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On $\mathbb{F}_{p^m}\mathfrak{R}$-additive skew negacyclic codes

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Abstract. Let $\mathfrak{R} = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ with $u^2 = 0$, where p is an odd prime and m is any positive integer. This article delves into the algebraic structure of skew negacyclic codes of length $2p^s$ over a finite field $\mathbb{F}_{p^m}$ and a finite chain ring $\mathfrak{R}$. The focus is on the classification and structural properties of these codes. Based on the different possible factorizations of $x^{2p^s} + 1$ over $\mathbb{F}_{p^m}$, a complete classification of the structural properties of skew negacyclic codes and their duals for length $2p^s$ over $\mathbb{F}_{p^m}$ and $\mathfrak{R}$ is provided. Furthermore, the algebraic structure of $\mathbb{F}_{p^m}\mathfrak{R}$-additive skew negacyclic codes with block length $(p^s, 2p^s)$ is discussed. The separability of $\mathbb{F}_{p^m}\mathfrak{R}$-additive skew negacyclic codes is also analyzed. To illustrate these results, several examples are presented, including the construction of Maximum Distance Separable (MDS) and near-MDS codes.

1. Introduction

Error-correcting codes are an integral part of modern communication systems, ensuring reliable data transmission even in the presence of noise or interference. Consider $\mathbb{F}_{p^m}$, a finite field with p^m elements, where p is an odd prime and m is a positive integer. A linear code is any non-empty subspace C of the finite-dimensional vector space $\mathbb{F}_{p^m}^n$ over $\mathbb{F}_{p^m}$. If C is an ideal of the quotient ring $\frac{\mathbb{F}_{p^m}[x]}{\langle x^n-\lambda\rangle}$, the code is called a constacyclic code of length n over $\mathbb{F}_{p^m}$, where $\lambda \in \mathbb{F}_{p^m} \setminus \{0\}$. A particularly important class of constacyclic codes arises when $\lambda = \pm 1$. Specifically, C is cyclic when $\lambda = 1$ and negacyclic when $\lambda = -1$. Negacyclic codes, introduced by Berlekamp in the early 1960s, form a significant subset of constacyclic codes [8]. These codes are noteworthy due to their algebraic structure, which offers advantages like simplicity in construction and ease of encoding and decoding.

Numerous researchers have studied linear and negacyclic codes over commutative rings (see [9, 16, 18, 24, 29, 32]). Boucher et al. [12] extended this study to non-commutative rings, exploring a broader class of codes. They introduced skew cyclic codes characterized by a generator polynomial in the non-commutative ring $\mathbb{F}_{p^m}[x;\pi]$, where π is a field automorphism on $\mathbb{F}_{p^m}$. These skew cyclic codes exhibited larger Hamming distances compared to previously known linear codes. Further research into skew cyclic codes over finite fields followed (see [14, 31]). Boucher et al. [13] and Jitman et al. [25] expanded the results to Galois

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rings and finite chain rings, respectively, providing the algebraic structure of these codes. Bagheri et al. [7] introduced skew cyclic codes of length p^s over $\mathbb{F}_{p^m}+u\mathbb{F}_{p^m}$, where $u^2=0$. Hesari et al. [22] later described the algebraic structure of the duals of skew constacyclic codes over this ring including self-dual codes. Several other studies have explored skew cyclic codes over various skew polynomial rings (see [4, 20, 23, 30, 33]).

In 1973, Delsarte [15] introduced the concept of codes over mixed alphabets. Any code is called an additive code if it is a subgroup of $\mathcal{F}^n$, where $\mathcal{F}$ is an Abelian group, and n is its direct product. Abualrub [1] and Borges [10] later introduced and examined $\mathbb{Z}_2\mathbb{Z}_4$-additive codes, which were subsequently generalized to $\mathbb{Z}_2\mathbb{Z}_{2^s}$ [6] and $\mathbb{Z}_2\mathbb{Z}_2[u]$ [2, 5]. Recently, Juan et al. [26] investigated linear skew cyclic codes over mixed alphabets $\mathbb{F}_qR$, where $R=\mathbb{F}_q+u\mathbb{F}_q$ with $u^2=0$, and determined the duals of separable skew cyclic codes. They established a sufficient condition for these codes to be self-dual over $\mathbb{F}_qR$. In 2024, Roghayeh et al. [21] studied skew cyclic codes of length p^s over $R_3=\mathbb{F}_{p^m}+u\mathbb{F}_{p^m}+u^2\mathbb{F}_{p^m}$ with $u^3=0$, classifying all possible generator polynomials of skew cyclic codes of length p^s over R_3. This result was later extended to mixed alphabets $\mathbb{F}_{p^m}R_3$.

Motivated by the literature above on skew cyclic codes, we investigate skew negacyclic codes of length $2p^s$ over the chain ring $\mathbb{F}_{p^m}+u\mathbb{F}_{p^m}$. In this article, we investigate skew negacyclic codes of length $2p^s$ over the finite field $\mathbb{F}_{p^m}$ and the chain ring $\mathfrak{R}=\mathbb{F}_{p^m}+u\mathbb{F}_{p^m}$, with $u^2=0$, where p is an odd prime, and m is any positive integer. We factor the polynomial $x^{2p^s}+1$ over $\mathbb{F}_{p^m}$ and $\mathfrak{R}$. By factoring the polynomial $x^{2p^s}+1$ under different conditions on p, we obtain various constituent codes for the skew negacyclic code of length $2p^s$. Using the algebraic structure over a single alphabet, we extend our exploration to skew negacyclic codes of block length $(p^s,2p^s)$ over mixed alphabets $\mathbb{F}_{p^m}\mathfrak{R}$. Furthermore, we derive the generator polynomials for skew negacyclic codes of block length $(p^s,2p^s)$ and separable skew negacyclic codes over mixed alphabets.

We present our article as follows: In Section 2, we discuss several useful results along with key definitions relevant to our study. Section 3 is divided into two subsections to examine the cases $p\equiv 1 \pmod 4$ and $p\equiv 3 \pmod 4$. In Subsection 3.1, we address the case $p\equiv 1 \pmod 4$ and show that skew negacyclic codes of length $2p^s$ decompose into a direct sum of a skew (π,α)-constacyclic code and a skew $(\pi,-\alpha)$-constacyclic code of length p^s over $\mathbb{F}_{p^m}$. The algebraic structure of skew negacyclic codes and their duals is discussed in Theorem 3.2. In Section 4, we extend the study from Section 3 to the ring $\mathfrak{R}=\mathbb{F}_{p^m}+u\mathbb{F}_{p^m}$, with $u^2=0$. This section is also split into the cases $p\equiv 1 \pmod 4$ and $p\equiv 3 \pmod 4$. In Subsection 4.1, we discuss the case $p\equiv 1 \pmod 4$ and study the algebraic structure of skew negacyclic codes and their duals, as shown in Proposition 4.1 and Theorem 4.2. In Subsection 4.2, we examine the case $p\equiv 3 \pmod 4$ and show that x^2+1 is irreducible over $\mathbb{F}_{p^m}$. Furthermore, in Theorem 4.5, we describe the complete structure of left ideals of $\mathcal{R}_2=\frac{\mathfrak{R}[x;\Pi]}{\langle (x^2+1)^{p^s}\rangle}$. In Theorem 4.8, we investigate the algebraic structure of the annihilators of the left ideals of $\mathcal{R}_2$ and derive the generator polynomial of the left ideals in Theorem 4.9. In Section 5, we extend our work to mixed alphabets $\mathbb{F}_{p^m}\mathfrak{R}$. We characterize all possible $\mathbb{F}_{p^m}\mathfrak{R}$-additive skew negacyclic codes of block length $(p^s,2p^s)$ (see Theorem 5.2). Furthermore, we investigate an important class of codes called separable codes. Using the definition of separable codes in Theorem 5.3, we demonstrate how separable $\mathbb{F}_{p^m}\mathfrak{R}$-additive skew negacyclic codes of block length $(p^s,2p^s)$ are a direct product of skew negacyclic codes C_1 and C_2, with lengths p^s and $2p^s$ over $\mathbb{F}_{p^m}$ and $\mathfrak{R}$, respectively. In Theorem 5.4, we derive the generator polynomial of separable $\mathbb{F}_{p^m}\mathfrak{R}$-additive skew negacyclic codes of block length $(p^s,2p^s)$. Section 6 concludes the paper.

2. Preliminaries

Consider the quotient ring $\mathbb{K}=\frac{\mathbb{F}_p[x]}{\langle a(x)\rangle}$. If an irreducible element $a(x)\in\mathbb{F}_p[x]$ has degree m, then the quotient ring $\mathbb{K}$ is known as a finite extension of $\mathbb{F}_p$. That is, $\mathbb{K}$ is a field of order p^m, and it has a subfield isomorphic to $\mathbb{F}_p$, so $\mathbb{K}$ forms a vector space over $\mathbb{F}_p$ with dimension m. This dimension, called the degree of the extension, is denoted as $[\mathbb{K}:\mathbb{F}_p]$. If $a(x)=s_0+s_1x+\cdots+s_mx^m$, where $s_i\in\mathbb{F}_p$, then, by [19, Theorem 20.1], if $\xi\in\mathbb{K}$ and $a(\xi)=0$, then ξ is a generator of $\mathbb{K}$. The elements of $\mathbb{K}$ can be expressed as polynomials in ξ with coefficients in $\mathbb{F}_p$. Hereafter, we refer to the finite field $\mathbb{F}_{p^m}=\mathbb{K}=\{0,\xi,\xi^2,\ldots,\xi^{p^m-2},\xi^{p^m-1}=1\}$.

Consider a ring $\mathfrak{R}$ with unity, then and suppose $\mathfrak{R}$ is referred to as a principal left (right) ideal ring if every left (right) ideal of $\mathfrak{R}$ can be generated by a single element. Furthermore, if $\mathfrak{R}$ has a unique maximal left (right) ideal, it is referred to as a local ring. If the set of all left (right) ideals of $\mathfrak{R}$ forms a chain, then $\mathfrak{R}$

is called a left (right) chain ring. A non-empty subset C of $\mathfrak{R}^n$ is referred to as a code of length n over $\mathfrak{R}$. If C forms an $\mathfrak{R}$-submodule of $\mathfrak{R}^n$, then it is called a linear code of length n over $\mathfrak{R}$.

