Filomat 39:20 (2025), 6899–6918
https://doi.org/10.2298/FIL2520899R

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Invariance of convex functions associated with a cardioid domain

Mohsan Raza[a,*], Amina Riaz[b], Derek K. Thomas[c]

[a]*Department of Mathematics, Government College University Faisalabad, Pakistan*
[b]*Department of Mathematics, COMSATS University Islamabad, Lahore Campus, Pakistan*
[c]*Swansea University, Bay Campus, Swansea, SA1 8EN, United Kingdom*

Abstract. Sharp bounds are given for some functional including the second and third Hankel determinants, and the Zalcman functional $a_2a_3 - a_4$ for convex functions related to a cardioid domain. We show that the sharp bounds for the third Hankel determinant for convex functions related to the cardioid domain and its inverse function are the same, but the corresponding sharp bounds for the second Hankel determinant and the Zalcman functional differ. The results presented therefore provide a new significant example of convex invariance, together with examples of non-invariance.

1. Introduction

Let $\mathcal{A}$ denote the class of functions f which are analytic in the open unit disc $\mathbb{D} := \{z : |z| < 1,\ z \in \mathbb{C}\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$ with Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D} \tag{1}$$

furthermore, let $\mathcal{S}$ represent the subclass of $\mathcal{A}$ which contains univalent functions in $\mathbb{D}$.

Let f and g be two functions which are analytic. Then, if there is a function w, also called the Schwarz function, with the condition that it is an analytic self map in $\mathbb{D}$ with $w(0) = 0$ such that $f(z) = g(w(z))$, then f is subordinated by g and symbolically expressed as $f \prec g$. In addition, the equivalent connection indicated below applies if g is one-to-one in $\mathbb{D}$:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{D}) \subseteq g(\mathbb{D}).$$

We denote by $\mathcal{B}$ the class of analytic functions w, known as Schwarz function, such that it maps $\mathbb{D}$ onto itself with $w(0) = 0$. Assuming that the function φ is symmetric about the real axis, analytic, univalent, and

2020 *Mathematics Subject Classification*. Primary 30C45; Secondary 30C50.
Keywords. Invariance of coefficients, convex functions, cardioid domain, Hankel determinant, Zalcman functional.
Received: 17 December 2024; Accepted: 25 May 2025
Communicated by Miodrag Mateljević
Research supported by HEC grant: 20-16367/NRPU/R&D/HEC/2021-2020.
* Corresponding author: Mohsan Raza
Email addresses: mohsan976@yahoo.com (Mohsan Raza), aymnariaz@gmail.com (Amina Riaz), d.k.thomas@swansea.ac.uk (Derek K. Thomas)
ORCID iDs: https://orcid.org/0000-0001-7466-7930 (Mohsan Raza), https://orcid.org/0000-0002-3236-0276 (Amina Riaz), https://orcid.org/0000-0002-9930-7357 (Derek K. Thomas)

starlike with respect to $\varphi(0) = 1$ with $\varphi'(0) > 0$ in $\mathbb{D}$. Ma and Minda [16] used the concept of subordination and the function φ to generalize the well-known classes of convex and starlike functions as follows:

$$\mathcal{S}^*(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}$$

and

$$C(\varphi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}.$$

The classes $\mathcal{S}^* = \mathcal{S}^*(\frac{1+z}{1-z})$ and $C = C(\frac{1+z}{1-z})$ are familiar classes of starlike and convex functions.

Sharma *et al.* [23] introduced the class of starlike functions associated with a cardioid domain, which is defined as

$$\mathcal{S}^*_{car} := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2 \right\}.$$

We provide here the sharp bounds for several functionals, such as the second and third Hankel determinants for the class C_{car} defined by

$$C_{car} := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2 \right\},$$

together with corresponding sharp bounds for the inverse coefficients of f.

Any univalent function f has an inverse function f^{-1} defined on some disc $|w| \leq 1/4 \leq r(f)$, with a Taylor series expansion

$$f^{-1}(w) = w + A_2 w^2 + A_3 w^3 + \cdots . \tag{2}$$

For analytic functions $f \in \mathcal{A}$, Pommerenke [17] introduced the qth Hankel determinant, which is given by

$$H_q(n)(f) := \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & \dots & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix},$$

where $n \geq 1$ and $q \geq 1$. The Hankel determinants $H_2(2)(f) = a_2a_4 - a_3^2$ and $H_2(3)(f) = a_3a_5 - a_4^2$ of order 2 have been examined by numerous authors (see e.g., [5, 7, 13, 22, 24, 25]), on the other hand, $H_2(1)(f) = a_2^2 - a_3$ is classical. Nonetheless, establishing precise bounds for

$$\left|H_3(1)(f)\right| = \left|2a_2a_3a_4 - a_3^3 - a_4^2 + a_3a_5 - a_2^2a_5\right| \tag{3}$$

of order 3 is a more challenging problem, and several authors have obtained non-sharp bounds for $|H_3(1)(f)|$, e.g., [1, 3, 19, 28]. We see that certain authors [2, 8–10, 12, 21] have found sharp bounds for $|H_3(1)(f)|$ for specific subclasses of univalent functions, based on a result in [11].

It was first time showed by Libera and Zlotkiewicz [14] that the inverse coefficients A_n preserves classical inequality $|a_n| \leq 1$ when $2 \leq n \leq 7$ for the class C of convex functions. In 2016, Thomas and Verma [27] showed some invariance properties amongst the class of strongly convex functions. Invariance property among the coefficients functionals of the subclass of convex functions related with sigmoid functions is studied in [20]. Recently, Thomas has given detailed discussion about this property for some coefficient functional for the class of convex functions and its subclasses. In this paper a question is raised about this property for the subclass of convex functions related with cardioid domain.

In this paper we find sharp bounds for $|H_2(2)(f)|$, $|H_2(2)(f^{-1})|$, $|H_3(1)(f)|$, $|H_3(1)(f^{-1})|$ and the Zalcman functionals $|a_2a_3 - a_4|$ and $|A_2A_3 - A_4|$ for the class $\mathcal{C}_{car}$, demonstrating some new and interesting invariance and non-invariance properties between corresponding functionals recently discussed in [26]. In particular, we show that for the class $\mathcal{C}_{car}$, bounds on second Hankel determinants $|H_2(2)(f)|$, $|H_2(2)(f^{-1})|$ and Zalcman functionals $|a_2a_3 - a_4|$ $|A_2A_3 - A_4|$ are variant but bound on third Hankel determinants $|H_3(1)(f)|$, $|H_3(1)(f^{-1})|$ are invariant. We give comparison at the end of the paper.

The class of analytic functions p defined for $z \in \mathbb{D}$, denoted by $\mathcal{P}$, is given by

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \tag{4}$$

with positive real part in $\mathbb{D}$.

We will use a general lemma by Choi et al. as well as the following two lemmas on the coefficients of functions in $\mathcal{P}$.

Lemma 1.1. *[11, 15] Let $p \in \mathcal{P}$, and be given by (4). Then*

$$\begin{aligned}
2c_2 &= c_1^2 + \xi(4 - c_1^2), &(5)\\
4c_3 &= c_1^3 + 2(4 - c_1^2)c_1\xi - (4 - c_1^2)c_1\xi^2 + 2(4 - c_1^2)(1 - |\xi|^2)\mu, &(6)\\
8c_4 &= c_1^4 + (4 - c_1^2)\xi(c_1^2(\xi^2 - 3\xi + 3) + 4\xi) - 4(4 - c_1^2)(1 - |\xi|^2)(c_1(\xi - 1)\mu &(7)\\
&\quad + \overline{\xi}\mu^2 - (1 - |\mu|^2)\gamma),
\end{aligned}$$

for some γ, ξ and μ such that $|\gamma| \le 1$, $|\xi| \le 1$ and $|\mu| \le 1$.

Lemma 1.2. *[4]Let $p \in \mathcal{P}$, and be given by (4) with $c_1 \ge 0$. Then*

$$\begin{aligned}
c_1 &= 2\zeta_1, &(8)\\
c_2 &= 2\zeta_1^2 + 2(1 - \zeta_1^2)\zeta_2, &(9)\\
c_3 &= 2\zeta_1^3 + 4(1 - \zeta_1^2)\zeta_1\zeta_2 - 2(1 - \zeta_1^2)\zeta_1\zeta_2^2 + 2(1 - \zeta_1^2)(1 - |\zeta_2|^2)\zeta_3, &(10)
\end{aligned}$$

for some $\zeta_1 \in [0, 1]$ and $\zeta_2, \zeta_3 \in \overline{\mathbb{D}}$.

