Filomat 39:20 (2025), 7161–7172
https://doi.org/10.2298/FIL2520161P

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Polynomial extension of the stronger Central Sets Theorem near zero

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Abstract. Furstenberg introduced the Central Sets Theorem in 1981. Later, in 1990, Bergelson and Hindman provided an algebraic proof of the Central Sets Theorem. In 2008, De, Hindman and Strauss proved a stronger version of Central Sets Theorem. On the other hand Hindman and Leader introduced in 1998, the Central Sets Theorem near zero for dense subsemigroup of $\mathbb{R}$. A stronger version of the Central Sets Theorem near zero was established by De and Hindman in [6]. Recently Goswami, Baglini and Patra developed a polynomial extension of the stronger Central Sets Theorem in 2023. In this article, we provide a polynomial version of the stronger Central Sets Theorem near zero.

1. Introduction and preliminaries

In [8] Furstenberg introduced the notion of central sets in $(\mathbb{N},+)$ in terms of topological dynamics. (Specifically, a subset A of $\mathbb{N}$ is central if and only if there exists a dynamical system (X,T), points x and y in X, and a neighborhood U of y such that y is uniformly recurrent, x and y are proximal, and $A=\{n\in\mathbb{N}:T_n(x)\in U\}$. (See [8] for the definitions of "dynamical system", "proximal"and "uniformly recurrent") He showed that if $\mathbb{N}$ is divided into finitely many classes, then one of them must be central, and he proved the following theorem.

Theorem 1.1. *Let $l\in\mathbb{N}$ and for each $i\in\{1,2,...,l\}$, let $(y_{i,n})_{n=1}^{\infty}$ be l-many sequences in $\mathbb{Z}$. Let $C\subseteq\mathbb{N}$ be a central set. Then there exist sequences $(a_n)_{n=1}^{\infty}$ in $\mathbb{N}$ and $(H_n)_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that*
(1) *for all n, $\max H_n<\min H_{n+1}$ and*
(2) *for all $F\in\mathcal{P}_f(\mathbb{N})$ and all $i\in\{1,2,...,l\}$,*

$$\sum_{n\in F}\left(a_n+\sum_{t\in H_n}y_{i,t}\right)\in C.$$

There are many extensions of this theorem in the literature in different directions. In one direction Bergelson, Moreira and Johnson [2] established a polynomial version of the above theorem.

2020 *Mathematics Subject Classification*. Primary 05D10; Secondary 22A15.
Keywords. Stone-Čech Compactification, Central sets theorem, Ultrafilter near zero.
Received: 13 December 2024; Revised: 17 May 2025; Accepted: 04 June 2025
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Theorem 1.2. *Let G be a countable abelian group, let $j \in \mathbb{N}$ and let $(y_\alpha)_{\alpha\in\mathcal{F}}$ be an IP-set in $\mathbb{Z}^j$. Let $F \subset P(\mathbb{Z}^j, \mathbb{Z})$ and $A \subset \mathbb{Z}$ be a central set in $\mathbb{Z}$. Then there exist an IP-set $(x_\alpha)_{\alpha\in\mathcal{F}}$ in $\mathbb{Z}$ and a sub-IP-set $(z_\alpha)_{\alpha\in\mathcal{F}}$ of $(y_\alpha)_{\alpha\in\mathcal{F}}$ such that*

$$\forall f \in F \qquad \forall \beta \in \mathcal{F} \qquad x_\beta + f(z_\beta) \in A.$$

Both theorems have natural generalizations over arbitrary countable commutative groups. In fact, Hindman, Maleki and Strauss [13] extended the theorem 1.1 for arbitrary semigroup considering countably many sequences at a time. Further De, Hindman and Strauss [7] extended theorem 1.1 considering arbitrary many sequences at a time. In this article, we present only the commutative version.

Theorem 1.3. *[7, Theorem 2.2] Let $(S, +)$ be a commutative semigroup and let C be a central subset of S. Then there exist functions $\alpha : \mathcal{P}_f\left(S^{\mathbb{N}}\right) \to S$ and $H : \mathcal{P}_f\left(S^{\mathbb{N}}\right) \to \mathcal{P}_f(\mathbb{N})$ such that*

(1) *If $F, G \in \mathcal{P}_f\left(S^{\mathbb{N}}\right)$ and $F \subsetneq G$ then* $\max H(F) < \min H(G)$ *and*

(2) *If $m \in \mathbb{N}, G_1, G_2, ..., G_m \in \mathcal{P}_f\left(S^{\mathbb{N}}\right)$;$G_1 \subsetneq G_2 \subsetneq .... \subsetneq G_m$; and for each $i \in \{1, 2, ..., m\}, (y_{i,n}) \in G_i$, then*

$$\sum_{i=1}^{m}\left(\alpha(G_i) + \sum_{t\in H(G_i)} y_{i,t}\right) \in C.$$

A very natural question arises here: does there exist a polynomial generalization of Theorem 1.3 in the direction of Theorem 1.2. In [9] Goswami, Baglini and Patra answered this question affirmatively.

Theorem 1.4. *[9, Theorem 11] Let A be a central set and $T \in \mathcal{P}_f(\mathbb{P})$. Then there exist functions $\alpha : \mathcal{P}_f\left(\mathbb{N}^{\mathbb{N}}\right) \to \mathbb{N}$ and $H : \mathcal{P}_f\left(\mathbb{N}^{\mathbb{N}}\right) \to \mathcal{P}_f(\mathbb{N})$, such that*

(1) *If $F \subsetneq G$ are in $\mathcal{P}_f\left(\mathbb{N}^{\mathbb{N}}\right)$, then $H(F) < H(G)$ and*

(2) *For any $n \in \mathbb{N}$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_n$ in $\mathcal{P}_f\left(\mathbb{N}^{\mathbb{N}}\right)$, we have for each $i \in \{1, 2, ..., n\}, f_i \in G_i$, and for all $P \in T$,*

$$\sum_{i=1}^{n}\alpha(G_i) + P\left(\sum_{i=1}^{n}\sum_{t\in H(G_i)} f_i(t)\right) \in A.$$

Another direction of the Central Sets Theorem was due to Hindman and Leader [12]. The authors introduced the notion of central set near zero for dense subsemigroups of $(\mathbb{R}, +)$. In fact, where as central sets live at infinity, central sets near zero live near zero and satisfy conclusions similar to those of central sets.

