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Solving dual quaternion matrix equation $AX - YB = C$

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Abstract. In robotics research, hand-eye calibration is a challenging problem, often represented by the matrix equation $AX = YB$. In this paper, by using the vector operators of dual quaternion matrices, the properties of the real representation of dual quaternion matrices and the properties of semi-tensor product (STP) of dual quaternion matrices, we aim to propose the general solution and the Hermitian solution of the dual quaternion matrix equation $AX - YB = C$, which is the general case of $AX = YB$, where X and Y are unknown dual quaternion matrices. Firstly, we can vectorize the matrix equation $AX - YB = C$ and combine it with STP and the real representation of dual quaternion matrices, to transform the dual quaternion matrix $AX - YB = C$ into a real linear system, thus we can get the necessary and sufficient condition for the solvability and the general solution expression of the dual quaternion matrix equation $AX - YB = C$. Based on this, we also get the Hermitian solution of the dual quaternion matrix equation $AX - YB = C$ by simplifying the complexity of computation with $\mathbb{GH}$-representation of special dual quaternion matrix. Additionally, we propose corresponding algorithms and provide the numerical examples to verify the effectiveness of the corresponding method.

1. Introduction

In this paper, $\mathbb{R}/\mathbb{Q}/\mathbb{D}/\mathbb{DQ}/\mathbb{R}_n/\mathbb{R}^n/\mathbb{Q}^n/\mathbb{DQ}_n/\mathbb{DQ}^n$ denote the sets of all real numbers, quaternions, dual numbers, dual quaternions, real row vectors, real column vectors, quaternion column vectors, dual quaternion row vectors, dual quaternion column vectors with n-dimension, respectively. $\mathbb{R}^{m\times n}/\mathbb{Q}^{m\times n}/\mathbb{DQ}^{m\times n}$ denote the sets of all $m \times n$ real matrices, quaternion matrices, dual quaternion matrices, respectively. $\mathbb{R}_S^{n\times n}/\mathbb{R}_A^{n\times n}/\mathbb{DQ}_H^{n\times n}$ denote the sets of all $n \times n$ real symmetric matrices, real anti-symmetric matrices, and dual quaternion Hermitian matrices, respectively. A^T represents the transpose of matrix A. $\overline{A}$ represents the conjugate of matrix A. A^H represents the conjugate transpose of matrix A. $A^\dagger$ represents Moore-Penrose

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inverse of A. $A \otimes B$, $A \ltimes B$ represent the Kronecker product and the semi-tensor product of matrices A and B, respectively. $\|\cdot\|_F$ represents the Frobenius norm of a real matrix or a quaternion matrix.

Dual quaternions were introduced by William Kingdon Clifford [3] in 1873. As a powerful tool, dual quaternions spread some properties of quaternions and dual numbers, and are widely used in many fields. For example, Wang et al. [11] constructed a new kinematic control model with unit dual quaternion descriptors, and explored Lie-group and Lie-algebra on unit dual quaternions and the approximate logarithms. Qi [8] proved that the dual quaternion optimization problems arising from the hand-eye calibration problem and the simultaneous localization and mapping (SLAM) problem are equality constrained standard dual quaternion optimization problems. Leclercq et al. [7] used dual quaternions to describe the 3D position and orientation of these objects, which can be used for position-only transformations (point transformations) or for combined position and orientation transformations (through line transformations). Based on the 2-norm of dual quaternion vectors, Chen et al. [5] proposed a new dual quaternion optimization method for the hand-eye calibration problem, letting the dual quaternion optimization problem decomposed to two quaternion optimization subproblems.

In recent years, dual quaternion matrix equations have been applied more and more in hand-eye calibration. Some dual quaternion matrix equations have been deeply studied by many scholars. Zhuang et al. [15] initially formulated the hand-eye calibration, and transformed the problem into the matrix equation $AX = YB$. Chen et al. [4] gave the necessary and sufficient conditions for the solvability of the dual quaternion matrix equation $AXB = C$, and presented the expression for the general solution. Xie et al. [12] surveyed the solutions of the matrix equation $AXB = C$ across various number systems and explored its application in color image processing, including a scheme for encrypting and decrypting two images. Xie et al. [13] utilized the M-P inverses and ranks of quaternion matrices, to derive the necessary and sufficient conditions for solving the system of dual quaternion matrix equations $(AX, XC) = (B, D)$. Wang et al. [10] reviewed theoretical research on the matrix equation system $AX = C$ and $XB = D$, exploring various solution methods and their applications in fields like control theory, optimization, image processing, and robotics. Xie et al. [14] established the conditions for solvability of this generalized hand-eye calibration dual quaternion matrix equation $AX - YB = C$ and provided a general expression for its solutions when it is solvable. Subsequent research has shown that the matrix equation $AX = YB$ can be converted into a dual quaternion equation $q_A q_X = q_Y q_B$ [6, 8, 17], demonstrating that the matrix equation $AX = YB$ is a specific case of the broader dual quaternion matrix equation

$$AX - YB = C. \tag{1}$$

In this paper, we propose a new method to solve (1) by equivalently transforming (1) into a real linear system, so that we can get the necessary and sufficient condition for the solvability, the general solution, the special solution and the minimal F^R-norm solution of (1). Specifically, we will solve the following two problems:

Problem 1.1. *Let $A \in \mathbb{DQ}^{m\times n}$, $B \in \mathbb{DQ}^{n\times p}$, $C \in \mathbb{DQ}^{m\times p}$, solving*

$$S_D = \{(X, Y) | X \in \mathbb{DQ}^{n\times p}, Y \in \mathbb{DQ}^{m\times n},\ AX - YB = C\}.$$

Find $(X_D, Y_D) \in S_D$ such that

$$\left\|\begin{pmatrix} X_D \\ Y_D \end{pmatrix}\right\|_{F^R} = \min_{X,Y\in S_D} \left\|\begin{pmatrix} X \\ Y \end{pmatrix}\right\|_{F^R}.$$

(X_D, Y_D) is called the minimal F^R-norm solution of $AX - YB = C$.