Lemma 1.3. *[6]. Let $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \le 1\}$, and for real numbers A, B, C, let*

$$Y(A, B, C) := \max\left\{|A + Bz + Cz^2| + 1 - |z|^2 : z \in \overline{\mathbb{D}}\right\}.$$

If $AC \ge 0$, then

$$Y(A, B, C) = \begin{cases} |A| + |B| + |C|, & |B| \ge 2(1 - |C|), \\ 1 + |A| + \dfrac{B^2}{4(1 - |C|)}, & |B| < 2(1 - |C|). \end{cases}$$

Lemma 1.4. *[18]Let w be a Schwarz function given by $w(z)=\sum\limits_{n=0}^{\infty} w_n z^n$, and*

$$\psi(\mu,\nu)=\left|w_3+\mu w_1 w_2+\nu w_1^3\right|.$$

Then $\psi(\mu,\nu)\leq \frac{2}{3}\left(|\mu|+1\right)\left(\frac{|\mu|+1}{3(|\mu|+1+\nu)}\right)^{1/2}$ *if* $(\mu,\nu)\in D_9$, *where*

$$D_9=\left\{(\mu,\nu):|\mu|\geq 2,\quad -\frac{2}{3}\left(|\mu|+1\right)\leq \nu\leq \frac{2|\mu|\left(|\mu|+1\right)}{\mu^2-2|\mu|+4}\right\}.$$

2. General coefficient functionals for the class of convex functions related with cardiod

Theorem 2.1. *Let $f\in C_{car}$ and be given by* (1). *Then*

$$\left|H_2(2)(f)\right|\leq \frac{40}{783}.$$

The inequality is sharp for the function

$$\begin{aligned} f_0(z) &= \int_0^z\left(\exp\int_0^s \frac{116\sqrt{174}-3712t+1682t^3}{3(-29+t\sqrt{174})^2}dt\right)ds \\ &= z+\frac{2}{87}\sqrt{174}z^2-\frac{8}{87}z^3-\frac{242}{22707}\sqrt{174}z^4-\cdots. \end{aligned} \tag{11}$$

Proof. Let $f\in C^*_{car}$. Then for $w\in\mathcal{B}$, we can write

$$1+\frac{zf''(z)}{f'(z)}=1+\frac{4}{3}w(z)+\frac{2}{3}w^2(z). \tag{12}$$

Let $p\in\mathcal{P}$, thus, utilising the subordination notion, we can have

$$w(z)=\frac{p(z)-1}{p(z)+1}. \tag{13}$$

Let p be of the form (4). From (12) and (13), comparing coefficients of like powers we obtain

$$a_2=\frac{1}{3}c_1, \tag{14}$$

$$a_3=\frac{1}{9}c_2+\frac{5}{108}c_1^2, \tag{15}$$

$$a_4=-\frac{1}{648}c_1^3+\frac{1}{36}c_1c_2+\frac{1}{18}c_3, \tag{16}$$

$$a_5=\frac{7}{9720}c_1^4-\frac{1}{180}c_2c_1^2+\frac{7}{540}c_1c_3+\frac{1}{360}c_2^2+\frac{1}{30}c_4. \tag{17}$$

Using (14) – (16), we have

$$H_2(2)(f)=-\frac{31c_1^4}{11664}-\frac{c_1^2c_2}{972}+\frac{c_1c_3}{54}-\frac{c_2^2}{81}.$$

We can suppose that $t:=\zeta_1$, so that $0\leq t\leq 1$, since the class C_{car} and the functional $H_2(2)(f)$ are rotationally invariant. Then using (8) – (10) and after simple computation we obtain

$$H_2(2)(f)=-\frac{19t^4}{729}+\frac{10}{243}t^2(1-t^2)\zeta_2-\frac{2}{81}\left(t^2+2\right)(1-t^2)\zeta_2^2+\frac{2}{27}t(1-t^2)(1-|\zeta_2|^2)\zeta_3=\varphi.$$

When $t = 0$, $\left|\varphi\right| \leq \frac{4}{81} < \frac{40}{783}$ and at $t = 1$, $\left|\varphi\right| = \frac{19}{729} < \frac{40}{783}$.

Then for $0 < t < 1$

$$\left|H_2(2)(f)\right| \leq \frac{2}{27}t(1-t^2)\left\{\left|A + B\zeta_2 + C\zeta_2^2\right| + (1 - |\zeta_2|^2\right\}, \quad \zeta_2 \in \overline{\mathbb{D}},$$

where $A = \frac{-19t^3}{54(1-t^2)}$, $B = \frac{5t}{9}$ and $C = -\frac{\left(t^2+2\right)}{3t}$, so $AC > 0$ for $t \in (0,1)$. Note that in Lemma 1.3

$$|B| - 2(1 - |C|) = \frac{11t^2 - 18t + 12}{9t} > 0, \quad 0 < t < 1.$$

Then

$$\frac{2}{27}t(1-t^2)\left(|A| + |B| + |C|\right) = \frac{-29t^4 + 12t^2 + 36}{729} = g(t).$$

We see that $\max\left\{g(t) : t \in [0,1]\right\} = g\left(\sqrt{\frac{6}{29}}\right) = \frac{40}{783}$, and so from Lemma 1.3, the result follows.

For sharpness, consider

$$p_0(z) = \frac{1 - z^2}{1 + 2t_0 z + z^2},$$

where $t_0 = \sqrt{\frac{6}{29}}$. Let

$$w_0(z) = \frac{p_0(z) - 1}{p_0(z) + 1} = \frac{z\left(29z - \sqrt{174}\right)}{\left(-29 + \sqrt{174}z\right)} = \frac{1}{29}\sqrt{174}z - \frac{23}{29}z^2 - \cdots$$

and

$$q_0(z) = 1 + \frac{4}{3}w_0(z) + \frac{2}{3}w_0^2(z).$$

We see that for $z \in \mathbb{D}$, $w_0(0) = 0$ and $|w_0(z)| < 1$. Then $f_0 \in C_{car}$ given by (11) and

$$f_0(z) = \int_0^z \left(\exp\int_0^s \frac{q_0(t) - 1}{t}dt\right)ds = z + \frac{2}{87}\sqrt{174}z^2 - \frac{8}{87}z^3 - \frac{242}{22707}\sqrt{174}z^4 - \cdots.$$

Here $a_2 = \frac{2}{87}\sqrt{174}$, $a_3 = -\frac{8}{87}$ and $a_4 = -\frac{242}{22707}\sqrt{174}$, and it is easy to see that $H_2(2)(f_0) = \frac{-40}{783}$. It brings the proof to a conclusion. □

Theorem 2.2. *Let $f \in C_{car}$ and be given by (1). Then*

$$|a_2 a_3 - a_4| \leq \frac{32}{81}\sqrt{\frac{2}{19}}.$$

The inequality is sharp for the function

$$\begin{aligned} f_1(z) &= \int_0^z \left(\exp \int_0^s \frac{2(76\sqrt{38}) - 874t + 361t^3)}{3(-19+2t\sqrt{38})^2} dt\right) ds \\ &= z + \frac{4}{57}\sqrt{38}z^2 + \frac{22}{513}z^3 - \frac{520}{29241}\sqrt{38}z^4 - \cdots . \end{aligned} \tag{18}$$

Proof. From (14) – (16) , we obtain

$$a_2a_3 - a_4 = \frac{11c_1^3}{648} + \frac{c_1c_2}{108} - \frac{c_3}{18}.$$

Since the class C_{car} and the functional $a_2a_3 - a_4$ are rotationally invariant, we again suppose that $t := \zeta_1$, so that $0 \le t \le 1$. Then using (8) – (10) and after simple computations we write

$$a_2a_3 - a_4 = \frac{5t^3}{81} - \frac{5}{27}t(1-t^2)\zeta_2 + \frac{1}{9}(1-t^2)t\zeta_2^2 - \frac{1}{9}(1-t^2)(1-|\zeta_2|^2)\zeta_3 = \varphi,$$

where ζ_2 and ζ_3 are such that $|\zeta_2| \le 1, |\zeta_3| \le 1$.