Theorem 1.5. *Let S be a dense subsemigroup of $(\mathbb{R}, +)$. For each $i \in \mathbb{N}$, let $(y_{i,n})_{n=1}^{\infty}$ be a sequence in S converging to zero in the usual topology of $\mathbb{R}$. Let C be a central set near zero in S. Then there exist sequences $(a_n)_{n=1}^{\infty}$ in S converging to zero in the usual topology of $\mathbb{R}$, and a sequence $(H_n)_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that*

(1) for all n, $\max H_n < \min H_{n+1}$ and

(2) for all $F \in \mathcal{P}_f(\mathbb{N})$ and all $i \in \{1, 2, ..., l\}$,

$$\sum_{n\in F}\left(a_n + \sum_{t\in H_n} y_{i,t}\right) \in C.$$

Proof. [12, Theorem 4.11]. □

The polynomial generalization of the above theorem was established in [5].

Theorem 1.6. *Let $(S,+)$ be a dense subsemigroup of $(\mathbb{R},+)$ containing 0 such that $(S\cap(0,1),\cdot)$ is a subsemigroup of $((0,1),\cdot)$. For each $i\in\mathbb{N}$, let $(y_{i,n})_{n=1}^{\infty}$ be a sequence in S converging to zero in the usual topology of $\mathbb{R}$. Let A be a central set near zero in S and $L\in\mathcal{P}_f(\mathbb{P}(S,S))$. Then for any $\delta>0$, there exist sequence $(a_n)_{n=1}^{\infty}$ in $S\cap(0,\delta)$ converging to zero in the usual topology of $\mathbb{R}$ and a sequence $(H_n)_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that*

(1) *for all n, $\max H_n<\min H_{n+1}$ and*

(2) *for all $F\in\mathcal{P}_f(\mathbb{N})$ and all $i\in\{1,2,...,l\}$,*

$$\left(\sum_{n\in F}a_n+P\left(\sum_{n\in F}\sum_{t\in H_n}y_{i,t}\right)\right)\in A.$$

for all $P\in L$.

In the present article, our aim is to provide a stronger form of the above theorem. The notation C_p^0 used in the following theorem will be defined in the second section.

Theorem 1.7. *Let $(S,+)$ be a dense subsemigroup of $(\mathbb{R},+)$ containing 0 such that $(S\cap(0,1),\cdot)$ is a subsemigroup of $((0,1),\cdot)$. Let A be a C_p^0 set in S and $L\in\mathcal{P}_f(\mathbb{P}(S,S))$. Then for each $\delta\in(0,1)$, there exist functions $\alpha_\delta:\mathcal{P}_f(\mathcal{T}_0)\to S$ and $H_\delta:\mathcal{P}_f(\mathcal{T}_0)\to\mathcal{P}_f(\mathbb{N})$ such that*

1. *$\alpha_\delta(F)<\delta$ for each $F\in\mathcal{P}_f(\mathcal{T}_0)$,*
2. *If $F,G\in\mathcal{P}_f(\mathcal{T}_0)$, $F\subsetneq G$, then $\max H_\delta(F)<\min H_\delta(G)$ and*
3. *If $n\in\mathbb{N}$ and $G_1,G_2,...,G_n\in\mathcal{P}_f(\mathcal{T}_0)$, $G_1\subsetneq G_2\subsetneq.....\subsetneq G_n$. For each $f_i\in G_i$, $i=1,2,...,n$, we have*

$$\sum_{i\in 1}^{n}\alpha_\delta(G_i)+P\left(\sum_{i\in 1}^{n}\sum_{t\in H_\delta(G_i)}f_i(t)\right)\in A.$$

for all $P\in L$, Where

$$\mathcal{T}_0=\left\{(x_n)_{n\in\mathbb{N}}\in S^{\mathbb{N}}: x_n \text{ converges to zero in usual topology of } \mathbb{R}\right\}.$$

Let $(S,\cdot)$ be any discrete semigroup and βS be the set of all ultrafilters on S, where the points of S are identified with the principal ultrafilters. Then $\left\{\overline{A}:A\subseteq S\right\}$, where $\overline{A}=\{p\in\beta S:A\in p\}$ forms a closed basis for the topology on βS. With this topology βS becomes a compact Hausdorff space in which S is dense, called the Stone-Čech compactification of S. The operation of S can be extended to βS making $(\beta S,\cdot)$ a compact, right topological semigroup with S contained in its topological center. That is, for all $p\in\beta S$ the function $\rho_p:\beta S\to\beta S$ is continuous, where $\rho_p(q)=q\cdot p$ and for all $x\in S$, the function $\lambda_x:\beta S\to\beta S$ is continuous, where $\lambda_x(q)=x\cdot q$. For $p,q\in\beta S$ and $A\subseteq S$, $A\in p\cdot q$ if and only if $\left\{x\in S:x^{-1}A\in q\right\}\in p$, where $x^{-1}A=\{y\in S:x\cdot y\in A\}$. One can see [14] for an elementary introduction to the semigroup $(\beta S,\cdot)$ and its combinatorial applications. An element $p\in\beta S$ is called idempotent if $p\cdot p=p$. A subset $A\subseteq S$ is called central if and only if A is an element of an idempotent ultrafilter p.

Here we will work with dense subsemigroups $((0,1),\cdot)$, in this case one can define

$$0^+=\bigcap_{\epsilon>0}cl_{\beta(0,1)_d}(0,\epsilon).$$

0^+ is two sided ideal of $(\beta(0,1)_d,\cdot)$, so contains the smallest ideal. It is also a subsemigroup of $(\beta\mathbb{R}_d,+)$. As a compact right topological semigroup, 0^+ has a smallest two sided ideal. $K(0^+)$ denotes the smallest ideal contained in 0^+. Central sets near zero are the elements from the idempotent in $K(0^+)$.

