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A uniqueness theorem related multiple values for holomorphic curves on annuli sharing hypersurfaces in subgeneral position

Phuong Tran Ha[a,*], Phommavong Chanthaphone[b]

[a]Department of Mathematics, Thai Nguyen University of Education, Luong Ngoc Quyen streets, Thai Nguyen City, VietNam
[b]PhD Student, Thai Nguyen University of Education, Luong Ngoc Quyen streets, Thai Nguyen City, VietNam

Abstract. In 2000, Y. Aihara proved a uniqueness theorem for linearly non-degenerate meromorphic mappings from $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$, which was given again in 2010 by Cao and Yi. In this paper, we will consider the same problem for holomorphic curves on an annulus to complex projective space in the case of hypersurfaces in subgeneral position, namely we give a uniqueness theorem related multiple values for algebraically non-degenerate holomorphic curves on an annulus sharing sufficiently many hypersurfaces.

1. Introduction

In 1975, H. Fujimoto (see [7]) proved that the linearly nondegenerate meromorphic maps from $\mathbb{C}^m$ to $\mathbb{P}^n(\mathbb{C})$ are uniquely determined by $3n + 2$ hyperplanes in general position with counting multiplicity. In 1983, Smiley ([16]) obtained an improvement with ignoring multiplicity. After that, this problem has been studied intensively by many authors. For example: Fujimoto ([8]), Ru ([15]), Yan-Chen ([5],[18]), Dethloff-Tan ([6]), An-Quang-Thai ([2]), Phuong-Minh ([12]), Phuong-Vilaisavanh ([14]) and others. When considering multiple values, in 2000, Y. Aihara (see [1]) proved a uniqueness theorem for linearly non-degenerate meromorphic mappings from $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$, which was given again in 2011 by Cao and Yi ([4]). In this paper, we give a uniqueness theorem related multiple values for algebraically non-degenerate holomorphic curves on an annulus sharing sufficiently many hypersurfaces. First of all, we introduce some notations.

Let $f = (f_0 : \cdots : f_n) : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve, where $f_0, \ldots, f_n$ are holomorphic functions and without common zeros on $\mathbb{C}^m$. Let D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d and Q is the homogeneous polynomial in $\mathbb{C}[z_0, \ldots, z_n]$ of degree d defining D. Assume

$$Q(z_0, \ldots, z_n) = \sum_{k=0}^{n_d} a_k z_0^{i_{k0}} \ldots z_n^{i_{kn}},$$

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* Corresponding author: Phuong Tran Ha
Email addresses: hatranphuong@yahoo.com, phuonght@tnue.edu.vn (Phuong Tran Ha), ctppmv@gmail.com (Phommavong Chanthaphone)
ORCID iDs: https://orcid.org/0000-0002-1359-0463 (Phuong Tran Ha), https://orcid.org/0009-0005-4809-9104 (Phommavong Chanthaphone)

where $n_d = \binom{n+d}{n} - 1$ and $i_{k0} + \cdots + i_{kn} = d$ for $k = 0, \ldots, n_d$. And we write

$$(f, D) = Q(f) = \sum_{k=0}^{n_d} a_k f_0^{i_{k0}} \ldots f_n^{i_{kn}}.$$

We say that (f, D) is free if $(f, D) \not\equiv 0$.

Under the assumption that (f, D) is free, we denote by $\nu_{(f,D)}$ the map from $\mathbb{C}^m$ to $\mathbb{Z}$ whose value $\nu_{(f,D)}(z)$ ($z \in \mathbb{C}^m$) is the intersection multiplicity of the images of f and D at $f(z)$. And for positive integers d, k, we set

$$\nu^d_{(f,D),\leqslant k} = \begin{cases} \min\{d, \nu_{(f,D)}\} & \text{if } 0 < \nu_{(f,D)} \leqslant k, \\ 0 & \text{otherwise.} \end{cases}$$

Now let $\mathcal{D} = \{D_1, \ldots, D_q\}$ be a collection of arbitrary hypersurfaces and Q_j be the homogeneous polynomial in $\mathbb{C}[z_0, \ldots, z_n]$ of degree d_j defining D_j for $j = 1, \ldots, q$. Let $d_{\mathcal{D}}$ is the least common multiple of the d_j, $j = 1, \ldots, q$ and denote

$$n_{\mathcal{D}} = \binom{n + d_{\mathcal{D}}}{n} - 1.$$

Let $N \geqslant n$ be an integer, we recall that the collection of hypersurfaces $\mathcal{D}$ is said to be in N-subgeneral position if $q > N$ and for any distinct $i_1, \ldots, i_{N+1} \in \{1, \ldots, q\}$, $\bigcap_{k=1}^{N+1} \operatorname{supp}(D_{i_k}) = \emptyset$. If $N = n$, the collection of hypersurfaces $\mathcal{D}$ is said to be in general position.

Let f and g be two linearly non-degenerate meromorphic maps of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$, and let $H_1, \ldots, H_q$ be q hyperplanes in general position such that

$$\dim f^{-1}(H_i \cap H_j) \leqslant m - 2 \quad \text{for } i \neq j.$$

Let $m_1, \ldots, m_q$ be positive integers or ∞ such that $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_q \geqslant n$,

$$\nu^1_{(f,H_j),\leqslant m_j} = \nu^1_{(g,H_j),\leqslant m_j}, \quad j = 1, 2 \ldots, q,$$

and $f(z) = g(z)$ for all $z \in \cup_{j=1}^q \{z \in \mathbb{C}^m : 0 < \nu_{(f,H_j)} \leqslant m_j\}$. In 2000, Y. Aihara (see [1]) proved $f = g$ if

$$\sum_{j=3}^{q} \frac{m_j}{m_j + 1} > \frac{qn - q + n + 1}{n} + \frac{m_1}{m_1 + 1} - \frac{m_2}{m_2 + 1}. \tag{1}$$

Note that condition (1) implies $q \geqslant 3n + 2$. In 2011, T. B. Cao and H. X. Yi ([4]) gave an improvement of Aihara's result and they proved $f = g$ if

$$\sum_{j=3}^{q} \frac{m_j}{m_j + 1} > \frac{qn - q + n + 1}{n} - (n - 1)\Big(\frac{1}{m_1 + 1} + \frac{1}{m_2 + 1}\Big). \tag{2}$$

Our idea here is to consider this problem for holomorphic mappings from an annulus into $\mathbb{P}^n(\mathbb{C})$ in the case of hypersurfaces in general position.