Firstly, take $t = 1$, then $\left|\varphi\right| = \dfrac{5}{81} < \frac{32}{81}\sqrt{\frac{2}{19}}$ and when $t = 0$, $\left|\varphi\right| \le \frac{1}{9} < \frac{32}{81}\sqrt{\frac{2}{19}}$. Next we take $t \in (0, 1)$ and applying Lemma 1.3 after using triangle inequality, we may write

$$|a_2a_3 - a_4| \le \frac{1}{9}(1-t^2)\left(\left|A + B\zeta_2 + C\zeta_2^2\right| + 1 - |\zeta_2|^2\right), \qquad \zeta_2 \in \overline{\mathbb{D}},$$

where $A = \dfrac{-5t^3}{9(1-t^2)}$, $B = \dfrac{5t}{3}$ and $C = -t$, then clearly $AC > 0$. Note now that

$$|B| - 2(1 - |C|) = \frac{11}{3}t - 2 \ge 0, \quad t \in \left[\frac{6}{11}, 1\right),$$

and so using Lemma 1.3, we have

$$|a_2a_3 - a_4| \le \frac{1}{9}(1-t^2)\left(|A| + |B| + |C|\right) = \frac{t(24-19t^2)}{81} := g(t).$$

Clearly $g'(t) = 0$ has two real roots $t_0 = -2\sqrt{\dfrac{2}{19}}$ and $t_1 = 2\sqrt{\dfrac{2}{19}}$. Hence $\max\{g(t) : t \in [0, 1]\} = g(t_1) = \dfrac{32}{81}\sqrt{\dfrac{2}{19}}$, which gives the required result.

For $t \in \left(0, \dfrac{6}{11}\right)$, we have $B < 2(1 - |C|)$, and using Lemma 1.3, we have

$$\left|\phi\right| \le \frac{1}{9}(1-t^2)\left(1 + |A| + \frac{B^2}{4(1-|C|)}\right) := g_1(t),$$

where

$$g_1(t) = \frac{45t^3 - 11t^2 + 36}{324}.$$

Now $g'(t) = \dfrac{135t^2 - 22t}{324}$ and $g'(t) = 0$ at $t_2 = 0$ and $t_3 = \dfrac{22}{135}$. We see that $g'(t) < 0$ for $t \in \left(0, \dfrac{22}{135}\right)$ and positive for $t \in \left(\dfrac{22}{135}, \dfrac{6}{11}\right)$, which implies that $\max\{g_1(t) : t \in [0,1]\} < g_1\left(\dfrac{6}{11}\right) = \dfrac{1480}{11979} < \dfrac{32}{81}\sqrt{\dfrac{2}{19}}$, as required.

For sharpness, consider the function

$$p_1(z) = \frac{1 - z^2}{1 + 2t_1 z + z^2},$$

where $t_1 = 2\sqrt{\dfrac{2}{19}}$. Let

$$w_1(z) = \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{z\left(19z - 2\sqrt{38}\right)}{\left(-19 + 2\sqrt{38}z\right)} = \frac{2}{19}\sqrt{38}z - \frac{11}{19}z^2 - \cdots$$

and

$$q_1(z) = 1 + \frac{4}{3}w_1(z) + \frac{2}{3}w_1^2(z).$$

We see that for $z \in \mathbb{D}$, $w_1(0) = 0$ and $|w_1(z)| < 1$. Then the function $f_1 \in C_{car}$ given by (18) and

$$f_1(z) = \int_0^z \left(\exp \int_0^s \frac{q_1(t) - 1}{t} dt\right) ds = z + \frac{4}{57}\sqrt{38}z^2 + \frac{22}{513}z^3 - \frac{520}{29241}\sqrt{38}z^4 - \cdots.$$

Here $a_2 = \dfrac{4}{57}\sqrt{38}$, $a_3 = \dfrac{22}{513}$ and $a_4 = -\dfrac{520}{29241}\sqrt{38}$. It is easy to see that $a_2 a_3 - a_4 = \dfrac{32}{81}\sqrt{\dfrac{2}{19}}$, and it brings the proof to a conclusion. □

Theorem 2.3. *Let $f \in C_{car}$ and be of the form* (1)*. Then*

$$\left|H_3(1)(f)\right| \le \frac{1}{81}. \tag{19}$$

The inequality is sharp for f_2 given by

$$f_2(z) = \int_0^z \left(\exp\left(\frac{4s^3}{9} + \frac{s^6}{9}\right)\right) ds = z + \frac{1}{9}z^4 + \cdots. \tag{20}$$

Proof. Since the class C_{car} is invariant under the rotation, we again assume that $c := c_1 \in [0,2]$ and substituting from (14)-(17) into (3) obtain

$$H_3(1)(f) = \frac{1}{12597120}\left[\begin{array}{l} 31104cc_2c_3 - 2468c^6 - 13392c_2^3 - 15444c^2c_2^2 + 6984c_2c^4 \\ +13176c^3c_3 - 27216c^2c_4 + 46656c_2c_4 - 38880c_3^2 \end{array}\right].$$

Using the equalities (5)-(7) and after some simplification, we obtain

$$H_3(1)(f) = \frac{1}{12597120}\left(v_1(c,\xi) + v_2(c,\xi)\mu + v_3(c,\xi)\mu^2 + \Psi(c,\xi,\mu)\gamma\right),$$

where $\gamma, \mu, \xi \in \overline{\mathbb{D}}$, and

$$\begin{aligned}
v_1(c,\xi) :=& -245c^6 + (4-c^2)((4-c^2)(486\xi^4c^2 - 2079\xi^2c^2 - 1242\xi^3c^2 \\
&+ 4968\xi^3) - 486c^4\xi^3 + 738c^4\xi - 864c^4\xi^2 - 1944\xi^2c^2),\\
v_2(c,\xi) :=& -108c(4-c^2)(1-|\xi|^2)(-c^2(25+18\xi) + 18(4-c^2)\xi^2),\\
v_3(c,\xi) :=& 1944(4-c^2)(1-|\xi|^2)(-(4-c^2)(5+|\xi|^2) + c^2\overline{\xi}),\\
\Psi(c,\xi,\mu) :=& 1944(4-c^2)(1-|\xi|^2)(1-|\mu|^2)(6\xi(4-c^2) - c^2).
\end{aligned}$$

Writing $s := |\xi|$, $t := |\mu|$ and using $|\gamma| \le 1$, we have

$$\begin{aligned}
|H_3(1)(f)| &\le \frac{1}{12597120}\left(|v_1(c,\xi)| + |v_2(c,\xi)|t + |v_3(c,\xi)|t^2 + |\Psi(c,\xi,\mu)|\right)\\
&\le E(c,s,t),
\end{aligned}$$

where

$$E(c,s,t) := \frac{1}{12597120}\left(g_1(c,s) + g_2(c,s)t + g_3(c,s)t^2 + g_4(c,s)(1-t^2)\right),$$

with

$$\begin{aligned}
g_1(c,s) :=& 245c^6 + (4-c^2)((4-c^2)(486s^4c^2 + 2079s^2c^2 + 1242s^3c^2\\
&+ 4968s^3) + 486c^4s^3 + 738c^4s + 864c^4s^2 + 1944s^2c^2),\\
g_2(c,s) :=& 108c(4-c^2)(1-s^2)(c^2(25+18s) + 18(4-c^2)s^2),\\
g_3(c,s) :=& 1944(4-c^2)(1-s^2)((4-c^2)(5+s^2) + c^2s),\\
g_4(c,s) :=& 1944(4-c^2)(1-s^2)(6s(4-c^2) + c^2).
\end{aligned}$$

Therefore, across the closed cuboid $\Lambda : [0,2] \times [0,1] \times [0,1]$, we must maximise $E(c,s,t)$. In order to accomplish this, we identify the maximum values on the twelve edges, inside Λ, and inside the six faces.

