



Inequalities for rational functions in the complex plane

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Abstract. This paper investigates inequalities for rational functions with prescribed poles and zeros located inside or outside a disk of radius k . We extend classical polynomial inequalities due to Govil (1973, 1980) to the setting of rational functions. In particular, we establish a relationship between the maximum modulus of a rational function on the circle of radius k (where $k \geq 1$) and on the unit circle. Our findings refine inequalities for rational functions previously obtained by Li, Mohapatra, and Rodriguez (1993), providing sharper bounds under certain conditions.

1. Introduction

Let $\mathcal{P}_n$ denote the space of complex polynomials $p(z) := \sum_{j=0}^n \alpha_j z^j$ of degree $n \geq 1$. For $a_j \in \mathbb{C}$ with $j = 1, 2, \dots, n$, let $w(z) = \prod_{j=1}^n (z - a_j)$ and define

$$B(z) := \frac{w^*(z)}{w(z)} = \prod_{j=1}^n \left(\frac{1 - \bar{a}_j z}{z - a_j} \right), \quad \mathcal{R}_n := \mathcal{R}_n(a_1, a_2, \dots, a_n) := \left\{ \frac{p(z)}{w(z)}, p \in \mathcal{P}_n \right\},$$

where $w^*(z) = z^n \overline{\left(\frac{1}{\bar{z}} \right)}$ is the conjugate transpose (reciprocal) of $w(z)$. Then $\mathcal{R}_n$ is the set of rational functions with poles $a_1, a_2, \dots, a_n$ at most and a finite limit at infinity. The function $B(z) \in \mathcal{R}_n$ is known as finite Blaschke Product.

In our discussion, we shall assume that the poles $a_1, a_2, \dots, a_n$ lie in the region $|z| > 1$, unless otherwise stated.

For $r(z) = \frac{p(z)}{w(z)} \in \mathcal{R}_n$, the conjugate transpose r^* of r is defined by

$$r^*(z) = B(z) \overline{r\left(\frac{1}{\bar{z}}\right)}.$$

Note that if $r(z) = \frac{p(z)}{w(z)} \in \mathcal{R}_n$, then $r^*(z) = \frac{p^*(z)}{w(z)}$ and hence $r^*(z) \in \mathcal{R}_n$.

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If $p \in \mathcal{P}_n$, then concerning the estimate of $|p'(z)|$ on the unit circle $|z| = 1$, we have the following well-known result due to Bernstein [6]:

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|. \quad (1)$$

The inequality (1) is sharp and equality holds for polynomials having all zeros at the origin.

Since equality in (1) holds if and only if $p(z)$ has all zeros at the origin, it is natural to expect a relationship between the bound n and the distance of the zeros of a polynomial from the origin. This fact was first observed by Erdős, who conjectured it; the conjecture was later proven by Lax [11].

If $p \in \mathcal{P}_n$ and $p(z) \neq 0$ in $|z| < 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|. \quad (2)$$

On the other hand, Turán [17] considered a polynomial $p(z)$ of degree n having all zeros inside $|z| \leq 1$ and proved that

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)|. \quad (3)$$

As an extension of inequalities (2) and (3), Malik [13] proved that if $p \in \mathcal{P}_n$ and $p(z) \neq 0$ in $|z| < k$, where $k \geq 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |p(z)|. \quad (4)$$

Whereas, if $p(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$, then

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k} \max_{|z|=1} |p(z)|. \quad (5)$$

As an analogue of inequality (4) for $k \leq 1$, Govil [9] demonstrated that if $p(z) \neq 0$ for $|z| < k$, with $k \leq 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{k^n + k^{n-1}} \max_{|z|=1} |p(z)|. \quad (6)$$

Whereas, as an analogue of (5) for $k \geq 1$, Govil [10] proved that if $p(z)$ has all its zeros in $|z| \leq k$, with $k \geq 1$, then

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k^n} \max_{|z|=1} |p(z)|. \quad (7)$$

Li, Mohapatra and Rodriguez [12] extended inequalities (2) and (3) to rational functions with prescribed poles and proved the following results.

