed point theorems of F.E. Browder [3, Theorem 1] and [4, Theorem 3] and established the following result in normed spaces: Given a non-empty compact and convex set K in a normed space X , and a continuous mapping $f : K \rightarrow X$, there exists a point $y_0 \in K$ such that

$$\|y_0 - f(y_0)\| = \min_{x \in K} \|x - f(y_0)\|.$$

If $f(K) \subset K$, then y_0 is a fixed point of f .

Inspired and motivated by the above works, in this paper we have established some existence of solutions for the extended generalized complementarity problems in the general setting of vector spaces in duality using the minimax inequality with a lopsided saddle point given by Ha [10]. We then proved some existence of solutions for the generalized complementarity problems and the associated regularized complementarity problems in reflexive Banach spaces with lack of coercivity assumptions. After which we restrict ourselves to the finite dimensional normed spaces in order to achieve the existence of solutions for the generalized complementarity problems using the notion of feasible points. The results in this paper extend and generalize the results of Sahu et al. [18], Yao and Chadli [20], Karamardian [15], Saigal [19] and many other results in the literature.

The rest of the paper is organized as follows. In Section 2, we state the extended generalized complementarity problem in the general setting of vector spaces in duality and provide some basic concepts and results that will be needed in the sequel. Section 3 is devoted to finding the existence of solutions for the extended generalized complementarity problems using the minimax inequality with a lopsided saddle point given by Ha [10]. In Section 4, we find some existence results for the generalized complementarity problems in reflexive Banach spaces with copositivity assumptions. Finally, in Section 4, we have used the concept of feasible points and found some existence of solutions for the generalized complementarity problems in the finite dimensional normed spaces.

2. Preliminaries

Let X and Y be Hausdorff topological vector spaces and $\langle \cdot, \cdot \rangle : X \times Y \rightarrow \mathbb{R}$ be a bilinear form between X and Y . A nonempty subset K of X is called a cone if $x \in K$ implies $\alpha x \in K$, for all reals $\alpha > 0$. The polar K^* of the cone K in Y is

$$K^* = \{y : y \in Y, \langle x, y \rangle \geq 0, \forall x \in K\}.$$

A collection Λ of subsets of a topological space X is said to have the finite intersection property if for every finite subcollection $\{C_1, C_2, \dots, C_n\}$ of Λ , the intersection $C_1 \cap C_2 \cap \dots \cap C_n$ is nonempty. A topological space

X is compact if and only if for every collection Λ of closed sets in X having the finite intersection property, the intersection $\bigcap_{C \in \Lambda} C$ of all the elements of Λ is nonempty. A real valued function f defined on a convex set X is called quasiconvex if, for each real number t , the set $\{x \in X : f(x) < t\}$ is convex. For a multivalued mapping F from X into Y , the graph of F denoted by $\mathcal{G}(F)$ is the set $\{(x, y) \in X \times Y : y \in F(x)\}$.

Let X and Y be Hausdorff topological vector spaces, K be a closed convex cone in X and K^* be the polar of K in Y . If $F : K \rightarrow 2^Y$ be a multivalued mapping, then the *extended generalized complementarity problem* (EGCP) is to find an $x \in X$ and a $y \in F(x)$ such that

$$x \in K, y \in K^* \text{ and } \langle x, y \rangle = 0. \quad (4)$$

The problem (4) was considered by Sahu et al. [18] in the year of 2021, in a general setting of vector spaces in duality. In this paper we shall find the existence of solutions for the problem (4) in a different approach, that is minimax inequality with a lopsided saddle point developed by Ha [10].

Example 2.1. Let $K = [0, \infty) \times 0$ and $F : K \rightarrow 2^{\mathbb{R}^2}$ be a multivalued map defined by

$$F(x) = \{y \in \mathbb{R}^2 : (x, y) \geq 0\}.$$

Then it is clear that $K^* = [0, \infty) \times \mathbb{R}$ and $F(x) = K^*, \forall x \in K$. For every $x \in K$ we can find a vector $y \in 0 \times \mathbb{R} \subset K^*$ such that $\langle x, y \rangle = 0$.

