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Infinite families of p-ary distance optimal linear codes for odd prime numbers p

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Abstract. In this paper, we consider infinite families of linear codes associated to down sets over the ring $R = \mathbb{F}_p + u\mathbb{F}_p$ for $u^2 = 0$ and p an odd prime number. We determine the Lee weight distributions for these codes when the down sets are generated by a single or two maximal elements. When applying the Gray map to such linear codes, we have identified $(p-1)(2p-1)$ classes of p-ary distance optimal linear codes, and some of which meet the Griesmer bound.

1. Introduction

An $[n, k, d]$ linear code C of length n over a finite field $\mathbb{F}_p$ is a k-dimensional subspace of $\mathbb{F}_p^n$ with minimum Hamming distance d. Given parameters n and k, we face the problem in coding theory to find the $[n, k, d]$ linear codes over $\mathbb{F}_p$ having the largest minimum distance d. It is well-known that for any $[n, k, d]$ linear code over $\mathbb{F}_p$, the length n must be at least the sum $\sum_{i=0}^{k-1}\lceil \frac{d}{p^i}\rceil$, which is known as the Griesmer bound (See [3]). We refer to a linear code as being optimal if it achieves the Griesmer bound. Furthermore, an $[n, k, d]$ code C is said to be distance optimal if no $[n, k, d+1]$ code exists, as detailed in ([4], Chapter 2). Consequently, an optimal code is distance optimal.

This paper delves into the construction and analysis of a family of linear codes

$$C_D = \{c_D(\mathbf{a}) = (\langle \mathbf{a}, \mathbf{l}\rangle)_{\mathbf{l}\in D} : \mathbf{a} \in E\}$$

over the ring $R = \mathbb{F}_p + u\mathbb{F}_p$, where $u^2 = 0$, D, E are subsets of $\mathbb{F}_p^m + u\mathbb{F}_p^m$ and p is an odd prime. s The sets D are link to down sets Δ of $\mathbb{F}_p^m$, which are subsets of $\mathbb{F}_p^m$ with partial order $\preceq$. The interested reader may consult [13] for more details. This generic construction was introduced by Ding et al. ([1],[2]). Many known few-weight linear codes could be produced by selecting specific commutative rings, their extensions and a proper defining set D (See [5] - [14]). In this paper, we are interested in the down sets Δ which is generated

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by a single maximal element or two maximal elements. By leveraging the Gray map ϕ, which preserves the Lee distance, we unveil several classes of (distance) optimal linear codes $\phi(C_D)$ over $\mathbb{F}_p$.

The rest of this paper is organized as follows. Section 2 deals with the preliminaries, introducing the necessary concepts and known results that form the basis of our work. In Section 3, we give the Lee weight distribution of $(p-1)(2p-1)$ classes of codes (Theorems 3.1 - 3.3). Section 4 employs the Gray map to derive $(p-1)(2p-1)$ families of p-ary distance optimal linear codes with some are optimal, as detailed in Theorems 4.1, 4.3 and 4.5. From these codes, we will compile Tables 4 - 6, which list distance optimal and optimal linear codes with relatively small dimensions and prime numbers. Finally, Section 5 concludes the paper.

2. Preliminaries

Let $R = \mathbb{F}_p + u\mathbb{F}_p$ be a ring with $u^2 = 0$. A linear code C of length m over R is an R-submodule of R^m. Given two vectors $x = (x_1, x_2, \dots, x_m)$ and $y = (y_1, y_2, \dots, y_m)$ in R^m, the inner product of x and y is

$$\langle x, y\rangle = \sum_{i=1}^{m} x_i y_i,$$

which ultimately belongs to R. Since R^m can be treated as $\mathbb{F}_p^m + u\mathbb{F}_p^m$, if we write x and y in the form of $x = a + ub$ and $y = c + ud$ where a, b, c, d are in $\mathbb{F}_p^m$, then

$$\langle x, y\rangle = a \cdot c + u(a \cdot d + b \cdot c).$$

The Gray map from R^m to $\mathbb{F}_p^{2m}$ is defined as $\phi : R^m \to \mathbb{F}_p^{2m}$, which maps $x = a + ub$ to $(b, a + b)$ with a, b being elements of $\mathbb{F}_p^m$. The Hamming weight $w_H(a)$ of a vector $a \in \mathbb{F}_p^m$ is the number of non-zero elements in a. The Lee weight $w_L(x)$ of a vector $x = a + ub$ in R^m is the Hamming weight of its image under the Gray map ϕ, which is

$$w_L(x) = w_H(\phi(x)) = w_H(b) + w_H(a + b).$$

The Lee distance $d_L(x, y)$ between two vectors x and y in R^m is defined as the Lee weight of their difference, which is $w_L(x-y)$. It's straightforward that we have an isometry from (R^m, d_L) to $(\mathbb{F}_p^{2m}, d_H)$, where d_H denotes the Hamming distance. Moreover, if we assume that C is a $\mathbb{F}_p$-submodule of R^m with parameters (n, p^k, d), then $\phi(C)$ is a linear code over $\mathbb{F}_p$ with parameters $[2n, k, d]$.

In what follows, let us assume that p is an odd prime number. We consider a down set Δ of the set $\mathbb{F}_p^m$, and two subsets $\mathbb{H} \subseteq \mathbb{F}_p^m$ and $\mathbb{T} \subseteq \mathbb{F}_p^m$. Define $\Delta^c = \mathbb{H} \setminus \Delta$ as the complement of Δ within $\mathbb{H}$, $L = \Delta^c + u\mathbb{F}_p^m$ and $L^c = \Delta + u\mathbb{F}_p^m$. We proceed to construct the codes as below

$$C_D = \left\{c_D(\mathbf{a}) = (\langle \mathbf{a}, \mathbf{l}\rangle)_{\mathbf{l}\in D} : \mathbf{a} \in \mathbb{F}_p^m + u\mathbb{T}\right\},\ D \in \{L, L^c\}.$$

Suppose $\mathbf{a} = \alpha + u\beta, \mathbf{f}_1 = s_1 + uy$ and $\mathbf{f}_2 = s_2 + uy$, where $\alpha \in \mathbb{F}_p^m, \beta \in \mathbb{T}, y \in \mathbb{F}_p^m, s_1 \in \Delta$ and $s_2 \in \Delta^c$ without any ambiguity that these elements are not in the bold typeface.