If $r \in \mathcal{R}_n$, such that $r(z) \neq 0$ for $|z| < 1$, then for $|z| = 1$,

$$|r'(z)| \leq \frac{|B'(z)|}{2} \max_{|z|=1} |r(z)|. \quad (8)$$

Whereas, if $r \in \mathcal{R}_n$, such that $r(z) \neq 0$ for $|z| > 1$ and $r(z)$ has exactly n poles at $a_1, a_2, \dots, a_n$, then for $|z| = 1$,

$$|r'(z)| \geq \frac{1}{2} [|B'(z)| - (n-m)] |r(z)|, \quad (9)$$

where m is the number of zeros of $r(z)$.

Aziz and Zargar [5] extended inequality (4) to rational functions and proved the following result:

Theorem A. If $r \in \mathcal{R}_n$, such that $r(z) \neq 0$ for $|z| < k$, with $k \geq 1$, then for $|z| = 1$,

$$|r'(z)| \leq \frac{1}{2} \left\{ |B'(z)| - \frac{n(k-1)}{k+1} \frac{|r(z)|^2}{(\max_{|z|=1} |r(z)|)^2} \right\} \max_{|z|=1} |r(z)|. \quad (10)$$

Aziz and Shah [4] extended inequality (5) to rational functions and proved:

Theorem B. If $r \in \mathcal{R}_n$, such that $r(z) \neq 0$ for $|z| > k$, with $k \leq 1$ and $r(z)$ has exactly n poles at $a_1, a_2, \dots, a_n$, then for $|z| = 1$,

$$|r'(z)| \geq \frac{1}{2} \left\{ |B'(z)| + \frac{2m - n(k+1)}{k+1} \right\} |r(z)|, \quad (11)$$

where m is the number of zeros of $r(z)$.

The inequalities concerning rational functions and their derivatives, which generalize the classical polynomial inequalities, originally emerged in the literature as tools to establish inverse theorems in approximation theory. These rational inequalities possess independent significance, as evidenced by numerous recent publications (see, for instance, [1, 14, 15]).

In this paper, we present new results for rational functions whose zeros lie entirely within $z \leq k$, where either $k \geq 1$ or $k \leq 1$. Our findings extend the inequalities (6) and (7) to the setting of rational functions.

Notations:

1. $B(kz) = \frac{[w(kz)]^*}{w(kz)} = \prod_{j=1}^n \left(\frac{1 - \overline{a_j} kz}{kz - a_j} \right)$; which is obtained when $B(z)$ is evaluated at kz .
2. $B_k(z) = \frac{w^*(kz)}{w(kz)} = \prod_{j=1}^n \left(\frac{k - \overline{a_j} z}{kz - a_j} \right)$; which is the Blaschke Product associated with $r(kz)$.

Note that $|B_k(z)| = 1$ for $|z| = 1$.

3. $B_k(kz) = \prod_{j=1}^n \left(\frac{k - \overline{a_j} zk}{k^2 z - a_j} \right)$; which is obtained by evaluating $B_k(z)$ at $z = kz$.
4. $B_{1/k}(z) = \prod_{j=1}^n \left(\frac{1 - \overline{a_j} zk}{z - a_j k} \right)$; which is Blaschke product associated with $r(z/k)$.
6. $r^*(kz) = B_k(z) \overline{r(k/\overline{z})} = \frac{p^*(kz)}{w(kz)}$; is the reciprocal of $r(kz)$.
7. $[r(kz)]^* = B(kz) \overline{r(1/\overline{kz})} = \frac{[p(kz)]^*}{w(kz)}$; is the reciprocal of $r(z)$ evaluated at kz .

2. Main Results

In this section we state our main results. Theorem 1 establishes a relationship between the maximum modulus of $r(z)$ on a circle of radius k , where $k \geq 1$ and the corresponding modulus taken over the unit circle.