Remark 2.2. If we take $X = Y = \mathbb{R}^n$ and the bilinear form $\langle \cdot, \cdot \rangle$ as the usual inner product in $\mathbb{R}^n$, then (4) reduces to the complementarity problem considered by Saigal [19]. If F is single valued function, then (4) reduces to the complementarity problem considered by Karamardian [14]. If we take $X = Y = \mathbb{R}^n$, the bilinear form $\langle \cdot, \cdot \rangle$ as the usual inner product in $\mathbb{R}^n$ and F is a single valued function, then (4) reduces to the complementarity problem considered by Habetler and Price [11].

The generalized variational inequality problem associated with the complementarity problem (4) is: Given a closed convex cone K in X and its polar K^* in Y . If $F : K \rightarrow 2^Y$ be a multivalued mapping, then the *generalized variational inequality problem* (GVI) is to find a vector $x \in K$ and a vector $y \in F(x)$ such that

$$\langle u - x, y \rangle \geq 0, \quad \forall u \in K. \quad (5)$$

The equivalence between problem (4) and problem (5) was established by Sahu et al. [18].

Theorem 2.3 ([18]). A vector $x \in K$ is a solution of the extended generalized complementarity problem (4) if and only if x is a solution of the generalized variational inequality problem (5).

Now we recall some basic concepts and results that will be needed in the sequel.

Definition 2.4. Let X and Y be the Hausdorff topological vector spaces. A multivalued mapping $F : X \rightarrow 2^Y$ is said to be

(a) *monotone* if for any $x, y \in X$,

$$\langle y^* - x^*, y - x \rangle \geq 0, \text{ for all } x^* \in F(x) \text{ and } y^* \in F(y);$$

(b) *strictly monotone* if for any $x, y \in X$ with $x \neq y$

$$\langle y^* - x^*, y - x \rangle > 0, \text{ for all } x^* \in F(x) \text{ and } y^* \in F(y);$$

(c) *pseudomonotone*, if for each $x, y \in X$

$$\langle x^*, y - x \rangle \geq 0 \implies \langle y^*, y - x \rangle \geq 0, \text{ for all } x^* \in F(x) \text{ and } y^* \in F(y).$$

Definition 2.5 (Aubin [1]). Let X and Y be two Hausdorff topological vector spaces. A multivalued mapping $F : X \rightarrow 2^Y$ is said to be upper semicontinuous at a point $x_0 \in X$ (for short, u.s.c. at x_0), if and only if, for any open set $O \subset Y$ such that $F(x_0) \subset O$, there exists a neighborhood U of x_0 such that $F(x) \subset O$ for every $x \in U$. We say that F is upper semicontinuous (for short, u.s.c.) if F is u.s.c. for every $x_0 \in X$.

Proposition 2.6 (Aubin [1]). The graph of an u.s.c. set-valued map F with closed values from a Hausdorff topological vector space X to a Hausdorff topological vector space Y is closed. If Y is compact and the graph $\mathcal{G}(F)$ of F is closed, then F is upper semicontinuous.

Ha [10] extended the minimax inequality of Fan [8] and given the following minimax inequality with a lopsided saddle point.

Lemma 2.7 (Ha [10]). Let X and Y be Hausdorff topological vector spaces and $K \subset X$, $L \subset Y$ be the nonempty compact convex subsets. If g is a continuous real valued bifunction defined on $K \times L$, and if F is an upper semicontinuous multivalued map defined on K such that

- (a) For each $y \in L$, $g(x, y)$ is quasiconvex in $x \in K$;
- (b) For each $x \in K$, $F(x)$ is a nonempty closed convex subset of L .

Then, there exist $x_0 \in K$ and $y_0 \in F(x_0)$ satisfying

$$g(x_0, y_0) \leq g(x, y_0), \quad \forall x \in K.$$

In order to achieve our goal, we need the following special forms of the Lemma 2.7. By taking $g(x, y) = \langle x, y \rangle$, we deduce the following result.

Corollary 2.8. Let X and Y be Hausdorff topological vector spaces, $K \subset X$, $L \subset Y$ be nonempty compact convex sets. Assume that the mapping $(u, v) \mapsto \langle u, v \rangle$ is continuous on $K \times L$ and let $F : K \rightarrow 2^Y$ be a multivalued mapping such that

- (i) F is upper semicontinuous on K ;
- (ii) For each $x \in K$, $F(x)$ is a nonempty closed convex subset of L .

Then there exists a point $\bar{x} \in K$ and a $y \in F(\bar{x})$ such that

$$\langle x - \bar{x}, y \rangle \geq 0, \quad \forall x \in K.$$

Further, by taking $X = Y$, $K = L$ and replacing the real valued bifunction $g(u, v)$ by the functional $\|u - v\|$, in Lemma 2.7, we derive the following result.