If $\mathbf{a} = \mathbf{0}$, then $w_L(c_L(\mathbf{a})) = w_L(c_{L^c}(\mathbf{a})) = 0$. Conversely, if $\mathbf{a} \neq \mathbf{0}$, then the Lee weight of the codeword $c_{L^c}(\mathbf{a})$ is determined by the following expression

$$\begin{aligned}
w_L(c_{L^c}(\mathbf{a})) =& w_L((\alpha \cdot s_1 + u(\alpha \cdot y + \beta \cdot s_1))_{s_1\in\Delta, y\in\mathbb{F}_p^m})\\
=& w_H((\alpha \cdot y + \beta \cdot s_1)_{s_1\in\Delta, y\in\mathbb{F}_p^m}) + w_H((\alpha \cdot y + (\alpha + \beta) \cdot s_1)_{s_1\in\Delta, y\in\mathbb{F}_p^m})\\
=& 2|L^c| - \frac{1}{p}\sum_{s_1\in\Delta}\sum_{y\in\mathbb{F}_p^m}\sum_{x\in\mathbb{F}_p}\zeta_p^{(\alpha\cdot y+\beta\cdot s_1)x} - \frac{1}{p}\sum_{s_1\in\Delta}\sum_{y\in\mathbb{F}_p^m}\sum_{x\in\mathbb{F}_p}\zeta_p^{(\alpha\cdot y+(\alpha+\beta)\cdot s_1)x}\\
=& 2|L^c|(1 - \frac{1}{p}) - \frac{1}{p}\sum_{x\in\mathbb{F}_p^*}\sum_{s_1\in\Delta}\zeta_p^{\beta\cdot s_1 x}\sum_{y\in\mathbb{F}_p^m}\zeta_p^{\alpha\cdot xy} - \frac{1}{p}\sum_{x\in\mathbb{F}_p^*}\sum_{s_1\in\Delta}\zeta_p^{(\alpha+\beta)\cdot s_1 x}\sum_{y\in\mathbb{F}_p^m}\zeta_p^{\alpha\cdot xy}
\end{aligned}$$

$$=2|L^c|(1-\frac{1}{p})-p^{m-1}\delta_{0,\alpha}(\sum_{x\in\mathbb{F}_p^*}\sum_{s_1\in\Delta}\zeta_p^{\beta\cdot s_1x}+\sum_{x\in\mathbb{F}_p^*}\sum_{s_1\in\Delta}\zeta_p^{(\alpha+\beta)\cdot s_1x}), \tag{2.1}$$

where ζ is a primitive p-th root of the unit and Δ is the Kronecker delta function. By the same token, we have

$$w_L(c_L(\mathbf{a}))=2|L|(1-\frac{1}{p})-p^{m-1}\delta_{0,\alpha}(\sum_{x\in\mathbb{F}_p^*}\sum_{s_2\in\Delta^c}\zeta_p^{\beta\cdot s_2x}+\sum_{x\in\mathbb{F}_p^*}\sum_{s_2\in\Delta^c}\zeta_p^{(\alpha+\beta)\cdot s_2x}). \tag{2.2}$$

It is important to note that the sum of the sizes of L and its complement L^c equals $p^m|\mathbb{H}|$. Consequently, the Lee weights of the codewords $c_{L^c}(a)$ and $c_L(a)$ satisfy the following relationship:

$$w_L(c_{L^c}(a))+w_L(c_L(a))=2p^{m-1}(p-1)|\mathbb{H}|-p^{m-1}(p-1)|\mathbb{H}|\delta_{0,\alpha}(\delta_{0,\beta}+\delta_{0,\alpha+\beta}). \tag{2.3}$$

3. The Lee weight distributions

By employing Equations (2.1),(2.2) and (2.3), we derive the Lee weight distribution for the code C_L when the down set is determined by a single maximal element in Theorem 3.1 or two maximal elements in Theorems 3.2 - 3.3.

Theorem 3.1. *Let $m\geq 2$ be an integer and p be an odd prime number. Consider the down set $\Delta=\langle(p-1,p-1,\dots,p-1,r)\rangle$ of $\mathbb{F}_p^m$ for $r\in[0,p-2]$. Additionally, let $\mathbb{T}=\mathbb{H}=\mathbb{F}_p^m$. Under these assumptions, the code C_L possesses a length of $p^{2m-1}(p-r-1)$, a size of p^{2m}, and its Lee weight distribution is detailed in the Table 1.*

Table 1: The weight distribution of C_L

Lee weight	Frequency
0	1
$2p^{2m-2}(p-1)(p-r-1)$	$p^{2m}-p$
$2p^{2m-1}(p-r-1)$	$p-1$

Proof. Since $\Delta^c=\{(p-1,\dots,p-1,j)\mid r<j<p\}$, we have $|\Delta^c|=p^{m-1}(p-r-1)$ and the length of the code C_L is $|L|=p^{2m-1}(p-r-1)$. Let $\mathbf{a}=\alpha+u\beta$ be a nonzero element, where $\alpha=(\alpha_1,\dots,\alpha_m)$ and $\beta=(\beta_1,\dots,\beta_m)$ are two vectors of $\mathbb{F}_p^m$. If $\alpha\neq 0$, then $w_L(c_L(\mathbf{a}))=2|L|(1-\frac{1}{p})=2p^{2m-2}(p-1)(p-r-1)$. If $\alpha=0$, then by Eq.(2.2), we have

$$w_L(c_L(\mathbf{a}))=2|L|(1-\frac{1}{p})-2p^{m-1}\sum_{x\in\mathbb{F}_p^*}\sum_{s_2\in\Delta^c}\zeta_p^{\beta\cdot s_2x}.$$