Problem 1.2. *Let $A, B, C \in \mathbb{DQ}^{n\times n}$, solving*

$$S_{DH} = \{(X, Y) | X \in \mathbb{DQ}_H^{n\times n}, Y \in \mathbb{DQ}_H^{n\times n},\ AX - YB = C\}.$$

Find $(\widehat{X_D}, \widehat{Y_D}) \in S_{DH}$ such that

$$\left\|\begin{pmatrix} \widehat{X_D} \\ \widehat{Y_D} \end{pmatrix}\right\|_{F^R} = \min_{X,Y\in S_{DH}} \left\|\begin{pmatrix} X \\ Y \end{pmatrix}\right\|_{F^R}.$$

$(\widehat{X_D}, \widehat{Y_D})$ is called the minimal F^R-norm Hermitian solution of $AX - YB = C$.

An outline of this paper is as follows:

In Section 2, we introduce the correlation properties of dual quaternion matrices, including the STP and vector operators of dual quaternion matrices. Meanwhile, we give the definition and properties of the real representation of dual quaternion matrices, and we prove the relationship between the F^R-norm of dual quaternion matrices and the Frobenius norm of its first column block of the corresponding real representation. In Section 3, under the condition when the equation has a solution, we establish the expression of the general solution and the Hermitian solution of the dual quaternion matrix equation $AX - YB = C$. In Section 4, we give numerical algorithms and examples. In Section 5, we summarize the paper and give conclusions.

2. Preliminary

A quaternion p can be represented by four-tuple notation:

$$p = p_0 + p_1 i + p_2 j + p_3 k,$$

while the orthonormal basis components i, j, k, as defined above satisfy the following well-known rules:

$$i^2 = j^2 = k^2 = ijk = -1.$$

It is essential to acknowledge that multiplication in the quaternion domain does not adhere to the commutative property.

A quaternion matrix P can be represented as

$$P = P_0 + P_1 i + P_2 j + P_3 k,$$

and we have the conjugate of P is defined as $\overline{P} = P_0 - P_1 i - P_2 j - P_3 k$; the conjugate transpose of P is defined as $P^H = P_0^T - P_1^T i - P_2^T j - P_3^T k$.

A dual quaternion a has the form

$$a = a_{st} + a_I \varepsilon,$$

where a_{st} and a_I are quaternions given below:

$$a_{st} = (a_0)_{st} + (a_1)_{st} i + (a_2)_{st} j + (a_3)_{st} k,$$

$$a_I = (a_0)_I + (a_1)_I i + (a_2)_I j + (a_3)_I k,$$

the quaternions a_{st} and a_I are the standard part and dual part of a, respectively.

For any two dual quaternions $a = a_{st} + a_I \varepsilon,\ b = b_{st} + b_I \varepsilon \in \mathbb{DQ}$, we have

$$a + b = (a_{st} + b_{st}) + (a_I + b_I)\varepsilon,$$

$$ab = a_{st} b_{st} + (a_{st} b_I + a_I b_{st})\varepsilon.$$

The set of n-dimentional dual quaternion vectors is denoted by

$$\mathbb{DQ}^n = \{x = x_{st} + x_I \varepsilon | x_{st}, x_I \in \mathbb{Q}^n\}.$$

A dual quaternion matrix $X \in \mathbb{DQ}^{m\times n}$ can be denoted as

$$X = X_{st} + X_I \varepsilon = (X_1)_{st} + (X_2)_{st} i + (X_3)_{st} j + (X_4)_{st} k + (X_1)_I \varepsilon + (X_2)_I i\varepsilon + (X_3)_I j\varepsilon + (X_4)_I k\varepsilon, \tag{2}$$

where $X_{st}, X_I \in \mathbb{Q}^{m\times n}$ are the standard part and dual part of X, respectively.

The conjugate of the dual quaternion matrix X is defined as $\overline{X} = \overline{X_{st}} + \overline{X_I}\varepsilon$. The conjugate transpose of the dual quaternion matrix X is defined as $X^H = X_{st}^H + X_I^H \varepsilon$. For $Y = Y_{st} + Y_I \varepsilon \in \mathbb{DQ}^{m\times n}$, by analogy, we have $X = Y$ if $X_{st} = Y_{st}$ and $X_I = Y_I$; furthermore,

$$X + Y = X_{st} + Y_{st} + (X_I + Y_I)\varepsilon,$$

$$XY = X_{st} Y_{st} + (X_{st} Y_I + X_I Y_{st})\varepsilon.$$

2.1. The real representation of dual quaternion matrices

Next we will give the definition and the relevant properties of the real representation of dual quaternion matrices, which can be combined with the vector operators of dual quaternion matrices, to further transform (1) into a real linear system.