Theorem 1. If $r(z) = \frac{p(z)}{w(z)} \in \mathcal{R}_n$ such that all the zeros of $r(z)$ lie in $|z| \leq k$, with $k \geq 1$, then for all z with $|z| = 1$,

$$\max_{|z|=1} |r(kz)| \geq \frac{2|B_k(kz)|}{1 + |B_k(kz)|} |r(z)|.$$

Equality holds for $k = 1$.

Next, we present the following analogue result, which extends Theorem B for $k \geq 1$.

Theorem 2. If $r(z) = p(z)/w(z)$, where $p(z) = \prod_{j=1}^n (z - z_j)$, be a rational function having all zeros in $|z| \leq k$, and all poles in $|z| \geq k$, where $k \geq 1$. Then, for $|z| = 1$,

$$\max_{|z|=1} |r'(z)| \geq \frac{1}{2|B(kz)|} \left[|B'_k(z)| + \sum_{j=1}^n \frac{k - |z_j|}{k + |z_j|} \right] |r(kz)|.$$

For $k = 1$, Theorem 2 reduces to the following corollary:

Corollary 1. If $r(z) = p(z)/w(z)$, where $p(z) = \prod_{j=1}^n (z - z_j)$, be a rational function having all zeros in $|z| \leq 1$, then, for $|z| = 1$,

$$\max_{|z|=1} |r'(z)| \geq \frac{1}{2} \left[|B'(z)| + \sum_{j=1}^n \frac{1 - |z_j|}{1 + |z_j|} \right] |r(z)|.$$

Since $|z_j| \leq 1$ for $j = 1, 2, \dots, n$, Corollary 1 provides a refinement of inequality (9).

Finally, we present the following result, which is analogous to Theorem A when $k \leq 1$.

Theorem 3. If $r(z) = p(z)/w(z) \in \mathcal{R}_n$, where $p(z) = \prod_{j=1}^n (z - z_j)$, and suppose $r(z)$ does not vanish in $|z| < k$, with all poles lying in $|z| \geq 1/k$, where $k \leq 1$, then

$$\max_{|z|=1} |r'(z)| \leq \left[\max_{|z|=1} |B'(z)| \max_{|z|=1} |r(z)| - A |r(kz)| \right], \quad (12)$$

provided that $|r'(z)|$ and $|r''(z)|$ becomes maximum at the same point on the circle $|z| = 1$, where

$$A = \frac{1}{2} \left[|B'_{1/k}(z)| + \sum_{j=1}^n \frac{1 - k|z_j|}{1 + k|z_j|} \right].$$

3. Lemmas

To prove these theorems, we require the following lemmas. The first lemma is due to Li, Mohapatra and Rodriguez [12].

Lemma 3.1. If $r \in \mathcal{R}_n$ and $|z| = 1$, then

$$|(r^*(z))'| + |r'(z)| \leq |B'(z)| \max_{|z|=1} |r(z)|.$$

Lemma 3.2. If $r \in \mathcal{R}_n$ and $G(z) = r(kz)$ for $k \geq 1$. Then, for $R \geq 1$ and $|z| = 1$,

$$|G(Rz)| + |G^*(Rz)| \leq (|B_k(Rz)| + 1) \max_{|z|=1} |G(z)|,$$

where $G^*(z) = B_k(z) \overline{G(1/\bar{z})}$.

Proof. The proof of this lemma is on the same lines as Theorem 1 of [3], so we omit it. $\square$

Lemma 3.3. If $r \in \mathcal{R}_n$. Then, for $k \geq 1$ and $|z| = 1$,

$$|(r(kz))'| \leq |B(kz)| \max_{|z|=1} |r'(z)|.$$

Proof. Let $M = \max_{|z|=1} |r'(z)|$, then

$$|r'(z)| \leq M \quad \text{for } |z| = 1.$$

For any complex number α with $|\alpha| < 1$, it follows that

$$|\alpha r'(z)| < |B(z)|M \quad \text{for } |z| = 1.$$

Since it is assumed that all the poles $a_1, a_2, \dots, a_n$ lie in $|z| > 1$, all the zeros of $B(z)$ lie in $|z| < 1$. Also, $r'(z)$ and $B(z)$ are analytic in $|z| < 1$. By Rouché's Theorem, all zeros of

$$F(z) = \alpha r'(z) + B(z)M \tag{13}$$

lie in $|z| < 1$.

