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Hypercyclicity and transitivity of random dynamical systems

Ahmed Zaou[a], Otmane Benchiheb[a,*], Mohamed Amouch[a]

[a]*Chouaib Doukkali University, Department of Mathematics, Faculty of science El Jadida, Morocco*

Abstract. In this paper, we introduce the notions of hypercyclicity and transitivity for random dynamical systems and we study specific properties related to these concepts. Additionally, we establish connections between them and irreducible Markov chains.

1. Introduction

Let $(X, \mathcal{B}, \mu)$ be a probability space. We say that a measurable map $T\colon (X, \mathcal{B}, \mu) \longrightarrow (X, \mathcal{B}, \mu)$ is a measure-preserving transformation, or that μ is T- invariant, if

$$\mu(T^{-1}(A)) = \mu(A) \text{ for all } A \in \mathcal{B}.$$

Measure-preserving transformations have some important dynamical properties. In particular, the famous **Poincaré recurrence theorem** asserts that if $T : (X, \mathcal{B}, \mu) \longrightarrow (X, \mathcal{B}, \mu)$ is measure-preserving then, for any measurable set A such that $\mu(A) > 0$, almost every point $x \in A$ is T-recurrent with respect to A, which means that,

$$T^n(x) \in A \text{ for infinitely many } n \in \mathbb{N}.$$

Another important dynamical property for measure-preserving transformations is the ergodicity. A measure-preserving transformation $T : (X, \mathcal{B}, \mu) \longrightarrow (X, \mathcal{B}, \mu)$ is called ergodic if it satisfies one of the following equivalent conditions:

1. Given any measurable sets A, B with positive measures, one can find an integer $n \geqslant 0$ such that

$$T^n(A) \cap B \neq \emptyset;$$

2. if $A \in \mathcal{B}$ satisfies $T(A) \subset A$, then $\mu(A) = 0$ or $\mu(A) = 1$.

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* Corresponding author: Otmane Benchiheb
Email addresses: zaou.a@ucd.ac.ma (Ahmed Zaou), otmane.benchiheb@gmail.com (Otmane Benchiheb), amouch.m@ucd.ac.ma (Mohamed Amouch)
ORCID iDs: https://orcid.org/0009-0005-0916-9616 (Ahmed Zaou), https://orcid.org/0000-0002-6759-2368 (Otmane Benchiheb), https://orcid.org/0000-0003-1440-2791 (Mohamed Amouch)

Recall that the starting point in the study of ergodic transformations is Birkhoff's ergodic theorem. If $T : (X, \mathcal{B}, \mu) \longrightarrow (X, \mathcal{B}, \mu)$ be a measure-preserving ergodic transformation, then for any $f \in L^1(X, \mu)$,

$$\frac{1}{N}\sum_{n=1}^{N} f(T^n x) \overset{N\to\infty}{\longrightarrow} \int_X f d\mu \quad \mu\text{-a.e.}$$

Now, let X be a Banach space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, with Borel σ-algebra $\mathcal{B}$. Let T be a bounded operator on X, and assume that we have been able to construct a probability measure μ on $(X, \mathcal{B})$ such that T is a measure-preserving ergodic transformation with respect to μ. Moreover, assume that μ has full support, which means that $\mu(U) > 0$ for every non-empty open set $U \subset X$. It follows from the definition of ergodicity that if T is ergodic then T is topologically transitive; that is, for each pair (U, V) of nonempty open subsets of X there exists some $n \in \mathbb{N}$ such that

$$T^n(U) \cap V \neq \emptyset.$$

Birkhoff in [13] provided an equivalent notion of the *topological transitivity* called hypercyclicity. Recall that T is said to be hypercyclic if there exists a vector x such that the orbit of x under T;

$$Orb(T, x) := \{T^n x : n \geq 0\},$$

is dense in X.
A similar notion in the linear dynamics of hypercyclicity is the supercyclicity. We say that T is supercyclic if there exists a vector x whose projective orbit;

$$\mathbb{K}.Orb(T, x) := \{\lambda T^n x : n \geq 0, \lambda \in \mathbb{K}\},$$

is a dense subset of X. As in the case of the hypercyclicity, if X is supposed to be separable, then T is supercyclic if and only if for each pair (U, V) of nonempty open subsets of X there exist some $n \in \mathbb{N}$ and some $\lambda \in \mathbb{K}$ such that

$$\lambda T^n(U) \cap V \neq \emptyset.$$

The notions of hypercyclicity and supercyclicity are well studied in the last few years, see for example K.G. Grosse-Erdmann and A. Peris's book [23] and F. Bayart and E. Matheron's book [7], and the survey article [21] by K.G. Grosse-Erdmann. In [2–6, 8, 40] it was studied the dynamics of a set of operators instead of a single operator. See also [1, 9, 26–29, 35–38] for more information and examples.

Another important notions in the linear topological dynamical systems are those of *mixing* and *weakly mixing* operators, see [22].

An operator T is said to be weakly mixing provided $T \oplus T$ is hypercyclic or equivalently topologically transitive on $X \oplus X$, see [22]. Clearly a weakly mixing operator is hypercyclic. However, the converse does not hold in general, see [17].

Finally, the operator is said to be *mixing* if for each pair (U, V) of nonempty open subsets of X, there exists $N \in \mathbb{N}$ such that

$$T^n(U) \cap V \neq \emptyset,$$

for all $n \geq N$, see [20].