Note that for $s_2=(s_{21},s_{22},\dots,s_{2m})$, we have

$$\begin{aligned}\sum_{x\in\mathbb{F}_p^*}\sum_{s_2\in\Delta^c}\zeta_p^{\beta\cdot s_2x}&=\sum_{x\in\mathbb{F}_p^*}\sum_{s_{21}\in\mathbb{F}_p}\zeta_p^{\beta_1s_{21}x}\cdots\sum_{s_{2,m-1}\in\mathbb{F}_p}\zeta_p^{\beta_{m-1}s_{2,m-1}x}\sum_{j=r+1}^{p-1}\zeta_p^{\beta_ms_{2m}x}\\&=\begin{cases}0, & \text{if } (\beta_1,\beta_2,\dots,\beta_{m-1})\neq 0,\\ -p^{m-1}(p-r-1), & \text{if } (\beta_1,\beta_2,\dots,\beta_{m-1})=0.\end{cases}\end{aligned}$$

Therefore,

$$w_L(c_L(\mathbf{a}))=\begin{cases}2p^{2m-2}(p-1)(p-r-1), & \text{if } (\beta_1,\beta_2,\dots,\beta_{m-1})\neq 0,\\ 2p^{2m-1}(p-r-1), & \text{if } (\beta_1,\beta_2,\dots,\beta_{m-1})=0.\end{cases}$$

□

Theorem 3.2. *Let $m \geq 3$ be an integer and p be an odd prime number. Consider the down set $\Delta = \langle (r, 0, \ldots, 0), (0, s, 0, \ldots, 0) \rangle$ of $\mathbb{F}_p^m$ for $r, s \in [1, p-1]$. Additionally, let $\mathbb{H} = \{(x_1, x_2, \ldots, x_m) \in \mathbb{F}_p^m \mid x_3 = 0\}$ and $\mathbb{T} = \{(x_1, x_2, \ldots, x_m) \in \mathbb{F}_p^m \mid x_2 = 0\}$. Under these assumptions, the code C_L possesses a length of $p^m(p^{m-1} - r - s - 1)$, a size of p^{2m-1}, and its Lee weight distribution is detailed in Table 2.*

Table 2: The weight distribution of C_L

Lee weight	Frequency
0	1
$2p^{m-1}(p-1)(p^{m-1} - r - s - 1)$	$p^{2m-1} - p^{m-1}$
$2p^{2m-2}(p-1)$	$p^{m-2} - 1$
$2p^m(p^{m-1} - p^{m-2} - r)$	$p^{m-1} - p^{m-2}$

Proof. Since $\Delta = \langle (r, 0, \ldots, 0), (0, s, 0, \ldots, 0) \rangle$ is a down set, then $|\Delta^c| = |\mathbb{H} \setminus \Delta| = p^{m-1} - r - s - 1$ and the length of the code C_L is $|L| = p^m(p^{m-1} - r - s - 1)$. Let $\mathbf{a} = \alpha + u\beta$ be a nonzero element, where $\alpha = (\alpha_1, \ldots, \alpha_m)$ and $\beta = (\beta_1, \ldots, \beta_m)$ are two vectors of $\mathbb{F}_p^m$. If $\alpha \neq 0$, then $w_L(c_L(\mathbf{a})) = 2|L|(1 - \frac{1}{p}) = 2p^{m-1}(p-1)(p^{m-1} - r - s - 1)$. If $\alpha = 0$, then by Eq.(2.1), we have

$$\begin{aligned} w_L(c_{L^c}(\mathbf{a})) =& 2|L^c|(1 - \frac{1}{p}) - 2p^{m-1} \sum_{x \in \mathbb{F}_p^*} \sum_{t_1 \in \Delta} \zeta_p^{\beta \cdot t_1 x} \\ =& \begin{cases} 0, & \text{if } \beta_1 = 0, \\ 2p^m r, & \text{if } \beta_1 \neq 0. \end{cases} \end{aligned}$$

Therefore,

$$\begin{aligned} w_L(c_L(\mathbf{a})) =& 2p^{2m-2}(p-1) - w_L(c_{L^c}(\mathbf{a})) \\ =& \begin{cases} 2p^{2m-2}(p-1), & \text{if } \beta_1 = 0, \\ 2p^m(p^{m-1} - p^{m-2} - r), & \text{if } \beta_1 \neq 0. \end{cases} \end{aligned}$$

□

Theorem 3.3. *Let $m \geq 4$ be an integer and p be an odd prime number. Consider the down set $\Delta = \langle (r, 0, \ldots, 0), (0, p-1, s, 0, \ldots, 0) \rangle$ of $\mathbb{F}_p^m$ for $r, s \in [1, p-1]$. Additionally, let $\mathbb{H} = \{(x_1, x_2, \ldots, x_m) \in \mathbb{F}_p^m \mid x_4 = 0\}$ and $\mathbb{T} = \{(x_1, x_2, \ldots, x_m) \in \mathbb{F}_p^m \mid x_1 = 0\}$. Under these assumptions, the code C_L possesses a length of $p^m(p^{m-1} - p(s+1) - r)$, a size of p^{2m-1}, and its Lee weight distribution is detailed in Table 3.*

Table 3: The weight distribution of C_L

Lee weight	Frequency
0	1
$2p^m(p^{m-1} - p^{m-2})$	$p^{m-3} - 1$
$2p^m(p^{m-1} - p^{m-2} - ps)$	$p^{m-3}(p-1)$
$2p^m(p^{m-1} - p^{m-2} - (p-1)(s+1))$	$p^{m-2}(p-1)$
$2p^{m-1}(p-1)(p^{m-1} - p(s+1) - r)$	$p^{m-1}(p^m - 1)$

Proof. Let $\Delta_1 = \langle (r, 0, \ldots, 0) \rangle$ and $\Delta_2 = \langle (0, p-1, s, 0, \ldots, 0) \rangle$. Then $\Delta = \Delta_1^* \sqcup \Delta_2$. It is easy to check that the length of the code C_L is $|L| = p^m(p^{m-1} - p(s+1) - r)$. Let $\mathbf{a} = \alpha + u\beta$ be a nonzero element, where