Definition 2.1. *Let $\pi : \mathbb{F}_{p^m} \to \mathbb{F}_{p^m}$. If π satisfies the following conditions, then it is called a ring homomorphism:*

1. $\pi(\xi_1 + \xi_2) = \pi(\xi_1) + \pi(\xi_2)$ for all $\xi_1, \xi_2 \in \mathbb{F}_{p^m}$,
2. $\pi(\xi_1\xi_2) = \pi(\xi_1)\pi(\xi_2)$ for all $\xi_1, \xi_2 \in \mathbb{F}_{p^m}$.

Further, if π is bijective, then π is called an automorphism on $\mathbb{F}_{p^m}$.

Let π be an automorphism on $\mathbb{F}_{p^m}$, and suppose π fixes each element of $\mathbb{F}_p$, then π is called an $\mathbb{F}_p$-automorphism, with $\mathbb{F}_p$ being the fixed field with respect to π. As we know, the set of all automorphisms of $\mathfrak{R}$, denoted by $\mathrm{Aut}(\mathfrak{R})$, forms a group with the binary operation of the composition of maps, then $\mathrm{Aut}(\mathfrak{R})$ has an identity automorphism $\mathbb{I}$. Let π be a non-identity automorphism on $\mathfrak{R}$. The smallest positive integer t is called the order of π if $\pi^t = \mathbb{I}$, and this is denoted by $\mathrm{ord}(\pi)$. In this context, π^t denotes the composition of π applied t times.

Definition 2.2. *Let $\mathfrak{R}$ be a finite commutative ring and τ be an automorphism on $\mathfrak{R}$. Then we define the set $\mathfrak{R}[x;\tau]$ as follows:*

$$\mathfrak{R}[x;\tau] = \{s_0 + s_1x + \cdots + s_nx^n : s_i \in \mathfrak{R}\},$$

where n is a non-negative integer.

*The concepts of addition and equality for polynomials are defined in the usual way, whereas multiplication is defined according to the underlying rule: $xs = \tau(s)x$ ($\forall s \in \mathfrak{R}$). This multiplication is associative and distributive, so using these two properties, we can easily extend this multiplication to every element of $\mathfrak{R}[x;\tau]$. The set $\mathfrak{R}[x;\tau]$ with the usual addition and the multiplication defined above forms a **non-commutative** ring, which is known as the **skew polynomial ring** over $\mathfrak{R}$ (see [12]). Moreover, any element of a skew polynomial ring is known as a **skew polynomial**. In particular, if τ is the identity automorphism on $\mathfrak{R}$, then the skew polynomial ring $\mathfrak{R}[x;\tau]$ becomes a **commutative** ring.*

Definition 2.3. *The center of any ring $\mathfrak{R}$, denoted by $\mathcal{Z}(\mathfrak{R})$, is defined by*

$$\mathcal{Z}(\mathfrak{R}) = \{a \in \mathfrak{R} \mid ab = ba\ \forall b \in \mathfrak{R}\}.$$

Note 2.4. *Let π be an automorphism on $\mathbb{F}_{p^m}$ of order t. We now define the center of the ring $\mathbb{F}_{p^m}[x;\pi]$. Let $\mathbb{F}_p$ be a subfield of $\mathbb{F}_{p^m}$ that remains fixed under the action of π. Then the center $\mathcal{Z}(\mathfrak{R})$ is precisely $\mathbb{F}_p[x^t]$, which represents a commutative polynomial ring. It can be readily shown that any element $f(x)$ in the center satisfies the commutative relations $f(x)x = xf(x)$ and $af(x) = f(x)a$ for all $a \in \mathbb{F}_{p^m}$. The element of $\mathbb{F}_p[x^t]$ is called a **central skew polynomial**.*

Consider $\mathfrak{R} = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ with $u^2 = 0$. According to [17], $\mathfrak{R}$ is a finite chain ring and $\mathfrak{R}(u)$ is the only maximal ideal of $\mathfrak{R}$. To define a skew polynomial structure over $\mathfrak{R}$, we first need to define an automorphism on $\mathfrak{R}$.

Lemma 2.5. *[3] Let $\pi \in \mathrm{Aut}(\mathbb{F}_{p^m})$ and $\eta \in \mathbb{F}_{p^m}^*$. Then the map $\Pi : \mathfrak{R} \to \mathfrak{R}$ given by*

$$\Pi_{\pi,\eta}(a + ub) = \pi(a) + u\eta\pi(b),$$

is an automorphism on $\mathfrak{R}$. Further, any automorphism on $\mathfrak{R}$ is of the form $\Pi_{\pi,\eta}$. Moreover, the collection of all automorphisms of $\mathfrak{R}$ is denoted by $\mathrm{Aut}(\mathfrak{R})$ and is given by

$$\mathrm{Aut}(\mathfrak{R}) = \{\Pi_{\pi,\eta} \mid \pi \in \mathrm{Aut}(\mathbb{F}_{p^m}) \text{ and } \eta \in \mathbb{F}_{p^m}^*\}.$$

From now on, we take $\eta = 1$ and for our convenience, we denote $\Pi_{\pi,1}$ as Π.

Proposition 2.6. *[25, Proposition 2.3] Let $p(x)$ and $s(x)$ be skew polynomials in $\mathfrak{R}[x;\Pi]$. If $p(x)s(x)$ is a monic polynomial in $\mathcal{Z}(\mathfrak{R})$, then $p(x)s(x) = s(x)p(x)$.*

Proposition 2.7. *[27, Right division algorithm] Let $p(x)$ and $s(x)$ be skew polynomials in $\mathbb{F}_{p^m}[x;\pi]$, where $p(x)$ is monic. Then there exist some polynomials $t(x)$ and $r(x)$ in $\mathbb{F}_{p^m}[x;\pi]$ such that*

$$s(x) = t(x)p(x) + r(x),$$

where $r(x) = 0$ or $\deg(r(x)) < \deg(p(x))$.

According to the above premise, $p(x)$ is a right divisor of $s(x)$ if and only if $r(x) = 0$. It is denoted as $p(x) \mid_r s(x)$. Similarly, $p(x)$ is referred to as a left divisor of $s(x)$ if $s(x) = p(x)t(x)$, and it is denoted as $p(x) \mid_l s(x)$, where $t(x)$ is any skew polynomial in $\mathbb{F}_{p^m}[x;\pi]$.

Example 2.8. *Suppose $\mathbb{F}_9 = \{0, \delta, \ldots, \delta^8 = 1\}$, where δ is a root of the polynomial $p(x) = x^2 + 1 \in \mathbb{F}_3[x]$ and let π be a Frobenius automorphism on $\mathbb{F}_9$ of order 2.*

$x^4 + 2\delta x^3 + \delta x + 1 = (x + 2\delta)(x + \delta + 2)(x + 2\delta)(x + \delta + 2)$. Therefore, $x + \delta + 2$ is a right divisor of $x^4 + 2\delta x^3 + \delta x + 1$. It can be easily verified that $x + \delta + 2$ cannot be a left divisor of $x^4 + 2\delta x^3 + \delta x + 1$ in $\mathbb{F}_9[x;\pi]$.

Definition 2.9. *[28] Consider skew polynomials $p(x)$ and $s(x)$ in $\mathbb{F}_{p^m}[x;\pi]$. We define the greatest common right divisor (abbreviated as $\gcd_r$) of $p(x)$ and $s(x)$ as the unique monic skew polynomial $d(x)$ in $\mathbb{F}_{p^m}[x;\pi]$ satisfying the following conditions:*

1. *$d(x)$ is a right divisor of $p(x)$ and $s(x)$.*
2. *Any other skew polynomial $d'(x)$ in $\mathbb{F}_{p^m}[x;\pi]$ that is a right divisor of both $p(x)$ and $s(x)$ must also be a right divisor of $d(x)$.*

Definition 2.10. *[28] Skew polynomials $p(x)$ and $s(x)$ in $\mathbb{F}_{p^m}[x;\pi]$ are called right relative prime if and only if $\gcd_r(p(x), s(x)) = 1$.*

Lemma 2.11. *[28, Right Bézout identity] If $d(x) = \gcd_r(p(x), s(x))$, then there exist $r_1(x)$ and $r_2(x)$ in $\mathbb{F}_{p^m}[x;\pi]$ such that*

$$d(x) = r_1(x)p(x) + r_2(x)s(x).$$

Definition 2.12. *[23] Let $p(x) \in \mathbb{F}_{p^m}[x;\pi]$ be a skew polynomial. If $p(x)$ is monic and irreducible, then it is called prime.*

Definition 2.13. *Let $\mathcal{I}$ be any non-empty subset of $\mathbb{F}_{p^m}[x;\pi]$. If $\mathcal{I}$ satisfies the following conditions, then $\mathcal{I}$ is called a left ideal of $\mathbb{F}_{p^m}[x;\pi]$*

1. *$a(x) + b(x) \in \mathcal{I}, \forall\ a(x), b(x) \in \mathcal{I}$,*
2. *$r(x)a(x) \in \mathcal{I}, \forall\ a(x) \in \mathcal{I}$ and $r(x) \in \mathbb{F}_{p^m}[x;\pi]$.*

Remark 2.14. *Every left (or right) ideal of $\mathbb{F}_{p^m}[x;\pi]$ is principally generated. Thus, $\mathbb{F}_{p^m}[x;\pi]$ is a left (or right) principal ideal domain. Further, from [12], it is well known that the ring $\mathbb{F}_{p^m}[x;\pi]$ is not a unique factorization domain.*

Example 2.15. *Suppose $\mathbb{F}_{p^m} = \mathbb{F}_{27} = \{0, \delta, \delta^2, \ldots, \delta^{25}, \delta^{26} = 1\}$, where δ is a root of the polynomial $p(x) = x^3 + 2x + 1 \in \mathbb{F}_3[x]$, and let $\pi^2 = I$. Consider a skew polynomial $x^3 - 1$ in $\mathbb{F}_{27}[x;\pi]$. We have,*

$$x^3 - 1 = (x - \delta^2)(x^2 + \delta^{18}x + \delta^{24}) = (x - \delta^6)(x^2 + \delta^2 x + \delta^{20}) = (x - 1)(x - 1)(x - 1).$$

That is, the skew polynomial $x^3 - 1$ in $\mathbb{F}_{27}[x;\pi]$ is not uniquely factorized.