In the upcoming discussion, $(\Omega, \mathcal{F}, \mathbb{P})$ refers to a probability space, $(X, \mathcal{B})$ denotes a measurable space, $\mathcal{P}(X)$ includes all subsets of X, and Θ is a non-empty set.
A random dynamical system on X is a pair (Γ, m), where $\Gamma = \{T_\theta : \theta \in \Theta\}$ is a family of maps from X into X, with Θ assumed to be endowed with a sigmafield Σ such that the map,

$$\begin{array}{rrcl} T: & (\Theta \times X, \Sigma \otimes \mathcal{B}) & \longrightarrow & (X, \mathcal{B}) \\ & (\theta, x) & \longmapsto & T_\theta x \end{array}$$

is measurable, and m be a probability measure on (Θ, Σ).
The evolution of the system is depicted informally as follows: Initially, the system is in a state $x \in X$. Randomly, according to the distribution m, an element θ_0 is chosen from the index set Θ. The system moves to the state $T_{\theta_0}(x)$ at time 1, where T_{θ_0} is the deterministic transformation associated with θ_0. Independently of the value of θ_0, another element θ_1 is randomly chosen from Θ according to the distribution m. The system then moves to the state $T_{\theta_1} \circ T_{\theta_0}(x)$ at time 2, where T_{θ_1} is the deterministic transformation associated with θ_1. This process continues indefinitely, with the system evolving according to the randomly chosen transformations T_{θ_i} for each time step i.
Also, we can define the random orbit of a state $x \in X$ as follows :

$$Orb(x, \Gamma, m) = \{T_{\theta_n} \circ T_{\theta_{n-1}} \circ ... \circ T_{\theta_0} x : n \in \mathbb{N}\},$$

where $(\theta_n)_{n \in \mathbb{N}}$ is a sequence of independent random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in (Θ, Σ) and identically distributed, according to the probability measure m; that is,

$$\mathbb{P}(\theta_n \in A) = m(A), \text{ for all } A \in \Sigma, \text{ and all } n \in \mathbb{N}.$$

We emphasize that the random orbit of a state $x \in X$ is a random set on X ; that is, $orb(x, \Gamma, m)$ is a set-valued map :

$$\begin{array}{rccl} Orb(x, \Gamma, m) : & \Omega & \longrightarrow & \mathcal{P}(X) \\ & \omega & \longmapsto & orb(x, \Gamma, m)(\omega), \end{array}$$

where for any ω of Ω,

$$Orb(x, \Gamma, m)(\omega) := \{T_{\theta_n(\omega)} \circ T_{\theta_{n-1}(\omega)} \circ ... \circ T_{\theta_0(\omega)} x : n \in \mathbb{N}\}.$$

For a comprehensive and thorough description of random sets, see [16] and [14].

In this paper, we extend the concepts of hypercyclicity and transitivity to random dynamical systems, aiming to generalize the notion of strong irreducibility of Markov chains to more general state spaces. More precisely, the contribution of the paper is two-fold . Firstly, we introduce the notions of hypercyclicity and transitivity for random dynamical systems and establish their connection with irreducible Markov chains. Secondly, we generalize the concept of strong irreducibility of Markov chains to encompass general state spaces, building upon the definition of transitivity for random dynamical systems.
The paper is organized as follows. In Section 1, we provide a summary of key notions and results that will be used in subsequent sections. In Section 2, we introduce the notion of hypercyclicity for random dynamical systems and establish their connection with the concept of irreducibility of Markov chains. In Section 3, we introduce the concept of transitivity for random dynamical systems and argue that it generalizes the notion of strong irreducibility of Markov chains to broader state spaces. In Section 4, we are attempting to study the relationship between the concepts of hypercyclicity and transitivity for random dynamical systems.

2. The basic set-up

In this section, we recall some fundamental results concerning homogeneous Markov chains and we give the relation between them and random dynamical systems. For a comprehensive and thorough description, see [18, 24, 30].

Definition 2.1. *(Markov kernel) Let $(X_1, \mathcal{B}_1)$ and $(X_2, \mathcal{B}_2)$ be two measurable spaces. A Markov kernel P on $X_1 \times \mathcal{B}_2$ is a mapping $P : X_1 \times \mathcal{B}_2 \to [0, 1]$ satisfying the following conditions:*

(i) for every $x \in X_1$, the mapping $P(x, .) : A \mapsto P(x, A)$ is a probability measure on $\mathcal{B}_2$,

(ii) for every $A \in \mathcal{B}_2$, the mapping $P(., A) : x \mapsto P(x, A)$ is a measurable function from $(X_1, \mathcal{B}_1)$ to $([0, 1], \mathcal{B}_{[0,1]})$.

Remark 2.2. *Suppose that X is a countable set. A Markov kernel on $X \times \mathcal{P}(X)$ where $\mathcal{P}(X)$ is the set of all subsets of X, is a (possibly infinite-dimensional) matrix*

$$P = (P(x, y) : (x, y) \in X \times X),$$

with nonnegative entries such that $\sum_{y\in Y} P(x, y) = 1$ for each $x \in X$.
In this case, we can associate to P the directed graph structure $G = (X, E)$, where

$$E = \{(x, y) \in X \times X : P(x, y) > 0\},$$

see [12].

Definition 2.3. *Let $(X_1, \mathcal{B}_1)$, $(X_2, \mathcal{B}_2)$ and $(X_3, \mathcal{B}_3)$ be tree measurable spaces. Let P_1 and P_2 be two Markov kernels on $X_1 \times \mathcal{B}_2$ and $X_2 \times \mathcal{B}_3$ respectively. The composition of P_1 and P_2, denoted P_1P_2 is a Markov kernel given by the formula:*

$$P_1P_2(x, A) = \int_{X_2} P_1(x, dy)P_2(y, A), \text{ for any } x \in X_1, \text{ and any } A \in \mathcal{B}_3.$$

Remark 2.4. *Let $(X, \mathcal{B})$ be a measurable space and P be a Markov kernel on $X \times \mathcal{B}$. The powers of P are Markov kernels, and given inductively, for each $x \in X$ and for each $A \in \mathcal{B}$, by $P^0(x, A) = \delta_x(A)$, and for each $n \geq 1$ by*

$$P^n(x, A) = \int_{y_1\in X} \cdots \int_{y_{n-1}\in X} P(x, dy_1)P(y_1, dy_2) \dots P(y_{n-1}, A).$$

Proposition 2.5. *Let $(X, \mathcal{B})$ be a measurable space and P be a Markov kernel on $X \times \mathcal{B}$. Then P verifies the Chapman-Kolmogorov equation :*

$$P^{n+m}(x, A) = \int_X P^n(x, dy)P^m(y, A), \quad \forall x \in X, \ \forall A \in \mathcal{B}, \ \forall n, m \in \mathbb{N}.$$

Remark 2.6.