In the case of a commutative semigroup S, a set $A\subseteq S$ is called J-set if for every $H\in\mathcal{P}_f\left(S^{\mathbb{N}}\right)$, there exist $a\in S$ and $\beta\in\mathcal{P}_f(\mathbb{N})$ such that for all $f\in H$,

$$a+\sum_{t\in\beta}f(t)\in A.$$

.

Here, we present the polynomial version of the J-set, named as J_p-set.

Definition 1.8. *[9, definition 3] Let $A \subseteq \mathbb{N}$. Then A is called a J_p-set if and only if for every $F \in \mathcal{P}_f(\mathbb{P})$ and every $H \in \mathcal{P}_f\left(\mathbb{N}^{\mathbb{N}}\right)$, there exist $a \in \mathbb{N}$ and $\beta \in \mathcal{P}_f(\mathbb{N})$ such that for all $P \in F$ and all $f \in H$,*

$$a + P\left(\sum_{t\in\beta} f(t)\right) \in A.$$

Theorem 1.9. *Let $l, m \in \mathbb{N}$, and $A \subseteq \mathbb{N}$ be a J_p-set. For each $i \in \{1, 2, ..., l\}$, let $\left(x^i_\alpha\right)_{\alpha\in\mathcal{P}_f(\mathbb{N})}$ be an IP-set in $\mathbb{N}$. Then for all finite $F \in \mathcal{P}_f(\mathbb{P})$, there exist $a \in \mathbb{N}$ and $\beta \in \mathcal{P}_f(\mathbb{N})$ such that $\min\beta > m$. We have*

$$a + P\left(x^i_\beta\right) \in A$$

for all $i \in \{1, 2, ..., l\}$ and $P \in F$.

Proof. [9, Lemma 10]. □

2. Main results

Let $(S, +)$ be a dense subsemigroup of $(\mathbb{R}, +)$ containing 0 such that $(S \cap (0, 1), \cdot)$ is a subsemigroup of $((0, 1), \cdot)$. In our work, we consider the set of polynomials $\mathbb{P}(S, S)$ from S to S, whose coefficients are in $\mathbb{Z}$ and $f(0) = 0$ for all $f \in \mathbb{P}$. In the following definition $\mathcal{T}_0$ denotes the set of all sequences in S that converge to zero.

Definition 2.1. *Let $(S, +)$ be a dense subsemigroup of $(\mathbb{R}, +)$ containing 0 such that $(S \cap (0, 1), \cdot)$ is a subsemigroup of $((0, 1), \cdot)$ and let $A \subseteq S$. Then A is a J-set near zero if and only if whenever $F \in P_f(\mathcal{T}_0)$ and $\delta > 0$, there exist $a \in S \cap (0, \delta)$ and $H \in P_f(\mathbb{N})$ such that for each $f \in F$,*

$$a + \sum_{t\in H} f(t) \in A.$$

The following is the polynomial version of J-sets near zero. We call it J_p-set near zero.

Definition 2.2. *Let $(S, +)$ be a dense subsemigroup of $(\mathbb{R}, +)$ containing 0 such that $(S \cap (0, 1), \cdot)$ is a subsemigroup of $((0, 1), \cdot)$. A set $A \subseteq S$ is called a J_p-set near zero whenever $F \in \mathcal{P}_f(\mathbb{P}(S, S)), H \in P_f(\mathcal{T}_0)$ and $\delta > 0$, there exist $a \in S \cap (0, \delta)$ and $\beta \in P_f(\mathbb{N})$ such that for each $f \in H$ and all $P \in F$,*

$$a + P\left(\sum_{t\in\beta} f(t)\right) \in A.$$

We can choose $\beta \in P_f(\mathbb{N})$ in the above definition to be greater than any presumed positive integers.

Lemma 2.3. *Let $(S, +)$ be a dense subsemigroup of $(\mathbb{R}, +)$ containing 0 such that $(S \cap (0, 1), \cdot)$ is a subsemigroup of $((0, 1), \cdot)$. Let $m \in \mathbb{N}$, and $A \subseteq S$ be a J_p-set near zero. Then for each $F \in \mathcal{P}_f(\mathbb{P}(S, S)), H \in P_f(\mathcal{T}_0)$ and $\delta > 0$, there exist $a \in S \cap (0, \delta)$ and $\beta \in P_f(\mathbb{N})$ with $\min\beta > m$, we have*

$$a + P\left(\sum_{t\in\beta} f(t)\right) \in A$$

for each $f \in H$ and for all $P \in F$.

Proof. Let $m \in \mathbb{N}$, $F \in \mathcal{P}_f(\mathbb{P}(S,S))$, $H \in P_f(\mathcal{T}_0)$ and $\delta > 0$, for each $f \in H$ define $g_f \in P_f(\mathcal{T}_0)$ by $g_f(t) = f(t+m)$, $t \in \mathbb{N}$. For this $K = \left\{g_f : f \in H\right\} \in P_f(\mathcal{T}_0)$, pick $a \in S \cap (0,\delta)$ and $\gamma \in P_f(\mathbb{N})$ such that

$$a + P\left(\sum_{t\in\gamma} g_f(t)\right) \in A$$

for each $f \in H$ and all $P \in F$. And let $\beta = m + \gamma$, we get our desire result. □

In the following discussion we shall continue to consider $(S,+)$ as a dense subsemigroup of $(\mathbb{R},+)$ containing 0 such that $(S \cap (0,1), \cdot)$ is a subsemigroup of $((0,1), \cdot)$. Let $(S_i)_{i=1}^{\infty}$ be a sequence in S converging to zero and $(S_\alpha)_{\alpha\in\mathcal{P}_f(\mathbb{N})}$ be the IP-set generated by $(S_i)_{i=1}^{\infty}$, where $S_\alpha = \sum_{i\in\alpha} S_i$, $\alpha \in \mathcal{P}_f(\mathbb{N})$. The following is an alternative version of the J_p-set near zero.