Definition 2.16. *A non-empty subset $\mathcal{I}$ of $\mathbb{F}_{p^m}[x;\pi]$ is called a two-sided ideal of $\mathbb{F}_{p^m}[x;\pi]$ if and only if it is generated by a central skew polynomial.*

Let $\Pi \in \mathrm{Aut}(\mathfrak{R})$ be an automorphism of order l. Note that any automorphism maps -1 to -1. The ideal $\langle x^{2p^s}+1\rangle$ is a two-sided ideal of $\mathfrak{R}[x;\Pi]$ if and only if $\mathrm{ord}(\Pi) = \mathrm{ord}(\pi) = l \mid p^s$ (see [25]). This implies that if $\mathrm{ord}(\Pi)$ divides n, then $\langle x^n+1\rangle$ is a two-sided ideal of $\mathfrak{R}[x;\Pi]$. From Proposition 2.6, we have that $x^{2p^s}+1$ is a monic central skew polynomial, and hence any right divisor of $x^{2p^s}+1$ is also a two-sided divisor. That is, if $x^{2p^s}+1 = y_1(x)y_2(x) = y_2(x)y_1(x)$, then $y_2(x) = \frac{x^{2p^s}+1}{y_1(x)}$, where $y_i(x) \in \mathfrak{R}[x;\Pi]$ for $i = 1, 2$.

Let $\Pi \in \mathrm{Aut}(\mathfrak{R})$ and Λ be a unit element of $\mathfrak{R}$. Let C be a linear code of length n over $\mathfrak{R}$. Then C is a skew (Π, Λ)-constacyclic code over $\mathfrak{R}$ if for $(c_0, c_1, \ldots, c_{n-1}) \in C$, it holds that $\Big(\Pi(\Lambda c_{n-1}), \Pi(c_0), \ldots, \Pi(c_{n-2})\Big) \in C$. In particular, for $\Lambda = 1$, C is a skew cyclic code, and for $\Lambda = -1$, C is a skew negacyclic code over $\mathfrak{R}$.

To establish an algebraic representation from the combinatorial structure of skew negacyclic codes, we examine the following correspondence: Let $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1}) \in C$ be a codeword. Then the skew polynomial representation of $\mathbf{c}$ is $c(x) = c_0 + c_1x + \cdots + c_{n-1}x^{n-1}$ in $\mathcal{R}_2 = \frac{\mathfrak{R}[x;\Pi]}{\langle x^n+1\rangle}$.

Now onward, we denote the quotient rings:

$$\mathcal{R}_1 = \frac{\mathbb{F}_{p^m}[x;\pi]}{\langle x^{p^s}+1\rangle}, \quad \mathcal{R}_1' = \frac{\mathbb{F}_{p^m}[x;\pi]}{\langle x^n+1\rangle} \text{ and } \mathcal{R}_2 = \frac{\mathfrak{R}[x;\Pi]}{\langle x^n+1\rangle},$$

where $n = 2p^s$ and we assume $\mathbb{F}_{p^m}$ is a field, with p being an odd prime, m and s being any positive integers, unless otherwise stated.

Proposition 2.17. *[25, Theorem 2.2] Let C be a linear code of length n over $\mathfrak{R}$. Then C is a skew negacyclic code if and only if its polynomial representation forms a left ideal of $\mathcal{R}_2$. Moreover, $C = \langle f(x)\rangle = \mathcal{R}_2(f(x))$, where $f(x)$ is a monic factor of x^n+1.*

Note 2.18. *We can define a ring epimorphism $\mu : \mathfrak{R} \to \mathbb{F}_{p^m}$ by $\mu(a+ub) = a$. This epimorphism can be extended from $\mathfrak{R}[x;\Pi]$ to $\mathbb{F}_{p^m}[x;\pi]$ as follows:*

$$\sum_{i=0}^{n}(a_i + ub_i)x^i \mapsto \sum_{i=0}^{n} a_i x^i.$$

Additionally, μ can be extended from $\mathcal{R}_2$ to $\mathcal{R}_1'$. Thus, the image of any left ideal $\mathcal{I}$ of $\mathcal{R}_2$, $\mu(\mathcal{I} :_{\mathcal{R}_2} u) = \mu(\{v \in \mathcal{R}_2 : vu \in \mathcal{I}\})$, is a left ideal of $\mathcal{R}_1'$. Since any skew negacyclic code $\mathcal{C}$ of length n over $\mathfrak{R}$ forms a left ideal of $\mathcal{R}_2$, the code $\mu(\mathcal{C} :_{\mathcal{R}_2} u)$ can be considered a skew negacyclic code over $\mathbb{F}_{p^m}$. This code is called the torsion code of C and is denoted by $Tor(C)$.

Now, we define a Gray map

$$\phi : \mathfrak{R} \to \mathbb{F}_{p^m}^2$$
$$\phi(a+ub) = (b, a+b), \text{ where } a, b \in \mathbb{F}_{p^m}.$$

Furthermore, we can verify that ϕ is an $\mathbb{F}_{p^m}$-linear map. The map ϕ can be naturally extended componentwise from $\mathfrak{R}^n$ to $\mathbb{F}_{p^m}^{2n}$ as follows:

$$(r_0, r_1, \ldots, r_{n-1}) \mapsto (b_0, a_0+b_0, \ldots, b_{n-1}, a_{n-1}+b_{n-1}),$$

where $r_i = a_i + ub_i \in \mathfrak{R}$, and $a_i, b_i \in \mathbb{F}_{p^m}$, for $i = 0, 1, \ldots, n-1$. The Lee weight of an element $r \in \mathfrak{R}$ is denoted by $w_L(r)$ and is given by $w_L(r) = w_H(\phi(r))$, where w_H is the Hamming weight. For any $\mathbf{r} = (r_0, r_1, \ldots, r_{n-1}) \in \mathfrak{R}^n$, the Lee weight of $\mathbf{r}$ is given by $w_L(\mathbf{r}) = \sum_{i=0}^{n-1} w_H(\phi(r_i))$. For any two elements $\mathbf{r}$ and $\mathbf{r}'$ of $\mathfrak{R}^n$, the Lee distance between them, denoted by d_L, is defined as $d_L(\mathbf{r}, \mathbf{r}') = w_L(\mathbf{r}-\mathbf{r}') = w_H(\phi(\mathbf{r}-\mathbf{r}'))$. The minimum Lee distance of C is defined as $d_L = d_L(C) = \min\{d_L(\mathbf{r}, \mathbf{r}') \mid \mathbf{r} \neq \mathbf{r}', \mathbf{r}, \mathbf{r}' \in \mathfrak{R}^n\}$. It follows from the definition of the Gray map that ϕ is a distance-preserving map from the Lee distance to the Hamming distance, and ϕ is a bijective map. Thus, $\phi(C)$ is a linear code over $\mathbb{F}_{p^m}$ with parameters $[2n, k, d_H]$, where $d_L = d_H$ and k is the dimension of $\phi(C)$.

Now, we define the dual of a linear code C over $\mathfrak{R}$. The dual code of C is denoted by $C^{\perp}$ and is given by

$$C^{\perp} = \{\mathbf{c}' \in \mathfrak{R}^n \mid \langle \mathbf{c}', \mathbf{c} \rangle = 0, \forall \mathbf{c} \in C\},$$

where $\langle \mathbf{c}', \mathbf{c} \rangle$ denotes the Euclidean inner product of $\mathbf{c}'$ and $\mathbf{c}$.

Definition 2.19. *Let C be a $[n, k, d]$-linear code over $\mathbb{F}_{p^m}$. Then the parameters satisfy*

$$d \leq n - k + 1.$$

If equality holds, the code C is called a maximum distance separable (MDS) code.

Proposition 2.20. *[17, Proposition 2.5] Let $\mathfrak{R}$ be a finite chain ring of cardinality p^q. The cardinality of a linear code C of length n over $\mathfrak{R}$ is p^k, where k is an integer such that $0 \leq k \leq nq$. Furthermore, the dual code $C^{\perp}$ has p^l codewords, where the sum of the dimensions of C and $C^{\perp}$ is nq, i.e., $k + l = nq$. Thus, the product of the cardinalities of C and $C^{\perp}$ equals the cardinality of $\mathfrak{R}^n$.*

Proposition 2.21. *[25] Let $\pi \in \mathrm{Aut}(\mathbb{F}_{p^m})$ and C be a linear code of length n over $\mathbb{F}_{p^m}$. If C is a skew (π, Λ)-constacyclic code over $\mathbb{F}_{p^m}$, then $C^{\perp}$ is a skew (π, Λ^{-1})-constacyclic code over $\mathbb{F}_{p^m}$.*

Proposition 2.22. *[12] Let $\psi : \mathfrak{R}[x; \Pi] \to \mathfrak{R}[x; \Pi]$ be the map defined by*

$$\psi\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} \Pi(a_i) x^i,$$

where $a_i \in \mathfrak{R}$ for $i = 0, 1, \ldots, n$. Then ψ is a surjective ring homomorphism.