$\alpha = (\alpha_1, \ldots, \alpha_m)$ is a vector of $\mathbb{F}_p^m$ and $\beta = (\beta_1, \ldots, \beta_m)$ is a vector of $\mathbb{T}$. If $\alpha \neq 0$, then $w_L(c_L(\mathbf{a})) = 2|L|(1 - \frac{1}{p}) = 2p^{m-1}(p-1)(p^{m-1} - p(s+1) - r)$. If $\alpha = 0$, then by Eq.(2.1), we have

$$\begin{aligned} w_L(c_{L^c}(\mathbf{a})) =& 2p^{m-1}(p-1)(p(s+1)+r) - 2p^{m-1} \sum_{x \in \mathbb{F}_p^*} \sum_{t_1 \in \Delta} \zeta_p^{\beta \cdot t_1 x} \\ =& 2p^{m-1}(p-1)(p(s+1)+r) - 2p^{m-1} \sum_{x \in \mathbb{F}_p^*} \left(\sum_{t_1 \in \Delta_1^*} \zeta_p^{\beta \cdot t_1 x} + \sum_{t_1 \in \Delta_2} \zeta_p^{\beta \cdot t_1 x} \right). \end{aligned}$$

Note that

$$\sum_{x \in \mathbb{F}_p^*} \sum_{t_1 \in \Delta_1^*} \zeta_p^{\beta \cdot t_1 x} = (p-1)r \text{ since } \beta_1 = 0$$

and

$$\sum_{x \in \mathbb{F}_p^*} \sum_{t_1 \in \Delta_2} \zeta_p^{\beta \cdot t_1 x} = \begin{cases} p(p-1)(s+1), & \text{if } \beta_2 = \beta_3 = 0, \\ p(p-1-s), & \text{if } \beta_2 = 0, \beta_3 \neq 0, \\ 0 & \text{if } \beta_2 \neq 0. \end{cases}$$

Therefore,

$$w_L(c_{L^c}(\mathbf{a})) = \begin{cases} 0, & \text{if } \beta_2 = \beta_3 = 0, \\ 2p^{m+1}s, & \text{if } \beta_2 = 0, \beta_3 \neq 0, \\ 2p^m(p-1)(s+1), & \text{if } \beta_2 \neq 0. \end{cases}$$

By Eq.(2.3), we have

$$\begin{aligned} w_L(c_L(\mathbf{a})) =& 2p^{2m-2}(p-1) - w_L(c_{L^c}(\mathbf{a})) \\ =& \begin{cases} 2p^m(p^{m-1} - p^{m-2}), & \text{if } \beta_2 = \beta_3 = 0, \\ 2p^m(p^{m-1} - p^{m-2} - ps), & \text{if } \beta_2 = 0, \beta_3 \neq 0, \\ 2p^m(p^{m-1} - p^{m-2} - (p-1)(s+1)), & \text{if } \beta_2 \neq 0, \end{cases} \end{aligned}$$

□

4. Families of (distance) optimal codes

Theorem 4.1. *Suppose $m \geq 2$ is an integer, and let p be an odd prime number. Define the down set Δ within $\mathbb{F}_p^m$ as $\Delta = \langle (p-1, p-1, \ldots, p-1, r) \rangle$, where $r \in [0, p-2]$. Furthermore, let $\mathbb{T} = \mathbb{H} = \mathbb{F}_p^m$. Under these conditions, the Gray image $\phi(C_L)$ of C_L is a distance optimal linear code over $\mathbb{F}_p$. In particular, if $\frac{p-2}{2} < r < p-1$, then $\phi(C_L)$ meets the Griesmer bound.*

Proof. According to Theorem 3.1, the parameters of the code $\phi(C_L)$ are:

$$n = 2p^{2m-1}(p-r-1), \quad k = 2m, \quad d = 2p^{2m-2}(p-1)(p-r-1).$$

First, we demonstrate that $\phi(C_L)$ is distance optimal. Assume that there exists an $[n, k, d+1]$-code. Then we have

$$\begin{aligned} & \sum_{i=0}^{2m-1} \left\lceil \frac{d+1}{p^i} \right\rceil \\ =& \sum_{i=0}^{2m-1} \left\lceil \frac{2p^{2m-2}(p-1)(p-r-1)+1}{p^i} \right\rceil \end{aligned}$$

$$
\begin{aligned}
&=\sum_{i=0}^{2m-2}\left\lceil \frac{2p^{2m-2}(p-1)(p-r-1)+1}{p^i}\right\rceil+\left\lceil \frac{2p^{2m-2}(p-1)(p-r-1)+1}{p^{2m-1}}\right\rceil\\
&=2p^{2m-1}(p-r-1)-2(p-r-1)+\sum_{i=0}^{2m-2}\left\lceil \frac{1}{p^i}\right\rceil+\left\lceil \frac{2(p-1)(p-r-1)}{p}+\frac{1}{p^{2m-1}}\right\rceil\\
&=2p^{2m-1}(p-r-1)+2m-1+\left\lceil \frac{-2(p-r-1)}{p}+\frac{1}{p^{2m-1}}\right\rceil\\
&=2p^{2m-1}(p-r-1)+2m-1-\left\lfloor \frac{2(p-r-1)}{p}-\frac{1}{p^{2m-1}}\right\rfloor
\end{aligned}
$$

Since $\frac{2(p-r-1)}{p}-\frac{1}{p^{2m-1}}<2$ and $m\geq 2$, we have $\left\lfloor \frac{2(p-r-1)}{p}-\frac{1}{p^{2m-1}}\right\rfloor\leq 1<2m-1$. Hence,

$$
\sum_{i=0}^{2m-1}\left\lceil \frac{d+1}{p^i}\right\rceil=\sum_{i=0}^{2m-1}\left\lceil \frac{2p^{2m-2}(p-1)(p-r-1)+1}{p^i}\right\rceil>2p^{2m-1}(p-r-1)=n
$$

which contracts to the Griesmer bound.