I. Firstly, we demonstrate that the interior of Λ does not include any critical points.

Let $(c,s,t) \in (0,2) \times (0,1) \times (0,1)$. Differentiating $E(c,s,t)$ with respect to t and after some computations we write

$$\begin{aligned}
\frac{\partial E}{\partial t} =& \frac{1}{116640}(4-c^2)(1-s^2)[36t(s-1)((4-c^2)(s-5) + c^2)\\
&+ c(18s^2(4-c^2) + c^2(18s+25))].
\end{aligned}$$

So that $\dfrac{\partial E}{\partial t} = 0$ when

$$t = \frac{c(18s^2(4-c^2) + c^2(18s+25))}{36(s-1)((4-c^2)(5-s) - c^2)} := t_0.$$

If t_0 is inside Λ, then $t_0 \in (0,1)$, which implies

$$c^3(18s+25) + 18cs^2(4-c^2) + 36(1-s)(4-c^2)(5-s) < 36c^2(1-s) \tag{21}$$

and

$$c^2 > \frac{4(s-5)}{s-6}. \tag{22}$$

Therefore, in order for critical points to occur, we need solutions which satisfy both inequalities (21) and (22).
Let $l(s) := 4(s-5)/(s-6)$. Since $l'(s) < 0$ for $(0,1)$, l is decreasing in $(0,1)$. Hence $c^2 > 16/5$. Now we show that (21) does not hold in $(4/\sqrt{5},2)\times(0,1)$. Consider the function

$$F(c,s) = c^3(18s+25) + 18cs^2(4-c^2) + 36(1-s)(4-c^2)(5-s) - 36c^2(1-s). \tag{23}$$

We show that $F(c,s) \geq 0$ in $(4/\sqrt{5},2)\times(0,1)$. Now

$$\begin{aligned}\frac{\partial F}{\partial s} &= 36(2-c)(c+2)^2 s + 18c^3 + 252c^2 - 864\\ &= k_1(c)\, s + k_0(c) \geq 0\end{aligned}$$

Indeed, since $c \in (4/\sqrt{5},2)$, it can be seen that $k_1(c) = 36(2-c)(c+2)^2 > 0$ and $k_0(c) = 18c^3 + 252c^2 - 864 > 0$. Thus we have

$$F(c,s) \geq F(c,0) = 25c^3 + 720 - 216c^2 > 0, \quad c \in (4/\sqrt{5},2).$$

This shows that (21) does not hold true for $s \in (0,1)$ and $c \in (4/\sqrt{5},2)$, thus E has no point of maxima in $(0,2)\times(0,1)\times(0,1)$.
II. Next, we examine the interior of the six faces of the cuboid Λ.

On the face $c = 0$, $E(c,s,t)$ reduces to

$$k_1(s,t) := E(0,s,t) = \frac{9(1-s^2)(t^2(s-1)(s-5)+6s)+23s^3}{3645}, \; s,\; t \in (0,1).$$

k_1 has no critical points in $(0,1)\times(0,1)$ since

$$\frac{\partial k_1}{\partial t} = \frac{2t(1-s^2)(s-1)(s-5)}{405} \neq 0, \; s,\; t \in (0,1).$$

On the face $c = 2$, $E(c,s,t)$ reduces to

$$E(2,s,t) = \frac{49}{39366}, \; s,\; t \in (0,1).$$

On the face $s = 0$, $E(c,s,t)$ reduces to $E(c,0,t)$, given by

$$k_2(c,t) := \frac{3888(3c^4-22c^2+40)t^2 + 2700c^3(4-c^2)t + 245c^6 - 1944c^4 + 7776c^2}{12597120},$$

where $c \in (0,2)$ and $t \in (0,1)$. We therefore solve $\dfrac{\partial k_2}{\partial t} = 0$ and $\dfrac{\partial k_2}{\partial c} = 0$ to find the critical points. The equation $\dfrac{\partial k_2}{\partial t} = 0$ gives

$$t = -\frac{25c^3}{72(10-3c^2)} =: t_1. \tag{24}$$

For the given range of t, t_1 must be in $(0,1)$, and it is only possible if $c > c_0$, where $c_0 \approx 1.82574$. From $\dfrac{\partial k_2}{\partial c} = 0$ we have

$$(7776c^2 - 28512)t^2 + c(5400 - 2250c^2)t + 245c^4 - 1296c^2 + 2592 = 0. \tag{25}$$

Putting (24) in (25) and simplifying, we obtain

$$3195c^8 - 65456c^6 + 427352c^4 - 1140480c^2 + 1036800 = 0. \tag{26}$$

A numerical computation shows that only solution c of (26) in $(0,2)$ with $c \approx 1.39280$. Thus, k_2 has no point of maxima in $(0,2) \times (0,1)$.
On the face $s = 1$, $E(c,s,t)$ can be written as

$$k_3(c,t) := E(c,1,t) = \frac{491c^6 - 4770c^4 + 7236c^2 + 19872}{3149280}, \ \ c \in (0,2).$$

Solving $\frac{\partial k_3}{\partial c} = 0$, the critical points are at $c =: c_0 = 0$ and $c =: c_1 = \frac{1}{491}\sqrt{780690 - 11784\sqrt{2333}} \approx 0.93667$. Thus, k_3 achieves its maxima $\frac{74656\sqrt{2333}}{878740245} + \frac{1234813}{390551220} \approx 0.00727$ at c_1.

On the face $t = 0$, $E(c,s,t)$ reduces to

$$k_4(c,s) := E(c,s,0) = \frac{1}{12597120}\begin{pmatrix} 245c^6 + (4-c^2)((4-c^2)(486s^4c^2 + 1242s^3c^2 \\ -6696s^3 + 11664s + 2079s^2c^2) + 486c^4s^3 \\ +738c^4s + 864c^4s^2 + 1944c^2) \end{pmatrix}.$$

The system of equations $\frac{\partial k_4}{\partial s} = 0$ and $\frac{\partial k_4}{\partial c} = 0$ has the following numerical solutions

$$\begin{cases} c_1 \approx 0.82165, \\ s_1 \approx 0.95391, \end{cases} \begin{cases} c_2 \approx 2.00000, \\ s_2 \approx 0.25230, \end{cases} \begin{cases} c_3 \approx -0.82165, \\ s_3 \approx 0.953910, \end{cases} \begin{cases} c_4 \approx -2.00000, \\ s_4 \approx 0.252300, \end{cases}$$
$$\begin{cases} c_5 \approx -1.90307, \\ s_5 \approx -1.50705, \end{cases} \begin{cases} c_6 \approx 1.903070, \\ s_6 \approx -1.50705. \end{cases}$$

The only solution (c_1, s_1) lies in $(0,2) \times (0,1)$. Therefore $k_4(c_1, s_1) = 0.00724 < 1/81$.

On the face $t = 1$, $E(c,s,t)$ reduces to

$$k_5(c,s) := E(c,s,1) = \frac{1}{12597120}\begin{pmatrix} 245c^6 + (4-c^2)((4-c^2)(2079s^2c^2 + 486s^4c^2 \\ +1944cs^2 + 1242c^2s^3 + 4968s^3 - 7776s^2 \\ -1944s^4 - 1944cs^4 + 9720) + 738c^4s + 864c^4s^2 \\ +1944s^2c^2 + 486c^4s^3 + 2700c^3 - 2700c^3s^2 \\ +1944c^3s - 1944c^3s^3 + 1944c^2s - 1944c^2s^3) \end{pmatrix},$$

The system of equations $\frac{\partial k_5}{\partial s} = 0$ and $\frac{\partial k_5}{\partial c} = 0$ has the folowing numerical solutions

$$\begin{cases} c_1 = 0, \\ s_1 = 0, \end{cases} \begin{cases} c_2 \approx 2.21123, \\ s_2 \approx 6.29092, \end{cases} \begin{cases} c_3 \approx 2.00000, \\ s_3 \approx 1.62729, \end{cases} \begin{cases} c_4 \approx 12.87555, \\ s_4 \approx 0.24484, \end{cases}$$
$$\begin{cases} c_5 \approx -0.72134, \\ s_5 \approx 0.944020, \end{cases} \begin{cases} c_6 \approx -2.00000, \\ s_6 \approx 0.744450, \end{cases} \begin{cases} c_7 \approx -1.77628, \\ s_7 \approx -0.05258, \end{cases} \begin{cases} c_8 \approx -1.67390, \\ s_8 \approx -0.82077, \end{cases}$$
$$\begin{cases} c_9 \approx -1.93498, \\ s_9 \approx -1.77777, \end{cases} \begin{cases} c_{10} \approx -2.31811, \\ s_{10} \approx -1.65533, \end{cases} \begin{cases} c_{11} \approx -1.78100, \\ s_{11} \approx -3.65886, \end{cases} \begin{cases} c_{12} \approx 2.00000, \\ s_{12} \approx -0.29096, \end{cases}$$
$$\begin{cases} c_{13} \approx 2.000000, \\ s_{13} \approx -1.33633. \end{cases}$$

This shows that the system has no solution in $(0,2) \times (0,1)$.