Corollary 2.9. Let X be a normed space and K be a nonempty compact and convex set in X . If $F : K \rightarrow X$ be a multivalued mapping such that

- (i) F is upper semicontinuous on K ;
- (ii) For each $x \in K$, $F(x)$ is a nonempty closed convex subset of K .

Then, there is a $\bar{x} \in K$ and $y \in F(\bar{x})$ such that

$$\|\bar{x} - y\| \leq \|x - y\|, \quad x \in K.$$

Remark 2.10. If F is a single valued mapping, then Theorem 2 of Fan [7] follows from the Corollary 2.9.

Definition 2.11. [18] Let X be a real reflexive Banach space with its dual X^* and the value of $x^* \in X^*$ at $x \in X$ be $\langle x, x^* \rangle$. Let K be a nonempty closed convex cone in X , and K^* be the polar of K in X^* . A multivalued mapping $F : K \rightarrow 2^{X^*}$ is said to be

- (i) copositive on K , if there is a $z^* \in F(0)$ such that for all $x \in K$ and $y^* \in F(x)$, $\langle x, y^* - z^* \rangle \geq 0$;

- (ii) strictly copositive on K , if there is a $z^* \in F(0)$ such that for all $0 \neq x \in K$ and $y^* \in F(x)$, $\langle x, y^* - z^* \rangle > 0$;
- (iii) strongly copositive on K , if there is a scalar $\alpha > 0$ and a vector $z^* \in F(0)$ such that for all $x \in K$ and $y^* \in F(x)$, $\langle x, y^* - z^* \rangle \geq \alpha \|x\|^2$.

Definition 2.12. Let X be a real reflexive Banach space with its dual X^* and the value of $x^* \in X^*$ at $x \in X$ be $\langle x, x^* \rangle$. Let K be a nonempty closed convex cone in X , and K^* be the polar of K in X^* . A multivalued monotone mapping $F : X \rightarrow 2^{X^*}$ is said to be proper at $x \in K$ if the set

$$B(x) = \{y \in K : y - x \in K, y^* - x^* \in K^* \text{ and } \langle y - x, y^* - x^* \rangle = 0 \text{ for all } x^* \in F(x) \text{ and } y^* \in F(y)\}$$

is bounded.

3. Existence of Solutions for Extended Generalized Complementarity Problems

Our principal aim in this section is to establish an existence of solution for the generalized complementarity problems in the general settings of vector spaces in duality.

Theorem 3.1. Let X and Y be Hausdorff topological vector spaces and K and L be closed convex cones of X and Y respectively. Assume that the mapping $(u, v) \mapsto \langle u, v \rangle$ is continuous on $K \times L$ and let $F : K \rightarrow 2^Y$ be a multivalued mapping such that

- (i) F is upper semicontinuous on K ;
- (ii) For each nonempty compact convex set C of K , there exists a nonempty compact convex set E of L , such that for each $x \in C$, $F(x)$ is nonempty closed convex subset of E ;
- (iii) There exists a nonempty compact and convex subset D of K , with the property that for every $x \in K \setminus D$, there is $z \in D$ such that

$$\langle x - z, y \rangle > 0, \quad \forall y \in F(x). \quad (6)$$

Then, the extended generalized complementarity problem (4) has a solution.

Proof. For every $u \in K$, consider the set

$$D_u = \{x \in D : \exists y \in F(x) \text{ such that } \langle u - x, y \rangle \geq 0\}.$$

Since F is upper semicontinuous on K and the mapping $(u, v) \mapsto \langle u, v \rangle$ is continuous on $K \times L$ by using condition (ii), we conclude from Lemma 3 of Sahu et al. [18] that the set D_u is closed in D for all $u \in K$. Now we wish to prove that the intersection of any finite number of the D_u 's is nonempty. That is, we want to prove that for arbitrary $u_1, u_2, \dots, u_m \in K$, $\bigcap_{i=1}^m D_{u_i} \neq \emptyset$. Let $\hat{D}$ be the convex closure of $D \cup \{u_1, u_2, \dots, u_m\}$. Since $\hat{D}$ is nonempty compact and convex subset in K . By Corollary 2.8, there exists a point $\hat{x} \in \hat{D}$ and a $y \in F(\hat{x})$ such that

$$\langle x - \hat{x}, y \rangle \geq 0, \quad \forall x \in \hat{D}. \quad (7)$$