Definition 2.23. *Consider a skew polynomial $p(x) = a_0 + a_1 x + \cdots + a_k x^k$ in $\mathfrak{R}[x; \Pi]$, where $a_k \neq 0$. Its reciprocal polynomial is denoted by $p^*(x)$ and is given by*

$$p^*(x) = a_k + \Pi(a_{k-1})x + \cdots + \Pi^k(a_0)x^k = \sum_{i=0}^{k} \Pi^i(a_{k-i})x^i.$$

Lemma 2.24. *Suppose $p(x)$ and $s(x)$ are elements of $\mathfrak{R}[x; \Pi]$. Then the following results hold:*

(*i*) *If $\deg(p(x)) \geq \deg(s(x))$, then $(p(x) + s(x))^* = p^*(x) + x^{\deg(p(x)) - \deg(s(x))} s^*(x)$,*
(*ii*) *$(p(x)s(x))^* = \psi^{\deg(p(x))} s^*(x) p^*(x)$,*
(*iii*) *If the constant coefficient of $p(x)$ is non-zero, then $(p^*(x))^* = \psi^k(p(x))$, where k is the degree of $p(x)$.*

Remark 2.25. *Let C be a skew negacyclic code over $\mathfrak{R}$ and let $\mathcal{I}$ be the polynomial representation of C, which forms a left ideal of $\mathcal{R}_2$. If $\mathcal{I}$ is a left ideal of $\mathcal{R}_2$, then $\mathcal{I}^* = \{p^*(x) \mid p(x) \in \mathcal{I}\}$ is a right ideal of $\mathcal{R}_2$. The annihilator of $\mathcal{I}$ is defined as:*

$$\mathcal{A}(\mathcal{I}) = \{s(x) \in \mathcal{R}_2 \mid p(x)s(x) = 0 \text{ for all } p(x) \in \mathcal{I}\},$$

and $\mathcal{A}(\mathcal{I})$ forms a right ideal of $\mathcal{R}_2$. Thus, if $\mathcal{I}$ is the left ideal associated with C, then the associated left ideal of the dual code $C^{\perp}$ is $\mathcal{A}(\mathcal{I})^*$.

3. The algebraic structure of skew negacyclic codes of length $2p^s$ over $\mathbb{F}_{p^m}$

This section provides a detailed study of the algebraic structure of skew negacyclic codes of length $2p^s$ over $\mathbb{F}_{p^m}$. The study is based on the factorization of $x^{2p^s} + 1$ under various conditions.

Note 3.1. *Since $p^m - 1 = (p-1)(p^{m-1} + p^{m-2} + \cdots + p + 1)$, where $(p^{m-1} + p^{m-2} + \cdots + p + 1)$ is an odd integer, it follows that 4 divides $p - 1$ if and only if 4 divides $p^m - 1$ for an odd prime p and an odd positive integer m.*

3.1. $p \equiv 1 \pmod 4$

Recall from Section 2 that $\mathbb{F}_{p^m}$ is a finite field, and

$$\mathbb{F}_{p^m} = \{0, \xi, \xi^2, \ldots, \xi^{p^m-2}, \xi^{p^m-1} = 1\},$$

where $\xi \in \mathbb{F}_{p^m}$ is a primitive root of an irreducible polynomial $a(x)$ over $\mathbb{F}_p[x]$.

Since $4 \mid p-1$, there exists an element $\alpha \in \mathbb{F}_p$ such that $\alpha = \zeta^{\frac{p-1}{4}}$, where ζ is a generator of $\mathbb{F}_p^* = \mathbb{F}_p \setminus \{0\}$. Let $\beta = \alpha^2 = \zeta^{\frac{p-1}{2}}$. Then $\beta \in \mathbb{F}_p^*$ and $\text{ord}(\beta) = 2$. Since $\mathbb{F}_p^*$ is cyclic, the unique element of order 2 in $\mathbb{F}_p^*$ is -1. Thus, $\beta = -1$.

Therefore, we can factor the polynomial $x^{2p^s} + 1$ as follows:

$$x^{2p^s} + 1 = x^{2p^s} - \alpha^2 = (x^{p^s} - \alpha)(x^{p^s} + \alpha).$$

Since $x^{2p^s} + 1$ is factorize in $\mathbb{F}_p$, and $\mathbb{F}_p$ is a subfield of $\mathbb{F}_{p^m}$, we can extend this factorization to $\mathbb{F}_{p^m}$, where $\alpha \in \mathbb{F}_p$ is embedded in $\mathbb{F}_{p^m}$.

$$x^{2p^s} + 1 = (x^{p^s} - \alpha)(x^{p^s} + \alpha).$$

Any automorphism of $\mathbb{F}_{p^m}$ fixes $\mathbb{F}_p$, so $\pi(\alpha) = \alpha$ for any $\pi \in \text{Aut}(\mathbb{F}_{p^m})$. By [25, Proposition 2.2], $\langle x^{p^s} \pm \alpha \rangle$ forms a two-sided ideal of $\mathbb{F}_{p^m}[x; \pi]$. Hence, by the Chinese Remainder Theorem (CRT), we have

$$\mathcal{R}_1' = \frac{\mathbb{F}_{p^m}[x;\pi]}{\langle x^{2p^s} + 1 \rangle} \cong \frac{\mathbb{F}_{p^m}[x;\pi]}{\langle x^{p^s} - \alpha \rangle} \oplus \frac{\mathbb{F}_{p^m}[x;\pi]}{\langle x^{p^s} + \alpha \rangle}.$$

From this decomposition of $\mathcal{R}_1'$, we deduce that every left ideal of $\mathcal{R}_1'$ can be uniquely expressed as the direct sum of left ideals of $\frac{\mathbb{F}_{p^m}[x;\pi]}{\langle x^{p^s} - \alpha \rangle}$ and $\frac{\mathbb{F}_{p^m}[x;\pi]}{\langle x^{p^s} + \alpha \rangle}$. Therefore, every skew negacyclic code of length $2p^s$ over $\mathbb{F}_{p^m}$ can be expressed as a direct sum of a skew (π, α)-constacyclic code and a skew $(\pi, -\alpha)$-constacyclic code, each of length p^s.

Now we investigate the decomposition of skew negacyclic codes of length $2p^s$ over $\mathbb{F}_{p^m}$. We can represent such a code C as follows:

$$C = C_1 \oplus C_2,$$

where C_1 and C_2 are left ideals of $\frac{\mathbb{F}_{p^m}[x;\pi]}{\langle x^{p^s} - \alpha \rangle}$ and $\frac{\mathbb{F}_{p^m}[x;\pi]}{\langle x^{p^s} + \alpha \rangle}$, respectively.

In the following theorem, we present the cardinality of the skew negacyclic code C of length $2p^s$ over $\mathbb{F}_{p^m}$ and the algebraic structure of the dual code of C.

Theorem 3.2. *Let $C = C_1 \oplus C_2$ be a skew negacyclic code of length $2p^s$ over $\mathbb{F}_{p^m}$. Then*

(*i*) $|C| = |C_1||C_2|$.

(*ii*) $C^\perp = C_1^\perp \oplus C_2^\perp$, *where $C_1^\perp$ and $C_2^\perp$ are left ideals of $\frac{\mathbb{F}_{p^m}[x;\pi]}{\langle x^{p^s} + \alpha \rangle}$ and $\frac{\mathbb{F}_{p^m}[x;\pi]}{\langle x^{p^s} - \alpha \rangle}$, respectively.*

Proof. The proof of (*i*) is straightforward, so we focus on proving (*ii*).

It is clear that $C_1^\perp \oplus C_2^\perp$ is a subset of $C^\perp$. Furthermore, the cardinality of $C_1^\perp \oplus C_2^\perp$ is given by the product of the cardinalities of $C_1^\perp$ and $C_2^\perp$, as follows:

$$|C_1^\perp \oplus C_2^\perp| = |C_1^\perp||C_2^\perp| = \frac{|\mathbb{F}_{p^m}|^{p^s}}{|C_1|} \cdot \frac{|\mathbb{F}_{p^m}|^{p^s}}{|C_2|} = \frac{|\mathbb{F}_{p^m}|^{2p^s}}{|C|} = |C^\perp|.$$

Thus, we conclude that $C^\perp = C_1^\perp \oplus C_2^\perp$.

Since $\alpha = \zeta^{\frac{p-1}{4}}$, we have $\alpha^{-1} = \zeta^{\frac{3p-3}{4}} = -\alpha$, and $(-\alpha)^{-1} = \alpha$. By Proposition 2.21, $C_1^\perp$ is a skew $(\pi, -\alpha)$-constacyclic code, and $C_2^\perp$ is a skew (π, α)-constacyclic code of length p^s over $\mathbb{F}_{p^m}$. Therefore, $C_1^\perp$ and $C_2^\perp$ are left ideals of $\frac{\mathbb{F}_{p^m}[x;\pi]}{\langle x^{p^s} + \alpha \rangle}$ and $\frac{\mathbb{F}_{p^m}[x;\pi]}{\langle x^{p^s} - \alpha \rangle}$, respectively. □

Example 3.3. *Let $\alpha = 2$, $\mathcal{R} = \frac{\mathbb{F}_{5^5}[x;\pi]}{\langle x^5-2\rangle}$, and $\pi \in \mathrm{Aut}(\mathbb{F}_{p^m})$ such that $\pi(a) = a^5$ for any $a \in \mathbb{F}_{5^5}$. Then $\mathrm{ord}(\pi) = 5$. Let w be a primitive root of an irreducible polynomial of degree 5 in $\mathbb{F}_5[x]$ of order $5^5 - 1$. The factorization of $x^5 - 2$ in $\mathbb{F}_{5^5}[x;\pi]$ is given by*

$$\begin{aligned} x^5 - 2 =&(x + 3w^4 + w^3 + 4w^2 + 3w + 3)(x + w^3 + 2w^2 + 2w + 3)(x + 2w^4 + 3w^3 + 2w^2 + w + 4)\\ &\times (x + 4w^4 + 4w^3 + 3w^2 + 3)(x + w^4 + w^3 + w^2 + 2w + 3). \end{aligned}$$

Consider $C = \mathcal{R}(f(x))$, where $f(x) = (x+3w^4+w^3+4w^2+3w+3)(x+w^3+2w^2+2w+3)(x+2w^4+3w^3+2w^2+w+4)$. This simplifies to:

$$f(x) = x^3 + (2w^3 + 2w^2 + 3w)x^2 + (4w^4 + 3w^3 + 3w^2 + 3w)x + 3w^4 + w^2 + 2w + 4.$$

Therefore, C has a generator matrix of the form:

$$\begin{bmatrix} 3w^4 + w^2 + 2w + 4 & 4w^4 + 3w^3 + 3w^2 + 3w & 2w^3 + 2w^2 + 3w & 1 & 0\\ 0 & 3w^{20} + w^{10} + 2w^5 + 4 & 4w^{20} + 3w^{15} + 3w^{10} + 3w^5 & 2w^{15} + 2w^{10} + 3w^5 & 1 \end{bmatrix}.$$

Hence, using the MAGMA algebra system [11], we find that C is an MDS code with parameters $[5, 2, 4]$.