Let $R_0 > 1$ be a fixed positive real number or $+\infty$, set

$$\Delta = \Big\{z \in \mathbb{C} : \frac{1}{R_0} < |z| < R_0\Big\},$$

be an annulus in $\mathbb{C}$. Let $f = (f_0 : \cdots : f_n) : \Delta \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve, where $f_0, \ldots, f_n$ are holomorphic functions and without common zeros on Δ. For $1 < r < R_0$, characteristic function $T_f(r)$ of f is defined by

$$T_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log \|f(re^{i\theta})\| d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log \|f(r^{-1}e^{i\theta})\| d\theta,$$

where $\|f(z)\| = \max\{|f_0(z)|, \dots, |f_n(z)|\}$. The above definition is independent, up to an additive constant, of the choice of the reduced representation of f. We set

$$O_f(r) = \begin{cases} O(\log r + \log T_f(r)) & \text{if } R_0 = +\infty \\ O(\log \dfrac{1}{R_0 - r} + \log T_f(r)) & \text{if } R_0 < +\infty. \end{cases}$$

Let D be a hypersurface such that (f, D) is free, we set $\nu_{(f,D)}$ the map from Δ to $\mathbb{Z}$ whose value $\nu_{(f,D)}(z)$ is the multiplicity of the zero of (f, D) at z. And

$$\nu^d_{(f,D),\leqslant k} = \begin{cases} \min\{d, \nu_{(f,D)}\} & \text{if } 0 < \nu_{(f,D)} \leqslant k, \\ 0 & \text{otherwise,} \end{cases}$$

for positive integers d, k. Out result are stated as follows:

Theorem 1. *Let f and g be algebraically non-degenerate holomorphic curves from Δ into $\mathbb{P}^n(\mathbb{C})$ such that $O_f(r) = o(T_f(r))$ and $O_g(r) = o(T_g(r))$. Let $\mathcal{D} = \{D_1, \dots, D_q\}$ be a collection of hypersurfaces in $N-$subgeneral position in $\mathbb{P}^n(\mathbb{C})$ such that $\overline{E}_f(D_i) \cap \overline{E}_f(D_j) = \emptyset$ for all $i \neq j \in \{1, \dots, q\}$. Take k_j $(j = 1, \dots, q)$ be positive integers or ∞ such that $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_q \geqslant n_{\mathcal{D}}$,*

$$\nu^1_{(f,D_j),\leqslant k_j} = \nu^1_{(g,D_j),\leqslant k_j}, \quad j = 1, 2 \dots, q,$$

and $f(z) = g(z)$ for all $z \in \cup_{j=1}^q \{z \in \Delta : 0 < \nu_{(f,D_j)} \leqslant k_j\}$. If

$$\sum_{j=3}^{q} \frac{k_j}{k_j + 1} > \frac{qn_{\mathcal{D}} - q}{n_{\mathcal{D}}} + \frac{(n_{\mathcal{D}} + 1)(2N - n + 1)}{(n + 1)n_{\mathcal{D}}} - (n_{\mathcal{D}} - 1)\Big(\frac{1}{k_2 + 1} + \frac{1}{k_1 + 1}\Big), \tag{3}$$

then $f \equiv g$.

Theorem 1 gives a uniqueness condition for algebraically non-degenerate holomorphic maps on an annulus. Note that, when $D_1, \dots, D_q$ are hyperplanes in general position then $N = n$, and $n_{\mathcal{D}} = n$, so the inequality (3) in Theorem 1 becomes the inequality (2), hence our theorem is an improvement of Cao-Yi's result ([4]). Furthermore the condition (3) implies $q > 2n_{\mathcal{D}} + \dfrac{(n_{\mathcal{D}} + 1)(2N - n + 1)}{n + 1}$, and when $D_1, \dots, D_q$ are hyperplanes in general position then $q \geqslant 3n + 2$, as in the Smiley's result ([16]).

2. Some Preparations

In this section, we introduce some notations and recall some results in Nevanlinna theory for meromorphic functions and holomorphic curves on annuli, which are necessary for proof of the our main result.

For any real number r such that $1 < r < R_0$, we denote

$$\Delta_{1,r} = \Big\{z \in \mathbb{C} : \frac{1}{r} < |z| \leqslant 1\Big\}, \ \Delta_{2,r} = \Big\{z \in \mathbb{C} : 1 < |z| < r\Big\},$$
$$\Delta_r = \Big\{z \in \mathbb{C} : \frac{1}{r} < |z| < r\Big\}.$$

Let f be a meromorphic function on Δ, $c \in \mathbb{C}$, we denote

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta; \quad m(r, \frac{1}{f - c}) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\theta}) - c|} d\theta.$$

Let

$$m_0(r, f) = m(r, f) + m(r^{-1}, f),$$

and

$$m_0(r, \frac{1}{f-c}) = m(r, \frac{1}{f-c}) + m(r^{-1}, \frac{1}{f-c}).$$

Denote by $n_1\Big(r, \dfrac{1}{f-c}\Big)$ the number of zeros of $f-c$ in $\Delta_{1,r}$, $n_2\Big(r, \dfrac{1}{f-c}\Big)$ the number of zeros of $f-c$ in $\Delta_{2,r}$, $n_1(r,\infty)$ the number of poles in $\Delta_{1,r}$ and $n_2(r,\infty)$ the number of poles of f in $\Delta_{2,r}$. Put

$$N_1\Big(r, \frac{1}{f-c}\Big) = \int_{1/r}^{1} \frac{n_1(t, \frac{1}{f-c})}{t} dt, \quad N_2\Big(r, \frac{1}{f-c}\Big) = \int_{1}^{r} \frac{n_2(t, \frac{1}{f-c})}{t} dt,$$

and

$$N_1(r, f) = N_1(r,\infty) = \int_{1/r}^{1} \frac{n_1(t,\infty)}{t} dt, \quad N_2(r, f) = N_2(r,\infty) = \int_{1}^{r} \frac{n_2(t,\infty)}{t} dt.$$

Set

$$N_0\Big(r, \frac{1}{f-c}\Big) = N_1\Big(r, \frac{1}{f-c}\Big) + N_2\Big(r, \frac{1}{f-c}\Big),$$

$$N_0(r, f) = N_1(r, f) + N_2(r, f).$$

The function

$$T_0(r, f) = m_0(r, f) + N_0(r, f) - 2m(1, f)$$

is called the *Nevanlinna characteristic* of f. It is easy to see that the functions $m_0(r, f), N_0(r, f)$ are non-negative and the function $T_0(r, f)$ is non-negative, continuous, non-decreasing and convex with respect to $\log r$.