In particular,

$$\langle u_i - \hat{x}, y \rangle \geq 0, \text{ for } i = 1, 2, \dots, m.$$

From the construction of D_u , it remains only to show that $\hat{x} \in D$. Suppose $\hat{x} \notin D$, but as $\hat{x} \in K \setminus D$, from relation (6) there exists an element $z \in D$ such that

$$\langle z - \hat{x}, y \rangle < 0, \quad \forall y \in F(\hat{x}),$$

which contradicts (7). Therefore, $\hat{x} \in D$ and $\bigcap_{i=1}^m D_{u_i} \neq \emptyset$. Hence by finite intersection property of compact sets, we have $\bigcap_{u \in K} D_u \neq \emptyset$. Thus, there exists an element $\bar{x} \in D$ and an element $y \in F(\bar{x})$ such that

$$\langle x - \bar{x}, y \rangle \geq 0, \quad \forall x \in K.$$

Therefore, $\bar{x}$ is a solution of generalized variational inequality problem (5) and Theorem 2.3, forces that $\bar{x}$ is a solution of extended generalized complementarity problem (4). $\square$

Remark 3.2. Theorem 3.1 improves Theorem 2 of Sahu et al. [18] in the way that the condition (ii) of Theorem 2 in [18] which is weaker than the pseudomonotonicity in the sense of Brézis have been avoided. Further, Theorem 3.1 generalizes Theorem 3.1 of Karamardian [14] from single valued case to multivalued case and Theorem 2.1 of Saigal [19] to the context of Hausdorff topological spaces.

4. Generalized Complementarity Problems in Reflexive Banach Spaces

In this section, we shall assume that X is a reflexive Banach space and X^* be its topological dual. Let the value of $x^* \in X^*$ at $x \in X$ be $\langle x, x^* \rangle$. Let K be a nonempty closed convex cone in X , and K^* be the polar of K in X^* that is

$$K^* = \{y^* \in X^*, \langle x, y^* \rangle \geq 0, \forall x \in K\}.$$

If $F : K \rightarrow 2^{X^*}$ be a multivalued mapping, then the complementarity problem (CP) in Banach spaces is to find an $x \in K$ and a $y^* \in F(x)$ such that

$$x \in K, y^* \in K^*, \text{ and } \langle x, y^* \rangle = 0. \quad (8)$$

Let $J : X \rightarrow 2^{X^*}$ be the duality mapping defined by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 \text{ and } \|x^*\| = \|x\|\}.$$

Clearly for any $x \in X$, $J(x) \neq \emptyset$ and since X and X^* are Banach spaces, the duality mapping J is single-valued and continuous. For $\varepsilon > 0$, let $F_\varepsilon : K \rightarrow 2^{X^*}$ be the multivalued mapping defined by $F_\varepsilon(x) := F(x) + \varepsilon J(x)$, for $x \in K$. For $\varepsilon > 0$, the regularized complementarity problem $(EGCP)_\varepsilon$ in X defined by Sahu et al. [18] is to find a vector $x_\varepsilon \in K$ and vector $y_\varepsilon^* \in F_\varepsilon(x)$ such that

$$x_\varepsilon \in K, y_\varepsilon^* \in K^* \text{ and } \langle x_\varepsilon, y_\varepsilon^* \rangle = 0. \quad (9)$$

Theorem 4.1. Let X be a reflexive Banach space and X^* be its topological dual. Let K and L be closed convex cones of X and X^* respectively and $F : K \rightarrow 2^{X^*}$ be a strongly copositive multivalued mapping such that

- (i) F is upper semicontinuous on K from weak topology $\sigma(X, X^*)$ of X to the weak topology $\sigma(X^*, X)$ of X^* ;
- (ii) For each nonempty compact convex set C of K , there exists a nonempty compact convex set E of L , such that for each $x \in C$, $F(x)$ is nonempty closed convex subset of E .

Then, the generalized complementarity problem (8) has a solution.

Proof. If we take $Y = X^*$, where X and X^* are endowed with the weak topology then the condition (i) and condition (ii) of Theorem 3.1 are satisfied. Since F is strongly copositive, by applying the similar procedure as in the proof of Theorem 4 of [18], we deduce that condition (iii) is satisfied with $z = 0$. Therefore from Theorem 3.1, x is a solution of the generalized complementarity problem (8). $\square$

If F is copositive instead of strongly copositive, we have the following result on the existence of solutions for the regularized complementarity problem (9).