Example 3.4. *As in Example 3.3, we consider $\alpha = -2$, and $\mathcal{R} = \frac{\mathbb{F}_{5^5}[x;\pi]}{\langle x^5+2\rangle}$. The factorization of $x^5 + 2$ in $\mathbb{F}_{5^5}[x;\pi]$ is given by*

$$\begin{aligned} x^5 + 2 =&(x + 2w^4 + w^3 + 2w^2 + w + 2)(x + 4w^4 + 4w^3 + 3w^2 + 4w + 4)(x + w^3 + w + 3)\\ &\times (x + 2w^4 + 2w + 3)(x + 2w^4 + 3w^2 + 4w + 4). \end{aligned}$$

Consider $C = \mathcal{R}(f(x))$, where $f(x) = (x+2w^4+w^3+2w^2+w+2)(x+4w^4+4w^3+3w^2+4w+4)$. This simplifies to:

$$f(x) = x^2 + (w^4 + 2w^3 + 3w + 2)x + 3w^4 + w^3 + w + 4.$$

Therefore, C has a generator matrix of the form:

$$\begin{bmatrix} 3w^4 + w^3 + w + 4 & w^4 + 2w^3 + 3w + 2 & 1 & 0 & 0\\ 0 & 3w^{20} + w^{15} + w^5 + 4 & w^{20} + 2w^{15} + 3w^5 + 2 & 1 & 0\\ 0 & 0 & 3w^{100} + w^{75} + w^{25} + 4 & w^{100} + 2w^{75} + 3w^{25} + 2 & 1 \end{bmatrix}.$$

Hence, using the MAGMA algebra system [11], we find that C is an MDS code with parameters $[5, 3, 3]$.

Example 3.5. *Consider $\mathcal{R}_1' = \frac{\mathbb{F}_{5^5}[x;\pi]}{\langle x^{10}+1\rangle}$ and $\pi \in \mathrm{Aut}(\mathbb{F}_{p^m})$ such that $\pi(a) = a^5$ for any $a \in \mathbb{F}_{5^5}$. Then $\mathrm{ord}(\pi) = 5$. Let w be a $(5^5 - 1)^{th}$-primitive root of an irreducible polynomial of degree 5 in $\mathbb{F}_5$. The factorization of $x^{10} + 1$ in $\mathbb{F}_{5^5}[x;\pi]$ is given by*

$$\begin{aligned} x^{10} + 1 =&(x + 3w^3 + 3w^2 + 4w + 3)(x + 4w^4 + 4w^3 + 2w^2 + 4w)(x + w^4 + 4w^2 + 3w + 3)\\ &\times (x + 4w^4 + 3w^3 + 3w^2 + 4w + 4)(x + w^4 + 4w^3 + 4w^2 + 2w + 1)(x + 4w^3 + 4w)\\ &\times (x + 4w^4 + 4w^3 + 4w + 2)(x + 4w^4 + w^2 + 2w + 1)(x + 2w^4 + 2w^3 + 3w^2 + 4w + 1)\\ &\times (x + w^3 + w + 3). \end{aligned}$$

Consider $C = \mathcal{R}_1'\big(f(x)\big)$, where

$$f(x) =(x + 3w^3 + 3w^2 + 4w + 3)(x + 4w^4 + 4w^3 + 2w^2 + 4w)(x + w^4 + 4w^3 + 4w^2 + 2w + 1)$$

$$\times (x + 4w^4 + 4w^3 + 4w + 2)(x + 4w^4 + w^2 + 2w + 1)(x + 2w^4 + 2w^3 + 3w^2 + 4w + 1).$$

$$\implies f(x) = x^7 + (4w^4 + 4w^3 + w^2 + 4w + 2)x^6 + (w^3 + 2w^2 + 4w + 1)x^5 + (3w^4 + 3w^3 + 2w^2 + 3w + 1)x^4 + (2w^4 + 4w^3 + 2)x^3 + (4w^4 + 3w^3 + w + 3)x^2 + (3w^4 + 3w^2 + w + 3)x + 3w^4 + w^3 + 2w + 2.$$

Therefore, C has a generator matrix whose first row is $3w^4 + w^3 + 2w + 2, 3w^4 + 3w^2 + w + 3, 4w^4 + 3w^3 + w + 3, 2w^4 + 4w^3 + 2, 3w^4 + 3w^3 + 2w^2 + 3w + 1, w^3 + 2w^2 + 4w + 1, 4w^4 + 4w^3 + w^2 + 4w + 2, 1, 0, 0$, and the $(i+1)^{th}$ row is given by the coefficients of $x^i f(x)$, where $i = 0, 1, 2$. Hence, using the MAGMA algebra system [11], we find that C is an MDS code with parameters $[10, 3, 8]$.

Example 3.6. *As in Example 3.5, we have a factorization of $x^{10} + 2$ in $\mathbb{F}_{5^5}[x; \pi]$. Consider $C = \mathcal{R}_1'\Big(f(x)\Big)$, where*

$$f(x) = (x + 4w^4 + 4w^3 + 2w^2 + 4w)(x + 4w^4 + 3w^3 + 3w^2 + 4w + 4)(x + w^4 + 4w^3 + 4w^2 + 2w + 1) \times (x + 4w^4 + 4w^3 + 4w + 2)(x + 2w^4 + 2w^3 + 3w^2 + 4w + 1).$$

$$\implies f(x) = x^5 + (2w^2 + 2)x^4 + (2w^4 + 4w + 4)x^3 + (4w^4 + 2w^3 + 4w^2 + 2)x^2 + (w^4 + 2w^3 + 2w^2 + 3w + 3)x + 2w^2 + 2w + 3.$$

Therefore, C has a generator matrix whose first row is $2w^2 + 2w + 3$, $w^4 + 2w^3 + 2w^2 + 3w + 3$, $4w^4 + 2w^3 + 4w^2 + 2$, $2w^4 + 4w + 4$, $2w^2 + 2$, 1, 0, 0, 0, 0, and the $(i+1)^{th}$ row is given by the coefficients of $x^i f(x)$, where $i = 0, 1, 2, 3, 4$. Hence, using the MAGMA algebra system [11], we find that C is an MDS code with parameters $[10, 5, 6]$.

Example 3.7. *Consider $\mathcal{R}_1 = \frac{\mathbb{F}_{13^{13}}[x;\pi]}{\langle x^{26}+1\rangle}$ and $\pi \in Aut(\mathbb{F}_{13^{13}})$ such that $\pi(a) = a^{13}$ for any $a \in \mathbb{F}_{13^{13}}$. Then $ord(\pi) = 13$. Let w be a $(13^{13} - 1)^{th}$-primitive root of an irreducible polynomial of degree 13 in $\mathbb{F}_{13}[x]$. We then have a factorization of $x^{26} + 1$ over $\mathbb{F}_{13^{13}}[x; \pi]$.*

Consider $C = \mathcal{R}_1\Big(f(x)\Big)$, where

$$\begin{aligned} f(x) =& (x + 10w^{12} + 11w^{11} + 11w^{10} + 6w^9 + 7w^8 + w^7 + 8w^6 + 2w^5 + 6w^4 + 9w^3 + 6w^2 + 12w \\ &+ 3)(x + 11w^{12} + w^{11} + 6w^{10} + 4w^9 + 12w^8 + 10w^6 + 6w^5 + 2w^4 + 10w^3 + 6w^2 + 4w \\ &+ 5)(x + 2w^{12} + 3w^{11} + 2w^{10} + 5w^9 + w^8 + 9w^7 + 4w^6 + 4w^5 + 11w^4 + 4w^3 + 7w^2 \\ &+ 11w + 11)(x + 2w^{12} + 7w^{11} + 11w^{10} + 2w^9 + 10w^8 + 3w^7 + 12w^6 + w^5 + w^2 + 10w \\ &+ 1)(x + 7w^{11} + 2w^{10} + 3w^9 + 4w^8 + 5w^7 + 11w^6 + 12w^5 + 12w^4 + 6w^3 + w^2 + 8w \\ &+ 9)(x + 9w^{12} + 8w^{11} + 6w^{10} + 4w^9 + 10w^8 + 9w^7 + 11w^6 + 5w^5 + w^4 + 9w^3 + 4w^2 \\ &+ 9w + 4)(x + 4w^{12} + 12w^{11} + 2w^{10} + 8w^9 + 2w^8 + 12w^7 + 7w^6 + 10w^4 + 12w^2 \\ &+ 12w + 3)(x + 11w^{12} + 9w^{11} + 4w^{10} + 5w^9 + 2w^8 + 6w^7 + 12w^6 + 5w^5 + 12w^4 \\ &+ w^3 + 5w^2 + 10w)(x + 7w^{12} + 2w^{10} + 11w^9 + 8w^8 + 6w^6 + 6w^5 + 7w^4 + 10w^2 \\ &+ 5w + 1)(x + 10w^{12} + w^{11} + 10w^{10} + 2w^9 + 5w^8 + 4w^7 + 6w^6 + 9w^5 + 2w^4 \\ &+ 7w^3 + 8w + 9)(x + 8w^{12} + 3w^{10} + 2w^9 + 2w^8 + 4w^7 + 9w^6 + 3w^5 + 4w^4 + 2w^3 \\ &+ 9w^2 + 5w)(x + 2w^{12} + 4w^{11} + 5w^{10} + 7w^9 + 6w^7 + 5w^6 + 10w^5 + 2w^4 + 12w^3 \\ &+ 12w^2 + 6w + 4)(x + 8w^{12} + 9w^{11} + 5w^{10} + 4w^9 + 2w^8 + 7w^7 + 6w^6 + 2w^5 \\ &+ 3w^4 + 2w^2 + w + 6)(x + 4w^{12} + 8w^{11} + 6w^{10} + 12w^9 + 3w^8 + 3w^7 + 6w^6 \\ &+ 6w^5 + 5w^4 + 8w^3 + 9w^2 + 12w + 7)(x + 7w^{12} + 3w^{11} + 12w^{10} + 11w^9 + 3w^8 \\ &+ 7w^7 + 5w^6 + 10w^5 + 8w^4 + 10w^3 + 5w^2 + 2w + 5)(x + 2w^{12} + 10w^{11} \\ &+ 9w^{10} + 11w^9 + 8w^8 + w^7 + 3w^6 + 4w^5 + 8w^4 + w^3 + w^2 + 12w + 12)(x + 4w^{11} \\ &+ 5w^{10} + 12w^9 + 6w^8 + 12w^7 + 7w^6 + 11w^5 + 10w^4 + 6w^3 + 5w + 6)(x + w^{12} + w^{10} \end{aligned}$$