Lemma 2.1. ([9]) *Let f be a non-constant meromorphic function on Δ. Then for any $r \in (1, R_0)$, we have*

$$N_0\Big(r, \frac{1}{f}\Big) - N_0(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log|f(r^{-1}e^{i\theta})|d\theta$$
$$- \frac{1}{\pi}\int_0^{2\pi} \log|f(e^{i\theta})|d\theta.$$

In this paper, a notation "$\|$" in the inequality is mean that for $R_0 = +\infty$, the inequality holds for $r \in (1, +\infty)$ outside a set Δ'_r satisfying $\int_{\Delta'_r} r^{\lambda-1}dr < +\infty$, and for $R_0 < +\infty$, the inequality holds for $r \in (1, R_0)$ outside a set Δ'_r satisfying $\int_{\Delta'_r} \dfrac{1}{(R_0-r)^{\lambda+1}}dr < +\infty$, where $\lambda \geqslant 0$.

Lemma 2.2. ([10]) *Let f be a non-constant meromorphic function on Δ and $\lambda \geqslant 0$. Then for any $r \in (1, R_0)$,*

(i) If $R_0 = +\infty$,

$$\| \quad m_0(r, \frac{f'}{f}) = O(\log r + \log T_0(r, f)).$$

(ii) If $R_0 < +\infty$,

$$\| \quad m_0(r, \frac{f'}{f}) = O(\log \frac{1}{R_0 - r} + \log T_0(r, f)).$$

Let $f = (f_0 : \cdots : f_n) : \Delta \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve on Δ, where $f_0, \ldots, f_n$ be holomorphic functions and without common zeros on Δ. Let D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d and let Q is the homogeneous polynomial of degree d defining D. For $1 < r < R_0$, under the assumption that $Q(f) \not\equiv 0$, the proximity function of f with respect to D is defined by

$$m_f(r, D) = \frac{1}{2\pi}\int_0^{2\pi} \log \frac{\|f(re^{i\theta})\|^d}{|Q(f)(re^{i\theta})|} d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log \frac{\|f(r^{-1}e^{i\theta})\|^d}{|Q(f)(r^{-1}e^{i\theta})|} d\theta,$$

where the above definition is independent, up to an additive constant, of the choice of the reduced representation of f.

Let $n_{1,f}(r,D)$ be the number of zeros of $Q(f)$ in $\Delta_{1,r}$, $n_{2,f}(r,D)$ be the number of zeros of $Q(f)$ in $\Delta_{2,r}$, counting multiplicity. The integrated counting functions are defined by

$$N_{1,f}(r,D)=\int_{r^{-1}}^{1}\frac{n_{1,f}(t,D)}{t}dt,\quad N_{2,f}(r,D)=\int_{1}^{r}\frac{n_{2,f}(t,D)}{t}dt,$$

and

$$N_f(r,D)=N_{1,f}(r,D)+N_{2,f}(r,D).$$

Let α be a positive integer, we denote by $n^{\alpha}_{1,f}(r,D)$ the number of zeros with multiplicity truncated by α of $Q(f)$ in $\Delta_{1,r}$, $n^{\alpha}_{2,f}(r,D)$ be the number of zeros of $Q(f)$ in $\Delta_{2,r}$ where any zero is counted with multiplicity if its multiplicity is less than or equal to α, and α times otherwise. The integrated truncated counting functions are defined by

$$N^{\alpha}_{1,f}(r,D)=\int_{r^{-1}}^{1}\frac{n^{\alpha}_{1,f}(t,D)}{t}dt,\quad N^{\alpha}_{2,f}(r,D)=\int_{1}^{r}\frac{n^{\alpha}_{2,f}(t,D)}{t}dt,$$

and

$$N^{\alpha}_f(r,D)=N^{\alpha}_{1,f}(r,D)+N^{\alpha}_{2,f}(r,D).$$

Let k be a positive integer, we denote by $n^{\alpha}_{1,f,\leqslant k}(r,D)$ (resp. $n^{\alpha}_{1,f,\geqslant k}(r,D)$) be the number of zeros having multiplicity $\leqslant k$ (resp. $\geqslant k$) of $Q(f)$ in $\Delta_{1,r}$, $n^{\alpha}_{2,f,\leqslant k}(r,D)$ (resp. $n^{\alpha}_{2,f,\geqslant k}(r,D)$) be the number of zeros having multiplicity $\leqslant k$ (resp. $\geqslant k$) of $Q(f)$ in $\Delta_{2,r}$, where any zero is counted with multiplicity if its multiplicity is less than or equal to α, and α times otherwise. The integrated counting functions are defined by

$$N^{\alpha}_{1,f,\leqslant k}(r,D)=\int_{r^{-1}}^{1}\frac{n^{\alpha}_{1,f,\leqslant k}(t,D)}{t}dt,\quad N^{\alpha}_{2,f,\leqslant k}(r,D)=\int_{1}^{r}\frac{n^{\alpha}_{2,f,\leqslant k}(t,D)}{t}dt;$$

$$N^{\alpha}_{1,f,\geqslant k}(r,D)=\int_{r^{-1}}^{1}\frac{n^{\alpha}_{1,f,\geqslant k}(t,D)}{t}dt,\quad N^{\alpha}_{2,f,\geqslant k}(r,D)=\int_{1}^{r}\frac{n^{\alpha}_{2,f,\geqslant k}(t,D)}{t}dt.$$

And

$$N^{\alpha}_{f,\leqslant k}(r,D)=N^{\alpha}_{1,f,\leqslant k}(r,D)+N^{\alpha}_{2,f,\leqslant k}(r,D);$$
$$N^{\alpha}_{f,\geqslant k}(r,D)=N^{\alpha}_{1,f,\geqslant k}(r,D)+N^{\alpha}_{2,f,\geqslant k}(r,D).$$

In 2021, H.T. Phuong and L. Vilaisavanh proved the following

Lemma 2.3. **([14])** *Let D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d and $f=(f_0:\cdots:f_n):\Delta\to\mathbb{P}^n(\mathbb{C})$ be a holomorphic curve whose image is not contained D. Then we have for any $1<r<R_0$,*

$$m_f(r,D)+N_f(r,D)=dT_f(r)+O(1).$$

Let $f_0,\dots,f_n$ be a holomorphic functions, we denote by $W(f_0,\dots,f_n)$ is the wronskian of the $f_0,\dots,f_n$, namely

$$W(f_0,\dots,f_n)=\begin{vmatrix} f_0 & f_1 & \cdot & f_n\\ f_0' & f_1' & \cdot & f_n'\\ \vdots & \vdots & \ddots & \vdots\\ f_0^{(n)} & f_1^{(n)} & \cdot & f_n^{(n)}\end{vmatrix}.$$

In 2023, H.T. Phuong and I. Padaphet proved the following

Lemma 2.4. **([13])** *Let $f:\Delta\to\mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate holomorphic curve and $(f_0:\cdots:f_n)$ is a reduced representative of f. Then we have for any $1<r<R_0$*

$$\| \quad m_0\Big(r,\frac{W(f_0,\dots,f_n)}{f_0\cdots f_n}\Big)=O_f(r). \tag{2.1}$$