1. *Let $(X, \mathcal{B})$ be a measurable space and P be a Markov kernel on $X \times \mathcal{B}$. For any $A, B \in \mathcal{B}$, for any $x \in X$ and for all $n, m \in \mathbb{N}^*$, we have*

$$P^{n+m}(x, A) \geqslant \int_B P^n(x, dy)P^m(y, A).$$

2. *When the state space of a Markov kernel is discrete, the Chapman-Kolmogorov equations can be expressed in terms of (possibly infinite-dimensional) matrix multiplication; that is, if X is a countable set and $P = (P(x, y) : (x, y) \in X \times X)$ a Markov kernel on $(X, \mathcal{P}(X))$, then the Chapman-Kolmogorov equation can be expressed as follows :*

$$P^{n+m} = P^nP^m, \quad \text{for each } n, m \in \mathbb{N}.$$

Definition 2.7. *Let $(X, \mathcal{B})$ be a measurable space. A stochastic process taking values in $(X, \mathcal{B})$ is a family of random variables $\{X_k : k \in \mathbb{N}\}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$, which take values in $(X, \mathcal{B})$.*

Definition 2.8. *(Markov chain). Let $(X, \mathcal{B})$ be a measurable space. A stochastic process $\{X_k : k \in \mathbb{N}\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with values on X is called a Markov chain if for all $A \in \mathcal{B}$ and all $k \in \mathbb{N}$,*

$$\mathbb{P}(X_{k+1} \in A/(X_0, X_1, \dots, X_k)) = P(X_{k+1} \in A/X_k) \quad \mathbb{P} - a.s.$$

Definition 2.9. *(Homogeneous Markov chain) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let P be a Markov kernel on a measurable space $(X, \mathcal{B})$. A stochastic process $\{X_k : k \in \mathbb{N}\}$ is called a homogeneous Markov chain with kernel P if for all $A \in \mathcal{B}$ and all $k \in \mathbb{N}$,*

$$\mathbb{P}(X_{k+1} \in A/(X_0, X_1, \dots, X_k)) = P(X_k, A) \quad \mathbb{P} - a.s.$$

The distribution of X_0 is called the initial distribution.

Remark 2.10. *Let $(X, \mathcal{B})$ be a measurable space and (Γ, m) be a random dynamical system on X. Then, we can define a Markov chain on X with kernel given by:*

$$P(x, A) = m\{\theta \in \Theta : T_\theta x \in A\}, \text{for any } x \in X, \text{and any } A \in \mathcal{B}.$$

The following theorem shows that under weak conditions on the structure of the state space X, any Markov kernel can be seen as a kernel induced by a random dynamical system.

Theorem 2.11. *Let $(X, \mathcal{B})$ be a measurable space, where $\mathcal{B}$ is assumed to be countably generated. Let P be a Markov kernel on $X \times \mathcal{B}$ and μ a probability measure on $\mathcal{B}$. Assume that $\{Z_k, : k \in \mathbb{N}\}$ is a sequence of independent random variables uniformly distributed on $[0, 1]$. Then there exist two measurable functions*

$$g : ([0, 1], \mathcal{B}_{[0,1]}) \longrightarrow (X, \mathcal{B})$$

and

$$f : ([0, 1] \times X, \mathcal{B}_{[0,1]} \otimes \mathcal{B}) \longrightarrow (X, \mathcal{B})$$

such that the process $\{X_k, k \in \mathbb{N}\}$ defined by

$$\begin{cases} X_0 = g(Z_0) \\ X_{k+1} = f_{Z_{k+1}}(X_k) \quad k \in \mathbb{N}. \end{cases}$$

is a Markov chain with kernel P and initial distribution μ.

Remark 2.12. *Under the same notations of the previous theorem, let $\mathcal{U}$ the uniform measure on $[0, 1]$, then the pair $(f, \mathcal{U})$ define a random dynamical system on X, called the random dynamical induced by the kernel P.*

3. Hypercyclicity of Random Dynamical Systems

In this section, introduce the notion of hypercyclicity for random dynamical systems and we give some interesting examples.

Let $(X, \mathcal{B})$ be a measurable space and (Γ, m) be a random dynamical system on X. For the sake of simplicity, in the subsequent discussion, for any sequence $(\theta_k)_{k\in\mathbb{N}}$ of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in Θ, distributed according to m, we will use the notation :

$$(T_{\theta_n} \circ T_{\theta_{n-1}} \circ \cdots \circ T_{\theta_0} x \in A) := \{\omega \in \Omega : T_{\theta_n(\omega)} \circ T_{\theta_{n-1}(\omega)} \circ \cdots \circ T_{\theta_0(\omega)} x \in A\},\ x \in X, A \in \mathcal{B}, n \in \mathbb{N}.$$

Definition 3.1. *Let $(X, \mathcal{B}, \mu)$ be a measure space and (Γ, m) be a random dynamical system on X. We say that (Γ, m) is hypercyclic if there exists a state $x \in X$ such that for all $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists some $n \in \mathbb{N}$ such that*

$$\mathbb{P}\ (\ T_{\theta_n} \circ T_{\theta_{n-1}} \circ \ldots \circ T_{\theta_0} x \in A) > 0.$$

In such a case, x is called a hypercyclic state for (Γ, m) and the set of all hypercyclic states for (Γ, m) is denoted by $HC(\Gamma, m)$.