Definition 2.1. *For dual quaternion matrix $X \in \mathbb{DQ}^{m\times n}$ as* (2). *Let O represents $m \times n$ zero matrix, the real representation of dual quaternion matrix X is defined as*

$$\psi(X) = \begin{bmatrix} (X_1)_{st} & -(X_2)_{st} & -(X_3)_{st} & -(X_4)_{st} & O & O & O & O \\ (X_2)_{st} & (X_1)_{st} & -(X_4)_{st} & (X_3)_{st} & O & O & O & O \\ (X_3)_{st} & (X_4)_{st} & (X_1)_{st} & -(X_2)_{st} & O & O & O & O \\ (X_4)_{st} & -(X_3)_{st} & (X_2)_{st} & (X_1)_{st} & O & O & O & O \\ (X_1)_I & -(X_2)_I & -(X_3)_I & -(X_4)_I & (X_1)_{st} & -(X_2)_{st} & -(X_3)_{st} & -(X_4)_{st} \\ (X_2)_I & (X_1)_I & -(X_4)_I & (X_3)_I & (X_2)_{st} & (X_1)_{st} & -(X_4)_{st} & (X_3)_{st} \\ (X_3)_I & (X_4)_I & (X_1)_I & -(X_2)_I & (X_3)_{st} & (X_4)_{st} & (X_1)_{st} & -(X_2)_{st} \\ (X_4)_I & -(X_3)_I & (X_2)_I & (X_1)_I & (X_4)_{st} & -(X_3)_{st} & (X_2)_{st} & (X_1)_{st} \end{bmatrix}.$$

The first column block of $\psi(X)$ is

$$\psi^c(X) = \begin{bmatrix} (X_1)_{st} \\ (X_2)_{st} \\ (X_3)_{st} \\ (X_4)_{st} \\ (X_1)_I \\ (X_2)_I \\ (X_3)_I \\ (X_4)_I \end{bmatrix},$$

it can be proved that the real representation $\psi(X)$ and its the first column block $\psi^c(X)$ have the following properties.

Theorem 2.2. *Let $X, Y \in \mathbb{DQ}^{m\times n}$, $Z \in \mathbb{DQ}^{n\times p}$, $\lambda \in \mathbb{R}$, then*

(1) $\psi(X + Y) = \psi(X) + \psi(Y)$; $\psi(\lambda X) = \lambda\psi(X)$; $\psi(XZ) = \psi(X)\psi(Z)$.

(2) $\psi^c(X + Y) = \psi^c(X) + \psi^c(Y)$; $\psi^c(\lambda X) = \lambda\psi^c(X)$; $\psi^c(XZ) = \psi(X)\psi^c(Z)$.

In order to facilitate numerical example and better illustrate the accuracy of the algorithms, we use the F^R-norm of dual quaternion matrix given in [2], where

$$\|X\|_{F^R} = \sqrt{\|X_{st}\|_F^2 + \|X_I\|_F^2}.$$

In attention, F^R-norm is not a norm, because it doesn't satisfy the scaling condition of norms. The specific content can be referred to [2].

A result is given below to illustrate the relation between the F^R-norm and the Frobenius norm of the first column block of the real representation of dual quaternion matrix.

Theorem 2.3. *Let $X = (X_1)_{st} + (X_2)_{st}i + (X_3)_{st}j + (X_4)_{st}k + (X_1)_I\varepsilon + (X_2)_Ii\varepsilon + (X_3)_Ij\varepsilon + (X_4)_Ik\varepsilon \in \mathbb{DQ}^{m\times n}$, then*

$$\|X\|_{F^R} = \left\|\psi^c(X)\right\|_F.$$

Proof. According to Definition 2.1,

$$\begin{aligned} &\|X\|_{F^R} \\ &= \sqrt{\|X_{st}\|_F^2 + \|X_I\|_F^2} \\ &= \sqrt{\|(X_1)_{st}\|_F^2 + \|(X_2)_{st}\|_F^2 + \|(X_3)_{st}\|_F^2 + \|(X_4)_{st}\|_F^2 + \|(X_1)_I\|_F^2 + \|(X_2)_I\|_F^2 + \|(X_3)_I\|_F^2 + \|(X_4)_I\|_F^2} \\ &= \left\|\psi^c(X)\right\|_F. \end{aligned}$$

□

2.2. The STP and vector operators of dual quaternion matrices

Based on the concept of STP of real matrices and STP of quaternion matrices, we will introduce the concept and properties of the STP of dual quaternion matrices in this section.

Definition 2.4. *([9]) Suppose $A \in \mathbb{DQ}^{m\times n}$, $B \in \mathbb{DQ}^{p\times q}$, and $t = lcm(n, p)$ is the least common multiple of n and p. Then the left STP of A and B is defined by*

$$A \ltimes B = (A \otimes I_{t/n})(B \otimes I_{t/p}).$$

The right STP of A and B is defined by

$$A \rtimes B = (I_{t/n} \otimes A)(I_{t/p} \otimes B).$$

If $n = p$, the left and the right STP of dual quaternion matrices reduce to the conventional matrix product. We use the symbol $\bowtie$ to express the left and right STP.

Lemma 2.5. *([9]) Let $A, B, C \in \mathbb{DQ}^{m\times n}$, a, b be dual numbers, then*
(a) $A \bowtie (aB + bC) = aA \bowtie B + bA \bowtie C$, $(aA + bB) \bowtie C = aA \bowtie C + bB \bowtie C$,
(b) $A \bowtie (B \bowtie C) = (A \bowtie B) \bowtie C$,
(c) $(A \bowtie B)^H = B^H \bowtie A^H$.