III. On the vertices of Λ, we have

$$E(0,0,0) = 0, \quad E(0,0,1) = \frac{1}{81}, \quad E(0,1,0) = \frac{23}{3645}, \quad E(0,1,1) = \frac{23}{3645},$$
$$E(2,0,0) = E(2,0,1) = E(2,1,0) = E(2,1,1) = \frac{49}{39366}.$$

IV. Finally we find the maxima of $G(c,s,t)$ on the 12 edges of Λ.

$$\begin{aligned} E(c,0,0) &= \frac{1}{12597120}(245c^6 - 1944c^4 + 7776c^2) \le E(2,0,0) \\ &= \frac{49}{39366} \approx 0.00124, \quad c \in (0,2). \end{aligned}$$

$$\begin{aligned} E(c,0,1) &= \frac{1}{2519424}(49c^6 - 540c^5 + 1944c^4 + 2160c^3 - 15552c^2 + 31104) \le E(0,0,1) \\ &= \frac{1}{81} \approx 0.01235, \quad c \in (0,2). \end{aligned}$$

$$\begin{aligned} E(c,1,0) &= \frac{1}{3149280}(491c^6 - 4770c^4 + 7236c^2 + 19872) \le E(\lambda_1,1,0) \\ &= \frac{74656\sqrt{2333}}{878740245} + \frac{1234813}{390551220} \approx 0.00727, \quad c \in (0,2), \end{aligned}$$

where

$$c := \lambda_1 = \frac{1}{491}\sqrt{780690 - 11784\sqrt{2333}} \approx 0.93666.$$

$$E(0,s,0) = \frac{1}{3645}(-31s^3 + 54s) \le E(0, \frac{3\sqrt{62}}{31}, 0) = \frac{4\sqrt{62}}{4185} \approx 0.0075259, \quad s \in (0,1)$$
$$E(0,s,1) = \frac{1}{3645}(-9s^4 + 23s^3 - 36s^2 + 45) \le E(0,0,1) = \frac{1}{81}, \quad s \in (0,1).$$

$$E(2,s,0) = \frac{49}{39366}, \quad s \in (0,1).$$
$$E(2,s,1) = \frac{49}{39366}, \quad s \in (0,1).$$
$$E(0,0,t) = \frac{1}{81}t^2 \le \frac{1}{81}, \quad t \in (0,1).$$
$$E(0,1,t) = \frac{23}{3645} \approx 0.00631, \quad t \in (0,1).$$
$$E(2,0,t) = \frac{49}{39366}, \quad t \in (0,1).$$
$$E(2,1,t) = \frac{49}{39366}, \quad t \in (0,1).$$

Since all cases have been dealt with, (19) holds.

To see that (19) is sharp, consider f_2 given in (20), which is equivalent to choosing $a_2 = a_3 = a_5 = 0$ and $a_4 = \frac{1}{9}$, which from (3) gives $|H_3(1)(f_2)| = \frac{1}{81}$. This completes the proof. □

3. Functionals for inverse coefficients of convex functions related with cardiod

In this section we find sharp bounds for the functionals considered in Section 2 for inverse coefficients when $f \in \mathcal{C}_{car}$.

We first note that since $f\left(f^{-1}(w)\right) = w$, using (2) it is easy to see that

$$A_2 = -a_2, \tag{27}$$
$$A_3 = 2a_2^2 - a_3, \tag{28}$$
$$A_4 = 5a_2a_3 - 5a_2^3 - a_4, \tag{29}$$
$$A_5 = 14a_2^4 - 21a_2^2a_3 + 6a_2a_4 + 3a_3^2 - a_5. \tag{30}$$

Using (14)-(17) in the above relations and equating coefficients, we obtain

$$A_2 = -\frac{1}{3}c_1, \tag{31}$$
$$A_3 = \frac{19}{108}c_1^2 - \frac{1}{9}c_2, \tag{32}$$
$$A_4 = \frac{17}{108}c_1c_2 - \frac{23}{216}c_1^3 - \frac{1}{18}c_3, \tag{33}$$
$$A_5 = \frac{437}{6480}c_1^4 - \frac{271}{1620}c_2c_1^2 + \frac{53}{540}c_1c_3 + \frac{37}{1080}c_2^2 - \frac{1}{30}c_4. \tag{34}$$

First we consider the second Hankel determinant and then Zalcman conjecture.

Theorem 3.1. *Let $f \in \mathcal{C}_{car}$ and be given by* (1)*. Then*

$$\left|H_2(2)(f^{-1})\right| \le \frac{76}{1431}.$$

The inequality is sharp for the function

$$f_3(z) = \int_0^z \left(\exp \int_0^s \frac{2\left(2\sqrt{159} + 53t\right)(106 + 6\sqrt{159}t + 53t^2)}{53(\sqrt{159} + 6t)^2} dt\right) ds.$$

Proof. Using (31) − (33), we have

$$H_2(2)(f^{-1}) = \frac{53c_1^4}{11664} - \frac{13c_1^2c_2}{972} + \frac{c_1c_3}{54} - \frac{c_2^2}{81}.$$

We can suppose that $t := \zeta_1$, so that $0 \le t \le 1$, since the class $\mathcal{C}_{car}$ and the functional $H_2(2)(f^{-1})$ are rotationally invariant. Then using (8) − (10) and after simple computation we obtain

$$H_2(2)(f^{-1}) = -\frac{7t^4}{729} - \frac{14}{243}t^2(1-t^2)\zeta_2 - \frac{2}{81}\left(t^2+2\right)(1-t^2)\zeta_2^2 + \frac{2}{27}t(1-t^2)(1-|\zeta_2|^2)\zeta_3 = \varphi.$$

When $t = 0$, $\left|\varphi\right| \le \dfrac{4}{81} < \dfrac{76}{1431}$ and at $t = 1$, $\left|\varphi\right| = \dfrac{7}{729} < \dfrac{76}{1431}$.

Then for $0 < t < 1$

$$\left|H_2(2)(f^{-1})\right| \le \frac{2}{27}t(1-t^2)\left\{\left|A + B\zeta_2 + C\zeta_2^2\right| + (1 - |\zeta_2|^2\right\}, \quad \zeta_2 \in \overline{\mathbb{D}},$$

where $A=\dfrac{-7t^3}{54(1-t^2)}$, $B=-\dfrac{7t}{9}$ and $C=-\dfrac{\left(t^2+2\right)}{3t}$, so $AC>0$ for $t\in(0,1)$. Note that in Lemma 1.3

$$|B|-2(1-|C|)=\frac{13t^2-18t+12}{9t}>0,\quad 0<t<1.$$

Then

$$\frac{2}{27}t(1-t^2)\left(|A|+|B|+|C|\right)=\frac{-53t^4+24t^2+36}{729}=g(t).$$

We see that $\max\{g(t):t\in[0,1]\}=g\left(2\sqrt{\dfrac{3}{53}}\right)=\dfrac{76}{1431}$, and so from Lemma 1.3, the result follows.

For sharpness, consider

$$p_3(z)=\frac{1+2t_3z+z^2}{1-z^2},$$

where $t_3=2\sqrt{\dfrac{3}{53}}$. Let

$$w_3(z)=\frac{p_3(z)-1}{p_3(z)+1}=\frac{z\left(53z+2\sqrt{159}\right)}{\left(53+2\sqrt{159}z\right)}=\frac{2}{53}\sqrt{159}z+\frac{41}{53}z^2-\cdots$$

and

$$q_3(z)=1+\frac{4}{3}w_3(z)+\frac{2}{3}w_3^2(z).$$

We see that for $z\in\mathbb{D}$, $w_3(0)=0$ and $|w_3(z)|<1$. Then $f_3\in\mathcal{C}_{car}$ given by (11) and

$$f_3(z)=\int_0^z\left(\exp\int_0^s\frac{q_3(t)-1}{t}dt\right)ds=z+\frac{4}{\sqrt{159}}z^2+\frac{14}{53}z^3+\frac{628}{75843}\sqrt{159}z^4+\cdots.$$

Here $a_2=\dfrac{4}{\sqrt{159}}$, $a_3=\frac{14}{53}$ and $a_4=\frac{628}{75843}\sqrt{159}$. Using (27) – (29), we have $A_2=-\dfrac{4}{\sqrt{159}}$, $A_3=-\frac{10}{159}$ and $A_4=\frac{932}{75843}\sqrt{159}$, and it is easy to see that $H_2(2)(f_3^{-1})=\dfrac{-76}{1431}$. It brings the proof to a conclusion. □