Remark 3.2. *Let $(X, \mathcal{B}, \mu)$ be a measure space and (Γ, m) be a random dynamical system on X. (Γ, m) is hypercyclic means that there exists a state $x \in X$ such that the orbit of x under (Γ, m) visits all non trivial parts of X with strictly positive probability; that is, $orb(x, \Gamma, m)$ visits all non trivial parts of X with strictly positive probability.*

Remark 3.3. *Let $(X, \mathcal{B}, \mu)$ be a measure space, where $\mathcal{B}$ is countably generated and P be a Markov kernel on X.*
Let $x \in X$ and $\{Z_k : k \in \mathbb{N}\}$ be a sequence of independent random variables uniformly distributed on $[0, 1]$. Then, by applying theorem 2.11, we can find a measurable function

$$f : ([0, 1] \times X, \mathcal{B}_{[0,1]} \otimes \mathcal{B}) \longrightarrow (X, \mathcal{B}),$$

such that

$$\begin{cases} X_0 = x \\ X_{k+1} = f_{Z_{k+1}}(X_k) = f_{Z_{k+1}} \circ f_{Z_k} \circ \cdots \circ f_{Z_0} x \quad \forall k \in \mathbb{N}. \end{cases}$$

is a Markov chain with kernel P and initial state x.
Let $\mathcal{U}$ the uniform measure on $[0,1]$ and $\Gamma = \{f_\theta : \theta \in [0,1]\}$, where

$$\begin{array}{rccl} f_\theta : & X & \longrightarrow & X \\ & x & \longmapsto & f(\theta, x). \end{array}$$

Then $(\Gamma, \mathcal{U})$ is the random dynamical system induced by the kernel P.
Now, assume that the Markov chain $(X_k, k \in \mathbb{N})$ is μ-irreducible; that is, for all set $B \in \mathcal{B}$, when $\mu(B) > 0$, for any $y \in X$ there exists $n > 0$ such that

$$P^n(y, B) > 0,$$

equivalently, for all set $B \in \mathcal{B}$, when $\mu(B) > 0$, for any $y \in X$ there exists $n > 0$ such that

$$\int_{y_1 \in X} \cdots \int_{y_{n-1} \in X} P(y, dy_1) P(y_1, dy_2) \ldots P(y_{n-1}, B) > 0.$$

With words, no matter the starting point is, the chain visits all non trivial sets of $\mathcal{B}$ with strictly positive probability, see [18] or [33]. Then

$$\mathbb{P}(\, f_{Z_n} \circ f_{Z_{n-1}} \circ \ldots \circ f_{Z_0} x \in B \,) > 0.$$

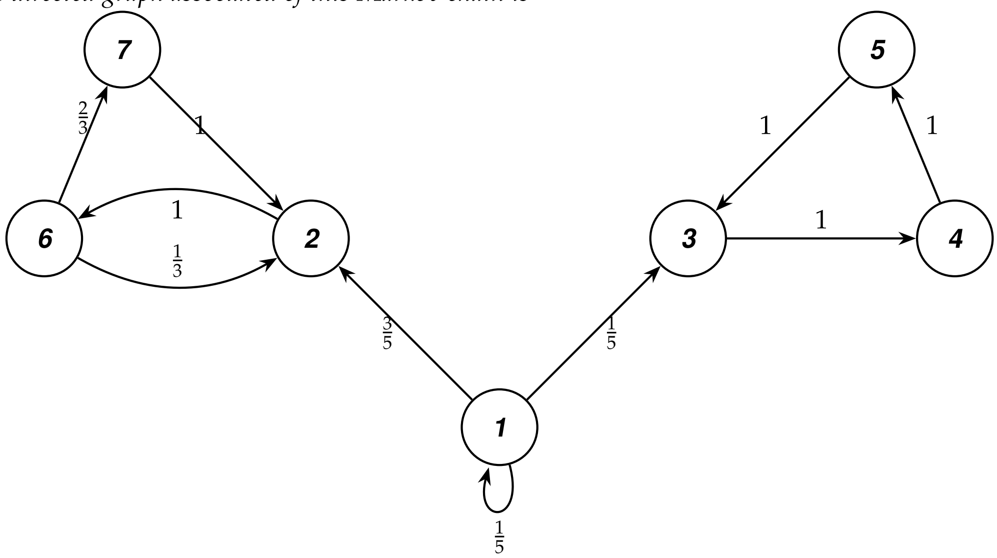
Then the random dynamical system $(\Gamma, \mathcal{U})$ (induced by the kernel P) is hypercyclic and

$$HC(\Gamma, m) = X.$$

Example 3.4. *Consider the Markov chain with transition matrix :*

$$P = \begin{pmatrix} \frac{1}{5} & \frac{3}{5} & \frac{1}{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The directed graph associated of this Markov chain is



In this case, we consider $X = \{1;2;3;4;5;6;7\}$, $\mathcal{B} = \mathcal{P}(X)$ and μ is the uniform measure on X.
Starting from the state 1, this Markov chain visits all state of X with strictly positive probability, then the random dynamical system (Γ, m) on $(X, \mathcal{P}(X), \mu)$ induced by this Markov kernel is hypercyclic and

$$HC(\Gamma, m) = \{1\}.$$

4. Transitivity of random dynamical systems

In this section, we define the notion of transitivity of random dynamical systems, and we give some results.
Let $(X, \mathcal{B})$ be a measurable space and (Γ, m) be a random dynamical system on X.

For any sequence $(\theta_k)_{k\in\mathbb{N}}$ of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with values on Θ, for any $B \in \mathcal{B}$ and for any $n \in \mathbb{N}$, we will use the following notations :

$$(T_{\theta_n(\omega)} \circ T_{\theta_{n-1}(\omega)} \circ \cdots \circ T_{\theta_0(\omega)} \in B) := \{x \in X : T_{\theta_n(\omega)} \circ T_{\theta_{n-1}(\omega)} \circ \cdots \circ T_{\theta_0(\omega)} x \in B\}, \forall \omega \in \Omega,$$

$$(A \cap (T_{\theta_n} \circ T_{\theta_{n-1}} \circ \cdots \circ T_{\theta_0} \in B) \neq \emptyset) := \{\omega \in \Omega : A \cap (T_{\theta_n(\omega)} \circ T_{\theta_{n-1}(\omega)} \circ \cdots \circ T_{\theta_0(\omega)} \in B) \neq \emptyset\}.$$

We give now the notion of transitivity for random dynamical systems.