Now let $D_1, \dots, D_q$ be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of the common degree d. Let $P_j(z_0, \dots, z_n) \in \mathbb{C}[z_0, \dots, z_n]$ be a homogeneous polynomial of degree d defining D_j, $j = 1, \dots, q$. We regard $\mathbb{C}[z_0, \dots, z_n]$ as a complex vector space. Then for any subset $R \subset \{1, \dots, q\}$ and define

$$\operatorname{rank}\{D_j\}_{j\in R} = \operatorname{rank}\{P_j\}_{j\in R}.$$

Lemma 2.5. **([2])** *Let $D_1, \dots, D_q$ be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of the common degree d. Put $M = \binom{n+d}{n} - 1$. Then there exist $(M - n)$ hypersurfaces $T_1, \dots, T_{M-n}$ of degree d in $\mathbb{P}^n(\mathbb{C})$ such that for any subset $R \subset \{1, \dots, q\}$ with*

$$\#R = \operatorname{rank}\{D_j\}_{j\in R} = n + 1,$$

then $\operatorname{rank}\{\{D_j\}_{j\in R} \cup \{T_j\}_{j=1}^{M-n}\} = M + 1$. *Here $\#R$ is number of elements of R.*

Let N be an integer number such that $N \geqslant n$, then we have:

Lemma 2.6. **([2])** *Let $D_1, \dots, D_q$ $(q > 2N - n + 1)$ be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of the common degree d in $N-$subgeneral position. Then there are positive rational constants w_j, $j = 1, \dots, q$, satisfying the following:*

(i) $0 < w_i \leqslant 1$, for all $i \in \{1, \dots, q\}$.

(ii) Setting $\tilde{w} = \max\{w_1, \dots, w_q\}$, one gets

$$\sum_{j=1}^{q} w_j = \tilde{w}(q - 2N + n - 1) + n + 1.$$

(iii) $\frac{n+1}{2N-n+1} \leqslant \tilde{w} \leqslant \frac{n}{N}$.

(iv) For any $R \subset \{1, \dots, q\}$ with $\#R = N + 1$ then $\sum_{j\in R} w_j \leqslant n + 1$.

(v) Let $E_i \geqslant 1, i = 1, \dots, q$, be arbitrarily given numbers. For $R \subset \{1, \dots, q\}$ with $\#R = N + 1$, there is a subset $R^o \subset R$ such that $\#R^o = \operatorname{rank}\{D_j\}_{j\in R^o} = n + 1$ and

$$\prod_{i\in R} E_i^{w_i} \leqslant \prod_{i\in R^o} E_i.$$

3. Proofs of Theorem 1

To prove our theorem, we need the following lemma which is a version of second main theorem for holomorphic curves on annulus.

Lemma 3.1. *Let $f : \Delta \to \mathbb{P}^n(\mathbb{C})$ be an algebraically non-degenerate holomorphic curve, and let $D_1, \dots, D_q$, be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of degree d_j in N-subgeneral position $(N \geqslant n)$. Let d is the least common multiple of the d_j and set $M = \binom{n+d}{n} - 1$. Assume that $q \geqslant \frac{(M+1)(2N-n+1)}{n+1}$, then for any $1 < r < R_0$, we have*

$$\| \quad (q - \frac{(M+1)(2N - n + 1)}{n + 1})T_f(r) \leqslant \sum_{j=1}^{q} \frac{1}{d} N_f^M(r, D_j) + O_f(r). \tag{3.1}$$

Proof. Let $(f_0, \dots, f_n)$ be the reduced representation of f and let Q_j be the homogeneous polynomial of degree d_j in $\mathbb{C}[z_0, \dots, z_n]$ defining D_j, $j = 1, \dots, q$.

First we consider the case of $d_1 = d_2 = \cdots = d_q = d$. Let $w_1, \dots, w_q$ are rational numbers which satisfy Lemma 2.6 with the hypersurfaces $D_1, \dots, D_q$. Let $S_1, \dots, S_{M-n}$ are hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ satisfying Lemma 2.5 and let P_j $(1 \leqslant j \leqslant M - n)$ is the homogeneous polynomial of degree d in $\mathbb{C}[z_0, \dots, z_n]$ defining hypersurface S_j. Let $\mathcal{I}_d$ is the set of all $(n+1)-$types of degree d, since $\#\mathcal{I}_d = M$, so we write $\mathcal{I}_d = \{I_0, \dots, I_M\}$. For any $j = 0, \dots, M$, we set

$$F_j = f^{I_j} = f_0^{i_{j0}} \dots f_n^{i_{jn}},$$

where $I_j = (i_{j0}, \dots, i_{jn}) \in \mathcal{I}_d$. Since f is algebraically non-degenerate holomorphic curve, $F_0, \dots, F_M$ are linearly independent on $\mathbb{C}$. So

$$W_F = W(F_0, \dots, F_M) \not\equiv 0.$$

Claim 1. *There is a positive constant $\alpha \geqslant 1$ such that for any distinct subset $\{l_1, \ldots, l_{N+1}\} \subset \{1, \ldots, q\}$, we have*

$$\|f(z)\|^d \leqslant \alpha \max_{j=1,\ldots,N+1} |Q_{l_j}(f)(z)| \tag{3.2}$$

for any $z \in \Delta$.

Proof. Indeed, let $\mathcal{R} = \{l_1, \ldots, l_{N+1}\} \subset \{1, \ldots, q\}$ be a distinct subset, from Lemma 2.6, there exists a subset $\mathcal{R}^o = \{t_1, \ldots, t_{n+1}\} \subset \mathcal{R}$ such that $\#\mathcal{R}^o = n + 1 = \operatorname{rank}\{D_j\}_{j\in R^o}$. So for any integer $k \in \{0, \ldots, n\}$, by Hilbert's Nullstellensatz [17], there is an integer $d_k \geqslant d$ and the homogeneous forms $L_{ji}, j = 1, \ldots, n + 1, i = 0, \ldots, n$, with coefficients in $\mathbb{C}$ such that

$$z_k^{d_k} = \sum_{j=1}^{n+1} L_{jk}(z_0, \ldots, z_n) Q_{t_j}(z_0, \ldots, z_n). \tag{3.3}$$