Definition 2.4. *Let $l \in \mathbb{N}$, $A \subseteq S$ be a J_p-set near zero if for each $i \in \{1,2,....,l\}$ and $\left(x_\alpha^i\right)_{\alpha\in\mathcal{P}_f(\mathbb{N})}$ an IP-set in $S \cap (0,1)$, we have for any finite $F \in \mathcal{P}_f(\mathbb{P}(S,S))$, there exist $a \in S$ and $\beta \in \mathcal{P}_f(\mathbb{N})$ such that*

$$a + P\left(x_\beta^i\right) \in A$$

for all $i \in \{1,2,....,l\}$ and $P \in F$.

Let us denote by $\mathcal{J}_p^0$ the set of all ultrafilters, whose members are J_p- set near zero, i.e,

$$\mathcal{J}_p^0 = \left\{p \in \beta S_d : \text{for all } A \in p, A \text{ is a } J_p\text{-set near zero}\right\}.$$

We shall denote by $E(\mathcal{J}_p^0)$ the set of all idempotents in $\mathcal{J}_p^0$. The following theorem shows that $\mathcal{J}_p^0$ is in fact non empty. The authors gratefully acknowledge Dr. Sayan Goswami for providing the proof of the following theorem.

Before we proceed to prove $\mathcal{J}_p^0 \neq \emptyset$, we need to recall the Polynomial Hales-Jewett Theorem.

2.1. Revisiting Polynomial Hales-Jewett Theorem

Now we pause to recall the Hales-Jewett theorem and it's polynomial extension. Let $\omega = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of positive integers. Given a nonempty set $\mathbb{A}$ called alphabet, a finite word is an expression of the form $w = a_1a_2\ldots a_n$ with $n \geq 1$ and $a_i \in \mathbb{A}$. The quantity n is called the length of w and denoted $|w|$. Let v (a variable) be a letter not belonging to $\mathbb{A}$. By a variable word over $\mathbb{A}$ we mean a word w over $\mathbb{A} \cup \{v\}$ that has at least one occurrence of v. For any variable word w, $w(a)$ is the result of replacing each occurrence of v by a.

The following theorem is known as Hales-Jewett theorem, is due to A. W. Hales and R. I. Jewett.

Theorem 2.5. [11, Hales-Jewett Theorem (1963)] *For all values $t, r \in \mathbb{N}$, there exists a number $HJ(r,t)$ such that, if $N \geq HJ(r,t)$ and $[t]^N$ is r colored then there exists a variable word w such that $\{w(a) : a \in [t]\}$ is monochromatic.*

The word space $[t]^N$ is called Hales-Jewett space or H-J space. The number HJ (r,t) is called Hales-Jewett number.

For $q, N \in \mathbb{N}$, $Q = [q]^N$, where $[q] = \{-q,\ldots,-1,0,1,\ldots,q\}$, $\emptyset \neq \gamma \subseteq [N]$ and $-q \leq x \leq q$, $a \oplus x\gamma$ is defined to be the vector b in Q obtained by setting $b_i = x$ if $i \in \gamma$ and $b_i = a_i$ otherwise.

In the statement of theorem [15, Polynomial Hales-Jewett Theorem], we have $a \in Q$ so that $a = \langle\vec{a}_1,\vec{a}_2,\ldots,\vec{a}_d\rangle$ where for $j \in \{1,2,\ldots d\}$, $\vec{a}_j \in [q]^{N^j}$ and we have $\gamma \subseteq [N] = \{1,2,\ldots,N\}$. Given $j \in \{1,2,\ldots,d\}$, let $\vec{a}_j = \langle a_{j,\vec{i}}\rangle_{\vec{i}\in N^j}$. Then $a \oplus x_1\gamma \oplus x_2(\gamma\times\gamma) \oplus \ldots \oplus x_d\gamma^d = b$ where $b = \langle\vec{b}_1,\vec{b}_2,\ldots,\vec{b}_d\rangle$ and for $j \in \{1,2,\ldots,d\}$, $\vec{b}_j = \langle b_{j,\vec{i}}\rangle_{\vec{i}\in N^i}$ where

$$b_{j,\vec{i}} = \begin{cases} x_j & \text{if } \vec{i} \in \gamma^i \\ a_{j,\vec{i}} & \text{otherwise.} \end{cases}$$

Theorem 2.6. [3, 15, Polynomial Hales-Jewett Theorem] *For any q, k, d there exists $N(q,k,d) \in \mathbb{N}$ such that whenever $Q = Q(N) = [q]^N \times [q]^{N\times N} \times \cdots \times [q]^{N^d}$ is k-colored there exist $a \in Q$ and $\gamma \subseteq [N]$ such that the set of points*

$$\left\{a \oplus x_1\gamma \oplus x_2(\gamma\times\gamma) \oplus \cdots \oplus x_d\gamma^d : x_i \in [q]\right\}$$

is monochromatic.

This N is said to be the P.H.J. number.

To prove $\mathcal{J}_p^0 \neq \emptyset$, we will use piecewise syndetic set near zero. So here we recall the defination and an important result.

Definition 2.7. *Let $(S,+)$ be a dense subsemigroup of $(\mathbb{R},+)$ containing 0 such that $(S\cap(0,1),\cdot)$ is a subsemigroup of $((0,1),\cdot)$. A subset A of S is piecewise syndetic near zero if there exist sequences $\langle F_n\rangle_{n=1}^\infty$ and $\langle\delta_n\rangle_{n=1}^\infty$ such that*

1. *For each $n \in \mathbb{N}$, $F_n \in \mathcal{P}_f\left(S\cap\left(0,\frac{1}{n}\right)\right)$ and $\delta_n \in \left(0,\frac{1}{n}\right)$.*
2. *for all $G \in \mathcal{P}_f(S)$ and all $\mu > 0$ there is some $x \in (0,\mu)\cap S$ such that for all $n \in \mathbb{N}$.*

$$(G\cap(0,\delta_n)) + x \subseteq \bigcup_{t\in F_n}(-t+A).$$

Theorem 2.8. *[12, Theorem 3.5] Let $(S,+)$ be a dense subsemigroup of $(\mathbb{R},+)$ containing 0 such that $(S\cap(0,1),\cdot)$ is a subsemigroup of $((0,1),\cdot)$. Let $A \subseteq S$, then $K\cap\overline{A} \neq \emptyset$ if and only if A is piecewise syndetic near zero.*

Now we are in the position to prove $\mathcal{J}_p^0 \neq \emptyset$. The proof is similar to the proof of [5, Theorem 9].