Definition 4.1. *Let $(X, \mathcal{B}, \mu)$ be a measure space and (Γ, m) be a random dynamical system on X. We say that (Γ, m) is transitive if for any pair (A, B) of $\mathcal{B}$ such that $\mu(A) > 0$ and $\mu(B) > 0$, there exists $n \in \mathbb{N}$ such that*

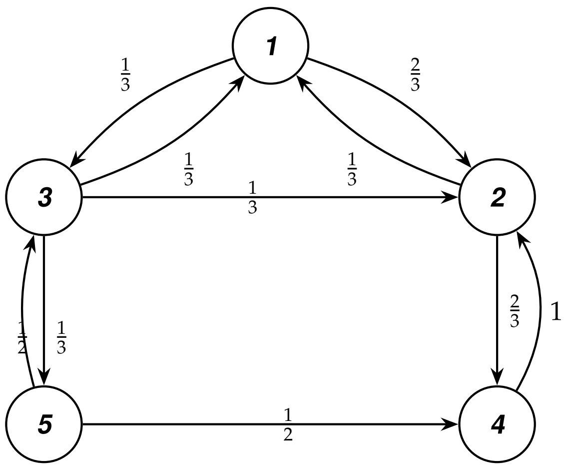
$$\mathbb{P}(A \cap (T_{\theta_n} \circ T_{\theta_{n-1}} \circ \cdots \circ T_{\theta_0} \in B) \neq \emptyset) > 0.$$

With words, (Γ, m) is transitive means that (Γ, m) connects all non trivial subsets of X with strictly positive probability.

Example 4.2. *We consider the Markov chain with transition matrix:*

$$P = \begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

The directed graph associated to this Markov chain,



In this case $X = \{1;2;3;4;5\}$ *and we assume that* $\mathcal{B} = \mathcal{P}(X)$ *and* μ *is the uniform measure on* X.
No matter the starting point is, this Markov chain visits the other points with strictly positive probability. Then the random dynamical system (Γ, m) *on* $(X, \mathcal{P}(X), \mu)$ *induced by this Markov chain is transitive.*

Remark 4.3.

1. *Suppose that* X *is a countable set and* P *be a Markov kernel on* $X \times \mathcal{P}(X)$. P *is called strongly irreducible if the homogeneous Markov chain on* X *with kernel* P *connects all the states of* X *with strictly positive probability, which is equivalent to the directed graph associated to* P *is connected, see [18].*
   In this case, we take a nonnegative measure μ *on* X *which loads the singletons; that is, for each* $x \in X$, $\mu(x) > 0$. *Then the random dynamical system on* X *induced by the kernel* P *is transitive if and only if* P *is strongly irreducible.*
2. *The transitivity of random dynamical systems can be seen as a generalization of the strong irreducibility of Markov chains on general state spaces.*
3. *Let* $(X, \mathcal{B}, \mu)$ *be a measure space and* (Γ, m) *be a random dynamical system on* X.
   For any $A, B, C \in \mathcal{B}$ *and* $n, m \in \mathbb{N}^*$, *by the Chapman-Kolmogorov equation, we can prove that*

$$\mathbb{P}(A \cap (T_{\theta_{p+n}} \circ T_{\theta_{p+n-1}} \circ \ldots \circ T_{\theta_0} \in B) \neq \emptyset) \geqslant \mathbb{P}(A \cap (T_{\theta_n} \circ T_{\theta_{n-1}} \circ \ldots \circ T_{\theta_0} \in C) \neq \emptyset)$$
$$\times \mathbb{P}(C \cap (T_{\theta_p} \circ T_{\theta_{p-1}} \circ \ldots \circ T_{\theta_0} \in B) \neq \emptyset).$$

   With words, the probability to connect A *and* B *is greater than or equal to the probability to connect* A *and* B *after visiting* C.

We now introduce the notion of communication between measurable subsets, which yields a classification of measurable subsets.

Definition 4.4. *Let* $(X, \mathcal{B}, \mu)$ *be a measure space and* (Γ, m) *be a random dynamical system on* X. *A measurable subset* A *leads to a measurable subset* B , *which we write* $A \longrightarrow B$, *if* $\mu(A) > 0$, $\mu(B) > 0$ *and there exists* $n \in \mathbb{N}$ *such that*

$$\mathbb{P}(A \cap (T_{\theta_n} \circ T_{\theta_{n-1}} \circ \cdots \circ T_{\theta_0} \in B) \neq \emptyset) > 0.$$

Two measurable subsets A *and* B *communicate, which we write* $A \longleftrightarrow B$, *if*
$A \longrightarrow B$ *and* $B \longrightarrow A$.

Proposition 4.5. *Let* $(X, \mathcal{B}, \mu)$ *be a measure space and* (Γ, m) *be random dynamical system on* X. *The relation of communication between measurable subsets of* $\mathcal{B}$ *is an equivalence relation.*

Proof. By definition, $A \longleftrightarrow A$ for all $A \in \mathcal{B}$, and $A \longleftrightarrow B$ if and only if $B \longleftrightarrow A$ for all $A, B \in \mathcal{B}$. Let $A, B, C \in \mathcal{B}$, if $A \longrightarrow B$ and $B \longrightarrow C$, then there exists $n, p \in \mathbb{N}$ such that

$$\mathbb{P}(A \cap (T_{\theta_n} \circ T_{\theta_{n-1}} \circ \cdots \circ T_{\theta_0} \in B) \neq \emptyset) > 0,$$

and

$$\mathbb{P}(B \cap (T_{\theta_p} \circ T_{\theta_{p-1}} \circ \cdots \circ T_{\theta_0} \in C) \neq \emptyset) > 0.$$