Easy to see that the degree of L_{ji} is $d_k - d$ for any $j = 1, \ldots, n + 1, i = 0, \ldots, n$, so there exists a constant $\alpha_{\mathcal{R}}$ which depends only on the coefficients of L_{jk}, namely depends only on the coefficients of $Q_1, \ldots, Q_q$ such that

$$|L_{jk}(f_0(z), \ldots, f_n(z))| \leqslant \alpha_{\mathcal{R}} \|f(z)\|^{d_k - d} \tag{3.4}$$

for any $z \in \Delta$. So from (3.3), we have

$$\begin{aligned} |f_k(z)|^{d_k} &\leqslant \alpha_{\mathcal{R}} \|f(z)\|^{d_k-d} \max\{|Q_{t_j}(f)(z)|, j = 1, \ldots, n + 1\} \\ &\leqslant \alpha_{\mathcal{R}} \|f(z)\|^{d_k-d} \max\{|Q_{l_j}(f)(z)|, j = 1, \ldots, N + 1\}. \end{aligned} \tag{3.5}$$

Note that, (3.5) holds for all $k = 0, \ldots, n$, so we have

$$\|f(z)\|^d \leqslant \alpha_{\mathcal{R}} \max\{|Q_{l_j}(f)(z)|, j = 1, \ldots, N + 1\}. \tag{3.6}$$

Put $\alpha = \max\{1, \max_{\mathcal{R}=\{i_1,\ldots,i_{N+1}\}\subset\{1,\ldots,q\}} \alpha_{\mathcal{R}}\}$, so α depends only on the coefficients of $Q_1, \ldots, Q_q$. From (3.6) we have (3.2) and the claim is proved. □

We next prove the lemma. Given $r : 1 < r < R_0$, let $x \in \Delta, |x| = r$ be fixed point, there exists renumbering $\{l_1, \ldots, l_q\}$ of $\{1, \ldots, q\}$ such that

$$|Q_{l_1}(f)(x)| \leqslant |Q_{l_2}(f)(x)| \leqslant \cdots \leqslant |Q_{l_q}(f)(x)|. \tag{3.7}$$

We set $\mathcal{R} = \{l_1, \ldots, l_{N+1}\}$ and $\mathcal{Q} = \{1, \ldots, q\} \backslash \mathcal{R}$. Then from Lemma 2.6, there exists a subset $\mathcal{R}^o = \{t_1, \ldots, t_{n+1}\} \subset \mathcal{R}$ such that $\#\mathcal{R}^o = n + 1 = \operatorname{rank}\{D_j\}_{j\in R^o}$. Put

$$W_{\mathcal{R}} = W(Q_{t_1}(f), \ldots, Q_{t_{n+1}}(f), P_1(f), \ldots, P_{M-n}(f)).$$

From Lemma 2.5 we have

$$\operatorname{rank}\{Q_{t_1}, \ldots, Q_{t_{n+1}}, P_1, \ldots, P_{M-n}\} = M + 1,$$

this implies that there exists a non-constant $C_1 \neq 0$ such that

$$W_{\mathcal{R}} = C_1 W_F. \tag{3.8}$$

Since the degree of $P_j(z)$ is d for any $j = 1, \ldots, M - n$, this implies that there exists a constant β which depends only on the coefficients of P_j such that

$$|P_j(f_0(z), \ldots, f_n(z))| = |P_j(f)(z)| \leqslant \beta \|f(z)\|^d \tag{3.9}$$

for any $j = 1, \ldots, M - n$ and for every $z \in \Delta$.

From Lemma 2.6, (3.2) in Claim 1, (3.8) and (3.9) we get

$$\begin{aligned}\frac{\|f(x)\|^{d(\sum_{j=1}^{q} w_j)}|W_F(x)|}{|Q_1(f)(x)|^{w_1}\dots|Q_q(f)(x)|^{w_q}} &= |W_F(x)|\prod_{j\in\mathcal{R}}\Big(\frac{\|f(x)\|^d}{|Q_j(f)(x)|}\Big)^{w_j}\prod_{j\in Q}\Big(\frac{\|f(x)\|^d}{|Q_j(f)(x)|}\Big)^{w_j}\\ &\leqslant \alpha^{q-N-1}.|W_F(x)|\prod_{j\in\mathcal{R}}\Big(\frac{\|f(x)\|^d}{|Q_j(f)(z)|}\Big)^{w_j}\\ &\leqslant \alpha^{q-N-1}.|W_F(x)|\prod_{j\in\mathcal{R}^o}\frac{\|f(x)\|^d}{|Q_j(f)(x)|}\\ &\leqslant \frac{\alpha^{q-N-1}\beta^{M-n}}{|C_1|}\frac{|W_{\mathcal{R}}(x)|\|f(x)\|^{d(M+1)}}{\prod\limits_{j\in R^o}|Q_j(f)(x)|\prod\limits_{j=1}^{M-n}|P_j(f)(x)|}.\end{aligned}$$

This implies that

$$\log\frac{\|f(x)\|^{d(\sum_{j=1}^{q} w_j)}|W_F(x)|}{|Q_1(f)(z)|^{w_1}\dots|Q_q(f)(x)|^{w_q}}\leqslant d(M+1)\log\|f(x)\|+S(x)+\log C, \tag{3.10}$$

where $C=\alpha^{q-N-1}\beta^{M-n}/|C_1|$ and

$$S(x)=\sum_{\mathcal{R}\subset\{1,\dots,q\}}\log^+\frac{|W_{\mathcal{R}}(x)|}{\prod\limits_{j\in\mathcal{R}^o}|Q_j(f)(x)|\prod_{j=1}^{M-n}|P_j(f)(x)|}.$$

By the same arguments as above for $y\in\Delta:|y|=1/r$, we have

$$\log\frac{\|f(y)\|^{d(\sum_{j=1}^{q} w_j)}|W_F(y)|}{|Q_1(f)(y)|^{w_1}\dots|Q_q(f)(y)|^{w_q}}\leqslant d(M+1)\log\|f(y)\|+S(y)+\log C. \tag{3.11}$$

Integrating both sides of the inequalities (3.10), (3.11) and adding sides by sides, from Lemma 2.1 and Lemma 2.6, we have

$$\begin{aligned}&d(q-2N+n-1-\frac{M-n}{\tilde{w}})T_f(r)\\ &\quad\leqslant\sum_{j=1}^{q}\frac{w_j}{\tilde{w}}N_f(r,D_j)-\frac{1}{\tilde{w}}N_0(r,\frac{1}{W_F})+\frac{1}{\tilde{w}}\int_0^{2\pi}\Big(S(re^{i\theta})+S(r^{-1}e^{i\theta})\Big)\frac{d\theta}{2\pi}+O(1).\end{aligned}\tag{3.12}$$