Then, we prove the optimality of $\phi(C_L)$. By computation, we have

$$
\begin{aligned}
&\sum_{i=0}^{2m-1}\left\lceil \frac{d}{p^i}\right\rceil\\
&=\sum_{i=0}^{2m-1}\left\lceil \frac{2p^{2m-2}(p-1)(p-r-1)}{p^i}\right\rceil\\
&=\sum_{i=0}^{2m-2}\left\lceil \frac{2p^{2m-2}(p-1)(p-r-1)}{p^i}\right\rceil+\left\lceil \frac{2p^{2m-2}(p-1)(p-r-1)}{p^{2m-1}}\right\rceil\\
&=2(p-r-1)(p^{2m-1}-1)+\left\lceil \frac{2(p-1)(p-r-1)}{p}\right\rceil.
\end{aligned}
$$

Note that

$$
\left\lceil \frac{2(p-1)(p-r-1)}{p}\right\rceil=2(p-r-1)+\left\lceil -\frac{2(p-r-1)}{p}\right\rceil=2(p-r-1)-\left\lfloor \frac{2(p-r-1)}{p}\right\rfloor.
$$

Since $\frac{p-2}{2}<r\leq p-1$, then $0\leq \frac{2(p-r-1)}{p}<1$ and

$$
\left\lceil \frac{2(p-1)(p-r-1)}{p}\right\rceil=2(p-r-1).
$$

It follows that

$$
\sum_{i=0}^{2m-1}\left\lceil \frac{d}{p^i}\right\rceil=2p^{2m-1}(p-r-1)=n,
$$

which shows if $\frac{p-2}{2}<r\leq p-1$, then $\phi(C_L)$ meets the Griesmer bound. $\square$

Tabel 4 is obtained by using Theorem 4.1.

Table 4: (Distance) optimal linear codes $\phi(C_L)$ in Theorem 4.1

p	n	k	d	Optimality	n	k	d	Optimality
3	972	6	648	Distance optimal	486	6	324	Optimal
3	8748	8	5832	Distance optimal	4374	8	2916	Optimal
5	25000	6	20000	Distance optimal	18750	6	15000	Distance optimal
5	12500	6	10000	Optimal	6250	6	5000	Optimal
5	625000	8	500000	Distance optimal	468750	8	375000	Distance optimal
5	312500	8	250000	Optimal	156250	8	125000	Optimal
7	201684	6	172872	Distance optimal	168070	6	144060	Distance optimal
7	134456	6	115248	Distance optimal	100842	6	86436	Optimal
7	67228	6	57624	Optimal	33614	6	28812	Optimal
7	9882516	8	8470728	Distance optimal	8235430	8	7058940	Distance optimal
7	6588344	8	5647152	Distance optimal	4941258	8	4235364	Optimal
7	3294172	8	2823576	Optimal	1647086	8	1411788	Optimal

Lemma 4.2. *The equation*

$$\left\lfloor \frac{-2(r+s+1)}{p} \right\rfloor + \left\lfloor \frac{2(r+s+1)(p-1)}{p^2} \right\rfloor = -2$$

holds if and only if $r+s \in \left\{\frac{p-1}{2}, p, \frac{3p-1}{2}, \frac{3p+1}{2}\right\}$.

Proof. Since $r+s \le 2p-2$, then

$$\frac{2}{p} - 4 \le \frac{-2(r+s+1)}{p}$$

and further

$$-4 \le \left\lfloor \frac{-2(r+s+1)}{p} \right\rfloor.$$

Observe that $\left\lfloor \frac{2(r+s+1)(p-1)}{p^2} \right\rfloor \ge 0$, there are three cases left:

$$\text{Case I.} \begin{cases} \left\lfloor \frac{-2(r+s+1)}{p} \right\rfloor = -2 \\ \left\lfloor \frac{2(r+s+1)(p-1)}{p^2} \right\rfloor = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{p}{2} < r+s+1 \le p \\ 0 \le r+s+1 < \frac{p^2}{2(p-1)} \end{cases} \Leftrightarrow \frac{p}{2} < r+s+1 < \frac{p^2}{2(p-1)} .$$

$$\text{Case II.} \begin{cases} \left\lfloor \frac{-2(r+s+1)}{p} \right\rfloor = -3 \\ \left\lfloor \frac{2(r+s+1)(p-1)}{p^2} \right\rfloor = 1 \end{cases} \Leftrightarrow \begin{cases} p < r+s+1 \le \frac{3p}{2} \\ \frac{p^2}{2(p-1)} \le r+s+1 < \frac{p^2}{p-1} \end{cases} \Leftrightarrow p < r+s+1 < \frac{p^2}{p-1} .$$

$$\text{Case III.} \begin{cases} \left\lfloor \frac{-2(r+s+1)}{p} \right\rfloor = -4 \\ \left\lfloor \frac{2(r+s+1)(p-1)}{p^2} \right\rfloor = 2 \end{cases} \Leftrightarrow \begin{cases} \frac{3p}{2} < r+s+1 \le 2p \\ \frac{p^2}{p-1} \le r+s+1 < \frac{3p^2}{2(p-1)} \end{cases} \Leftrightarrow \frac{3p}{2} < r+s+1 < min\left\{2p, \frac{3p^2}{2(p-1)}\right\} .$$

Summarized by Cases I-III, we conclude that $r+s \in \left\{p, \frac{3p-1}{2}\right\}$ if $p=3$ and $r+s \in \left\{\frac{p-1}{2}, p, \frac{3p-1}{2}, \frac{3p+1}{2}\right\}$ if $p > 3$. □

Theorem 4.3. *Suppose $m \ge 3$ is an integer, and let p be an odd prime number. Define the down set $\Delta = \langle (r,0,\ldots,0),(0,s,0,\ldots,0)\rangle$ of $\mathbb{F}_p^m$ for $r,s \in [1,p-1]$. Furthermore,let $\mathbb{H} = \{(x_1,x_2,\ldots,x_m) \in \mathbb{F}_p^m \mid x_3 = 0\}$ and $\mathbb{T} = \{(x_1,x_2,\ldots,x_m) \in \mathbb{F}_p^m \mid x_2 = 0\}$. Under these conditions, the Gray image $\phi(C_L)$ of C_L is a distance optimal linear code over $\mathbb{F}_p$. In particular, if $r+s \in \left\{\frac{p-1}{2}, p, \frac{3p-1}{2}, \frac{3p+1}{2}\right\}$, then $\phi(C_L)$ meets the Griesmer bound.*