Theorem 3.2. *Let $f\in\mathcal{C}_{car}$ and be given by* (1). *Then*

$$|A_2A_3-A_4|\le\frac{20}{81}\sqrt{\frac{10}{31}}.$$

The inequality is sharp for the function

$$f_4(z)=\int_0^z\left(\exp\int_0^s\frac{2\left(\sqrt{310}+31t\right)(62+3\sqrt{310}t+31t^2)}{3(31+\sqrt{310}t)^2}dt\right)ds.$$

Proof. Using (31) − (33), we have

$$A_2A_3 - A_4 = \frac{31}{648}c_1^3 - \frac{13}{108}c_1c_2 + \frac{1}{18}c_3. \tag{35}$$

We consider a function $p \in \mathcal{P}$ of the form (4) and a Schwarz function $w(z) = \sum_{n=0}^{\infty} w_n z^n$, such that

$$p(z) = \frac{1 + w(z)}{1 - w(z)}.$$

A simple computation shows that

$$c_1 = 2w_1, \;\; c_2 = 2w_2 + 2w_1^2, \quad c_3 = 2w_3 + 4w_1w_2 + 2w_1^3.$$

Using above in (35), we have

$$|A_2A_3 - A_4| = \frac{1}{9}\left|w_3 + \mu w_1w_2 + \nu w_1^3\right|,$$

where $\mu = -7/3,\ \nu = 1/9$. Now using Lemma 1.4, we have $\left|\mu\right| > 2$, $-\frac{2}{3}\left(\left|\mu\right| + 1\right) = -\frac{20}{9}$, $\frac{2|\mu|(|\mu|+1)}{\mu^2-2|\mu|+4} = \frac{140}{127}$. This shows that $-\frac{2}{3}\left(\left|\mu\right| + 1\right) \leq \nu \leq \frac{2|\mu|(|\mu|+1)}{\mu^2-2|\mu|+4}$. Therefore $|A_2A_3 - A_4| \leq \frac{20}{81}\sqrt{\frac{10}{31}}$.

For sharpness, consider

$$p_3(z) = \frac{1 + 2t_4z + z^2}{1 - z^2},$$

where $t_4 = \sqrt{\frac{10}{31}}$. Let

$$w_4(z) = \frac{p_4(z) - 1}{p_4(z) + 1} = \frac{z\left(31z + \sqrt{310}\right)}{\left(31 + \sqrt{310}z\right)} = \sqrt{\frac{10}{31}}z + \frac{21}{31}z^2 - \cdots$$

and

$$q_4(z) = 1 + \frac{4}{3}w_4(z) + \frac{2}{3}w_4^2(z).$$

We see that for $z \in \mathbb{D}$, $w_4(0) = 0$ and $|w_4(z)| < 1$. Then $f_4 \in C_{car}$ given by (11) and

$$f_4(z) = \int_0^z \left(\exp \int_0^s \frac{q_4(t) - 1}{t}dt\right)ds = z + \frac{2}{93}\sqrt{310}z^2 + \frac{236}{837}z^3 + \frac{548}{77841}\sqrt{310}z^4 + \cdots.$$

Here $a_2 = \frac{2}{93}\sqrt{310}$, $a_3 = \frac{236}{837}$ and $a_4 = \frac{548}{77841}\sqrt{310}$. Using (27) − (29), we have $A_2 = -\frac{2}{93}\sqrt{310}$, $A_3 = \frac{4}{837}$ and $A_4 = \frac{68}{8649}\sqrt{310}$, and it is easy to see that $A_2A_3 - A_4 = -\frac{20}{81}\sqrt{\frac{10}{31}}$. This completes the result. □

Theorem 3.3. *If $f \in C_{car}$ and is given by* (1). *Then*

$$|H_3(1)(f^{-1})| \leq \frac{1}{81}. \tag{36}$$

The inequality is sharp for the function f_2 given in (20).

Proof. The third Hankel determinant for the inverse function can be written as

$$H_3(1)(f^{-1}) = 2A_2A_3A_4 - A_2^2A_5 - A_3^3 + A_3A_5 - A_4^2. \tag{37}$$

Since the class $\mathcal{P}$ is invariant under the rotation, the value of c_1 can be assumed to be the interval $[0,2]$. Let $c := c_1$, then substituting the values of A_i's from (31)-(34) in (37), we obtain

$$H_3(1)(f^{-1}) = \frac{1}{12597120}\left[\begin{array}{l} 31104cc_2c_3 - 10656c_2c^4 + 13176c^3c_3 \\ +14796c^2c_2^2 - 27216c^2c_4 + 46656c_2c_4 \\ -38880c_3^2 + 962c^6 - 30672c_2^3 \end{array}\right].$$

Using the equalities (5)-(7) after some simplification, we are able to obtain

$$H_3(1)(f^{-1}) = \frac{1}{12597120}\left(v_1(c,\xi) + v_2(c,\xi)\mu + v_3(c,\xi)\mu^2 + \Psi(c,\xi,\mu)\gamma\right),$$

where $\gamma,\ \mu,\ \xi \in \overline{\mathbb{D}}$,

$$\begin{aligned} v_1(c,\xi) :=& -235c^6 + (4-c^2)((4-c^2)(918\xi^3c^2 - 999\xi^2c^2 + 486\xi^4c^2 \\ & - 3672\xi^3) - 486c^4\xi^3 - 1944\xi^2c^2 + 558c^4\xi - 864c^4\xi^2), \\ v_2(c,\xi) :=& -108c(4-c^2)(1-|\xi|^2)(-c^2(25+18\xi) + 18(4-c^2)\xi^2), \\ v_3(c,\xi) :=& 1944(4-c^2)(1-|\xi|^2)(-(4-c^2)(5+|\xi|^2) + c^2\overline{\xi}), \\ \Psi(c,\xi,\mu) :=& 1944(4-c^2)(1-|\xi|^2)(1-|\mu|^2)(6\xi(4-c^2) - c^2). \end{aligned}$$

Taking $s := |\xi|$, $t := |\mu|$ and using $|\gamma| \le 1$, we have

$$\begin{aligned} |H_3(1)(f^{-1})| &\le \frac{1}{12597120}\left(|v_1(c,\xi)| + |v_2(c,\xi)|t + |v_3(c,\xi)|t^2 + |\Psi(c,\xi,\mu)|\right) \\ &\le G(c,s,t), \end{aligned}$$

where

$$G(c,s,t) := \frac{1}{12597120}\left(g_1(c,s) + g_2(c,s)t + g_3(c,s)t^2 + g_4(c,s)(1-t^2)\right),$$

with

$$\begin{aligned} g_1(c,s) :=& 235c^6 + (4-c^2)((4-c^2)(918s^3c^2 + 999s^2c^2 + 486s^4c^2 \\ & + 3672s^3) + 486c^4s^3 + 1944s^2c^2 + 558c^4s + 864c^4s^2), \\ g_2(c,s) :=& 108c(4-c^2)(1-s^2)(c^2(25+18s) + 18(4-c^2)s^2), \\ g_3(c,s) :=& 1944(4-c^2)(1-s^2)((4-c^2)(5+s^2) + c^2s), \\ g_4(c,s) :=& 1944(4-c^2)(1-s^2)(6s(4-c^2) + c^2). \end{aligned}$$

Let the closed cuboid be $\Lambda : [0,2]\times[0,1]\times[0,1]$, so in order to maximize G we find points of maxima on the twelve edges, inside Λ and inside the six faces of Λ.

I. First, we demonstrate that there aren't any critical points within Λ. Let $(c,s,t) \in (0,2)\times(0,1)\times(0,1)$. Now

$$\begin{aligned} \frac{\partial G}{\partial t} =& \frac{1}{116640}(4-c^2)(1-s^2)[36t(s-1)((4-c^2)(s-5) + c^2) \\ & + c(18s^2(4-c^2) + c^2(18s+25))]. \end{aligned}$$

The equation $\frac{\partial G}{\partial t} = 0$ yields

$$t = \frac{c(18s^2(4-c^2) + c^2(18s+25))}{36(s-1)((4-c^2)(5-s)-c^2)} := t_0.$$

For critical points, t_0 must belong to $(0,1)$ which is possible only if

$$c^3(18s+25) + 18cs^2(4-c^2) + 36(1-s)(4-c^2)(5-s) < 36c^2(1-s) \tag{38}$$

and

$$c^2 > \frac{4(s-5)}{s-6}. \tag{39}$$

Consequently, in order for the critical points to exist, we must find solutions that meet both inequalities (38) and (39).