From Lemma 2.4, we have

$$\begin{aligned}\|\quad\frac{1}{2\pi}\int_0^{2\pi}\Big(S(re^{i\theta})+S(r^{-1}e^{i\theta})\Big)d\theta &=\sum_{\mathcal{R}\subset\{1,\dots,q\}}m_0\Big(r,\frac{W_{\mathcal{R}}}{\prod\limits_{j\in\mathcal{R}}|Q_j(f)|\prod_{j=1}^{M-n}|P_j(f)|}\Big)\\ &=O_f(r).\end{aligned}\tag{3.13}$$

We will now estimate $\sum_{j=1}^{q} w_jN_f(r,D_j)-N_0(r,\dfrac{1}{W_F})$. For each $z_0\in\Delta_{1,r}$, there are integers $\beta_j\geqslant 0$, $1\leqslant j\leqslant q$, and nowhere vanishing holomorphic functions g_j in a neighborhood U of z_0 such that

$$Q_j(f)(z)=(z-z_0)^{\beta_j}g_j,$$

for $j = 1, \dots, q$, where $\beta_j = 0$ if $Q_j(f)$ is not vanishing at z_0. Since the hypersurfaces $D_1, \dots, D_q$ are in $N-$subgeneral position, there are at most N index $j \in \{1, \dots, q\}$ such that $\beta_j > 0$. Without loss of generality, we assume that $\beta_j > 0$ for $1 \leqslant j \leqslant k \leqslant N$ and $\beta_j = 0$ for $j > k$.

Let $\mathcal{R} = \{1, \dots, N+1\}$, then from Lemma 2.6, there exists a subset $R^o = \{t_1, \dots, t_{n+1}\} \subset \mathcal{R}$ such that $\mathrm{rank}\{D_j, j \in R^o\} = n+1$. Setting

$$W_{\mathcal{R}} = W(Q_{t_1}, \dots, Q_{t_{n+1}}, P_1, \dots, P_{M-n}).$$

Then there is a non-zero constant C_2 such that $W_F = C_2.W_{\mathcal{R}}$. So we have that W_F vanishes at z_0 with order at least

$$\sum_{j\in\mathcal{R}^o} \max\{0, \beta_j - M\} \leqslant \sum_{j=1}^{q} \max\{0, \beta_j - M\}.$$

This implies that

$$\begin{aligned}\sum_{j=1}^{q} w_j\Big(N_{1,f}(r, D_j) - N^M_{1,f}(r, D_j)\Big) &\leqslant \sum_{j=1}^{q} \tilde{w}\Big(N_{1,f}(r, D_j) - N^M_{1,f}(r, D_j)\Big)\\ &\leqslant \tilde{w} N_1(r, \frac{1}{W_F}) \leqslant N_1(r, \frac{1}{W_F}).\end{aligned}$$

Therefore,

$$\sum_{j=1}^{q} w_j N_{1,f}(r, D_j) - N_1(r, \frac{1}{W_F}) \leqslant \sum_{j=1}^{q} w_j N^M_{1,f}(r, D_j).$$

By the same arguments as above, we have

$$\sum_{j=1}^{q} w_j N_{2,f}(r, D_j) - N_2(r, \frac{1}{W_F}) \leqslant \sum_{j=1}^{q} w_j N^M_{2,f}(r, D_j).$$

So

$$\sum_{j=1}^{q} w_j N_f(r, D_j) - N_0(r, \frac{1}{W_F}) \leqslant \sum_{j=1}^{q} w_j N^M_f(r, D_j). \tag{3.14}$$

Combining (3.12), (3.13) and (3.14) we have

$$\begin{aligned}\| \quad d(q - 2N + n - 1 - \frac{M-n}{\tilde{w}})T_f(r) &\leqslant \sum_{j=1}^{q} \frac{w_j}{\tilde{w}} N^M_f(r, D_j) + O_f(r)\\ &\leqslant \sum_{j=1}^{q} N^M_f(r, D_j) + O_f(r). \end{aligned}\tag{3.15}$$

Note that $\tilde{w} \geqslant \dfrac{n+1}{2N-n+1}$, so from (3.15) we obtain (3.1).

Next we consider the case that $d_1, \dots, d_q$ are not the same. For any $j \in \{1, \dots, q\}$, we set $Q^*_j = Q_j^{d/d_j}$, where d is the least common multiple of the $d_1, \dots, d_q$. Then $Q^*_1, \dots, Q^*_q$ have the same degree d. Let D^*_j is hypersurface defining by Q^*_j for $j = 1, \dots, q$. Since (3.15) and $\tilde{w} \geqslant \dfrac{n+1}{2N-n+1}$ we have

$$\| \quad d(q - \frac{(M+1)(2N-n+1)}{n+1})T_f(r) \leqslant \sum_{j=1}^{q} N^M_f(r, D^*_j) + O_f(r). \tag{3.16}$$

Note that if $z \in \Delta$ is the zero of $Q_j(f)$ with multiplicity k then z is zero of $Q_j^{d/d_j}(f)$ with multiplicity kd/d_j. This implies that

$$\begin{aligned}N_f^M(r, D_j^*) &= N_{1,f}^M(r, D_j^*) + N_{2,f}^M(r, D_j^*)\\ &\leqslant \frac{d}{d_j}N_{1,f}^M(r, D_j) + \frac{d}{d_j}N_{2,f}^M(r, D_j) = \frac{d}{d_j}N_f^M(r, D_j).\end{aligned}$$

This implies (3.1) from (3.16). So the lemma is proved. □

Now we proof of Theorem 1. Assume for the sake contradiction that $f \not\equiv g$. Let $(f_0, \dots, f_n)$ and $(g_0, \dots, g_n)$ are reduced representations of f and g, respectively. Then since $f \neq g$, there exist two numbers $\alpha, \beta \in \{0, \dots, n\}$, $\alpha \neq \beta$ such that $f_\alpha g_\beta \not\equiv f_\beta g_\alpha$. We set

$$\Phi = f_\alpha g_\beta - f_\beta g_\alpha.$$

Now for any $z \in \Delta$, we have

$$\begin{aligned}\log|\Phi(z)| &= \log|(f_\alpha g_\beta - f_\beta g_\alpha)(z)|\\ &\leqslant \log(2.\max\{|f_\alpha(z)g_\beta(z)|, |f_\beta(z)g_\alpha(z)|\})\\ &= \max\{\log|f_\alpha(z)| + \log|g_\beta(z)|, \log|f_\beta(z)| + \log|g_\alpha(z)|\} + \log 2\\ &\leqslant \max\{\log|f_\alpha(z)|, \log|f_\beta(z)|\} + \max\{\log|g_\alpha(z)|, \log|g_\beta(z)|\} + \log 2\\ &= \log\max\{|f_\alpha(z)|, |f_\beta(z)|\} + \log\max\{|g_\alpha(z)|, |g_\beta(z)|\} + \log 2\\ &\leqslant \log\|f(z)\| + \log\|g(z)\| + \log 2.\end{aligned}$$