$$+ 10w^9 + 12w^8 + 10w^7 + 11w^6 + 7w^5 + 12w^4 + 2w^3 + 4w^2 + 12w + 6)(x + 7w^{12}$$
$$+ 9w^{11} + 2w^9 + 9w^8 + 4w^7 + 7w^6 + w^5 + 2w^4 + 4w^3 + 6w^2 + 7w + 9)(x + w^{12}$$
$$+ 10w^{11} + 5w^9 + 12w^8 + 2w^7 + 4w^6 + 11w^4 + 9w^3 + 11w^2 + 2w)(x + 11w^{12}$$
$$+ 2w^{11} + 11w^{10} + 4w^9 + 4w^7 + 6w^6 + 12w^5 + 2w^4 + 12w^3 + 4w^2 + 3w + 8)$$
$$\times (x + 4w^{12} + 7w^{11} + 7w^{10} + 10w^8 + 11w^7 + 12w^6 + 9w^5 + 10w^4$$
$$+ 8w^3 + 5w^2 + 9w + 10)(x + 2w^{12} + 4w^{10} + 8w^9 + 5w^8 + 7w^6 + 8w^5 + 4w^4$$
$$+ 2w^3 + 12w^2 + 7w + 12)(x + 10w^{12} + w^{11} + 7w^{10} + 4w^9 + 5w^8 + 7w^7$$
$$+ 2w^6 + 6w^5 + 12w^4 + 11w^2 + 9w + 3)(x + 7w^{12} + 6w^{11} + 5w^9 + 3w^8 + 2w^7$$
$$+ 12w^6 + 6w^5 + 8w^4 + 9w^3 + 12w^2 + 2w + 7)(x + 9w^{12} + 2w^{11} + 12w^{10}$$
$$+ 7w^9 + 8w^8 + 6w^7 + 5w^6 + 11w^5 + 4w^4 + 7w^3 + 8w^2 + 2w)(x + 7w^{12} + 12w^{11}$$
$$+ 6w^{10} + 10w^9 + 9w^8 + 8w^7 + 4w^6 + 6w^5 + 3w^4 + 11w^3 + 6w^2 + 7w + 10)(x$$
$$+ 2w^{12} + 4w^{11} + 4w^{10} + 6w^9 + 8w^8 + 3w^7 + 10w^6 + 7w^5 + 9w^4 + 10w^3$$
$$+ 2w^2 + 11w + 3)(x + 6w^{12} + 10w^{11} + 9w^{10} + 11w^9 + 4w^8 + 2w^7 + 11w^6$$
$$+ 8w^5 + 6w^4 + 5w^3 + 6w^2 + 9w + 11)(x + 5w^{12} + 2w^{11} + 7w^{10} + 2w^9$$
$$+ 7w^8 + 3w^7 + 6w^6 + 11w^5 + 12w^4 + 5w^3 + 9w^2 + 6w + 1)(x + 3w^{12} + 11w^{11}$$
$$+ 9w^{10} + 5w^9 + 10w^8 + 11w^7 + 12w^6 + 8w^5 + 10w^4 + 7w^2 + 6w)(x + 8w^{12}$$
$$+ 7w^{11} + 11w^{10} + 2w^9 + 10w^8 + 11w^7 + 12w^6 + 6w^5 + 6w^4 + 12w^3 + 3w^2$$
$$+ 10w)(x + 4w^{12} + 11w^{11} + 12w^{10} + 6w^9 + 10w^8 + 2w^7 + 11w^6 + 5w^5$$
$$+ 7w^4 + 5w^3 + 6w^2 + 8w + 9)(x + 3w^{12} + 8w^{11} + 10w^{10} + 12w^9 + 12w^8 +$$
$$2w^7 + 12w^6 + 2w^5 + 10w^4 + 6w^3 + 5w^2 + 2w + 11)(x + 4w^{12} + 5w^{11} + 7w^{10}$$
$$+ 3w^9 + 6w^8 + 10w^7 + 9w^6 + 12w^5 + 8w^4 + 7w^3 + 5w^2 + 11w + 4)$$
$$\times (x + 10w^{12} + 6w^{11} + 9w^{10} + 11w^9 + 11w^8 + 4w^7 + 7w^6 + 5w^5$$
$$+ 12w^4 + 9w^3 + 2w^2 + 3w + 12)(x + 2w^{12} + 5w^{11} + 10w^{10} + 6w^9 + 2w^8$$
$$+ 4w^7 + 8w^6 + 6w^5 + 11w^4 + 12w^3 + 12w^2 + 10w + 2)(x + 11w^{12} + 4w^{11}$$
$$+ 9w^{10} + 10w^9 + 7w^8 + 8w^7 + 5w^6 + 9w^5 + 2w^4 + 7w^3 + 5w^2 + 12w + 8)(x$$
$$+ 7w^{12} + 12w^{11} + 2w^{10} + 4w^9 + 10w^8 + 8w^7 + 4w^6 + 9w^5 + 11w^4 + 12w^3$$
$$+ 2w^2 + 5w + 6)(x + 12w^{12} + 11w^{11} + 3w^{10} + 6w^9 + 8w^8 + 4w^7 + 12w^6$$
$$+ 12w^5 + 9w^4 + 7w^3 + 10w^2 + 7w + 1)(x + 12w^{12} + 2w^{11} + 6w^{10} + 10w^9$$
$$+ 9w^8 + 4w^7 + 5w^6 + 12w^5 + 2w^4 + 8w^3 + 11w^2 + 7w + 10)(x + 12w^{12} + 10w^{11}$$
$$+ 12w^{10} + 4w^9 + 3w^8 + 11w^7 + 9w^6 + 7w^5 + 11w^4 + 12w^2 + 5w)(x + 5w^{12}$$
$$+ 11w^{11} + 6w^{10} + 4w^9 + 11w^8 + 2w^7 + 7w^6 + 12w^5 + 9w^4 + 10w^3 + 6w^2 + 7w)$$
$$\times (x + 11w^{12} + 3w^{11} + 6w^{10} + 7w^9 + 4w^8 + 12w^7 + 2w^6 + 9w^5$$
$$+ 8w^4 + 10w^3 + 12w^2 + 6w + 2)(x + 9w^{12} + 5w^{11} + 8w^{10} + 12w^9 + 4w^8$$
$$+ 10w^7 + 6w^6 + 5w^5 + 7w^4 + 2w^3 + 2w^2 + 3w + 1)$$

is an MDS skew negacyclic code of length 26 *over* $\mathbb{F}_{13}$.

3.2. $p \equiv 3 \pmod 4$

In this subsection, we consistently assume that $p \equiv 3 \pmod 4$. In this class of primes, there does not exist any $a \in \mathbb{F}_{p^m}$ such that $a^2 = -1$. Hence, $x^2 + 1$ is an irreducible polynomial in $\mathbb{F}_{p^m}[x; \pi]$. Since $\mathbb{F}_{p^m}$ has

characteristic p, we can further simplify this to:

$$x^{2p^s} + 1 = (x^2 + 1)^{p^s}.$$

Lemma 3.8. *The ideal $\langle (x^2+1)^{p^s} \rangle$ is a two-sided ideal of $\mathbb{F}_{p^m}[x;\pi]$.*

Proof. As $\mathbb{F}_p$ is fixed by $\pi \in \mathrm{Aut}(\mathbb{F}_{p^m})$, we have $\pi(-1) = -1$, and $\mathrm{ord}(\pi) \mid p^s$. Hence, by [25, Proposition 2.2], the ideal $\langle (x^2+1)^{p^s} \rangle$ is a two-sided ideal of $\mathbb{F}_{p^m}[x;\pi]$. □

Define a set

$$\mathcal{B} := \left\{ f(x) \mid f(x) \text{ is a monic divisor of } (x^2+1)^{p^s} \right\}.$$

Proposition 3.9. *The quotient ring $\mathcal{R}_1' = \frac{\mathbb{F}_{p^m}[x;\pi]}{\langle (x^2+1)^{p^s} \rangle}$ is a principal left ideal ring. Moreover, the generator of any left ideal is of the form $f(x) + \langle (x^2+1)^{p^s} \rangle$, where $f(x) \in \mathcal{B}$.*

Proof. Let $\mathcal{I} = \frac{\mathfrak{I}}{\langle (x^2+1)^{p^s} \rangle}$ be a left ideal of $\mathcal{R}_1'$, where $\mathfrak{I}$ is a left ideal of $\mathbb{F}_{p^m}[x;\pi]$ containing $\langle (x^2+1)^{p^s} \rangle$. Since $\mathbb{F}_{p^m}[x;\pi]$ is a principal left ideal ring, $\mathfrak{I} = \langle f(x) \rangle$ for some $f(x) \in \mathbb{F}_{p^m}[x;\pi]$, where $f(x)$ is a divisor of $(x^2+1)^{p^s}$. Thus, we infer that $f(x) + \langle (x^2+1)^{p^s} \rangle$ is a generator of $\mathcal{I}$. □