Let $g(s) := 4(s-5)/(s-6)$. Since $g'(s) < 0$ for $(0,1)$, g is decreasing in $(0,1)$. Hence $c^2 > 16/5$. Since (21) and (38) are same therefore (21) does not hold in $(4/\sqrt{5}, 2) \times (0,1)$. Thus G has no critical point in $(0,2) \times (0,1) \times (0,1)$.

II. Next, we address each of the six faces of Λ separately to determine the points of maxima within of them.

On the face $c = 0$, $G(c,s,t)$ can be written as

$$k_1(s,t) := G(0,s,t) = \frac{9(1-s^2)(t^2(s-1)(s-5)+6s) + 17s^3}{3645}, \; s, \; t \in (0,1).$$

k_1 has no point of maxima in $(0,1) \times (0,1)$ since

$$\frac{\partial k_1}{\partial t} = \frac{2t(1-s^2)(s-1)(s-5)}{405} \neq 0, \; s, \; t \in (0,1).$$

On the face $c = 2$, $G(c,s,t)$ takes the form

$$G(2,s,t) = \frac{47}{39366}, \; s, \; t \in (0,1).$$

On the face $s = 0$, $G(c,s,t)$ can be written as $G(c,0,t)$, given by

$$k_2(c,t) := \frac{3888(3c^4 - 22c^2 + 40)t^2 + 2700c^3(4-c^2)t + 235c^6 - 1944c^4 + 7776c^2}{12597120},$$

where $c \in (0,2)$ and $t \in (0,1)$. In order to find the points of maxima, wee solve $\frac{\partial k_2}{\partial t} = 0$ and $\frac{\partial k_2}{\partial c} = 0$ simultanously. From $\frac{\partial k_2}{\partial t} = 0$, we obtain

$$t = -\frac{25c^3}{72(10-3c^2)} =: t_1. \tag{40}$$

For the given range of t, t_1 to be in $(0,1)$, if and only if $c > c_0$, $c_0 \approx 1.82574$. A calculation shows that $\frac{\partial k_2}{\partial c} = 0$ implies

$$(7776c^2 - 28512)t^2 + c(5400 - 2250c^2)t + 235c^4 - 1296c^2 + 2592 = 0. \tag{41}$$

By putting (40) in (41) and simplifying, we get

$$2835c^8 - 63056c^6 + 423352c^4 - 1140480c^2 + 1036800 = 0. \tag{42}$$

After some calculations (42) gives a solution in $(0,2)$ as $c \approx 1.385493688$. Thus we conclude that k_2 has no point of maxima in $(0,2)\times(0,1)$.

On the face $s = 1$, $G(c,s,t)$ reduces to

$$k_3(c,t) := G(c,1,t) = \frac{365c^6 - 4932c^4 + 8424c^2 + 29376}{6298560}, \quad c \in (0,2).$$

Solving $\frac{\partial k_3}{\partial c} = 0$, we obtain critical points at $c =: c_0 = 0$ and

$$c =: c_1 = \frac{1}{365}\sqrt{600060 - 2190\sqrt{46606}} \approx 0.9774086938.$$

Thus k_3 achieves its maxima $\frac{1}{971210250}(23303\sqrt{46606} + 94823) \approx 0.005277511404$ at c_1 .

On the face $t = 0$, $G(c,s,t)$ can be written as

$$k_4(c,s) := G(c,s,0) = \frac{1}{12597120}\left(\begin{array}{l} 235c^6 + (4-c^2)((4-c^2)(918s^3c^2 + 486s^4c^2 \\ -7992s^3 + 11664s + 999s^2c^2) + 486c^4s^3 \\ +558c^4s + 864c^4s^2 + 1944c^2) \end{array}\right).$$

The system $\dfrac{\partial k_4}{\partial s} = 0$ and $\dfrac{\partial k_4}{\partial c} = 0$ has following numerical solutions

$$\left\{\begin{array}{l} c_1 \approx 2.00000, \\ s_1 \approx 0.26626, \end{array}\right. \quad \left\{\begin{array}{l} c_2 \approx 1.27289, \\ s_2 \approx 1.12021, \end{array}\right. \quad \left\{\begin{array}{l} c_3 \approx -1.27289, \\ s_3 \approx 1.12021, \end{array}\right. \quad \left\{\begin{array}{l} c_4 \approx -2.00000, \\ s_4 \approx 0.26626, \end{array}\right.$$
$$\left\{\begin{array}{l} c_5 \approx -1.92123, \\ s_5 \approx -1.48470, \end{array}\right. \quad \left\{\begin{array}{l} c_6 \approx 1.921230, \\ s_6 \approx -1.48470. \end{array}\right.$$

This shows that the system has no solution in $(0,2)\times(0,1)$.

On the face $t = 1$, $G(c,s,t)$ reduces to

$$k_5(c,s) := G(c,s,1) = \frac{1}{12597120}\left(\begin{array}{l} 235c^6 + (4-c^2)((4-c^2)(999s^2c^2 + 918s^3c^2 \\ +1944s^2c + 486s^4c^2 + 3672s^3 - 7776s^2 \\ -1944s^4 - 1944s^4c + 9720) + 1944s^2c^2 + 558c^4s \\ +864c^4s^2 + 486c^4s^3 + 2700c^3 - 2700c^3s^2 \\ +1944c^3s - 1944c^3s^3 + 1944sc^2 - 1944s^3c^2) \end{array}\right).$$

The system $\dfrac{\partial k_5}{\partial s} = 0$ and $\dfrac{\partial k_5}{\partial c} = 0$ has the following numerical solutions

$$\left\{\begin{array}{l} c_1 = 0, \\ s_1 = 0, \end{array}\right. \quad \left\{\begin{array}{l} c_2 \approx 2.359773, \\ s_2 \approx 3.52180, \end{array}\right. \quad \left\{\begin{array}{l} c_3 \approx 19.24887, \\ s_3 \approx 0.40454, \end{array}\right. \quad \left\{\begin{array}{l} c_4 \approx 2.00000, \\ s_4 \approx 1.92420, \end{array}\right.$$
$$\left\{\begin{array}{l} c_5 \approx -0.94546, \\ s_5 \approx 0.993190, \end{array}\right. \quad \left\{\begin{array}{l} c_6 \approx -2.00000, \\ s_6 \approx 0.713520, \end{array}\right. \quad \left\{\begin{array}{l} c_7 \approx -1.73100, \\ s_7 \approx -3.75467, \end{array}\right. \quad \left\{\begin{array}{l} c_8 \approx 1.642290, \\ s_8 \approx -0.96818, \end{array}\right.$$
$$\left\{\begin{array}{l} c_9 \approx 2.235200, \\ s_9 \approx -0.19876. \end{array}\right.$$

This shows that the system has no solution in $(0,2)\times(0,1)$.

III. On the vertices of Λ, we have

$$G(0,0,1) = 0, \quad G(0,0,0) = \frac{1}{81}, \quad G(0,1,0) = \frac{17}{3645}, \quad G(0,1,1) = \frac{17}{3645},$$
$$G(2,0,1) = G(2,0,0) = G(2,1,1) = G(2,1,0) = \frac{47}{39366}.$$