Theorem 4.2. Let X be a reflexive Banach space and X^* be its topological dual. Let K and L be closed convex cones of X and X^* respectively and $F : K \rightarrow 2^{X^*}$ be a copositive multivalued mapping such that

- (i) F is upper semicontinuous on K from weak topology $\sigma(X, X^*)$ of X to the weak topology $\sigma(X^*, X)$ of X^* ;
- (ii) For each nonempty compact convex set C of K , there exists a nonempty compact convex set E of L , such that for each $x \in C$, $F(x)$ is nonempty closed convex subset of E .

Then, the regularized complementarity problem (9) has a solution.

Proof. Since F is upper semicontinuous on K from weak topology $\sigma(X, X^*)$ of X to the weak topology $\sigma(X^*, X)$ of X^* and duality mapping J is continuous, we conclude that the multi-valued mapping F_ε is upper semicontinuous on K from weak topology $\sigma(X, X^*)$ of X to the weak topology $\sigma(X^*, X)$ of X^* . Since J is single-valued, condition (ii) holds for F_ε . Since F is copositive on K , there is a $z^* \in F(0)$ such that for all $x \in K$ and $y^* \in F(x)$, we have

$$\langle x, y^* - z^* \rangle \geq 0.$$

We can rewrite the above expression as

$$\langle x, y^* - z^* \rangle + \varepsilon \langle x, J(x) - J(0) \rangle \geq \varepsilon \langle x, J(x) - J(0) \rangle. \quad (10)$$

Since $\langle \cdot, \cdot \rangle$ is linear in second argument, from relation (10) we have

$$\langle x, y^* - z^* \rangle + \langle x, \varepsilon J(x) - \varepsilon J(0) \rangle \geq \varepsilon \langle x, J(x) - J(0) \rangle.$$

Again by linearity of $\langle \cdot, \cdot \rangle$ in second argument, we have

$$\langle x, y^* + \varepsilon J(x) - (z^* + \varepsilon J(0)) \rangle \geq \varepsilon \langle x, J(x) - J(0) \rangle.$$

Since $J(0) = 0$ and $\langle x, J(x) \rangle = \|x\|^2$ by the definition of J , we have

$$\langle x, y^* + \varepsilon J(x) - (z^* + \varepsilon J(0)) \rangle \geq \varepsilon \|x\|^2. \quad (11)$$

Since J is single valued, $y^* + \varepsilon J(x) \in F(x) + \varepsilon J(x)$ and $z^* + \varepsilon J(0) \in F(0) + \varepsilon J(0)$. Taking $y^* + \varepsilon J(x) = y_\varepsilon^*$ and $z^* + \varepsilon J(0) = z_\varepsilon^*$, from relation (11), we have

$$\langle x, y_\varepsilon^* - z_\varepsilon^* \rangle \geq \varepsilon \|x\|^2.$$

Thus, F_ε is strongly copositive on K with constant ε and therefore, the result follows from Theorem 4.1. $\square$

Theorem 4.3. Let X be a reflexive Banach space and X^* be its topological dual. Let K and L be closed convex cones of X and X^* respectively and $F : K \rightarrow 2^{X^*}$ be a multivalued mapping such that

- (i) F is upper semicontinuous on K from weak topology $\sigma(X, X^*)$ of X to the weak topology $\sigma(X^*, X)$ of X^* ;
- (ii) For each nonempty compact convex set C of K , there exists a nonempty compact convex set E of L , such that for each $x \in C$, $F(x)$ is nonempty closed convex subset of E ;
- (iii) There exists a nonempty compact and convex subset D of K , with the property that for every $\varepsilon > 0$ and for every $x \in K \setminus D$, there is $z \in D$ such that

$$\langle x - z, y \rangle + \varepsilon \langle x - z, J(x) \rangle > 0, \quad \forall y \in F(x). \quad (12)$$

Then, the regularized complementarity problem (9) has a solution.