We claim that for $0 \leq \theta < 2\pi$ and $k \geq 1$,

$$|(r(ke^{i\theta}))'| \leq |B(ke^{i\theta})|M, \tag{14}$$

for points $e^{i\theta}$ that are not zeros of $(r(kz))'$. For if (14) is not true, then

$$|(r(ke^{i\theta}))'| > |B(ke^{i\theta})|M, \tag{15}$$

for a point $e^{i\theta}$ which is not a zero of $(r(kz))'$. Let $\alpha = -\frac{B(ke^{i\theta})M}{(r(ke^{i\theta}))'}$. Then, by (15) $|\alpha| < 1$, and from (13), we get

$$F(ke^{i\theta}) = 0 \quad \text{for } k \geq 1.$$

This is a contradiction to the fact that all the zeros of $F(z)$ lie in $|z| < 1$. This establishes (14) for points $e^{i\theta}$ that are not zeros of $(r(kz))'$. Since (14) is trivially true for points $e^{i\theta}$ which are zeros of $(r(kz))'$, we conclude that for $k \geq 1$ and $|z| = 1$,

$$|(r(kz))'| \leq |B(kz)| \max_{|z|=1} |r'(z)|.$$

$\square$

Lemma 3.4. For $w = re^{i\phi}$, with $r \leq k$,

$$\operatorname{Re} \left(\frac{ke^{i\theta}}{ke^{i\theta} - w} \right) \geq \frac{k}{k+r}.$$

Proof. An easy computation gives

$$\operatorname{Re}\left(\frac{ke^{i\theta}}{ke^{i\theta} - w}\right) = \frac{k^2 - rk \cos(\phi - \theta)}{k^2 + r^2 - 2kr \cos(\phi - \theta)}. \quad (16)$$

For $r \leq k$, we observe that:

$$kr(k - r)(\cos(\phi - \theta) + 1) \geq 0.$$

This inequality implies:

$$-kr^2 \cos(\phi - \theta) + k^2 r \geq kr^2 - k^2 r \cos(\phi - \theta).$$

Adding k^3 and subtracting $k^2 r \cos(\phi - \theta)$ on both sides yields:

$$\begin{aligned} k^3 - k^2 r \cos(\phi - \theta) - kr^2 \cos(\phi - \theta) + k^2 r &\geq k^3 + kr^2 - 2k^2 r \cos(\phi - \theta) \\ \Rightarrow (k + r)(k^2 - rk \cos(\phi - \theta)) &\geq k(k^2 + r^2 - 2kr \cos(\phi - \theta)) \\ \Rightarrow \frac{k^2 - rk \cos(\phi - \theta)}{k^2 + r^2 - 2kr \cos(\phi - \theta)} &\geq \frac{k}{k + r}. \end{aligned}$$

Substituting this into equation (16) completes the proof. $\square$

Lemma 3.5. If $|a_j| \geq k$ for all j , then for $|z| = 1$,

$$\frac{zB'_k(z)}{B_k(z)} = |B'_k(z)|.$$

Proof. We have,

$$B_k(z) = \prod_{j=1}^n \frac{k - \bar{a}_j z}{kz - a_j}.$$

Taking the logarithmic derivative, we obtain:

$$\frac{zB'_k(z)}{B_k(z)} = \sum_{j=1}^n \left[\frac{-\bar{a}_j z}{k - \bar{a}_j z} - \frac{kz}{kz - a_j} \right].$$

For $|z| = 1$, this simplifies to:

$$\frac{zB'_k(z)}{B_k(z)} = \sum_{j=1}^n \frac{|a_j|^2 - k^2}{|zk - a_j|^2}.$$

Since $|a_j| \geq k$ for $j = 1, 2, \dots, n$, the right-hand side is non-negative. Thus, for $|z| = 1$,

$$\frac{zB'_k(z)}{B_k(z)} = \left| \frac{zB'_k(z)}{B_k(z)} \right| = |B'_k(z)|.$$