Since, by the Chapman-Kolmogorov equation,

$$\mathbb{P}\left(A \cap (T_{\theta_{p+n}} \circ T_{\theta_{p+n-1}} \circ \ldots \circ T_{\theta_0} \in C) \neq \emptyset\right) \geqslant \mathbb{P}\left(A \cap (T_{\theta_n} \circ T_{\theta_{n-1}} \circ \ldots \circ T_{\theta_0} \in B) \neq \emptyset\right)$$
$$\times \mathbb{P}\left(B \cap (T_{\theta_p} \circ T_{\theta_{p-1}} \circ \ldots \circ T_{\theta_0} \in C) \neq \emptyset\right),$$

it follows that

$$\mathbb{P}\left(A \cap (T_{\theta_{p+n}} \circ T_{\theta_{p+n-1}} \circ \ldots \circ T_{\theta_0} \in C) \neq \emptyset\right) > 0.$$

This proves that $A \longrightarrow C$.
In a similar way we show that if $C \longrightarrow B$ and $B \longrightarrow A$, then $C \longrightarrow A$. $\square$

Remark 4.6. *Let $(X, \mathcal{B}, \mu)$ be a measure space and (Γ, m) be a random dynamical system on X. The sigmafield $\mathcal{B}$ may be partitioned into equivalence classes for the communication relation. The equivalence class of the measurable subset A is denoted by $C(A)$ i.e.,*

$$C(A) = \{B \in \mathcal{B} : A \longleftrightarrow B\}.$$

Note that by definition, a measurable subset communicates with itself.

Proposition 4.7. *Let $(X, \mathcal{B}, \mu)$ be a measure space and (Γ, m) be a random dynamical system on X. Then (Γ, m) is transitive if and only if for any $A \in \mathcal{B}$ such that $\mu(A) > 0$,*

$$C(A) = \{B \in \mathcal{B} : \mu(B) > 0\}.$$

Proof. Suppose that (Γ, m) is transitive. Let $A \in \mathcal{B}$ such that $\mu(A) > 0$, then for any $B \in \mathcal{B}$ such that $\mu(B) > 0$, there exists $n, p \in \mathbb{N}$ such that

$$\mathbb{P}(A \cap (T_{\theta_n} \circ T_{\theta_{n-1}} \circ \cdots \circ T_{\theta_0} \in B) \neq \emptyset) > 0,$$

and

$$\mathbb{P}(B \cap (T_{\theta_p} \circ T_{\theta_{p-1}} \circ \cdots \circ T_{\theta_0} \in A) \neq \emptyset) > 0.$$

Thus $A \longrightarrow B$ and $B \longrightarrow A$. Hence $B \in C(A)$.
Vice versa, it is clear that if for any $A \in \mathcal{B}$ such that $\mu(A) > 0$, $C(A) = \{B \in \mathcal{B} : \mu(B) > 0\}$, then (Γ, m) is transitive. $\square$

Remark 4.8. *Let $(X, \mathcal{B}, \mu)$ be a measure space, where $\mathcal{B}$ is assumed countably generated by a family $\mathcal{A}$, and (Γ, m) be a random dynamical system on X. We can associate to (Γ, m) the graph structure $G = (\mathcal{A}, \mathcal{E})$, where*

$$\mathcal{E} = \{(A, B) \in \mathcal{A} \times \mathcal{A} : A \longleftrightarrow B\}.$$

5. Some properties of hypercyclicity and transitivity for random dynamical systems

Most of the results on Markov chains on general state spaces are obtained by means of the simplifying assumption that the state space contains an accessible atom. An atom is a set of states out of which the chain exits under a distribution common to all its individual states; that is, a set C is an atom if each time the chain visits C, it regenerates, i.e., it leaves C under a probability distribution that is constant over C. A set is called accessible if the chains eventually enter this set wherever it starts from with positive probability, see [33] or [18].
In a similar way, we define the notions of atom and accessibility for random dynamical systems.

Definition 5.1. *Let $(X, \mathcal{B}, \mu)$ be a measure space and (Γ, m) be a random dynamical system on X. A subset $\alpha \in \mathcal{B}$ is called an atom if $\mu(\alpha) > 0$ and there exists a nonnegative measure ν on $\mathcal{B}$ such that for any $A \in \mathcal{B}$,*

$$m\{\theta \in \Theta : T_\theta x \in A\} = \nu(A), \text{for all } x \in \alpha.$$

Remark 5.2. *Suppose that X is a countable set endowed with the σ-algebra $\mathcal{P}(X)$, where $\mathcal{P}(X)$ denote the set of all subsets of X, and let (Γ, m) be a random dynamical system on X, then any singleton of X is an atom of (Γ, m).*

Definition 5.3. *Let $(X, \mathcal{B}, \mu)$ be a measure space and (Γ, m) be a random dynamical system on X. We say that a subset $A \in \mathcal{B}$ is accessible if $\mu(A) > 0$ and for all $B \in \mathcal{B}$ such that $\mu(B) > 0$, there exists $n \in \mathbb{N}$ such that*

$$\mathbb{P}(B \cap (T_{\theta_n} \circ T_{\theta_{n-1}} \circ \ldots \circ T_{\theta_0} \in A) \neq \emptyset) > 0.$$

Remark 5.4. *Let $(X, \mathcal{B}, \mu)$ be a measure space and (Γ, m) be a random dynamical system on X. If (Γ, m) is transitive, then any measurable subset A such that $\mu(A) > 0$ is accessible.*

According to the Birkhoff transitivity theorem any operator on a separable complete metric space without isolated points is hypercyclic if only if it is topologically transitive, see [10]. In the next results, we give a similar result for random dynamical systems.