Proof. According to Theorem 3.2, the parameters of the code $\phi(C_L)$ are:

$$n = 2p^m(p^{m-1} - r - s - 1), \quad k = 2m - 1, \quad d = 2p^{m-1}(p-1)(p^{m-1} - r - s - 1).$$

First, we demonstrate that $\phi(C_L)$ is distance optimal. Assume that there exists an $[n, k, d+1]$-code. Then we have

$$\begin{aligned}
&\sum_{i=0}^{2m-2} \left\lceil \frac{d+1}{p^i} \right\rceil \\
=&\sum_{i=0}^{2m-2} \left\lceil \frac{2p^{m-1}(p-1)(p^{m-1}-r-s-1)+1}{p^i} \right\rceil \\
=&\sum_{i=0}^{m-1} \left\lceil \frac{2p^{m-1}(p-1)(p^{m-1}-r-s-1)+1}{p^i} \right\rceil + \sum_{i=m}^{2m-2} \left\lceil \frac{2p^{2m-2}(p-1)}{p^i} + \frac{1-2p^{m-1}(p-1)(r+s+1)}{p^i} \right\rceil \\
=&2(p^{m-1}-r-s-1)(p^m-1) + m + \sum_{0}^{m-2} 2p^i(p-1) + \sum_{i=1}^{m-1} \left\lceil \frac{1}{p^{m-1+i}} - \frac{2(p-1)(r+s+1)}{p^i} \right\rceil
\end{aligned}$$

Since

$$\begin{aligned}
&\left\lceil \frac{1}{p^m} - \frac{2(p-1)(r+s+1)}{p} \right\rceil + \left\lceil \frac{1}{p^{m+1}} - \frac{2(p-1)(r+s+1)}{p^2} \right\rceil \\
=&-2(r+s+1) + \left\lceil \frac{1}{p^m} + \frac{2(r+s+1)}{p} \right\rceil + \left\lceil \frac{1}{p^{m+1}} - \frac{2(r+s+1)}{p} + \frac{2(r+s+1)}{p^2} \right\rceil \\
\geq&-2(r+s+1) + \left\lceil \frac{1}{p^m} + \frac{1}{p^{m+1}} + \frac{2(r+s+1)}{p^2} \right\rceil \\
\geq&-2(r+s+1) + 1
\end{aligned}$$

and when $i \geq 3$,

$$\left\lceil \frac{1}{p^{m-1+i}} - \frac{2(p-1)(r+s+1)}{p^i} \right\rceil \geq \left\lceil \frac{1-2p^{m-1}(p-1)(2p-1)}{p^{m-1+i}} \right\rceil = 0.$$

It follows that

$$\sum_{i=1}^{m-1} \left\lceil \frac{1}{p^{m-1+i}} - \frac{2(p-1)(r+s+1)}{p^i} \right\rceil \geq -2(r+s+1) + 1,$$

and so

$$\begin{aligned}
\sum_{i=0}^{2m-2} \left\lceil \frac{d+1}{p^i} \right\rceil &\geq 2(p^{m-1}-r-s-1)(p^m-1) + m + 2(p^{m-1}-1) - 2(r+s+1) + 1 \\
&= n + m - 1 > n,
\end{aligned}$$

which contracts to the Griesmer bound.

Then, we prove the optimality of $\phi(C_L)$. By computation, we have

$$\sum_{i=0}^{2m-2} \left\lceil \frac{d}{p^i} \right\rceil$$

$$
\begin{aligned}
&= \sum_{i=0}^{2m-2} \left\lceil \frac{2p^{m-1}(p-1)(p^{m-1}-r-s-1)}{p^i} \right\rceil \\
&= 2(p^m-1)(p^{m-1}-r-s-1) + \sum_{i=m}^{2m-2} \left\lceil \frac{2p^{m-1}(p-1)(p^{m-1}-r-s-1)}{p^i} \right\rceil \\
&= n - 2(p^{m-1}-r-s-1) + \sum_{i=1}^{m-1} \left\lceil \frac{2(p-1)(p^{m-1}-r-s-1)}{p^i} \right\rceil \\
&= n - 2(p^{m-1}-r-s-1) + \sum_{i=1}^{m-1} \left\lceil \frac{2p^{m-1}(p-1)}{p^i} - \frac{2(p-1)(r+s+1)}{p^i} \right\rceil \\
&= n - 2(p^{m-1}-r-s-1) + 2(p^{m-1}-1) + \sum_{i=1}^{m-1} \left\lceil -\frac{2(p-1)(r+s+1)}{p^i} \right\rceil \\
&= n + 2(r+s) - 2(r+s+1) + \left\lceil \frac{2(r+s+1)}{p} \right\rceil + \sum_{i=2}^{m-1} \left\lceil \frac{2(r+s+1)}{p^i} - \frac{2(r+s+1)}{p^{i-1}} \right\rceil \\
&= n - 2 + \left\lceil \frac{2(r+s+1)}{p} \right\rceil + \left\lceil \frac{2(r+s+1)}{p^2} - \frac{2(r+s+1)}{p} \right\rceil
\end{aligned}
$$

By Lemma 4.2, when $r+s \in \left\{\frac{p-1}{2}, p, \frac{3p-1}{2}, \frac{3p+1}{2}\right\}$, we have

$$
\begin{aligned}
&\left\lceil \frac{2(r+s+1)}{p} \right\rceil + \left\lceil \frac{2(r+s+1)}{p^2} - \frac{2(r+s+1)}{p} \right\rceil \\
&= -\left\lfloor \frac{-2(r+s+1)}{p} \right\rfloor - \left\lfloor \frac{2(r+s+1)(p-1)}{p^2} \right\rfloor = 2
\end{aligned}
$$

then $\sum_{i=0}^{2m-2} \left\lceil \frac{d}{p^i} \right\rceil = n$ and $\phi(C_L)$ meets the Griesmer bound. □

Tabel 5 is obtained by using Theorem 4.3.