$\square$

Lemma 3.6. If $|a_j| \geq k$ for all j , then for $|z| = 1$,

$$\operatorname{Re}\left(\frac{z[w(kz)]'}{w(kz)}\right) = \frac{n - |B'_k(z)|}{2}.$$

Proof. We have

$$B_k(z) = \frac{w^*(kz)}{w(kz)}.$$

Taking the logarithmic derivative yields:

$$\frac{zB'_k(z)}{B_k(z)} = \frac{z[w^*(kz)]'}{w^*(kz)} - \frac{z[w(kz)]'}{w(kz)}.$$

Using Lemma 3.5, we get

$$\operatorname{Re} \left[\frac{z \{w^*(kz)\}'}{w^*(kz)} \right] - \operatorname{Re} \left[\frac{z \{w(kz)\}'}{w(kz)} \right] = |B'_k(z)|. \quad (17)$$

Also,

$$w^*(kz) = z^n \overline{w\left(\frac{k}{z}\right)}.$$

So that,

$$[w^*(kz)]' = nz^{n-1} \overline{w\left(\frac{k}{z}\right)} - z^n \overline{w'\left(\frac{k}{z}\right)} \frac{k}{z^2}.$$

For $|z| = 1$, a straightforward calculation shows:

$$\operatorname{Re} \left[\frac{z \{w^*(kz)\}'}{w^*(kz)} \right] = n - \operatorname{Re} \left[\frac{z \{w(kz)\}'}{w(kz)} \right]. \quad (18)$$

Combining equation (17) and (18) gives the required result. $\square$

Lemma 3.7. For $|z| = 1$,

$$\left| B_{1/k} \left(\frac{z}{k} \right) \right| = \frac{1}{|B_{1/k}(kz)|}, \quad (19)$$

and

$$\left| B \left(\frac{z}{k} \right) \right| = \frac{1}{|B(kz)|}. \quad (20)$$

Proof. We have

$$\begin{aligned} \left| B_{1/k} \left(\frac{z}{k} \right) \right| &= \prod_{j=1}^n \left| \frac{1 - \bar{a}_j k(z/k)}{(z/k) - a_j k} \right| \\ &= \prod_{j=1}^n \left| \frac{k - \bar{a}_j z k}{z - a_j k^2} \right|. \end{aligned}$$

For $|z| = 1$, this simplifies to:

$$\left| B_{1/k} \left(\frac{z}{k} \right) \right| = \prod_{j=1}^n \left| \frac{kz - a_j k}{1 - \bar{a}_j k^2 z} \right| = \frac{1}{|B_{1/k}(kz)|}.$$

Equation (20) follows by a similar argument, so we omit the proof. $\square$

4. Proofs of the Theorems

Proof of Theorem 1. Since $r(z)$ has all its zeros in $|z| \leq k$, where $k \geq 1$, it can be written as

$$r(z) = \frac{\prod_{j=1}^n (z - z_j)}{w(z)}, \quad |z_j| \leq k, \quad k \geq 1.$$

For $z_j = r_j e^{i\theta_j}$ with $r_j \leq k$, $j = 1, 2, \dots, n$ and for points $e^{i\theta}$ that are not zeros of $r(z)$, we have

$$\left| \frac{r(k^2 e^{i\theta})}{r(e^{i\theta})} \right| = \prod_{j=1}^n \left| \frac{k^2 e^{i\theta} - r_j e^{i\theta_j}}{e^{i\theta} - r_j e^{i\theta_j}} \right| \cdot \left| \frac{w(e^{i\theta})}{w(k^2 e^{i\theta})} \right|. \quad (21)$$

We claim that for $|z| = 1$,

$$\left| \frac{w(z)}{w(k^2 z)} \right| = \frac{1}{k^n} |B_k(kz)|, \quad (22)$$

with equality if and only if $k = 1$. Observe that

$$|1 - \overline{a_j} z| = |\overline{1 - \overline{a_j} z}| = |1 - a_j \overline{z}|.$$

For $|z| = 1$, this becomes

$$\frac{1}{k} \left| \frac{k - \overline{a_j} z}{k^2 z - a_j} \right| = \left| \frac{z - a_j}{k^2 z - a_j} \right|.$$

By definition of $B_k(kz)$ and $w(z)$, this simplifies to:

$$\frac{1}{k^n} |B_k(kz)| = \left| \frac{w(z)}{w(k^2 z)} \right| \quad \text{for } |z| = 1.$$

Equality holds if and only if $k = 1$, completing the proof of (22).