Proof. Since F is upper semicontinuous on K from weak topology $\sigma(X, X^*)$ of X to the weak topology $\sigma(X^*, X)$ of X^* and duality mapping J is continuous, we conclude that the multi-valued mapping F_ε is upper semicontinuous on K from weak topology $\sigma(X, X^*)$ of X to the weak topology $\sigma(X^*, X)$ of X^* and thus the condition (i) of Theorem 3.1 is satisfied for F_ε . Since J is single-valued, condition (ii) of Theorem 3.1 also holds for F_ε . Finally from (iii), we see that condition (iii) of Theorem 3.1 holds for F_ε and therefore the conclusion follows from Theorem 3.1.

$\square$

5. Generalized Complementarity Problems in Finite Dimensional Normed Spaces

In this section we shall assume that X be a finite dimensional normed space and X^* be its topological dual. Let the value of $x^* \in X^*$ at $x \in X$ be denoted by $\langle x, x^* \rangle$. Let K be a nonempty closed convex cone in X , and K^* be the polar of K in X^* . Let F be a multi-valued mapping from K into X^* , then the complementarity problem in finite dimensional normed spaces is to find an $x \in K$ and an $x^* \in F(x)$ such that

$$x \in K, \quad x^* \in K^*, \quad \text{and} \quad \langle x, x^* \rangle = 0. \quad (13)$$

A point $x \in K$ is said to be feasible to the problem (13) if $x \in K$ and $F(x) \cap K^* \neq \emptyset$ and strict feasible to (13) if $x \in K$ and $F(x) \cap \text{int}(K^*) \neq \emptyset$.

The following lemma is a generalization of Lemma 2.1 of Karamardian [15], whose proof is similar to that of Lemma 2.1 of Karamardian [15] and therefore omitted.

Lemma 5.1. *Let K be a closed pointed solid and convex cone in X and $x^* \in X^*$ be any element. Then,*

- (a) $x^* \in \text{int}(K^*)$ if and only if $\langle x, x^* \rangle > 0$ for all $0 \neq x \in K$;
- (b) $x^* \in \text{int}(K^*)$ if and only if for any real number $r > 0$, the set $D = \{x : x \in K, 0 \leq \langle x, x^* \rangle \leq r\}$ is compact.

Theorem 5.2. *Let X be a finite dimensional normed space and X^* be its topological dual. Let K and L be closed convex cones of X and X^* respectively and $F : K \rightarrow 2^{X^*}$ be a multivalued mapping such that*

- (i) F is upper semicontinuous and pseudomonotone on K from weak topology $\sigma(X, X^*)$ of X to the weak topology $\sigma(X^*, X)$ of X^* ;
- (ii) For each nonempty compact convex set C of K , there exists a nonempty compact convex set E of L , such that for each $x \in C$, $F(x)$ is nonempty closed convex subset of E .

If there exists a strict feasible point $x \in K$ to (13), then the problem (13) has a solution.

Proof. Since $x \in K$ is the strict feasible point to the problem (13), we have $F(x) \cap \text{int}(K^*) \neq \emptyset$. Let $x^* \in F(x) \cap \text{int}(K^*)$. Consider the set

$$D = \{y \in K : \langle y - x, x^* \rangle \leq 0\}.$$

If $\langle x, x^* \rangle = 0$, then x is the solution of problem (13). Therefore, we assume that $\langle x, x^* \rangle > 0$. Since $x^* \in \text{int}(K^*)$ by Lemma 5.1, D is compact. Let $x \in K \setminus D$, then we have

$$\langle y - x, \xi \rangle > 0, \quad \text{for all } \xi \in F(x). \quad (14)$$

Thus assumption (iii) of Theorem 3.1 is satisfied. Therefore, the results follows from Theorem 3.1.

□

Remark 5.3. *Theorem 5.2 generalizes Theorem 12.5 of Yao and Chadli [20] and Theorem 3.2 of Saigal [19].*

Theorem 5.4. *Let X be a finite dimensional normed space and X^* be its topological dual. Let K and L be closed convex cones of X and X^* respectively and $F : K \rightarrow 2^{X^*}$ be a multivalued mapping such that*

- (i) F is upper semicontinuous and monotone on K from weak topology $\sigma(X, X^*)$ of X to the weak topology $\sigma(X^*, X)$ of X^* ;
- (ii) For each nonempty compact convex set C of K , there exists a nonempty compact convex set E of L , such that for each $x \in C$, $F(x)$ is nonempty closed convex subset of E .

If there exists a feasible point $x \in K$ to (13) at which F is proper, then the problem (13) has a solution.