Lemma 3.10. *Let $p(x)$ be a non-zero skew polynomial in $\mathcal{R}_1'$. Then $p(x)$ has a left inverse in $\mathcal{R}_1'$ if and only if $\mathrm{gcd_r}\left(p(x), (x^2+1)^{p^s}\right) = 1$.*

Proof. Assume $\mathrm{gcd_r}\left(p(x), (x^2+1)^{p^s}\right) = 1$. Let $\frac{\mathfrak{I}}{\langle (x^2+1)^{p^s} \rangle}$ be a left ideal of $\mathcal{R}_1'$, where $\mathfrak{I}$ is a left ideal of $\mathbb{F}_{p^m}[x;\pi]$ containing $\langle (x^2+1)^{p^s} \rangle$. The left ideal of $\mathbb{F}_{p^m}[x;\pi]$ generated by $p(x)$ is given by $\mathbb{F}_{p^m}[x;\pi](p(x))$. Hence, $\langle (x^2+1)^{p^s} \rangle \subset \mathbb{F}_{p^m}[x;\pi](p(x)) + \langle (x^2+1)^{p^s} \rangle$ is a left ideal of $\mathbb{F}_{p^m}[x;\pi]$. This implies that $\frac{\mathbb{F}_{p^m}[x;\pi](p(x)) + \langle (x^2+1)^{p^s} \rangle}{\langle (x^2+1)^{p^s} \rangle}$ is a left ideal of $\mathcal{R}_1'$. Since $\mathbb{F}_{p^m}[x;\pi]$ is a principal ideal ring, we have $\mathbb{F}_{p^m}[x;\pi](a(x)) = \mathbb{F}_{p^m}[x;\pi](p(x)) + \langle (x^2+1)^{p^s} \rangle$ for some $a(x) \in \mathbb{F}_{p^m}[x;\pi]$. Therefore, there exist polynomials $m_1(x)$ and $t_1(x)$ in $\mathbb{F}_{p^m}[x;\pi]$ such that

$$a(x) = m_1(x)p(x) + t_1(x)(x^2+1)^{p^s}.$$

Similarly, there exist polynomials $m_2(x)$ and $t_2(x)$ in $\mathbb{F}_{p^m}[x;\pi]$ such that

$$p(x) = m_2(x)a(x) \quad \text{and} \quad (x^2+1)^{p^s} = t_2(x)a(x).$$

This implies that $a(x)$ is a right divisor of both $p(x)$ and $(x^2+1)^{p^s}$. Thus, we conclude that $a(x) = 1$. Further, we have

$$\begin{aligned} 1 + \langle (x^2+1)^{p^s} \rangle &= m_1(x)p(x) + \langle (x^2+1)^{p^s} \rangle \\ &= \left(m_1(x) + \langle (x^2+1)^{p^s} \rangle \right) \left(p(x) + \langle (x^2+1)^{p^s} \rangle \right). \end{aligned}$$

This implies $m_1(x)$ is a left inverse of $p(x)$ in $\mathcal{R}_1'$.

Conversely, suppose $m(x)$ is a left inverse of $p(x)$. Let $a(x)$ be a common right divisor of $p(x)$ and $(x^2+1)^{p^s}$. Thus, there exists a polynomial $b(x) \in \mathbb{F}_{p^m}[x;\pi]$ such that $p(x) = b(x)a(x)$. Therefore,

$$m(x)b(x)a(x) = m(x)p(x) = 1 \text{ in } \mathcal{R}_1'.$$

Hence, $a(x)$ has a left inverse in $\mathcal{R}_1'$. Since $a(x)$ is a monic polynomial in $\mathbb{F}_{p^m}[x;\pi]$, it implies $a(x) = 1$. Thus,

$$\mathrm{gcd_r}\left(p(x), (x^2+1)^{p^s}\right) = 1.$$

□

Example 3.11. *Consider* $\mathcal{R}_1' = \frac{\mathbb{F}_{3^3}[x;\pi]}{\langle (x^2+1)^3\rangle}$ *and* $\pi \in \mathrm{Aut}(\mathbb{F}_{3^3})$ *such that* $\pi(a) = a^3$ *for any* $a \in \mathbb{F}_{3^3}$. *Then* $\mathrm{ord}(\pi) = 3$. *Let* w *be a* $(3^3-1)^{th}$*-primitive root of an irreducible polynomial of degree* 3 *in* $\mathbb{F}_3[x]$. *The factorization of* x^6+1 *in* $\mathbb{F}_{3^3}[x;\pi]$ *is given by*

$$x^6+1 = (x^2+w^2x+w^2)(x^2+x+2w^2+2w+1)(x^2+(2w^2+w+1)x+1).$$

Consider $C = \mathcal{R}_1'(f(x))$, *where*

$$f(x) = (x^2+x+2w^2+2w+1)(x^2+(2w^2+w+1)x+1)$$
$$\implies f(x) = x^4+(2w^2+2w+2)x^3+(w^2+2w+1)x^2+(w^2+2w+2)x+2w^2+2w+1.$$

Therefore, C *has a generator matrix of the form*

$$\begin{bmatrix} 2w^2+2w+1 & w^2+2w+2 & w^2+2w+1 & 2w^2+2w+2 & 1 & 0 \\ 0 & 2w^6+2w^3+1 & w^6+2w^3+2 & w^6+2w^3+1 & 2w^6+2w^3+2 & 1 \end{bmatrix}.$$

Hence, by the MAGMA algebra system [11], we find that C *is an MDS code with parameters* $[6,2,5]$.

4. The algebraic structure of skew negacyclic codes of length $2p^s$ over $\mathfrak{R}$

Consider $\mathfrak{R} = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ with $u^2 = 0$. It is well known that $\mathfrak{R}$ is a finite chain ring with nilpotency index 2, and the ideal generated by u is the only maximal ideal of $\mathfrak{R}$.

This section delves into the algebraic structure of skew negacyclic codes of length $2p^s$ over $\mathfrak{R}$. We assume that $\mathrm{ord}(\Pi)$ divides p^s, where $\Pi \in \mathrm{Aut}(\mathfrak{R})$. Let $\alpha + u\beta \in \mathfrak{R}$. If α is a non-zero element of $\mathbb{F}_{p^m}$, then $\alpha + u\beta$ is a unit in $\mathfrak{R}$. Furthermore, a code C of length $2p^s$ over $\mathfrak{R}$ is called a skew negacyclic code if and only if the polynomial representation of C forms a left ideal of $\mathcal{R}_2 = \frac{\mathfrak{R}[x;\Pi]}{\langle x^{2p^s}+1\rangle}$.

4.1. $p \equiv 1 \pmod 4$

From Subsection 3.1, we have the factorization of $x^{2p^s}+1$ over $\mathbb{F}_{p^m}$ as follows:

$$\begin{aligned} x^{2p^s}+1 &= x^{2p^s} - \alpha^2 \\ &= (x^{p^s}-\alpha)(x^{p^s}+\alpha), \end{aligned}$$

where $\alpha = \zeta^{\frac{p-1}{4}}$ is an element of $\mathbb{F}_p$, and ζ is a generator of $\mathbb{F}_p^*$. Since $\mathbb{F}_{p^m}$ is a subring of $\mathfrak{R}$, we can consider this factorization over $\mathfrak{R}$. For any $\Pi \in \mathrm{Aut}(\mathfrak{R})$, we have $\pi(\alpha) = \alpha$, where $\pi \in \mathrm{Aut}(\mathbb{F}_{p^m})$. Thus, by Lemma 2.5, for any $\Pi \in \mathrm{Aut}(\mathfrak{R})$, we have $\Pi(\alpha) = \alpha$. From [25, Proposition 2.2], $\langle x^{p^s} \pm \alpha\rangle$ are two-sided ideals of $\mathfrak{R}[x;\Pi]$. Hence, by the Chinese Remainder Theorem (CRT), we have

$$\mathcal{R}_2 = \frac{\mathfrak{R}[x;\Pi]}{\langle x^{2p^s}+1\rangle} \cong \frac{\mathfrak{R}[x;\Pi]}{\langle x^{p^s}-\alpha\rangle} \oplus \frac{\mathfrak{R}[x;\Pi]}{\langle x^{p^s}+\alpha\rangle}.$$

Following this decomposition of $\mathcal{R}_2$, we can see that every left ideal of $\mathcal{R}_2$ can be uniquely expressed as the direct sum of left ideals of $\frac{\mathfrak{R}[x;\Pi]}{\langle x^{p^s}-\alpha\rangle}$ and $\frac{\mathfrak{R}[x;\Pi]}{\langle x^{p^s}+\alpha\rangle}$, respectively. Thus, any skew negacyclic code of length $2p^s$ over $\mathfrak{R}$ can be written as the direct sum of two skew codes over $\mathfrak{R}$: namely, a skew (Π,α)-constacyclic code and a skew $(\Pi,-\alpha)$-constacyclic code. We can determine the algebraic structure of skew negacyclic codes of length $2p^s$ over $\mathfrak{R}$ by using the study conducted on the algebraic structures of skew constacyclic codes of length p^s over $\mathfrak{R}$ as performed in [7]. The number of codewords in skew constacyclic codes of length p^s over $\mathfrak{R}$ is examined in [22]. Based on this, we can determine the number of codewords in skew negacyclic codes of length $2p^s$ over $\mathfrak{R}$.

By a similar argument as given in Subsection 3.1, we can represent the skew negacyclic code of length $2p^s$ over $\mathfrak{R}$ as the direct sum of two constituent codes, i.e.,

$$C = C_1 \oplus C_2,$$

where C_1 and C_2 are left ideals of $\frac{\mathfrak{R}[x;\Pi]}{\langle x^{p^s}-\alpha\rangle}$ and $\frac{\mathfrak{R}[x;\Pi]}{\langle x^{p^s}+\alpha\rangle}$, respectively.