We also claim that for $k \geq 1$ and $r_j \leq k$, $j = 1, 2, \dots, n$,

$$\left| \frac{k^2 e^{i\theta} - r_j e^{i\theta_j}}{e^{i\theta} - r_j e^{i\theta_j}} \right| \geq k, \quad (23)$$

with equality if and only if $k = 1$.

To prove our claim, we need to minimize the function

$$h(\phi) = |k^2 e^{i\theta} - r_j e^{i\theta_j}|^2 - k^2 |e^{i\theta} - r_j e^{i\theta_j}|^2, \quad k \geq 1, \quad r_j \leq k \quad (j = 1, 2, \dots, n).$$

A straightforward computation yields:

$$h(\phi) = (k^2 - 1)(k^2 - r_j^2).$$

For $k \geq 1$, and $r_j \leq k$ ($j = 1, 2, \dots, n$), we have

$$(k^2 - 1)(k^2 - r_j^2) \geq 0,$$

with equality if and only if $k = 1$. This establishes (23).

Therefore, from equation (21), we obtain

$$|r(k^2 e^{i\theta})| \geq |B_k(k e^{i\theta})| |r(e^{i\theta})|, \quad (24)$$

for all points $e^{i\theta}$ with $0 \leq \theta < 2\pi$ that are not zeros of $r(z)$. Since inequality (24) is trivially true for points $e^{i\theta}$ that are zeros of $r(z)$, we conclude that for $|z| = 1$,

$$|r(k^2z)| \geq |B_k(kz)||r(z)|. \quad (25)$$

Let $G(z) = r(kz)$ and $H(z) = B_k(z)G\left(\frac{1}{z}\right)$. By Lemma 3.2, for $R \geq 1$ and $|z| = 1$,

$$|G(Rz)| + |H(Rz)| \leq (|B_k(Rz)| + 1) \max_{|z|=1} |G(z)|.$$

In particular, taking $R = k \geq 1$, we obtain for $|z| = 1$,

$$|G(kz)| + |H(kz)| \leq (|B_k(kz)| + 1) \max_{|z|=1} |G(z)|.$$

This implies for $|z| = 1$,

$$|r(k^2z)| + |B_k(kz)||r(z)| \leq (|B_k(kz)| + 1) \max_{|z|=1} |r(kz)|.$$

Using inequality (25), we obtain for $|z| = 1$,

$$2|B_k(kz)||r(z)| \leq (|B_k(kz)| + 1) \max_{|z|=1} |r(kz)|.$$

This completes the proof of Theorem 1. $\square$

Proof of Theorem 2. Let $G(z) = r(kz)$. Then, $G(z)$ has all its zeros in $|z| \leq 1$ and can be expressed as:

$$G(z) = \frac{\prod_{j=1}^n (kz - z_j)}{w(kz)}, \quad |z_j| \leq k \text{ for } j = 1, 2, 3, \dots, n.$$

By logarithmic differentiation, we obtain

$$\frac{G'(z)}{G(z)} = \sum_{j=1}^n \frac{k}{kz - z_j} - \frac{(w(kz))'}{w(kz)}.$$

For points $z = e^{i\theta}$ ($0 \leq \theta < 2\pi$), that are not zeros of $G(z)$, we have

$$\operatorname{Re} \left(e^{i\theta} \frac{G'(e^{i\theta})}{G(e^{i\theta})} \right) = \sum_{j=1}^n \operatorname{Re} \left(k \frac{e^{i\theta}}{ke^{i\theta} - z_j} \right) - \operatorname{Re} \left(\frac{e^{i\theta} (w(ke^{i\theta}))'}{w(ke^{i\theta})} \right).$$