So from Lemma 2.1 we have for any $r : 1 < r < R_0$,

$$\begin{aligned}N_0\Big(r, \frac{1}{\Phi}\Big) &= \frac{1}{2\pi}\int_0^{2\pi}\log|\Phi(re^{i\theta})|d\theta + \frac{1}{2\pi}\int_0^{2\pi}\log|\Phi(r^{-1}e^{i\theta})|d\theta\\ &\leqslant \frac{1}{2\pi}\int_0^{2\pi}\log\|f(re^{i\theta})\|d\theta + \frac{1}{2\pi}\int_0^{2\pi}\log\|f(r^{-1}e^{i\theta})\|d\theta\\ &\quad + \frac{1}{2\pi}\int_0^{2\pi}\log\|g(re^{i\theta})\|d\theta + \frac{1}{2\pi}\int_0^{2\pi}\log\|g(r^{-1}e^{i\theta})\|d\theta + O(1).\\ &= T_f(r) + T_g(r) + O(1) = T(r) + O(1),\end{aligned} \tag{3.17}$$

where $T(r) = T_f(r) + T_g(r)$.

Let $Q_j, 1 \leqslant j \leqslant q$, be the homogeneous polynomials in $\mathbb{C}[z_0, \dots, z_n]$ of degree d_j defining D_j. Of course we may assume that $q \geqslant n_{\mathcal{D}} + 1$. For any $j = 1, 2, \dots, q$, we set $Q_j^* = Q_j^{d_{\mathcal{D}}/d_j}$ so that $Q_1^*, \dots, Q_q^*$ have a same degree of $d_{\mathcal{D}}$. Let D_j^* is hypersurface defining by Q_j^* for $j = 1, \dots, q$. Applying Lemma 3.1 to holomorphic maps $f : \Delta \to \mathbb{P}^n(\mathbb{C})$ and the hypersurfaces D_j^*, $j = 1, \dots, q$, we have

$$\| \quad (q - \frac{(n_{\mathcal{D}} + 1)(2N - n + 1)}{n + 1})T_f(r) \leqslant \frac{1}{d_{\mathcal{D}}}\sum_{j=1}^q N_f^{n_{\mathcal{D}}}(r, D_j^*) + O_f(r).$$

Similarly for g, we have

$$\| \quad (q - \frac{(n_{\mathcal{D}} + 1)(2N - n + 1)}{n + 1})T_g(r) \leqslant \frac{1}{d_{\mathcal{D}}}\sum_{j=1}^q N_g^{n_{\mathcal{D}}}(r, D_j^*) + O_g(r).$$

So we get

$$\begin{aligned} \| \quad & (q - \frac{(n_{\mathcal{D}}+1)(2N-n+1)}{n+1})T(r) \\ & \leqslant \frac{1}{d_{\mathcal{D}}} \sum_{j=1}^{q} (N_f^{n_{\mathcal{D}}}(r, D_j^*) + N_g^{n_{\mathcal{D}}}(r, D_j^*)) + o(T(r)). \end{aligned} \tag{3.18}$$

Now for any $j = 1, \ldots, q$, we have

$$\begin{aligned} N_f^{n_{\mathcal{D}}}(r, D_j^*) &= N_{f,\leqslant k_j}^{n_{\mathcal{D}}}(r, D_j^*) + N_{f,\geqslant k_j+1}^{n_{\mathcal{D}}}(r, D_j^*) \\ &\leqslant N_{f,\leqslant k_j}^{n_{\mathcal{D}}}(r, D_j^*) + \frac{n_{\mathcal{D}}}{k_j+1} N_{f,\geqslant k_j+1}(r, D_j^*) \\ &\leqslant N_{f,\leqslant k_j}^{n_{\mathcal{D}}}(r, D_j^*) + \frac{n_{\mathcal{D}}}{k_j+1} (N_f(r, D_j^*) - N_{f,\leqslant k_j}^{n_{\mathcal{D}}}(r, D_j^*)) \\ &= \Big(1 - \frac{n_{\mathcal{D}}}{k_j+1}\Big) N_{f,\leqslant k_j}^{n_{\mathcal{D}}}(r, D_j^*) + \frac{n_{\mathcal{D}}}{k_j+1} N_f(r, D_j^*) \\ &\leqslant n_{\mathcal{D}}\Big(1 - \frac{n_{\mathcal{D}}}{k_j+1}\Big) N_{f,\leqslant k_j}^{1}(r, D_j^*) + \frac{n_{\mathcal{D}} d_{\mathcal{D}}}{k_j+1} T_f(r) + o(T_f(r)). \end{aligned} \tag{3.19}$$

Similarly for g we have

$$N_g^{n_{\mathcal{D}}}(r, D_j^*) \leqslant n_{\mathcal{D}}\Big(1 - \frac{n_{\mathcal{D}}}{k_j+1}\Big) N_{g,\leqslant k_j}^{1}(r, D_j^*) + \frac{n_{\mathcal{D}} d_{\mathcal{D}}}{k_j+1} T_g(r) + o(T_g(r)). \tag{3.20}$$

Combining (3.19) and (3.20), we have

$$\begin{aligned} N_f^{n_{\mathcal{D}}}(r, D_j^*) + N_g^{n_{\mathcal{D}}}(r, D_j^*) &\leqslant n_{\mathcal{D}}\Big(1 - \frac{n_{\mathcal{D}}}{k_j+1}\Big)\Big(N_{f,\leqslant k_j}^{1}(r, D_j^*) + N_{g,\leqslant k_j}^{1}(r, D_j^*)\Big) \\ &\qquad + \frac{n_{\mathcal{D}} d_{\mathcal{D}}}{k_j+1} T(r) + o(T(r)). \end{aligned}$$

This implies from (3.18),

$$\begin{aligned} \| \quad & (q - \frac{(n_{\mathcal{D}}+1)(2N-n+1)}{n+1})T(r) \\ & \leqslant \frac{n_{\mathcal{D}}}{d_{\mathcal{D}}} \sum_{j=1}^{q} \Big(1 - \frac{n_{\mathcal{D}}}{k_j+1}\Big)\Big(N_{f,\leqslant k_j}^{1}(r, D_j^*) + N_{g,\leqslant k_j}^{1}(r, D_j^*)\Big) \\ & \quad + n_{\mathcal{D}} \sum_{j=1}^{q} \frac{1}{k_j+1} T(r) + o(T(r)). \end{aligned} \tag{3.21}$$