Theorem 5.5. *Let $(X, \mathcal{B}, \mu)$ be a measure space, and let (Γ, m) be a random dynamical system on X that admits a countable atom α. If (Γ, m) is transitive on X, then it is hypercyclic and $\alpha \subset HC(\Gamma, m)$. Furthermore*

$$\bigcup_{\beta \in \mathcal{A}} \beta \subset HC(\Gamma, m),$$

where $\mathcal{A} = \{\beta \in \mathcal{B} / \beta \text{ a countable atom}\}$.

Proof. Since α is an atom, then there exist a nonnegative measure ν on $\mathcal{B}$ such that

$$m\{\theta \in \Theta : T_\theta x \in A\} = \nu(A) = m\{\theta \in \Theta : T_\theta y \in A\}, \quad \text{for all } x, y \in \alpha.$$

Moreover, the sequence $(\theta_p)_{p \in \mathbb{N}}$ consists of independent and identically distributed random variables with respect to the probability measure m, which implies that

$$\mathbb{P}(T_{\theta_n} \circ T_{\theta_{n-1}} \circ \cdots \circ T_{\theta_0} x \in A) = \mathbb{P}(T_{\theta_n} \circ T_{\theta_{n-1}} \circ \cdots \circ T_{\theta_0} y \in A),$$

for all $n \in \mathbb{N}$ and $x, y \in \alpha$.
Suppose that (Γ, m) is transitive. Then, for any measurable subset $A \subset \mathcal{B}$ with $\mu(A) > 0$, there exists $n \in \mathbb{N}$ such that

$$\mathbb{P}(\alpha \cap (T_{\theta_n} \circ T_{\theta_{n-1}} \circ ... \circ T_{\theta_0} \in A) \neq \emptyset) > 0,$$

then,

$$\mathbb{P}(\bigcup_{x \in \alpha} \{T_{\theta_n} \circ T_{\theta_{n-1}} \circ ... \circ T_{\theta_0} x \in A\}) > 0,$$

thus,

$$\mathbb{P}(T_{\theta_n} \circ T_{\theta_{n-1}} \circ ... \circ T_{\theta_0} x_0 \in A) > 0 \text{ for some } x_0 \in \alpha,$$

and hence,

$$\mathbb{P}(T_{\theta_n} \circ T_{\theta_{n-1}} \circ ... \circ T_{\theta_0} x \in A) > 0 \text{ for any } x \in \alpha.$$

It follows that the random dynamical system (Γ, m) is hypercyclic and $\alpha \subset HC(\Gamma, m)$. □

Theorem 5.6. *Let $(X, \mathcal{B}, \mu)$ be a measure space and (Γ, m) be a random dynamical system on X that admits an accessible atom α. If (Γ, m) has a hypercyclic state x with $x \in \alpha$, then it is transitive.*

Proof. Suppose that (Γ, m) has a hypercyclic state $x \in \alpha$. Let $A, B \in \mathcal{B}$ such that $\mu(A) > 0$ and $\mu(B) > 0$. For all $p, n \in \mathbb{N}$, by the Chapman-Kolmogorov equation, we get

$$\begin{aligned}\mathbb{P}\left(A \cap (T_{\theta_{p+n}} \circ T_{\theta_{p+n-1}} \circ ... \circ T_{\theta_0} \in B) \neq \emptyset\right) &\geqslant \mathbb{P}\left(A \cap (T_{\theta_n} \circ T_{\theta_{n-1}} \circ ... \circ T_{\theta_0} \in \alpha) \neq \emptyset\right) \\ &\times \mathbb{P}\left(\alpha \cap (T_{\theta_p} \circ T_{\theta_{p-1}} \circ ... \circ T_{\theta_0} \in B) \neq \emptyset\right).\end{aligned}$$

Since x is a hypercyclic state for (Γ, m), then there exists $n \in \mathbb{N}$ such that

$$\mathbb{P}(T_{\theta_n} \circ T_{\theta_{n-1}} \circ ... \circ T_{\theta_0} x \in B) > 0,$$

hence

$$\mathbb{P}(\alpha \cap (T_{\theta_n} \circ T_{\theta_{n-1}} \circ ... \circ T_{\theta_0} \in B) \neq \emptyset) > 0.$$

On the other hand, since α is accessible, there exists $p \in \mathbb{N}$ such that

$$\mathbb{P}(A \cap (T_{\theta_p} \circ T_{\theta_{p-1}} \circ ... \circ T_{\theta_0} \in \alpha) \neq \emptyset) > 0,$$

Thus

$$\mathbb{P}(A \cap (T_{\theta_{p+n}} \circ T_{\theta_{p+n-1}} \circ ... \circ T_{\theta_0} \in B) \neq \emptyset) > 0.$$

Hence (Γ, m) is transitive. □

Remark 5.7. *Let $(X, \mathcal{B}, \mu)$ be a measure space and (Γ, m) be a random dynamical system on X. Assume that (Γ, m) is hypercyclic and $HC(\Gamma, m) = X$, then (Γ, m) is transitive.*

When the state space is not discrete, many Markov chains do not admit accessible atoms. This is the case for random dynamical systems. If the state space does not possess an atom, we may require instead that the chain restart anew from A with some fixed probability (strictly less than one) that is constant over A and this property is satisfied by many Markov chains. Such sets are called small sets.
In a similar way, we define the notion of small sets for random dynamical systems.

Definition 5.8. *Let $(X, \mathcal{B}, \mu)$ be a measure space and (Γ, m) be a random dynamical system on X. A set $C \in \mathcal{B}$ is called a small set if there exist $n \in \mathbb{N}$ and a nonnegative measure ν such that for all $A \in \mathcal{B}$,*

$$\mathbb{P}(C \cap (T_{\theta_n} \circ T_{\theta_{n-1}} \circ ... \circ T_{\theta_0} \in A) \neq \emptyset) \geqslant \nu(A).$$

More precisely, the set C is called (n, ν)-small set.