Theorem 2.9. *Let $(S,+)$ be a dense subsemigroup of $(\mathbb{R},+)$ containing 0 such that $(S\cap(0,1),\cdot)$ is a subsemigroup of $((0,1),\cdot)$.*

Then $K(0^+) \subseteq \mathcal{J}_p^0$, i.e. $\mathcal{J}_p^0 \neq \emptyset$.

Proof. We shall show that in fact every piecewise syndetic set near zero is a J_p set near zero. Let A be a piecewise syndetic set near zero. We assume that $\langle F_n\rangle_{n=1}^\infty$ and $\langle\delta_n\rangle_{n=1}^\infty$ be the sequences for the set A as per definition 2.7. Fixed a n, and let $\mid F_n \mid= r$. Let $F \in \mathcal{P}_f(S,S)$ be the given set of polynomials, and let $[q]$ be the set of coefficients of F. Let $\left\{\left\langle S_i^j\right\rangle_{i=1}^\infty : j = 1,2,\ldots,l\right\}$ be a collection of sequences converging to 0. Now we apply a shadow alphabet argument.

For each $j = 1,2,\ldots,l$, define $[q_j]$ to be a copy of $[q]$ {i.e., $[q_j] = \{c_j : c \in [q]\}$}. Now redefine $[q] = \bigcup_{j=1}^l \left[q_j\right]$.

Let F be any given system of finitely many polynomials each of which vanishes at 0 and let $\left\{\left\langle S_i^j\right\rangle_i^\infty : j = 1,2,\ldots,l\right\}$ be a collection of sequences converging to 0.

As above $[q]$ be the set of coefficients of the polynomials in F and $\left\{\left[q_j\right] : j = 1,2,\ldots,l\right\}$ be a sequence of finite sets such that $[q] = \cup_{j=1}^l \left[q_j\right]$.

Let $N = N([q],r,d)$ be the P.H.J number guaranteed by Theorem [PHJ] and let

$$Q = [q]^N \times [q]^{N\times N} \times \ldots \times [q]^{N^d}.$$

Now define a map $C : Q \to S$ by

$$\begin{aligned} &C\left(\langle a_i\rangle_{i=1}^N, \left\langle a_i,a_j\right\rangle_{i,j=1,1}^{N.N}, \ldots, \langle a_{i_1},\ldots,a_{i_d}\rangle_{i_1,\ldots,i_d=1,\ldots,1}^{N,\ldots,N}\right) \\ &= \textstyle\sum_{i=1}^N a_iS_i' + \sum_{i_1,i_2=1}^N a_{i_1}a_{i_2}S_{i_1}'S_{i_2}' + \ldots + \sum_{i_1,\ldots,i_d=1}^N a_{i_1}\ldots a_{i_d}S_{i_1}'\ldots S_{i_d}' \end{aligned}$$

where $S_k' = S_k^i$ whenever $a_k \in [q_i]$.

Then $C(Q)$ is a finite set and hence for any $\epsilon > 0$, there exists an element $x \in S \cap (0, \epsilon)$ such that $(C(Q) \cap (0, \delta_n)) + x \subseteq \cup_{t \in F_n} - t + A$.

Then for any $c \in Q$, $C(c) + x \in \cup_{t \in F_n} - t + A$, provided $C(c) \leq \delta_n$.

Let $\chi : Q \to \{1, 2, \dots, r\}$ be a coloring of Q such that

$\chi(c) = min\{i : C(c) + x \in -t_i + A, t_i \in F_n, C(c) < \delta_n\}$

i,e, $\chi(c) = min\{i : C(c) + x + t_i \in A, t_i \in F_n, C(c) < \delta_n\}$

Where $F_n = \{t_1, \dots, t_r\} \in \mathcal{P}_f \left(S \cap \left(0, \frac{1}{n}\right)\right)$.

Hence there exists $t \in F_n$ such that by Theorem [PHJ], there exist $a \in Q$ and $\gamma \in \mathcal{P}_f(\mathbb{N})$ such that

$$x + t + \left(C\left(\left\{a \oplus x_1\gamma \oplus x_2(\gamma \times \gamma) \oplus \dots \oplus x_d\gamma^d : x_i \in [q]\right\}\right) \cap (0, \delta_n)\right) \subseteq A.$$

Now for suitably choice of $x_i's$ leads to the conclusion {Note that for each choice of $x_1, x_2, \dots, x_d \in [q_i]$ gives us different IP sets $(S^i_\alpha)_{\alpha \in \mathcal{P}_f(\mathbb{N})}$.}

$$\left\{b + P\left(\sum_{t \in \gamma} S^j_t\right) \cap (0, \delta_n) : j \in \{1, 2, \dots, l\}, P \in F\right\} \subseteq A$$

for some suitable constant $b \in S$.

Since the polynomials from F vanishes at 0. And $\left\{\left\langle S^j_i \right\rangle_{i=1}^{\infty} : j = 1, 2, \dots, l\right\}$ converging to 0. Then we can assume for the $\delta_n > 0$, $P\left(\sum_{t \in \gamma} S^j_t\right) \subseteq (0, \delta_n)$.

For all $j \in \{1, 2, \dots, l\}, P \in F$.

Therefore

$$\left\{b + P\left(\sum_{t \in \gamma} S^j_t\right) : j \in \{1, 2, \dots, l\}, P \in F\right\} \subseteq A$$

for some suitable constant $b \in S$.

We have if A be an piecewise syndetic set near zero then, there exists $b \in S$ such that $b + P\left(\sum_{t \in \gamma} S^j_t\right) = b + P\left(S^j_\gamma\right) \in A$, for all $j \in \{1, 2, \dots, l\}, P \in F$.