Definition 2.6. *([9]) For $A \in \mathbb{DQ}^{m\times n}$, $Col_v(A)(1 \le v \le n)$ and $Row_u(A)(1 \le u \le m)$ represent the v-th column and the u-th row of A, respectively. Denote*

$$V_c(A) = [(Col_1(A))^T, (Col_2(A))^T, \cdots, (Col_n(A))^T]^T,$$
$$V_r(A) = [Row_1(A), Row_2(A), \cdots, Row_m(A)]^T.$$

$V_c(A)$ and $V_r(A)$ are called the column vector representation and the row vector representation of dual quaternion matrix A, respectively. The column vector representation and the row vector representation of matrices are collectively referred as the vector operators of matrices.

The next two lemmas are the core to transform the dual quaternion matrices equation into the real linear system. For the proof of properties of vector operators on dual quaternion matrices, please refer to [9].

Lemma 2.7. *Let $A \in \mathbb{DQ}^{m\times n}, X \in \mathbb{DQ}^{n\times p}$, then*

$$\begin{aligned} &V_r(AX) = A \ltimes V_r(X),\\ &V_c(AX) = A \rtimes V_c(X). \end{aligned} \tag{3}$$

Lemma 2.8. *Let $B \in \mathbb{DQ}^{m\times n}, Y \in \mathbb{DQ}^{q\times m}$, then*

$$\begin{aligned} &V_c(\overline{YB}) = B^H \ltimes V_c(\overline{Y}),\\ &V_r(\overline{YB}) = B^H \rtimes V_r(\overline{Y}). \end{aligned} \tag{4}$$

Subsequently, we give the necessary and sufficient condition for the solvability and the general solution expression of the real linear system.

Lemma 2.9. *([1]) Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Then the linear system $Ax = b$ has a solution if and only if $AA^\dagger b = b$. In this case, the general solution can be written in following parametric form*

$$x = A^\dagger b + (I - A^\dagger A)y,$$

where $y \in \mathbb{R}^n$ is an arbitrary vector. $A^\dagger b$ is the minimal norm solution of the linear system $Ax = b$.

3. The solutions of Problem 1.1 and Problem 1.2

First, we will introduce the $\mathcal{H}$-representation of special real matrices, which will be used to reduce the complexity of solving matrix equations.

Definition 3.1. *([16]) Consider a p-dimensional real matrix subspace $\mathbb{X} \subset \mathbb{R}^{n\times n}$. Assume $e_1, e_2, \cdots, e_p$ form the bases of $\mathbb{X}$, which means that for any $X \in \mathbb{X}$, we have $X = x_1e_1 + x_2e_2 + \cdots + x_pe_p$, and define $H = [V_r(e_1), V_r(e_2), \cdots, V_r(e_p)]$. Then*

$$V_r(X) = H\widetilde{X}$$

and $\widetilde{X} = [x_1, x_2, \cdots, x_p]^T$, $H\widetilde{X}$ is called an $\mathcal{H}$-representation of $V_r(X)$, and H is called an $\mathcal{H}$-representation matrix of $V_r(X)$.

Lemma 3.2. *([9]) Let $X_1 \in \mathbb{R}_S^{n\times n}$, $X_2 \in \mathbb{R}_A^{n\times n}$. Then*

$$V_r(X_1) = H_1\widetilde{X_1},$$

$$V_r(X_2) = H_2\widetilde{X_2},$$

where

$$H_1 = (V_r(P_{11}), \cdots, V_r(P_{1n}), V_r(P_{22}), \cdots, V_r(P_{2n}), \cdots, V_r(P_{nn})),$$

and $P_{uv} = (p_{lk})_{n\times n}$ with $p_{uv} = p_{vu} = 1$, the other entries being zero, and

$$\widetilde{X_1} = (x_{11}, \cdots, x_{1n}, x_{22}, \cdots, x_{2n}, \cdots, x_{nn})^T.$$

$$H_2 = (V_r(Q_{12}), \cdots, V_r(Q_{1n}), V_r(Q_{23}), \cdots, V_r(Q_{2n}), \cdots, V_r(Q_{(n-1),n})),$$

and $Q_{uv} = (q_{lk})_{n\times n}$ with $q_{uv} = -q_{vu} = 1$, the other entries being zero, and

$$\widetilde{X_2} = (x_{12}, \cdots, x_{1n}, x_{23}, \cdots, x_{2n}, \cdots, x_{(n-1),n})^T.$$

$\mathcal{H}$-representation can separate independent elements from a special structure matrix. In the following part, we will give $\mathbb{GH}$-representation of dual quaternion matrices with special structures.

Definition 3.3. *Consider a p-dimensional dual quaternion matrix subspace $\mathbb{Y} \subset \mathbb{DQ}^{n\times n}$. For each $X = (X_1)_{st} + (X_2)_{st}i + (X_3)_{st}j + (X_4)_{st}k + (X_1)_I\varepsilon + (X_2)_Ii\varepsilon + (X_3)_Ij\varepsilon + (X_4)_Ik\varepsilon \in \mathbb{Y}$, if we express*

$$V_r\left(\begin{bmatrix}(X_1)_{st}\\(X_2)_{st}\\(X_3)_{st}\\(X_4)_{st}\\(X_1)_I\\(X_2)_I\\(X_3)_I\\(X_4)_I\end{bmatrix}\right) = H_D\widehat{X},$$

then $H_D\widehat{X}$ *is called the* $\mathbb{GH}$*-representation of* $V_r\left(\begin{bmatrix}(X_1)_{st}\\(X_2)_{st}\\(X_3)_{st}\\(X_4)_{st}\\(X_1)_I\\(X_2)_I\\(X_3)_I\\(X_4)_I\end{bmatrix}\right)$, *where*

$$H_D=\begin{bmatrix}H_{(X_1)_{st}}&&&&&&&\\&H_{(X_2)_{st}}&&&&&&\\&&H_{(X_3)_{st}}&&&&&\\&&&H_{(X_4)_{st}}&&&&\\&&&&H_{(X_1)_I}&&&\\&&&&&H_{(X_2)_I}&&\\&&&&&&H_{(X_3)_I}&\\&&&&&&&H_{(X_4)_I}\end{bmatrix},\ \widehat{X}=\begin{bmatrix}\widetilde{(X_1)_{st}}\\\widetilde{(X_2)_{st}}\\\widetilde{(X_3)_{st}}\\\widetilde{(X_4)_{st}}\\\widetilde{(X_1)_I}\\\widetilde{(X_2)_I}\\\widetilde{(X_3)_I}\\\widetilde{(X_4)_I}\end{bmatrix},$$