Table 5: (Distance) optimal linear codes $\phi(C_L)$ in Theorem 4.3

p	n	k	d	Optimality	n	k	d	Optimality
3	324	5	216	Distance optimal	270	5	180	Optimal
3	216	5	144	Optimal	3888	7	2592	Distance optimal
3	3726	7	2484	Optimal	3564	7	2376	Optimal
5	5500	5	4400	Optimal	5250	5	4200	Distance optimal
5	5000	5	4000	Distance optimal	4750	5	3800	Optimal
5	4500	5	3600	Distance optimal	4250	5	3400	Optimal
5	4000	5	3200	Optimal	152500	7	122000	Optimal
5	151250	7	121000	Distance optimal	150000	7	120000	Distance optimal
5	148750	7	119000	Optimal	147500	7	118000	Distance optimal
5	146250	7	117000	Optimal	145000	7	116000	Optimal
7	31556	5	27048	Distance optimal	30870	5	26460	Optimal
7	30184	5	25872	Distance optimal	29498	5	25284	Distance optimal
7	28812	5	24696	Distance optimal	28126	5	24108	Optimal

Lemma 4.4. *The equation*

$$
\left\lceil \frac{2r}{p} \right\rceil + \left\lceil \frac{2(s+1-r)}{p} + \frac{2r}{p^2} \right\rceil + \left\lceil -\frac{2(s+1)}{p} + \frac{2(s+1-r)}{p^2} + \frac{2r}{p^3} \right\rceil = 2
$$

holds if and only if $r \geq \frac{p+1}{2}, r - \frac{p+3}{2} \leq s \leq \frac{p-3}{2}$ *or* $r \leq \frac{p-1}{2}, r-1 \leq s \leq \frac{p-3}{2}$ *or* $r \geq \frac{p+1}{2}, r-1 \leq s \leq r + \frac{p-5}{2}$ *or* $r \leq \frac{p-1}{2}, s \geq r + \frac{p-1}{2}$.

Proof. Since $2 \geq \left\lceil \frac{2r}{p} \right\rceil \geq 1, 0 \geq \left\lceil -\frac{2(s+1)}{p} + \frac{2(s+1-r)}{p^2} + \frac{2r}{p^3} \right\rceil \geq -1$, in order to make the equation hold, there are the following four situations.

Case 1. $\left\lceil \frac{2r}{p} \right\rceil = 2, \left\lceil \frac{2(s+1-r)}{p} + \frac{2r}{p^2} \right\rceil = 0, \left\lceil -\frac{2(s+1)}{p} + \frac{2(s+1-r)}{p^2} + \frac{2r}{p^3} \right\rceil = 0$. It follows that $r \geq \frac{p+1}{2}, r - \frac{p+3}{2} \leq s \leq \frac{p-3}{2}$.

Case 2. $\left\lceil \frac{2r}{p} \right\rceil = 1, \left\lceil \frac{2(s+1-r)}{p} + \frac{2r}{p^2} \right\rceil = 1, \left\lceil -\frac{2(s+1)}{p} + \frac{2(s+1-r)}{p^2} + \frac{2r}{p^3} \right\rceil = 0$. It follows that $r \leq \frac{p-1}{2}, r-1 \leq s \leq \frac{p-3}{2}$.

Case 3. $\left\lceil \frac{2r}{p} \right\rceil = 2, \left\lceil \frac{2(s+1-r)}{p} + \frac{2r}{p^2} \right\rceil = 1, \left\lceil -\frac{2(s+1)}{p} + \frac{2(s+1-r)}{p^2} + \frac{2r}{p^3} \right\rceil = -1$. It follows that $r \geq \frac{p+1}{2}, r-1 \leq s \leq r + \frac{p-5}{2}$.

Case 4. $\left\lceil \frac{2r}{p} \right\rceil = 1, \left\lceil \frac{2(s+1-r)}{p} + \frac{2r}{p^2} \right\rceil = 2, \left\lceil -\frac{2(s+1)}{p} + \frac{2(s+1-r)}{p^2} + \frac{2r}{p^3} \right\rceil = -1$. It follows that $r \leq \frac{p-1}{2}, s \geq r + \frac{p-1}{2}$. □

Theorem 4.5. *Suppose* $m \geq 4$ *is an integer and let* p *be an odd prime number. Define the down set* $\Delta = \langle (r, 0, \ldots, 0), (0, p-1, s, 0, \ldots, 0) \rangle$ *of* $\mathbb{F}_p$ *for* $r, s \in [1, p-1]$. *Furthermore, let* $\mathbb{H} = \{(x_1, x_2, \ldots, x_m) \in \mathbb{F}_p^m \mid x_4 = 0\}$ *and* $\mathbb{T} = \{(x_1, x_2, \ldots, x_m) \in \mathbb{F}_p^m \mid x_1 = 0\}$. *Under these conditions, the Gray image* $\phi(C_L)$ *of* C_L *is a distance optimal linear code. In particular, if* $r \geq \frac{p+1}{2}, r - \frac{p+3}{2} \leq s \leq \frac{p-3}{2}$ *or* $r \leq \frac{p-1}{2}, r-1 \leq s \leq \frac{p-3}{2}$ *or* $r \geq \frac{p+1}{2}, r-1 \leq s \leq r + \frac{p-5}{2}$ *or* $r \leq \frac{p-1}{2}, s \geq r + \frac{p-1}{2}$, *then* $\phi(C_L)$ *meets the Griesmer bound.*

Proof. According to Theorem 3.3, the parameters of the code $\phi(C_L)$ are:

$$n = 2p^m(p^{m-1} - p(s+1) - r), \quad k = 2m-1, \quad d = 2p^{m-1}(p-1)(p^{m-1} - p(s+1) - r).$$

First, we demonstrate that $\phi(C_L)$ is distance optimal. Assume that there exists an $[n, k, d+1]$-code. Then we have

$$\begin{aligned}
\sum_{i=0}^{2m-2} \left\lceil \frac{d+1}{p^i} \right\rceil = \sum_{i=0}^{2m-2} \left\lceil \frac{2p^{m-1}(p-1)(p^{m-1} - p(s+1) - r) + 1}{p^i} \right\rceil \\
&= 2(p^m - 1)(p^{m-1} - p(s+1) - r) + m + 2(p^{m-1} - 1) + \sum_{i=m}^{2m-2} \left\lceil \frac{1 - 2p^{m-1}(p-1)(p(s+1)+r)}{p^i} \right\rceil \\
&= n + 2(p(s+1)+r) + m - 2 + \sum_{i=1}^{m-1} \left\lceil \frac{1}{p^{m-1+i}} - \frac{2(p-1)(p(s+1)+r)}{p^i} \right\rceil
\end{aligned}$$