By theorem 2.8, $\overline{A} \cap K(0^+(S)) \neq \emptyset$, then there exists $p \in \overline{A} \cap K(0^+(S))$ so $p \in \mathcal{J}^0_p$. □

We know that

$$J(S) = \{p \in \beta S : \forall A \in p, A \text{ is a J-set}\}$$

is a two sided ideal of βS.

In [4, Theorem 3.9] Bayatmanesh, Tootkaboni proved that

$$J_0(S) = \{p \in \beta S : \forall A \in p, A \text{ is a J-set near zero}\}$$

is also a two sided ideal of 0^+.

After defining J_p-set, Goswami, Baglini and Patra demonstated that

$$\mathcal{J}_p = \left\{p \in \beta\mathbb{N} : \text{for all } A \in p, A \text{ is a } J_p\text{-set}\right\}$$

is also a two sided ideal of $(\beta\mathbb{N}, +)$[9, Theorem 8].

So we expect that $\mathcal{J}^0_p$ also a two sided ideal of 0^+. The proof is in the following.

Theorem 2.10. *Let $(S, +)$ be a dense subsemigroup of $(\mathbb{R}, +)$ containing 0 such that $(S \cap (0, 1), \cdot)$ is a subsemigroup of $((0, 1), \cdot)$. Then $\mathcal{J}^0_p$ is a two sided ideal of 0^+.*

Proof. Let $p \in \mathcal{J}_p^0$ and $q \in 0^+(S)$. We want to show $p+q, q+p \in \mathcal{J}_p^0$.

To show, $p+q \in \mathcal{J}_p^0$, let $A \in p+q$, so that $B = \{x \in S : -x + A \in q\} \in p$. Hence B is a J_p-set near zero. Then for any $F \in \mathcal{P}_f(\mathbb{P}(S,S))$, $H \in P_f(\mathcal{T}_0)$ and $\delta > 0$, there exist $a \in S \cap \left(0, \frac{\delta}{2}\right)$ and $\beta \in P_f(\mathbb{N})$ such that for each $f \in H$ and all $P \in F$,

$$a + P\left(\sum_{t\in\beta} f(t)\right) \in B.$$

. Then

$$\bigcap_{f\in F}\left\{-\left(a + P\left(\sum_{t\in\beta} f(t)\right)\right) + A\right\} \in q.$$

Let us choose

$$y \in \left[\bigcap_{f\in F}\left\{-\left(a + P\left(\sum_{t\in\beta} f(t)\right)\right) + A\right\}\right] \cap \left(0, \frac{\delta}{2}\right).$$

So, for all $f \in F$, and all $P \in F$,

$$y + a + P\left(\sum_{t\in\beta} f(t)\right) \in A,$$

Hence for any $F \in \mathcal{P}_f(\mathbb{P}(S,S))$, $H \in P_f(\mathcal{T}_0)$ and $\delta > 0$, there exist $x = y + a \in S \cap (0,\delta)$ and $\beta \in P_f(\mathbb{N})$ such that for each $f \in H$ and all $P \in F$,

$$x + P\left(\sum_{t\in\beta} f(t)\right) \in A.$$

Therefore A is a J_p-set near zero. Since A is arbitrary element from $p+q$. So, $p+q \in \mathcal{J}_p^0$. Now, If $A \in q+p$, then $B = \{x \in S : -x + A \in p\} \in q$. Choose $x \in B \cap \left(0, \frac{\delta}{2}\right)$, $-x + A \in p$. So, $-x + A$ is a J_p-set near zero.

Therefore for any $F \in \mathcal{P}_f(\mathbb{P}(S,S))$, $H \in P_f(\mathcal{T}_0)$ and $\delta > 0$, there exist $a \in S \cap \left(0, \frac{\delta}{2}\right)$ and $\beta \in P_f(\mathbb{N})$ such that for each $f \in H$ and all $P \in F$,

$$a + P\left(\sum_{t\in\beta} f(t)\right) \in -x + A.$$

Hence for all $f \in F$, and all $P \in F$,

$$x + a + P\left(\sum_{t\in\beta} f(t)\right) \in A.$$

Let us define, $z = x + a \in S \cap (0,\delta)$

So for any $F \in \mathcal{P}_f(\mathbb{P}(S,S))$, $H \in P_f(\mathcal{T}_0)$ and $\delta > 0$, there exist $z \in S \cap (0,\delta)$ and $\beta \in P_f(\mathbb{N})$ such that for each $f \in H$ and all $P \in F$,

$$z + P\left(\sum_{t\in\beta} f(t)\right) \in A.$$

Therefore A is a J_p-set near zero. Since A is arbitrary element from $q+p$. So $q+p \in \mathcal{J}_p^0$. □

Lemma 2.11. *Let $(S,+)$ be a dense subsemigroup of $(\mathbb{R},+)$ containing 0 such that $(S\cap(0,1),\cdot)$ is a subsemigroup of $((0,1),\cdot)$. Then $\mathcal{J}_p^0$ is closed subset of 0^+.*

Proof. Let $p\in 0^+\setminus\mathcal{J}_p^0$, then there exists $A\in p$ such that A is not a J_p-set. Then $\bar{A}\cap\mathcal{J}_p^0=\emptyset$ and $p\in\overline{A}$. So p is not a limit point of $\mathcal{J}_p^0$. So the proof is done.

Where $\overline{A}=\{p\in\beta S : A\in p\}$. □

By Ellis' theorem [14, Corollary 2.39], a straightforward consequence of Theorem 2.10 and lemma 2.11 is that there are idempotent ultrafilters, and even minimal idempotent ultrafilters, in $\mathcal{J}_p^0$. We denote the set of all idempotents in $\mathcal{J}_p^0$ by $E\left(\mathcal{J}_p^0\right)$.

Definition 2.12. *$A\subseteq S$ is said to be a C_p^0 set in S if $A\in p$ for some ultrafilter $p\in E\left(\mathcal{J}_p^0\right)$.*

The central sets theorem near zero originally proved by Hindman and Leader in [12]. Recently the Central Set Theorem was extended by Goswami, Baglini and Patra for polynomials in [9].