$H_{(X_m)st}$ *represents the* $\mathcal{H}$*-representation matrix of* $(X_m)_{st}$ *and* $H_{(X_m)I}$ *represents the* $\mathcal{H}$*-representation matrix of* $(X_m)_I$, $(m = 1, 2, 3, 4)$*, respectively.*

Based on Definition 3.3, the $\mathbb{GH}$*-representation of dual quaternion Hermitian matrix is given in the following Theorem.*

Theorem 3.4. *Let* $X = (X_1)_{st} + (X_2)_{st}i + (X_3)_{st}j + (X_4)_{st}k + (X_1)_I\varepsilon + (X_2)_I\varepsilon i + (X_3)_I\varepsilon j + (X_4)_I\varepsilon k \in \mathbb{DQ}_H^{n\times n}$*, then*

$$\psi^c(V_r(X))=\begin{bmatrix}V_r(X_1)_{st}\\V_r(X_2)_{st}\\V_r(X_3)_{st}\\V_r(X_4)_{st}\\V_r(X_1)_I\\V_r(X_2)_I\\V_r(X_3)_I\\V_r(X_4)_I\end{bmatrix}=\begin{bmatrix}H_1&&&&&&&\\&H_2&&&&&&\\&&H_2&&&&&\\&&&H_2&&&&\\&&&&H_1&&&\\&&&&&H_2&&\\&&&&&&H_2&\\&&&&&&&H_2\end{bmatrix}\begin{bmatrix}\widetilde{(X_1)_{st}}\\\widetilde{(X_2)_{st}}\\\widetilde{(X_3)_{st}}\\\widetilde{(X_4)_{st}}\\\widetilde{(X_1)_I}\\\widetilde{(X_2)_I}\\\widetilde{(X_3)_I}\\\widetilde{(X_4)_I}\end{bmatrix}=\mathcal{H}_{\mathcal{D}}\widehat{X},$$

where H_1, H_2 *are matrices defined in Lemma 3.2.*

According to the real representation, the vector operators and STP of dual quaternion matrices, we can convert Problem 1 into the corresponding problem of solving the real linear system.

Theorem 3.5. *Let* $A \in \mathbb{DQ}^{m\times n}$, $B \in \mathbb{DQ}^{n\times p}$, $C \in \mathbb{DQ}^{m\times p}$*. Denote*

$$\begin{aligned}&U = \psi(A \otimes I_p),\\&V = -N_1\psi(I_m \otimes B^H)N_2,\\&N_1 = diag[I_{mp}, -I_{mp}, -I_{mp}, -I_{mp}, I_{mp}, -I_{mp}, -I_{mp}, -I_{mp}],\\&N_2 = diag[I_{mn}, -I_{mn}, -I_{mn}, -I_{mn}, I_{mn}, -I_{mn}, -I_{mn}, -I_{mn}].\end{aligned}$$

Dual quaternion matrix equation (1) *has a solution if and only if*

$$\left(\begin{bmatrix}U & V\end{bmatrix}\begin{bmatrix}U & V\end{bmatrix}^{\dagger} - I_{8mp}\right)\psi^c(V_r(C)) = 0. \tag{5}$$

In this case, the solution set in Problem 1 can be represented as

$$S_D = \left\{ (X, Y) | \begin{pmatrix} \psi^c(V_r(X)) \\ \psi^c(V_r(Y)) \end{pmatrix} = \begin{bmatrix} U & V \end{bmatrix}^\dagger \psi^c(V_r(C)) + (I_{8n(p+m)} - \begin{bmatrix} U & V \end{bmatrix}^\dagger \begin{bmatrix} U & V \end{bmatrix}) y \right\}, \tag{6}$$

where $y \in \mathbb{R}^{8n(p+m)}$ *is an arbitrary vector.*

Then, the minimal F^R*-norm solution* (X_D, Y_D) *of* (1) *satisfies*

$$\begin{pmatrix} \psi^c(V_r(X_D)) \\ \psi^c(V_r(Y_D)) \end{pmatrix} = \begin{bmatrix} U & V \end{bmatrix}^\dagger \psi^c(V_r(C)). \tag{7}$$