Applying Lemma 3.4 and 3.6, we derive the lower bound

$$\operatorname{Re} \left(e^{i\theta} \frac{G'(e^{i\theta})}{G(e^{i\theta})} \right) \geq \sum_{j=1}^n \frac{k}{k + |z_j|} - \frac{n - |B'_k(e^{i\theta})|}{2}.$$

Now,

$$\left| e^{i\theta} \frac{G'(e^{i\theta})}{G(e^{i\theta})} \right| \geq \operatorname{Re} \left(e^{i\theta} \frac{G'(e^{i\theta})}{G(e^{i\theta})} \right) \geq \sum_{j=1}^n \frac{k}{k + |z_j|} - \frac{n - |B'_k(e^{i\theta})|}{2}.$$

This inequality immediately implies,

$$|G'(e^{i\theta})| \geq \left\{ \sum_{j=1}^n \frac{k}{k + |z_j|} - \frac{n - |B'_k(e^{i\theta})|}{2} \right\} |G(e^{i\theta})|,$$

for points $e^{i\theta}$ ($0 \leq \theta < 2\pi$) that are not zeros of $G(z)$. But above inequality is trivially true for points $e^{i\theta}$ that are zeros of $G(z)$. Consequently, for all z with $|z| = 1$, we have the uniform lower bound:

$$|G'(z)| \geq \left\{ \sum_{j=1}^n \frac{k}{k + |z_j|} - \frac{n - |B'_k(z)|}{2} \right\} |G(z)|.$$

Using the value of $G(z)$, we obtain for all z with $|z| = 1$,

$$|(r(kz))'| \geq \left\{ \sum_{j=1}^n \frac{k}{k + |z_j|} - \frac{n - |B'_k(z)|}{2} \right\} |r(kz)|.$$

Applying Lemma 3.3, we obtain for $|z| = 1$,

$$|B(kz)| \max_{|z|=1} |r'(z)| \geq \left\{ \sum_{j=1}^n \frac{k}{k + |z_j|} - \frac{n - |B'_k(z)|}{2} \right\} |r(kz)|.$$

Rearranging terms, we arrive at the final lower bound for $|z| = 1$:

$$\max_{|z|=1} |r'(z)| \geq \frac{1}{2|B(kz)|} \left[|B'_k(z)| + \sum_{j=1}^n \frac{k - |z_j|}{k + |z_j|} \right] |r(kz)|. \quad \square$$

Proof of Theorem 3. Since all the zeros of $r(z)$ lie in $|z| \geq k$, all the zeros of $r^*(z)$ lie in $|z| \leq 1/k$, where $1/k \geq 1$. Applying Theorem 2 to $r^*(z)$, we obtain for $|z| = 1$,

$$\max_{|z|=1} |r^{**}(z)| \geq \frac{1}{2|B(z/k)|} \left\{ |B'_{1/k}(z)| + \sum_{j=1}^n \frac{1 - k|z_j|}{1 + k|z_j|} \right\} |r^*(z/k)|.$$

Using Lemma 3.7, this simplifies to for $|z| = 1$,

$$\max_{|z|=1} |r^{**}(z)| \geq \frac{|B(kz)|}{2} \left\{ |B'_{1/k}(z)| + \sum_{j=1}^n \frac{1 - k|z_j|}{1 + k|z_j|} \right\} |B(z/k)| |r(kz)| = \frac{1}{2} \left\{ |B'_{1/k}(z)| + \sum_{j=1}^n \frac{1 - k|z_j|}{1 + k|z_j|} \right\} |r(kz)|. \quad (26)$$

By hypothesis $|r'(z)|$ and $|r^{**}(z)|$ becomes maximum at the same point on $|z| = 1$. Let $\max_{|z|=1} |r'(z)| = |r'(e^{i\zeta})|$, for some $0 \leq \zeta < 2\pi$. Then, $\max_{|z|=1} |r^{**}(z)| = |r^{**}(e^{i\zeta})|$ and hence by Lemma 3.1,

$$|r'(e^{i\zeta})| + |r^{**}(e^{i\zeta})| \leq |B'(e^{i\zeta})| \max_{|z|=1} |r(z)|.$$