Proposition 5.9. *Let $(X, \mathcal{B}, \mu)$ be a measure space and (Γ, m) be a random dynamical system on X. If (Γ, m) admits an accessible (k, ν)-small set C, then the set $\{A \in \mathcal{B} : \mu(A) > 0\}$ is a countable union of families of small sets.*

Proof. Let C be an accessible small set. For any $(p, q) \in \mathbb{N} \times \mathbb{N}^*$, define

$$C_{p,q} = \{A \in \mathcal{B},\ \mu(A) > 0 : \mathbb{P}(A \cap (T_{\theta_p} \circ T_{\theta_{p-1}} \circ ... \circ T_{\theta_0} \in C) \neq \emptyset) > \frac{1}{q}\}.$$

Since C is accessible, for every $A \in \mathcal{B}$, $\mu(A) > 0$ there exists $p \in \mathbb{N}$ such that

$$\mathbb{P}(A \cap (T_{\theta_p} \circ T_{\theta_{p-1}} \circ ... \circ T_{\theta_0} \in C)) > 0,$$

thus

$$\{A \in \mathcal{B} : \mu(A) > 0\} = \bigcup_{(p,q) \in \mathbb{N} \times \mathbb{N}^*} C_{p,q}.$$

Moreover, each element of $C_{p,q}$ is $(p + k, \frac{1}{q}\nu)$-small set, indeed, for any $A \in C_{p,q}$ and for any $B \in \mathcal{B}$,

$$\mathbb{P}(A \cap (T_{\theta_{p+k}} \circ T_{\theta_{p+k-1}} \circ ... \circ T_{\theta_0} \in B) \neq \emptyset) \geqslant \mathbb{P}(A \cap (T_{\theta_p} \circ ... \circ T_{\theta_0} \in C) \neq \emptyset) \times \mathbb{P}(C \cap (T_{\theta_k} \circ ... \circ T_{\theta_0} \in B) \neq \emptyset),$$

Then,

$$\mathbb{P}(A \cap (T_{\theta_{p+k}} \circ T_{\theta_{p+k-1}} \circ ... \circ T_{\theta_0} \in B) \neq \emptyset) \geqslant \mathbb{P}(A \cap (T_{\theta_p} \circ T_{\theta_{p-1}} \circ ... \circ T_{\theta_0} \in C) \neq \emptyset) \times \nu(B) \geqslant \frac{1}{q}\nu(B).$$

□

Theorem 5.10. *Let $(X, \mathcal{B}, \mu)$ be a measure space and (Γ, m) be a random dynamical system on X. If (Γ, m) admits an accessible (k, μ)-small set C, then it is transitive.*

Proof. Let $A, B \in \mathcal{B}$, such that $\mu(A) > 0$ and $\mu(B) > 0$. Since C is accessible, there exists $n \in \mathbb{N}$ such that

$$\mathbb{P}(A \cap (T_{\theta_n} \circ T_{\theta_{n-1}} \circ ... \circ T_{\theta_0} \in C) \neq \emptyset) > 0.$$

On the other hand,

$$\mathbb{P}\,(A\cap(T_{\theta_{k+n}}\circ T_{\theta_{k+n-1}}\circ\ldots\circ T_{\theta_0}\in B)\neq\emptyset)\geqslant\mathbb{P}\,(A\cap(T_{\theta_n}\circ T_{\theta_{n-1}}\circ\ldots\circ T_{\theta_0}\in C)\neq\emptyset)\times\mathbb{P}\,(C\cap(T_{\theta_k}\circ T_{\theta_{k-1}}\circ\ldots\circ T_{\theta_0}\in B)\neq\emptyset).$$

Thus

$$\mathbb{P}(A\cap(T_{\theta_{k+n}}\circ T_{\theta_{k+n-1}}\circ\ldots\circ T_{\theta_0}\in B)\neq\emptyset)\geqslant\mathbb{P}(A\cap(T_{\theta_n}\circ T_{\theta_{n-1}}\circ\ldots\circ T_{\theta_0}\in C)\neq\emptyset)\times\mu(B).$$

Hence

$$\mathbb{P}(A\cap(T_{\theta_{k+n}}\circ T_{\theta_{k+n-1}}\circ\cdots\circ T_{\theta_0}\in B)\neq\emptyset)>0.$$

□

Theorem 5.11. *Let $(X,\mathcal{B},\mu)$ be a measure space and (Γ,m) be a random dynamical system on X that admits a (k,μ)-small set C. Suppose that there exist $x\in X$ and $n\in\mathbb{N}$, such that*

$$\mathbb{P}(T_{\theta_n}\circ T_{\theta_{n-1}}\circ\ldots\circ T_{\theta_0}x\in C)>0.$$

Then (Γ,m) is hypercyclic.

Proof. Let $A\in\mathcal{B}$, such that $\mu(A)>0$. Let $x\in X$ and $n\in\mathbb{N}$, such that

$$\mathbb{P}(T_{\theta_n}\circ T_{\theta_{n-1}}\circ\ldots\circ T_{\theta_0}x\in C)>0.$$

Since C is (k,μ)-small set, then

$$\mathbb{P}(C\cap(T_{\theta_k}\circ T_{\theta_{k-1}}\circ\ldots\circ T_{\theta_0}\in A)\neq\emptyset)\geqslant\mu(A).$$

Thus

$$\begin{aligned}\mathbb{P}((T_{\theta_{k+n}}\circ T_{\theta_{k+n-1}}\circ\ldots\circ T_{\theta_0}x\in A))&\geqslant\mathbb{P}(T_{\theta_n}\circ T_{\theta_{n-1}}\circ\ldots\circ T_{\theta_0}x\in C)\times\mathbb{P}(C\cap(T_{\theta_k}\circ T_{\theta_{k-1}}\circ\ldots\circ T_{\theta_0}\in A)\neq\emptyset)\\&\geqslant\mathbb{P}(T_{\theta_n}\circ T_{\theta_{n-1}}\circ\ldots\circ T_{\theta_0}x\in C)\times\mu(A)\\&>0.\end{aligned}$$

Hence (Γ,m) is hypercyclic and $x\in HC(\Gamma,m)$. □

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