In [9] the authors introduce C_p-set by defining that, C_p-set is the member of the ultrafilters in $E\left(\mathcal{J}_p\right)=\left\{p\in\mathcal{J}_p : p \text{ is an idempotent ultrafilter}\right\}$.

Theorem 2.13. *Let A be a C_p-set and let $F\in\mathcal{P}_f(\mathbb{P})$. There exist functions $\alpha:\mathcal{P}_f\left(\mathbb{N}^{\mathbb{N}}\right)\to S$ and $H:\mathcal{P}_f\left(\mathbb{N}^{\mathbb{N}}\right)\to\mathcal{P}_f(\mathbb{N})$ such that*

1. *If $G,K\in\mathcal{P}_f\left(\mathbb{N}^{\mathbb{N}}\right)$ and $G\subsetneq K$ then $\max H(G)<\min H(K)$ and*
2. *If $n\in\mathbb{N}, G_1,G_2,....,G_n\in\mathcal{P}_f\left(\mathbb{N}^{\mathbb{N}}\right)$;$G_1\subsetneq G_2\subsetneq ....\subsetneq G_n$; and for each $i\in\{1,2,....,n\}$, $f_i\in G_i$, then for all $P\in F$,*

$$\sum_{i=1}^{n}\alpha(G_i)+P\left(\sum_{i=1}^{n}\sum_{t\in H(G_i)}f_i(t)\right)\in A.$$

Proof. [9, Theorem 11]. □

Following them we proved the Stronger Polynomial Central sets theorem near zero. Before that let's recall some notions.

Definition 2.14. *Let $(S,+)$ be a dense subsemigroup of $(\mathbb{R},+)$ containing 0 such that $(S\cap(0,1),\cdot)$ is a subsemigroup of $((0,1),\cdot)$.*

1. $\mathcal{J}_p^0=\left\{p\in\beta S_d : \textit{for all } A\in p, A \textit{ is a } J_p \textit{ -set near zero}\right\}$.
2. $E\left(\mathcal{J}_p^0\right)=\left\{p\in\mathcal{J}_p^0 : p \textit{ is an idempotent ultrafilter}\right\}$.
3. *A set $A\subseteq S$ is called C_p^0 set if $A\in p\in E\left(\mathcal{J}_p^0\right)$.*

We are now ready to prove our main result, Theorem 1.7. Since $K(0^+)\subseteq\mathcal{J}_p^0$, every central set near zero is also a C_p^0 set.

Theorem 2.15. *Let $(S,+)$ be a dense subsemigroup of $(\mathbb{R},+)$ containing 0 such that $(S\cap(0,1),\cdot)$ is a subsemigroup of $((0,1),\cdot)$. Let A be a C_p^0 set in S is particularly central set near zero, and $L\in\mathcal{P}_f(\mathbb{P}(S,S))$. Then for each $\delta\in(0,1)$, there exist functions $\alpha_\delta:\mathcal{P}_f(\mathcal{T}_0)\to S$ and $H_\delta:\mathcal{P}_f(\mathcal{T}_0)\to\mathcal{P}_f(\mathbb{N})$ such that*

1. *$\alpha_\delta(F)<\delta$ for each $F\in\mathcal{P}_f(\mathcal{T}_0)$,*
2. *if $F,G\in\mathcal{P}_f(\mathcal{T}_0)$ and $F\subsetneq G$, then $\max H_\delta(F)<\min H_\delta(G)$ and*

3. *If $n \in \mathbb{N}$ and $G_1, G_2, ..., G_n \in \mathcal{P}_f(\mathcal{T}_0)$, $G_1 \subsetneq G_2 \subsetneq ..... \subsetneq G_n$ and $f_i \in G_i, i = 1, 2, ..., n$. then*

$$\sum_{i=1}^{n} \alpha_\delta(G_i) + P\left(\sum_{i=1}^{n} \sum_{t \in H_\delta(G_i)} f_i(t)\right) \in A$$

for all $P \in L$.

Proof. Choose an idempotent $p \in \mathcal{J}_p^0$ with $A \in p$. For $\delta > 0$ and $F \in \mathcal{P}_f(\mathcal{T}_0)$, we shall use induction on $| F |$ and define $\alpha_\delta(F) \in S$ and $H_\delta(F) \in \mathcal{P}_f(\mathbb{N})$ for witnessing (1),(2),(3).

At first, let $F = \{f\}$. As p is idempotent, the set $A^* = \{x \in A : -x + A \in p\}$ belongs to p [14, corollary 4.14], hence it is a J_p- set near zero. So for $\delta > 0$ there exist $H \in \mathcal{P}_f(\mathbb{N})$ and $a \in S \cap (0, \delta)$ such that

$$\forall P \in L,\ a + P\left(\sum_{t \in H} f(t)\right) \in A^*.$$

By setting $\alpha_\delta(\{f\}) = a$ and $H_\delta(\{f\}) = H$, conditions (1),(2),(3) are satisfied.

Now assume that $| F |> 1$ and $\alpha_\delta(G)$ and $H_\delta(G)$ have been defined for all proper subsets G of F. Let $K_\delta = \bigcup \{H_\delta(G) : \emptyset \neq G \subsetneq F\} \in \mathcal{P}_f(\mathbb{N})$, $m = \max K_\delta$ and

Let

$$R = \left\{ \begin{array}{c} \sum_{i=1}^{n} \sum_{t \in H_\delta(G_i)} f_i(t) \mid n \in \mathbb{N}, \\ \emptyset \neq G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_n \subsetneq F, \\ f_i \in G_i, \forall i = 1, 2, .., n. \end{array} \right\}$$

$$M_\delta = \left\{ \begin{array}{c} \sum_{i=1}^{n} \alpha_\delta(G_i) + P\left(\sum_{i=1}^{n} \sum_{t \in H_\delta(G_i)} f_i(t)\right) \mid n \in \mathbb{N}, \\ \emptyset \neq G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_n \subsetneq F, \\ f_i \in G_i, \forall i = 1, 2, .., n,\ P \in L. \end{array} \right\}$$

Then R and M_δ are finite subsets of S and by inductive hypothesis, $M_\delta \subseteq A^*$.