Combining this with (26), we deduce

$$\begin{aligned} \max_{|z|=1} |B'(z)| \max_{|z|=1} |r(z)| &\geq \max_{|z|=1} |r'(z)| + \max_{|z|=1} |r^{**}(z)| \\ &\geq \max_{|z|=1} |r'(z)| + \frac{1}{2} \left\{ |B'_{1/k}(z)| + \sum_{j=1}^n \frac{1 - k|z_j|}{1 + k|z_j|} \right\} |r(kz)|. \end{aligned}$$

This gives after simplification,

$$\max_{|z|=1} |r'(z)| \leq \left[\max_{|z|=1} |B'(z)| \max_{|z|=1} |r(z)| - A |r(kz)| \right], \quad (27)$$

where

$$A = \frac{1}{2} \left[|B'_{1/k}(z)| + \sum_{j=1}^n \frac{1 - k|z_j|}{1 + k|z_j|} \right].$$

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References

- [1] I. Ahmad, Ab Liman, W.M. Shah, A Note on Rational Functions with Prescribed Poles and Restricted Zeros, *Vietnam J. Math.* 43 (2015) 159–162.
- [2] A. Aziz, Inequalities for the derivative of a polynomial, *Proc. Amer. Math. Soc.* 89(2) (1983) 259–266.
- [3] A. Aziz and N.A. Rather, Growth of maximum modulus of rational functions with prescribed poles, *Math. Inequal. Appl.* 2 (1999) 165–173.
- [4] A. Aziz and W. M. Shah, Some refinements of Bernstein-Type inequalities for rational functions, *Glas. Mat.* 32(52) (1997) 29–37.
- [5] A. Aziz and B. A. Zarger, Some properties of rational functions with prescribed poles, *Canad. Math. Bull.* 44 (1999) 417–426.
- [6] S. N. Bernstein, Sur l'ordre de la meilleure approximation des fonctions continues par des polynômes de degré donné, *Mem. Acad. R. Belg.* 4 (1912) 1–103.
- [7] R.B. Gardner, N.K. Govil, G.V. Milovanović, Extremal Problems and Inequalities of Markov-Bernstein Type for Algebraic Polynomials, *Mathematical Analysis and Its Applications*. Elsevier/Academic Press, London, 2022.
- [8] A. Giroux, Q.I. Rahman and G. Schmeisser, On Bernstein's Inequality, *Can. J. Math.* 31(2)(1979) 347–353.
- [9] N.K. Govil, On a theorem of S. Bernstein, *J. Math. Phys. Sci.* 14 (1980) 183–187.
- [10] N.K. Govil, On the derivative of a polynomial, *Proc. Amer. Math. Soc.* 41 (1973) 543–546.
- [11] P. D. Lax, Proof of a conjecture of P. Erdős on the derivative of a polynomial, *Bull. Amer. Math. Soc. (N.S)* 50 (1944) 509–513.
- [12] Xin Li, R. N. Mohapatra and R. S. Rodriguez, Bernstein inequalities for rational functions with prescribed poles, *J. London Math. Soc.* 51 (1995) 523–531.
- [13] M.A. Malik, On the derivative of a polynomial, *J. London Math. Soc.* 1 (1969) 57–60.
- [14] A. Mir and S. Hans, Inequalities concerning rational functions in the complex domain, *Sib. Math. J.* 63 (5) (2022) 1012–1022.
- [15] I. Qasim, A. Liman, Bernstein type Inequalities for Rational Functions, *Indian J. Pure Appl. Math.* 46(3) (2015) 337–348.
- [16] W.M. Shah, A Generalization of a Theorem of Paul Turán, *Journal of Ramanujan Society* 1 (1996) 29–35.
- [17] P. Turán, Über die ableitung von polynomen, *Compos. Math.* 7 (1939) 89–95.