Since

$$\begin{aligned}
&\left\lceil \frac{1}{p^m} - \frac{2(p-1)(p(s+1)+r)}{p} \right\rceil + \left\lceil \frac{1}{p^{m+1}} - \frac{2(p-1)(p(s+1)+r)}{p^2} \right\rceil \\
&= -2(p(s+1)+r) + \left\lceil \frac{1}{p^m} + \frac{2(p(s+1)+r)}{p} \right\rceil + \left\lceil \frac{1}{p^{m+1}} - \frac{2(p(s+1)+r)}{p} + \frac{2(p(s+1)+r)}{p^2} \right\rceil \\
&\geq -2(p(s+1)+r) + \left\lceil \frac{1}{p^m} + \frac{1}{p^{m+1}} + \frac{2(p(s+1)+r)}{p^2} \right\rceil \\
&\geq -2(p(s+1)+r) + 1
\end{aligned}$$

and

$$\sum_{i=3}^{m-1} \left\lceil \frac{1}{p^{m-1+i}} - \frac{2(p-1)(p(s+1)+r)}{p^i} \right\rceil \geq -1$$

Hence,

$$\sum_{i=0}^{2m-1}\left\lceil\frac{d+1}{p^i}\right\rceil \geq n+2(p(s+1)+r)+m-2-2(p(s+1)+r)+1-1=n+m-2>n$$

which contracts to the Griesmer bound.

Then, we prove the optimality of $\phi(C_L)$. By computation, We have

$$\begin{aligned}\sum_{i=0}^{2m-2}\left\lceil\frac{d}{p^i}\right\rceil=&\sum_{i=0}^{2m-2}\left\lceil\frac{2p^{m-1}(p-1)(p^{m-1}-p(s+1)-r)}{p^i}\right\rceil\\
=&2(p^m-1)(p^{m-1}-p(s+1)-r)+2(p^{m-1}-1)+\sum_{i=1}^{m-1}\left\lceil\frac{2(p-1)(-p(s+1)-r)}{p^i}\right\rceil\\
=&n-2+2(s+1)+\left\lceil\frac{2r}{p}\right\rceil-2(s+1)+\left\lceil\frac{2(s+1-r)}{p}+\frac{2r}{p^2}\right\rceil+\left\lceil-\frac{2(s+1)}{p}+\frac{2(s+1-r)}{p^2}+\frac{2r}{p^3}\right\rceil\\
=&n-2+\left\lceil\frac{2r}{p}\right\rceil+\left\lceil\frac{2(s+1-r)}{p}+\frac{2r}{p^2}\right\rceil+\left\lceil-\frac{2(s+1)}{p}+\frac{2(s+1-r)}{p^2}+\frac{2r}{p^3}\right\rceil\end{aligned}$$

By Lemma 4.4, if $r\geq\frac{p+1}{2}, r-\frac{p+3}{2}\leq s\leq\frac{p-3}{2}$ or $r\leq\frac{p-1}{2}, r-1\leq s\leq\frac{p-3}{2}$ or $r\geq\frac{p+1}{2}, r-1\leq s\leq r+\frac{p-5}{2}$ or $r\leq\frac{p-1}{2}, s\geq r+\frac{p-1}{2}$, then

$$\left\lceil\frac{2r}{p}\right\rceil+\left\lceil\frac{2(s+1-r)}{p}+\frac{2r}{p^2}\right\rceil+\left\lceil-\frac{2(s+1)}{p}+\frac{2(s+1-r)}{p^2}+\frac{2r}{p^3}\right\rceil=2$$

which shows $\phi(C_L)$ meets the Griesmer bound. □

Tabel 6 is obtained by using Theorem 4.5.

Table 6: (Distance) optimal linear codes $\phi(C_L)$ in Theorem 4.5

p	n	k	d	Optimality	n	k	d	Optimality
3	3240	7	2160	Distance optimal	2754	7	1836	Optimal
5	142500	7	114000	Optimal	136250	7	109000	Distance optimal
5	130000	7	104000	Optimal	123750	7	99000	Optimal
5	141250	7	113000	Optimal	135000	7	108000	Distance optimal
5	128750	7	103000	Distance optimal	122500	7	98000	Optimal
5	140000	7	112000	Optimal	133750	7	107000	Optimal
5	138750	7	111000	Optimal	132500	7	106000	Distance optimal
5	126250	7	101000	Optimal	120000	7	96000	Optimal
7	1575056	7	1350048	Optimal	1541442	7	1321236	Optimal
7	1507828	7	1292424	Distance optimal	1474214	7	1263612	Optimal
7	1440600	7	1234800	Optimal	1406986	7	1205988	Optimal
7	1570254	7	1345932	Optimal	1536640	7	1317120	Optimal
7	1503026	7	1288308	Distance optimal	1469412	7	1259496	Distance optimal
7	1435798	7	1230684	Optimal	1402184	7	1201872	Optimal
7	1555848	7	1333584	Optimal	1522234	7	1304772	Optimal
7	1488620	7	1275960	Distance optimal	1455006	7	1247148	Optimal
7	1421392	7	1218336	Optimal	1387778	7	1189524	Optimal

5. Conclusions

The principal contributions of this paper are outlined as follows:

- The construction of linear codes C_D over the ring $\mathbb{F}_p + u\mathbb{F}_p$ associated with down sets, where $u^2 = 0$ and p is an odd prime number.
- The determination of the Lee weight distributions for the codes C_D over $\mathbb{F}_p + u\mathbb{F}_p$ when the down sets are generated by a single maximal or two maximal elements (Theorems 3.1 - 3.3).
- The derivation of several infinite families of p-ary optimal linear codes from the Gray images of the codes C_D over $\mathbb{F}_p + u\mathbb{F}_p$ (Theorems 4.1 , 4.3 and 4.5).

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