By hypothesis that $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_q \geqslant n_{\mathcal{D}}$, we have

$$\begin{aligned}
&\sum_{j=1}^{q}\Big(1-\frac{n_{\mathcal{D}}}{k_j+1}\Big)N^1_{f,\leqslant k_j}(r,D_j^*)\\
&\quad=\Big(1-\frac{n_{\mathcal{D}}}{k_1+1}\Big)N^1_{f,\leqslant k_1}(r,D_1^*)+\sum_{j=2}^{q}\Big(1-\frac{n_{\mathcal{D}}}{k_j+1}\Big)N^1_{f,\leqslant k_j}(r,D_j^*)\\
&\quad\leqslant\Big(1-\frac{n_{\mathcal{D}}}{k_1+1}\Big)N^1_{f,\leqslant k_1}(r,D_1^*)+\sum_{j=2}^{q}\Big(1-\frac{n_{\mathcal{D}}}{k_2+1}\Big)N^1_{f,\leqslant k_j}(r,D_j^*)\\
&\quad=\Big(\frac{n_{\mathcal{D}}}{k_2+1}-\frac{n_{\mathcal{D}}}{k_1+1}\Big)N^1_{f,\leqslant k_1}(r,D_1^*)+\sum_{j=1}^{q}\Big(1-\frac{n_{\mathcal{D}}}{k_2+1}\Big)N^1_{f,\leqslant k_j}(r,D_j^*)\\
&\quad=\Big(\frac{n_{\mathcal{D}}}{k_2+1}-\frac{n_{\mathcal{D}}}{k_1+1}\Big)d_{\mathcal{D}}T_f(r)+\sum_{j=1}^{q}\Big(1-\frac{n_{\mathcal{D}}}{k_2+1}\Big)N^1_{f,\leqslant k_j}(r,D_j^*)+O(1). \qquad (3.22)
\end{aligned}$$

Similarly for g, we get

$$\begin{aligned}
&\sum_{j=1}^{q}\Big(1-\frac{n_{\mathcal{D}}}{k_j+1}\Big)N^1_{g,\leqslant k_j}(r,D_j^*)\\
&\quad=\Big(\frac{n_{\mathcal{D}}}{k_2+1}-\frac{n_{\mathcal{D}}}{k_1+1}\Big)d_{\mathcal{D}}T_g(r)+\sum_{j=1}^{q}\Big(1-\frac{n_{\mathcal{D}}}{k_2+1}\Big)N^1_{g,\leqslant k_j}(r,D_j^*)+O(1). \qquad (3.23)
\end{aligned}$$

Hence, we deduce from the inequalities (3.21), (3.22), (3.23),

$$\begin{aligned}
\| \quad &(q-\frac{(n_{\mathcal{D}}+1)(2N-n+1)}{n+1})T(r)\\
&\leqslant n_{\mathcal{D}}^2\Big(\frac{1}{k_2+1}-\frac{1}{k_1+1}\Big)T(r)+n_{\mathcal{D}}\sum_{j=1}^{q}\frac{1}{k_j+1}T(r)\\
&\quad+\frac{n_{\mathcal{D}}}{d_{\mathcal{D}}}\Big(1-\frac{n_{\mathcal{D}}}{k_2+1}\Big)\sum_{j=1}^{q}\Big(N^1_{f,\leqslant k_j}(r,D_j^*)+N^1_{g,\leqslant k_j}(r,D_j^*)\Big)+o(T(r)). \qquad (3.24)
\end{aligned}$$

We see that, if $z_0 \in \Delta$ is a zero of $Q_j^*(f)$ with multiplicity $\alpha \leqslant k_j$, then z_0 is a zero of $Q_j(f)$ with multiplicity $\alpha d_j/d_{\mathcal{D}} \leqslant k_j$. Since $f(z) = g(z)$ on $\cup_{j=1}^{q}\{z \in \Delta : 0 < \nu_{(f,D_j)}(z) \leqslant k_j\}$, we have $g(z_0) = f(z_0)$, so $f_\alpha(z_0)g_\beta(z_0) = f_\beta(z_0)g_\alpha(z_0)$. Namely z_0 is a zero of the function Φ. And if $z_0 \in \Delta$ is a zero of $Q_j^*(g)$ with multiplicity $\beta \leqslant k_j$, then z_0 is a zero of $Q_j(g)$ with multiplicity $\beta d_j/d_{\mathcal{D}} \leqslant k_j$, since $\nu^1_{(f,D_j),\leqslant k_j} = \nu^1_{(g,D_j),\leqslant k_j}$, we have z_0 is a zero of $Q_j(f)$ with multiplicity $\leqslant k_j$. So according to the above argument, we have z_0 is a zero of the function Φ. By the hypothesis $\overline{E}_f(D_i) \cap \overline{E}_f(D_j) = \emptyset$, we have $\overline{E}_g(D_i) \cap \overline{E}_g(D_j) = \emptyset$ for any pair $i \neq j \in \{1, \ldots, q\}$, so z_0 will be not a zero of $Q_i(f)$ and $Q_i(g)$ for all $i \in \{1, \ldots, q\}$ and $i \neq j$. This implies that

$$\sum_{j=1}^{q}N^1_{f,\leqslant k_j}(r,D_j^*)+\sum_{j=1}^{q}N^1_{g,\leqslant k_j}(r,D_j^*)\leqslant 2N_0\Big(r,\frac{1}{\Phi}\Big). \qquad (3.25)$$

So from (3.17) we obtain

$$\sum_{j=1}^{q}(N^1_{f,\leqslant k_j}(r,D_j^*)+N^1_{g,\leqslant k_j}(r,D_j^*)\Big)\leqslant 2d_{\mathcal{D}}T(r)+O(1).$$

Hence we get from (3.24),

$$\begin{aligned} \| \quad & (q - \frac{(n_{\mathcal{D}}+1)(2N-n+1)}{n+1})T(r) \\ & \leqslant 2n_{\mathcal{D}}T(r) - n_{\mathcal{D}}^2\Big(\frac{1}{k_2+1} + \frac{1}{k_1+1}\Big)T(r) \\ & \quad + n_{\mathcal{D}}\sum_{j=1}^{q}\frac{1}{k_j+1}T(r) + o(T(r)). \end{aligned}$$

This implies that

$$\begin{aligned} \Big(\sum_{j=3}^{q}\frac{k_j}{k_j+1} - \frac{qn_{\mathcal{D}}-q}{n_{\mathcal{D}}} - \frac{(n_{\mathcal{D}}+1)(2N-n+1)}{(n+1)n_{\mathcal{D}}} \\ + (n_{\mathcal{D}}-1)\Big(\frac{1}{k_2+1} + \frac{1}{k_1+1}\Big)\Big)T(\rho) \leqslant o(T_f(\rho)). \end{aligned}$$

This is a contradiction. The theorem is proved.

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