Proof. According to the properties of the vector operators, the real representation and STP of dual quaternion matrices, we have

$$\begin{aligned}
& AX - YB = C \\
\Longleftrightarrow & V_r(AX) - V_r(YB) = V_r(C) \\
\Longleftrightarrow & A \ltimes V_r(X) - \overline{V_r(\overline{YB})} = V_r(C) \\
\Longleftrightarrow & A \ltimes V_r(X) - \overline{B^H \rtimes V_r(\overline{Y})} = V_r(C) \\
\Longleftrightarrow & (A \otimes I_p) V_r(X) - \overline{(I_m \otimes B^H) V_r(\overline{Y})} = V_r(C) \\
\Longleftrightarrow & \psi^c((A \otimes I_p) V_r(X) - \overline{(I_m \otimes B^H) V_r(\overline{Y})}) = \psi^c(V_r(C)) \\
\Longleftrightarrow & \psi(A \otimes I_p) \psi^c(V_r(X)) - \psi^c(\overline{(I_m \otimes B^H) V_r(\overline{Y})}) = \psi^c(V_r(C)) \\
\Longleftrightarrow & \psi(A \otimes I_p) \psi^c(V_r(X)) - N_1 \psi^c((I_m \otimes B^H) V_r(\overline{Y})) = \psi^c(V_r(C)) \\
\Longleftrightarrow & \psi(A \otimes I_p) \psi^c(V_r(X)) - N_1 \psi(I_m \otimes B^H) \psi^c(V_r(\overline{Y})) = \psi^c(V_r(C)) \\
\Longleftrightarrow & \psi(A \otimes I_p) \psi^c(V_r(X)) - N_1 \psi(I_m \otimes B^H) N_2 \psi^c(V_r(Y)) = \psi^c(V_r(C)) \\
\Longleftrightarrow & \begin{bmatrix} \psi(A \otimes I_p) & -N_1 \psi(I_m \otimes B^H) N_2 \end{bmatrix} \begin{pmatrix} \psi^c(V_r(X)) \\ \psi^c(V_r(Y)) \end{pmatrix} = \psi^c(V_r(C)) \\
\Longleftrightarrow & \begin{bmatrix} U & V \end{bmatrix} \begin{pmatrix} \psi^c(V_r(X)) \\ \psi^c(V_r(Y)) \end{pmatrix} = \psi^c(V_r(C)).
\end{aligned}$$

Then we can get the general solution (X, Y) of the dual quaternion matrix equation (1) through Lemma 2.9, which satisfies

$$\begin{pmatrix} \psi^c(V_r(X)) \\ \psi^c(V_r(Y)) \end{pmatrix} = \begin{bmatrix} U & V \end{bmatrix}^\dagger \psi^c(V_r(C)) + (I_{8n(p+m)} - \begin{bmatrix} U & V \end{bmatrix}^\dagger \begin{bmatrix} U & V \end{bmatrix}) y, \tag{8}$$

where $y \in \mathbb{R}^{8n(p+m)}$ is an arbitrary vector.

For (8), we can also get the minimal F^R-norm solution (X_D, Y_D) of (1), which satisfies

$$\begin{pmatrix} \psi^c(V_r(X_D)) \\ \psi^c(V_r(Y_D)) \end{pmatrix} = \begin{bmatrix} U & V \end{bmatrix}^\dagger \psi^c(V_r(C)).$$

□

Based on Theorem 3.5, we can further get the Hermitian solution of (1).

Theorem 3.6. *Let* $A, B \in \mathbb{DQ}^{n \times n}$, $X, Y \in \mathbb{DQ}_H^{n \times n}$. *Denote*

$$\begin{aligned}
U &= \psi(A \otimes I_n), \\
V &= -N_3 \psi(I_n \otimes B^H) N_3, \\
N_3 &= diag[I_{n^2}, -I_{n^2}, -I_{n^2}, -I_{n^2}, I_{n^2}, -I_{n^2}, -I_{n^2}, -I_{n^2}],
\end{aligned}$$

(1) *has a Hermitian solution if and only if*

$$\left(\begin{bmatrix} U\mathcal{H}_{\mathcal{D}} & V\mathcal{H}_{\mathcal{D}} \end{bmatrix}\begin{bmatrix} U\mathcal{H}_{\mathcal{D}} & V\mathcal{H}_{\mathcal{D}} \end{bmatrix}^{\dagger} - I_{8n^2}\right)\psi^{c}(V_r(C)) = 0. \tag{9}$$

In this case, the Hermitian solution set of (1) *can be represented as*

$$S_{DH} = \left\{(X,Y) \middle| \begin{pmatrix} \widehat{X} \\ \widehat{Y} \end{pmatrix} = \begin{bmatrix} U\mathcal{H}_{\mathcal{D}} & V\mathcal{H}_{\mathcal{D}} \end{bmatrix}^{\dagger}\psi^{c}(V_r(C)) + \left(I_{16n^2} - \begin{bmatrix} U\mathcal{H}_{\mathcal{D}} & V\mathcal{H}_{\mathcal{D}} \end{bmatrix}^{\dagger}\begin{bmatrix} U\mathcal{H}_{\mathcal{D}} & V\mathcal{H}_{\mathcal{D}} \end{bmatrix}\right)y\right\}, \tag{10}$$

where $\widehat{X}, \widehat{Y}, \mathcal{H}_{\mathcal{D}}$ are defined in Theorem 3.4, and $y \in \mathbb{R}^{16n^2}$ is an arbitrary vector.