IV. Finally we find the points of maxima of $G(c,s,t)$ on the 12 edges of Λ.

$$\begin{aligned} G(c,0,0) &= \frac{235c^6 - 1944c^4 + 7776c^2}{12597120} \leq G(2,0,0) \\ &= \frac{47}{39366} \approx 0.001193923690, \quad c \in (0,2). \end{aligned}$$

$$\begin{aligned} G(c,0,1) &= \frac{47c^6 - 540c^5 + 1944c^4 + 2160c^3 - 15552c^2 + 31104}{2519424} \leq G(0,0,1) \\ &= \frac{1}{81} \approx 0.01234567901, \quad c \in (0,2). \end{aligned}$$

$$\begin{aligned} G(c,1,0) &= \frac{365c^6 - 4932c^4 + 8424c^2 + 29376}{6298560} \leq G(\lambda_1,1,0) \\ &= \frac{1}{971210250}(23303\sqrt{46606} + 94823) \approx 0.005277511404, \quad c \in (0,2), \end{aligned}$$

where

$$c := \lambda_1 = \frac{1}{365}\sqrt{600060 - 2190\sqrt{46606}} \approx 0.9774086938.$$

$$G(0,s,0) = \frac{-37s^3 + 54s}{3645} \leq G(0, \frac{3\sqrt{74}}{37}, 0) = \frac{4\sqrt{74}}{4995} \approx 0.006888748963, \quad s \in (0,1)$$

$$G(0,s,1) = \frac{-9s^4 + 17s^3 - 36s^2 + 45}{3645} \leq G(0,0,1) = \frac{1}{81}, \quad s \in (0,1).$$

$$\begin{aligned} G(2,s,0) &= \frac{47}{39366} \approx 0.001193923690, \quad s \in (0,1). \\ G(2,s,1) &= \frac{47}{39366} \approx 0.001193923690, \quad s \in (0,1). \\ G(0,0,t) &= \frac{1}{81}t^2 \leq \frac{1}{81} \approx 0.01234567901, \quad t \in (0,1). \\ G(0,1,t) &= \frac{17}{3645} \approx 0.004663923182, \quad t \in (0,1). \\ G(2,0,t) &= \frac{47}{39366} \approx 0.001193923690, \quad t \in (0,1). \\ G(2,1,t) &= \frac{47}{39366} \approx 0.001193923690, \quad t \in (0,1). \end{aligned}$$

Thus all the above cases show that (36) holds.

Finally we note that (36) is sharp for $f_2 \in \mathcal{C}_{car}$ given in (20). Here we have $a_2 = a_3 = a_5 = 0$ and $a_4 = 1/9$. So from (27) – (30) it implies that $A_2 = A_3 = A_5 = 0$ and $A_4 = -1/9$ and by using equation (37), we have

$$|H_3(1)(f_2^{-1})| = \frac{1}{81}.$$

Hence the proof is complete. □

We end by noting that the results in the paper give the following curious examples of invariance and non-invariance, and that a recent survey of invariance and non-invariance among functionals of various classes of convex functions was presented at Congressio VIII (see [26]). Thus Theorem 3.4 adds new examples to those given in [26]. We also note the curious result that there is non-invariance between the second Hankel determinants $\left|H_2(2)(f)\right|$ and $\left|H_2(2)(f^{-1})\right|$, but invariance between $|H_3(1)(f)|$ and $|H_3(1)(f^{-1})|$.

Theorem 3.4. *Let $f \in C_{car}$ and be given by* (1)*, then the following sharp inequalities hold.*

$$\left|H_2(2)(f)\right|, \left|H_2(2)(f^{-1})\right| \leq \frac{40}{783}, \frac{76}{1431}.$$

$$|a_2a_3 - a_4|, |A_2A_3 - A_4 \leq \frac{32}{81}\sqrt{\frac{2}{19}}, \frac{20}{81}\sqrt{\frac{10}{31}},$$

$$|H_3(1)(f)|, |H_3(1)(f^{-1})| \leq \frac{1}{81}.$$

References

[1] K. O. Babalola, *On H3(1) Hankel determinants for some classes of univalent functions*, Ineq. Theor. Appl. **6** (2010), 1–7.

[2] S. Banga, S. S. Kumar, *The sharp bounds of the second and third Hankel determinants for the class $\mathcal{SL}^*$*, Math. Slovaca **70** (2020), 849-862.

[3] D. Bansal, S. Maharana, J. K. Prajapat, *Third order Hankel determinant for certain univalent functions*, J. Korean Math. Soc. **52** (2015), no. 6, 1139–1148.

[4] N. E. Cho, B. Kowalczyk, A. Lecko, *Sharp bounds for some coefficient functionals over the class of functions convex in the direction of the imaginary axis*, Bull. Aust. Math. Soc. **100** (2019), 86–96.

[5] N. E. Cho, B. Kowalczyk, O. S. Kwon, A. Lecko, Y. J. Sim, *The bounds of some determinants for starlike functions of order alpha*, Bull. Malay. Math. Sci. Soc. **41** (2018), 523–535.

[6] J. H. Choi, Y. C. Kim, T. Sugawa, *A general approach to the Fekete-Szegö problem*, J. Math. Soc. Japan **59** (2007), no. 3, 707–727.

[7] A. Janteng, S. A. Halim, M. Darus, *Coeffcient inequality for a function whose derivative has a positive real part*, J. Inequal. Pure Appl. Math. **7** (2006), no. 2, 50.

[8] B. Kowalczyk, A. Lecko, Y. J. Sim, *The sharp bound of the Hankel determinant of the third kind for convex functions*, Bull. Austr. Math. Soc. **97** (2018), 435–445.

[9] B. Kowalczyk, A. Lecko, M. Lecko, Y. J. Sim, *The sharp bound of the third Hankel determinant for some classes of analytic functions*, Bull. Korean Math. Soc. **55** (2018), 1859–1868.

[10] O. S. Kwon, A. Lecko, Y. J. Sim, *The bound of the Hankel determinant of the third kind for starlike functions*, Bull. Malays. Math. Sci. Soc. **42** (2019), 767–780.

[11] O. S. Kwon, A. Lecko, Y. J. Sim, *On the fourth coefficient of functions in the Carathéodory class*, Comput. Methods Funct. Theory **18** (2018), 307–314.

[12] A. Lecko, Y. J. Sim, B. Smiarowska, *The sharp bound of the Hankel determinant of the third kind for starlike functions of order 1/2*, Complex Anal. Oper. Theory **13** (2019), 2231–2238.

[13] S. K. Lee, V. Ravichandran, S. Supramanian, *Bound for the second Hankel determinant of certain univalent functions*, J. Ineq. Appl. **2013** (2013), 281.

[14] R. J. Libera, E. J. Zlotkiewicz, *Coefficient bounds for the inverse of a function with derivative in $\mathcal{P}$*, Proc. Amer. Math. Soc. **87** (1983), no. 2, 251–257.

[15] R. J. Libera, E. J. Zlotkiewicz, *Early coefficients of the inverse of a regular convex functions*, Proc. Amer. Math. Soc. **85** (1982), 225–230.

[16] W. Ma, D. Minda, *A unified treatment of some special classes of univalent functions*, Proc. Confer. Complex Anal. (Tianjin, 1992), 157–169.

[17] C. Pommerenke, *On the coefficients and Hankel determinants of univalent functions*, J. London Math. Soc. **14** (1966), 111–122.

[18] D. V. Prokhorov, J. Szynal, *Inverse coefficients for (α, β)-convex functions*, Ann. Univ. Mariae Curie-Sklodowska Sect. A **35** (1981), 125–143

[19] M. Raza, S. N. Malik, *Upper bound of the third Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli*, J. Inequal. Appl. **2013** (2013), 412.

[20] M. Raza, D. K. Thomas, A. Riaz, *Coefficient estimates for starlike and convex functions related to sigmoid functions*, Ukr. Math. J. **75** (2023), 782–799.

[21] A. Riaz, M. Raza, D. K. Thomas, *Hankel determinants for starlike and convex functions associated with sigmoid functions*, Forum Math. **34** (2022), 137-156.

[22] D. Raducanu, P. Zaprawa, *Second Hankel determinant for the close-to-convex functions*, C. R., Math., Acad. Sci. Paris **355** (2017), no. 10, 1063–1071.

[23] K. Sharma, N. K. Jain, V. Ravichandran, *Starlike functions associated with a cardioid*, Afrika Mat. **27** (2016), 923–939.
[24] Y. J. Sim, A. Lecko, D. K. Thomas, *The second Hankel determinant for strongly convex and Ozaki close-to-convex functions*, Annali di Math. **200** (2021), 2515–2533.
[25] J. Sokól, D. K. Thomas, *The second Hankel determinant for alpha-convex functions*, Lithuanian Math. J. **58** (2018), 212–218.
[26] D. K. Thomas, *Coefficient invariances for convex functions*, In A. Lecko, D. K. Thomas ed. Current Research in Mathematical and Computer Sciences IV, Olsztyn, Uniwersytet Warmińsko-Mazurski w Olsztynie, (2023), 133–140.
[27] D. K. Thomas and S. Verma, *Invariance of the coefficients of strongly convex functions*, Bull. Aust. Math. Soc. **95** (2017), no. 3, 436–445.
[28] P. Zaprawa, *Third Hankel determinants for classes of univalent functions*, Medit. J. Math. **14** (2017), 19.