Proof. Since $x \in K$ is the feasible point to the problem (13) at which F is proper, we have $F(x) \cap K^* \neq \emptyset$. Let $x^* \in F(x) \cap K^*$ and for $z^* \in \text{int}(K^*)$ and for every real $r > 0$, consider the set

$$D_r = \{y \in K : y - x \in K, \langle y - x, z^* \rangle = r\}.$$

Clearly D_r is nonempty and convex. Since $z^* \in \text{int}(K^*)$ and by Lemma 5.1, D_r is compact. Since the multivalued mapping $F(y) - x^*$ is upper semicontinuous on D_r for all $r > 0$, by Corollary 2.8, for all $r > 0$ there exists an $x_r \in D_r$ and an $x_r^* \in F(x_r)$ such that

$$\langle y - x_r, x_r^* - x^* \rangle \geq 0, \text{ for all } y \in D_r.$$

Which can be written as

$$\langle y - x - (x_r - x), x_r^* - x^* \rangle \geq 0, \text{ for all } y \in D_r.$$

Since $\langle \cdot, \cdot \rangle$ is linear in first argument, we have

$$\langle y - x, x_r^* - x^* \rangle - \langle x_r - x, x_r^* - x^* \rangle \geq 0, \text{ for all } y \in D_r.$$

Which can be written as

$$\langle y - x, x_r^* - x^* \rangle \geq \langle x_r - x, x_r^* - x^* \rangle, \text{ for all } y \in D_r. \quad (15)$$

Since F is monotone, we have

$$\langle x_r - x, x_r^* - x^* \rangle \geq 0. \quad (16)$$

From relations (15) and (16), we deduce that

$$\langle y - x, x_r^* - x^* \rangle \geq 0, \text{ for all } y \in D_r.$$

Thus, $x_r^* - x^* \in K^*$ and the properness of F at x implies that there exists an $\bar{r} > 0$ and an $x_{\bar{r}}^* \in F(x_{\bar{r}})$, such that

$$\langle y - x, x_{\bar{r}}^* - x^* \rangle \geq \langle x_{\bar{r}} - x, x_{\bar{r}}^* - x^* \rangle > 0, \text{ for all } y \in D_r.$$

By Lemma 5.1, $x_{\bar{r}}^* - x^* \in \text{int}(K^*)$ and therefore the conclusion follows from Theorem 5.2. $\square$

Corollary 5.5. Let X be a finite dimensional normed space and X^* be its topological dual. Let K and L be closed convex cones of X and X^* respectively and $F : K \rightarrow 2^{X^*}$ be a multivalued mapping such that

- (i) F is upper semicontinuous and strictly monotone on K from weak topology $\sigma(X, X^*)$ of X to the weak topology $\sigma(X^*, X)$ of X^* ;
- (ii) For each nonempty compact convex set C of K , there exists a nonempty compact convex set E of L , such that for each $x \in C$, $F(x)$ is nonempty closed convex subset of E .

If there exists a feasible point $x \in K$ to (13), then the problem (13) has a unique solution.

Proof. Since F is strictly monotone, $B(x) = \{x\}$, for all $x \in K$ and therefore, F is proper at any $x \in K$. Thus, from Theorem 5.4 the problem (13) has a solution. To prove uniqueness, let there are two distinct solutions x_1 and x_2 . Then, for all $x_1^* \in F(x_1)$ and $x_2^* \in F(x_2)$, we have

$$0 < \langle x_1 - x_2, x_1^* - x_2^* \rangle = -\langle x_1, x_2^* \rangle - \langle x_2, x_1^* \rangle \leq 0,$$

which is a contradiction and hence the solution is unique.

$\square$

Remark 5.6. Theorems 5.2, 5.4 and Corollary 5.5 generalize respectively Theorems 4.1, 4.2 and Corollary 4.1 of Karamardian [15].

Conclusion

We have established some existence of solutions for the extended generalized complementarity problems in the context of topological vector spaces using the minimax inequality with a lopsided saddle point. We then proved some existence results for generalized complementarity problems in reflexive Banach spaces with the lack of coercivity assumptions using the notion of copositivity. Further, we used the concept of feasible and strictly feasible points and derived some existence of solutions for the generalized complementarity problems in the finite dimensional normed spaces. Our future aim is to investigate, under what more assumptions, the results that we have established in this paper in the finite dimensional normed spaces context to the context of general reflexive Banach spaces.

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