Then, the minimal F^R-norm Hermitian solution $(\widehat{X_D}, \widehat{Y_D})$ of (1) *satisfies*

$$\begin{pmatrix} \widehat{X_D} \\ \widehat{Y_D} \end{pmatrix} = \begin{bmatrix} U\mathcal{H}_{\mathcal{D}} & V\mathcal{H}_{\mathcal{D}} \end{bmatrix}^{\dagger}\psi^{c}(V_r(C)). \tag{11}$$

Proof. From Theorem 3.5, we can get the following equation which is equivalent to (1),

$$\begin{bmatrix} U & V \end{bmatrix}\begin{pmatrix} \psi^{c}(V_r(X)) \\ \psi^{c}(V_r(Y)) \end{pmatrix} = \psi^{c}(V_r(C)).$$

By using Theorem 3.4, we can get

$$\psi^{c}(V_r(X)) = \mathcal{H}_{\mathcal{D}}\widehat{X},$$
$$\psi^{c}(V_r(Y)) = \mathcal{H}_{\mathcal{D}}\widehat{Y},$$

then, we obtain

$$\begin{bmatrix} U\mathcal{H}_{\mathcal{D}} & V\mathcal{H}_{\mathcal{D}} \end{bmatrix}\begin{pmatrix} \widehat{X} \\ \widehat{Y} \end{pmatrix} = \psi^{c}(V_r(C)),$$

thus, we can obtain the Hermitian solution (X, Y) of (1) by using Lemma 2.9, which satisfies

$$\begin{pmatrix} \widehat{X} \\ \widehat{Y} \end{pmatrix} = \begin{bmatrix} U\mathcal{H}_{\mathcal{D}} & V\mathcal{H}_{\mathcal{D}} \end{bmatrix}^{\dagger}\psi^{c}(V_r(C)) + \left(I_{16n^2} - \begin{bmatrix} U\mathcal{H}_{\mathcal{D}} & V\mathcal{H}_{\mathcal{D}} \end{bmatrix}^{\dagger}\begin{bmatrix} U\mathcal{H}_{\mathcal{D}} & V\mathcal{H}_{\mathcal{D}} \end{bmatrix}\right)y. \tag{12}$$

From (12), we can get the minimal F^R-norm Hermitian solution (X_D, Y_D) of (1), which satisfies

$$\begin{pmatrix} \widehat{X_D} \\ \widehat{Y_D} \end{pmatrix} = \begin{bmatrix} U\mathcal{H}_{\mathcal{D}} & V\mathcal{H}_{\mathcal{D}} \end{bmatrix}^{\dagger}\psi^{c}(V_r(C)).$$

□

4. Numerical Algorithm and Example

In this section, by using the discussion in Section 3, we propose Algorithm 4.1 and Algorithm 4.3 to solve Problem 1.1 and Problem 1.2, respectively. Two numerical examples are given to show the effectiveness of algorithms.

Algorithm 4.1. *Calculate the minimal F^R-norm solution.*
Input : $A \in \mathbb{DQ}^{m\times n}, B \in \mathbb{DQ}^{n\times p}, C \in \mathbb{DQ}^{m\times p}$;
Output : $\begin{pmatrix} \psi^{c}(V_r(X_D)) \\ \psi^{c}(V_r(Y_D)) \end{pmatrix}$
(1): Fix the form of ψ satisfying Definition 2.1;
(2): Calculate $A \otimes I_p, I_m \otimes B^H$;
(3): Calculate the matrix $\begin{bmatrix} U & V \end{bmatrix}$;
(4): If (5) *holds, then calculate the minimal F^R-norm solution according to* (7).

Example 4.2. *Given* $A \in \mathbb{DQ}^{m\times n}, B \in \mathbb{DQ}^{n\times p}, C \in \mathbb{DQ}^{m\times p}$*, where*

$$A = \begin{bmatrix} 0.83 - 0.37i + 0.16j - 0.39k & -0.64 - 0.64i + 0j - 0.43k \\ -0.76 - 0.25i + 0.58j - 0.14k & -0.27 - 0.96i - 0.03j + 0k \end{bmatrix} + \begin{bmatrix} 0.34 + 0.35i + 0.70j + 0.52k & 0.41 - 0.29i - 0.76j - 0.41k \\ -0.02 - 0.65i + 0.72j + 0.24k & 0.33 + 0.50i + 0.34j - 0.73k \end{bmatrix}\varepsilon,$$

$$B = \begin{bmatrix} 0.54 + 0.10i - 0.13j - 0.82k & -0.29 - 0.81i + 0.45j + 0.24k \\ -0.55 - 0.83i + 0.08j - 0.05k & 0.43 + 0.53i + 0.13j - 0.72k \end{bmatrix} + \begin{bmatrix} -0.95 + 0.19i + 0.12j + 0.21k & -0.60 + 0.44i - 0.45j - 0.50k \\ -0.64 + 0.09i - 0.27j - 0.71k & 0.40 + 0.39i + 0.11j - 0.82k \end{bmatrix}\varepsilon,$$

$$C = \begin{bmatrix} 0.79 + 0.30i + 0.53j - 0.08k & -0.08 + 0.18i + 0.59j - 0.78k \\ 0.53 - 0.35i + 0.63j + 0.45k & 0.57 - 0.70i + 0.31j - 0.30k \end{bmatrix} + \begin{bmatrix} 0.06 + 0.93i + 0.12j - 0.34k & -0.92 + 0.24i + 0.11j + 0.28k \\ -0.33 - 0.67i - 0.28j - 0.60k & 0.62 + 0.06i + 0.02j - 0.78k \end{bmatrix}\varepsilon,$$