Let

$$B = A^* \cap \left(\bigcap_{x \in M_\delta} (-x + A^*)\right) \in p.$$

For $P \in L$ and $d \in R$, let us define the polynomial $Q_{P,d} \in \mathbb{P}(S, S)$ by

$$Q_{P,d}(y) = P(y + d) - P(d)$$

the coefficients of P come from $\mathbb{Z}$.

Let $M = L \cup \{Q_{P,d} \mid P \in L \text{ and } d \in R\}$.

From Lemma 2.3, there exist $\gamma \in \mathcal{P}_f(\mathbb{N})$ with $\min(\gamma) > m$ and $a \in S \cap (0, \delta)$ such that

$$\forall Q \in M, f \in F,\ a + Q\left(\sum_{t \in \gamma} f(t)\right) \in B.$$

We set $\alpha_\delta(F) = a < \delta$ and $H_\delta(F) = \gamma$. So (1) is satisfies immediately. Now we are left to verify conditions (2) and (3).

Since $\min(\gamma) > m$, (2) is satisfies.

to verify (3), let $n \in \mathbb{N}$ and $G_1, G_2, ..., G_n \in \mathcal{P}_f(\mathcal{T}_0)$, $G_1 \subsetneq G_2 \subsetneq ..... \subsetneq G_n = F$ and $f_i \in G_i, i = 1, 2, ..., n$ and let $P \in L$.

For $n = 1$, then $\alpha_\delta(G_n) + P\left(\sum_{t \in H_\delta(G_n)} f_n(t)\right) = a + P\left(\sum_{t \in \gamma} f(t)\right) \in B \subseteq A^*$.

If $n > 1$, then

$$\sum_{i=1}^{n} \alpha_\delta(G_i) + P\left(\sum_{i=1}^{n} \sum_{t \in H_\delta(G_i)} f_i(t)\right)$$

$$= \alpha_\delta(G_n) + \sum_{i=1}^{n-1} \alpha_\delta(G_i) + P\left(\sum_{t \in H_\delta(G_n)} f_i(t) + \sum_{i=1}^{n-1} \sum_{t \in H_\delta(G_i)} f_i(t)\right)$$

$$= a + \sum_{i=1}^{n-1} \alpha_\delta(G_i) + P\left(\sum_{t \in \gamma} f_n(t) + \sum_{i=1}^{n-1} \sum_{t \in H_\delta(G_i)} f_i(t)\right)$$

(Since $G_n = F$ and $\alpha_\delta(F) = a, H_\delta(F) = \gamma$.)

$$= a + \sum_{i=1}^{n-1} \alpha_\delta(G_i) + P\left(\sum_{i=1}^{n-1} \sum_{t \in H_\delta(G_i)} f_i(t)\right) +$$

$$P\left(\sum_{t \in \gamma} f_n(t) + \sum_{i=1}^{n-1} \sum_{t \in H_\delta(G_i)} f_i(t)\right) - P\left(\sum_{i=1}^{n-1} \sum_{t \in H_\delta(G_i)} f_i(t)\right)$$

$$= a + y + Q_{P,d}\left(\sum_{t \in \gamma} f_n(t)\right)$$

where $y = \sum_{i=1}^{n-1} \alpha_\delta(G_i) + P\left(\sum_{i=1}^{n-1} \sum_{t \in H_\delta(G_i)} f_i(t)\right) \in M_\delta$,
$d = \sum_{i=1}^{n-1} \sum_{t \in H_\delta(G_i)} f_i(t) \in R$, and $P \in L$ so $Q_{P,d} \in M$.
So we have

$$a + Q_{p,d}\left(\sum_{t \in \gamma} f_n(t)\right) \in B \in -y + A^*$$

Therefore

$$\sum_{i=1}^{n} \alpha_\delta(G_i) + P\left(\sum_{i=1}^{n} \sum_{t \in H_\delta(G_i)} f_i(t)\right) \in A^*.$$

This completes the induction argument, hence the proof. □

We can also generalize this theorem along Phulara's way easily.

Theorem 2.16. *Let $(S, +)$ be a dense subsemigroup of $(\mathbb{R}, +)$ containing 0 such that $(S \cap (0, 1), \cdot)$ is a subsemigroup of $((0, 1), \cdot)$. Let $\langle C_n \rangle_{n \in \mathbb{N}}$ be decreasing family of C_p^0 sets in S such that all $C_i \in p \in E\left(\mathcal{J}_p^0\right)$ and $L \in \mathcal{P}_f(\mathbb{P}(S, S))$. Then for each $\delta \in (0, 1)$, there exist functions $\alpha_\delta : \mathcal{P}_f(\mathcal{T}_0) \to S$ and $H_\delta : \mathcal{P}_f(\mathcal{T}_0) \to \mathcal{P}_f(\mathbb{N})$ such that*

1. *$\alpha_\delta(F) < \delta$ for each $F \in \mathcal{P}_f(\mathcal{T}_0)$,*

2. *if* $F, G \in \mathcal{P}_f(\mathcal{T}_0)$ *and* $F \subsetneq G$, *then* $\max H_\delta(F) < \min H_\delta(G)$ *and*
3. *If* $n \in \mathbb{N}$ *and* $G_1, G_2, ..., G_n \in \mathcal{P}_f(\mathcal{T}_0)$, $G_1 \subsetneq G_2 \subsetneq ..... \subsetneq G_n$ *and* $f_i \in G_i, i = 1, 2, ..., n.$ *with* $|G_1| = k$ *then*

$$\sum_{i \in 1}^{n} \alpha_\delta(G_i) + P\left(\sum_{i \in 1}^{n} \sum_{t \in H_\delta(G_i)} f_i(t)\right) \in C_k.$$

Acknowledgement. The second author acknowledge the Grant CSIR-UGC NET fellowship with file No. 09/106(0202)/2020-EMR-I. The authors also acknowledge the valuable suggestions and guidance of their supervisor, Prof. Dibyendu De.

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