By using Algorithm 4.1, it is easy to get the minimal F^R*-norm solution of the equation* (1)*, where*

$$X = \begin{bmatrix} -0.01 + 0.35i - 0.11j + 0.15k & -0.25 + 0.26i + 0.27j - 0.03k \\ -0.04 + 0.17i - 0.36j + 0.32k & 0.24 + 0.32i - 0.09j + 0.67k \end{bmatrix} + \begin{bmatrix} 0.43 + 0.30i - 0.03j - 0.41k & -0.05 + 0.15i - 0.12j - 0.29k \\ -0.49 - 0.27i + 0.23j + 0.02k & -0.21 - 0.24i - 0.07j + 0.04k \end{bmatrix}\varepsilon,$$

$$Y = \begin{bmatrix} -0.27 - 0.07i + 0.47j + 0.06k & -0.04 - 0.38i + 0.35j + 0.43k \\ 0.12 - 0.44i - 0.12j - 0.24k & -0.15 - 0.24i - 0.42j - 0.12k \end{bmatrix} + \begin{bmatrix} -0.14 + 0.22i + 0.48j - 0.17k & 0.40 - 0.08i + 0.35j - 0.31k \\ 0.37 - 0.19i - 0.08j + 0.30k & 0.18 + 0.16i - 0.22j + 0.11k \end{bmatrix}\varepsilon,$$

and, $\|AX - YB - C\|_{F^R}$ *is* $6.80e - 15$.

Algorithm 4.3. *Calculate the minimal* F^R*-norm Hermitian solution.*
Input : $A, B, C \in \mathbb{DQ}^{n\times n}$;
Output : $\begin{pmatrix} \widehat{X_D} \\ \widehat{Y_D} \end{pmatrix}$
(1): Fix the form of ψ *satisfying Definition 2.1;*
(2): Calculate $A \otimes I_n, I_n \otimes B^H$;
(3): Calculate the matrix $\begin{bmatrix} U\mathcal{H}_{\mathcal{D}} & V\mathcal{H}_{\mathcal{D}} \end{bmatrix}$;
(4): If (9) *holds, then calculate the minimal* F^R*-norm Hermitian solution according to* (11).

Example 4.4. *Given* $A, B, C \in \mathbb{DQ}^{n\times n}$*, where*

$$A = \begin{bmatrix} 0.70 - 1.58i - 1.33j + 0.02k & -0.35 + 0.28i + 0.35j - 1.75k \\ -2.05 + 0.51i + 1.13j - 0.26k & -0.82 + 0.03i - 0.30j - 0.29k \end{bmatrix} + \begin{bmatrix} -0.83 - 2.00i - 0.04j - 0.72k & -1.16 + 0.52i + 1.02j - 0.23k \\ -0.98 + 0.96i - 0.80j + 1.35k & -0.53 - 0.02i - 0.13j - 0.59k \end{bmatrix}\varepsilon,$$

$$B = \begin{bmatrix} -0.29 + 1.66i - 0.18j - 1.45k & -1.12 - 1.26i - 1.33j + 0.39k \\ -0.85 + 0.31i + 0.79j + 0.33k & 2.53 - 0.87i - 2.33j + 0.45k \end{bmatrix}$$
$$+ \begin{bmatrix} -0.13 - 1.36i + 0.55j + 0.66k & -0.48 - 0.85i - 1.12j - 0.20k \\ 0.18 + 0.46i + 1.04j - 0.07k & 0.86 - 0.34i + 1.26j - 0.22k \end{bmatrix} \varepsilon,$$

$$C = \begin{bmatrix} 0.06 + 2.22i - 0.19j - 1.45k & -0.07 - 5.05i - 1.17j - 6.23k \\ 2.39 - 0.27i + 2.15j - 1.06k & -3.15 + 2.51i - 3.58j - 0.37k \end{bmatrix}$$
$$+ \begin{bmatrix} 1.58 + 0.19i + 1.10j - 2.55k & -4.84 + 1.24i + 1.46j - 5.03k \\ -2.13 - 3.52i - 0.02j + 1.32k & 0.46 - 0.21i - 4.46j - 0.80k \end{bmatrix} \varepsilon,$$

By using the Algorithm 4.3, it is easy to get the minimal F^R-norm Hermitian solution of (1), *where*

$$X = \begin{bmatrix} -0.31 + 0.09i - 0.12j - 0.16k & 0.28 - 0.55i + 0.53j + 0.15k \\ 0.40 + 0.46i - 0.70j + 0.50k & 0.85 + 0.48i - 0.39j + 0.23k \end{bmatrix}$$
$$+ \begin{bmatrix} 0.05 + 0.13i + 0.34j - 0.54k & -0.13 - 0.02i + 0.14j - 0.08k \\ 0.34 - 0.06i - 0.46j - 0.09k & 0.22 - 0.05i - 0.07j - 0.24k \end{bmatrix} \varepsilon,$$

$$Y = \begin{bmatrix} -0.98 - 0.20i + 0.24j - 0.18k & -0.18 - 0.08i - 0.02j + 0.96k \\ -0.40 + 0.20i - 0.10j - 1.19k & -0.03 + 0.02i + 0.22j - 0.14k \end{bmatrix}$$
$$+ \begin{bmatrix} -0.06 - 0.19i + 0.03j - 0.38k & 0.51 + 0.21i + 0.16j - 0.07k \\ 0.46 - 0.40i + 0.10j + 0.20k & -0.12 + 0.08i + 0.42j - 0.05k \end{bmatrix} \varepsilon,$$

and, $\|AX - YB - C\|_{F^R}$ *is* $1.67e - 14$.

5. Conclusion

In this paper, we studied the general solution and the Hermitian solution of dual quaternion matrix equation $AX - YB = C$. Firstly, we defined the concept of the real representation of dual quaternion matrices. Then by using the vector operators, the properties of the STP and the $\mathbb{GH}$-representation of dual quaternion matrices, we transformed the dual quaternion matrix equation $AX - YB = C$ into a real linear system equivalently, thus obtaining the general solution and the Hermitian solution of equation(1). Corresponding numerical examples were provided to verify the effectiveness of the methods.

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