Proposition 4.1. *Let $C = C_1 \oplus C_2$ be a skew negacyclic code of length $2p^s$ over $\mathfrak{R}$. Then $|C| = |C_1||C_2|$.*

In [22, Theorem 3.15], the authors determined the algebraic structure of the dual of skew (Π, λ)-constacyclic codes of length p^s over $\mathfrak{R}$. Using this result, we also examine the algebraic structure of the dual of skew negacyclic codes of length $2p^s$ over $\mathfrak{R}$. The following theorem presents the algebraic structure of the dual code of length $2p^s$ over $\mathfrak{R}$.

Theorem 4.2. *Let $C = C_1 \oplus C_2$ be a skew negacyclic code of length $2p^s$ over $\mathfrak{R}$. Then its dual is given by:*

$$C^{\perp} = C_1^{\perp} \oplus C_2^{\perp},$$

where $C_1^{\perp}$ and $C_2^{\perp}$ are left ideals of $\frac{\mathfrak{R}[x;\Pi]}{\langle x^{p^s}+\alpha\rangle}$ and $\frac{\mathfrak{R}[x;\Pi]}{\langle x^{p^s}-\alpha\rangle}$, respectively.

Proof. Since $C_1^{\perp} \oplus C_2^{\perp} \subseteq C^{\perp}$, the cardinality of $C_1^{\perp} \oplus C_2^{\perp}$ is given by the product of the cardinalities of $C_1^{\perp}$ and $C_2^{\perp}$, i.e.,

$$|C_1^{\perp} \oplus C_2^{\perp}| = |C_1^{\perp}||C_2^{\perp}| = \frac{|\mathfrak{R}|^{p^s}}{|C_1|} \cdot \frac{|\mathfrak{R}|^{p^s}}{|C_2|} = \frac{|\mathfrak{R}|^{2p^s}}{|C|} = |C^{\perp}|.$$

Thus, we conclude that $C^{\perp} = C_1^{\perp} \oplus C_2^{\perp}$.

Since $\alpha = \zeta^{\frac{p-1}{4}}$, we have $\alpha^{-1} = \zeta^{\frac{3p-3}{4}} = -\alpha$ and $(-\alpha)^{-1} = \alpha$ in $\mathbb{F}_{p^m} \subset \mathfrak{R}$. From Proposition 2.21, $C_1^{\perp}$ is a skew $(\Pi, -\alpha)$-constacyclic code and $C_2^{\perp}$ is a skew (Π, α)-constacyclic code of length p^s over $\mathfrak{R}$. Hence, $C_1^{\perp}$ is a left ideal of $\frac{\mathfrak{R}[x;\Pi]}{\langle x^{p^s}+\alpha\rangle}$ and $C_2^{\perp}$ is a left ideal of $\frac{\mathfrak{R}[x;\Pi]}{\langle x^{p^s}-\alpha\rangle}$. □

By considering the above discussion, we now present an example.

Example 4.3. *Let $\alpha = 2$, $\mathfrak{R} = \mathbb{F}_{5^5} + u\mathbb{F}_{5^5}$ with $u^2 = 0$, $\mathcal{R}_2 = \frac{\mathfrak{R}[x;\Pi]}{\langle x^5-2\rangle}$, and $\Pi \in \mathrm{Aut}(\mathfrak{R})$ such that $\Pi(a) = a^5$ for any $a \in \mathfrak{R}$. Then $\mathrm{ord}(\Pi) = 5$. Let w be a $(5^5-1)^{th}$ primitive root of an irreducible polynomial of degree 5 in $\mathbb{F}_5[x]$. The factorization of $x^5 - 2$ in $\mathfrak{R}[x;\Pi]$ is given by*

$$\begin{aligned} x^5 - 2 = {} & (x + 3w^4 + w^3 + 4^2 + 3w + 3)(x + w^3 + 2w^2 + 2w + 3)(x + 2w^4 + 3w^3 + 2w^2 + w + 4) \\ & \times (x + 4w^4 + 4w^3 + 3w^2 + 3)(x + w^4 + w^3 + w^2 + 2w + 3). \end{aligned}$$

Consider $C = \mathcal{R}_2(f(x))$, where

$$f(x) = (x + 3w^4 + w^3 + 4w^2 + 3w + 3)(x + w^3 + 2w^2 + 2w + 3) + u(x + 3w^4 + w^3 + 4w^2 + 3w + 3).$$

This simplifies to

$$f(x) = x^2 + (3w^4 + 2w^3 + 2w^2 + 1 + u)x + 2w^4 + 3w^3 + 2w^2 + w + 3 + u(3w^4 + w^3 + 4w^2 + 3w + 3).$$

Therefore, C has a generator matrix G with its rows given as follows:

First row: $2w^4 + 3w^3 + 2w^2 + w + 3 + u(3w^4 + w^3 + 4w^2 + 3w + 3), 3w^4 + 2w^3 + 2w^2 + 1 + u, 1, 0, 0.$

Second row: $0, 2w^{20} + 3w^{15} + 2w^{10} + w^5 + 3 + u(3w^{20} + w^{15} + 4w^{10} + 3w^5 + 3), 3w^{20} + 2w^{15} + 2w^{10} + 1 + u, 1, 0.$

Third row: $0, 0, 2w^{100} + 3w^{75} + 2w^{50} + w^{25} + 3 + u(3w^{100} + w^{75} + 4w^{50} + 3w^{25} + 3), 3w^{100} + 2w^{75} + 2w^{50} + 1 + u, 1.$

The generator matrix of $\phi(C)$ is given by

$$\phi(C) = \begin{pmatrix} \phi(G) \\ \phi(uG) \end{pmatrix},$$

with all six rows of the matrix $\phi(C)$ given as follows:

First row: $3w^4 + w^3 + 4w^2 + 3w + 3, 3w^4 + w^3 + 4w^2 + 3w + 3 + 2w^4 + 3w^3 + 2w^2 + w + 3, 1, 1 + 3w^4 + 2w^3 + 2w^2 + 1, 0, 1, 0, 0, 0, 0.$

Second row: $0,0,3w^{20}+w^{15}+4w^{10}+3w^5+3, 3w^{20}+w^{15}+4w^{10}+3w^5+3+2w^{20}+3w^{15}+2w^{10}+w^5+3, 1, 1+3w^{20}+2w^{15}+2w^{10}+1, 0, 1, 0, 0.$

Third row: $0,0,0,0,3w^{100}+w^{75}+4w^{50}+3w^{25}+3, 3w^{100}+w^{75}+4w^{50}+3w^{25}+3+2w^{100}+3w^{75}+2w^{50}+w^{25}+3, 1, 1+3w^{100}+2w^{75}+2w^{50}+1, 0, 1.$

Fourth row: $2w^4+3w^3+2w^2+w+3, 2w^4+3w^3+2w^2+w+3, 3w^4+2w^3+2w^2+1, 3w^4+2w^3+2w^2+1, 1, 1, 0, 0, 0, 0.$

Fifth row: $0,0,2w^{20}+3w^{15}+2w^{10}+w^5+3, 2w^{20}+3w^{15}+2w^{10}+w^5+3, 3w^{20}+2w^{15}+2w^{10}+1, 3w^{20}+2w^{15}+2w^{10}+1, 1, 1, 0, 0.$

Sixth row: $0,0,0,0,2w^{100}+3w^{75}+2w^{50}+w^{25}+3, 2w^{100}+3w^{75}+2w^{50}+w^{25}+3, 3w^{100}+2w^{75}+2w^{50}+1, 3w^{100}+2w^{75}+2w^{50}+1, 1, 1.$

Hence, by the MAGMA algebra system [11], we get that C is an MDS code with parameters $[10, 6, 5]$.

Example 4.4. *As in Example 4.3, similarly, we consider $\alpha = -2$, and*

$$\mathcal{R}_2 = \frac{\Re[x;\Pi]}{\langle x^5+2\rangle}.$$

We have a factorization of x^5+2 in $\Re[x;\Pi]$ as follows:

$$\begin{aligned} x^5+2 =&(x+2w^4+w^3+2w^2+w+2)(x+4w^4+4w^3+3w^2+4w+4)\\ &(x+w^3+w+3)(x+2w^4+2w+3)(x+2w^4+3w^2+4w+4). \end{aligned}$$

Consider $C = \mathcal{R}_2(f(x))$, where

$$f(x) = (x+2w^4+w^3+2w^2+w+2)(x+4w^4+4w^3+3w^2+4w+4)(x+2w^4+3w^2+4w+4)$$

$$+u(x+2w^4+w^3+2w^2+w+2).$$

Expanding $f(x)$, we obtain

$$f(x) = x^3+(3w^4+4w^3+3w+2)x^2+(3w^4+2w^3+3w+2+u)x+w^3+4w^2+u(2w^4+w^3+2w^2+w+2).$$

Therefore, C has a generator matrix G, with its rows given as follows:

First row: $w^3+4w^2+u(2w^4+w^3+2w^2+w+2), 3w^4+2w^3+3w+2+u, 3w^4+4w^3+3w+2, 1, 0.$

Second row: $0, w^{15}+4w^{10}+u(2w^{20}+w^{15}+2w^{10}+w^5+2), 3w^{20}+2w^{15}+3w^5+2+u, 3w^{20}+4w^{15}+3w^5+2, 1.$

The generator matrix of $\phi(C)$ is given by

$$\phi(C) = \begin{pmatrix} \phi(G) \\ \phi(uG) \end{pmatrix},$$

and all four rows of the matrix $\phi(C)$ can be obtained similarly to those in Example 4.3. Hence, by the MAGMA algebra system [11], we get that $\phi(C)$ is a near-MDS code with parameters $[10, 4, 6]$.

4.2. $p \equiv 3 \pmod 4$

This section consistently assumes that $p \equiv 3 \pmod 4$, i.e., $4 \mid (p-3)$. This implies that either $8 \mid (p-3)$ or $8 \mid (p-7)$. Moreover, in both cases, x^2+1 is irreducible over $\Re$. Thus, by considering the arguments given in Proposition 3.9 and Lemma 3.10, we can discuss the algebraic structure of left ideals of $\frac{\Re[x;\Pi]}{\langle (x^2+1)^{p^s}\rangle}$. For convenience, we denote

$$\mathcal{R}_2 = \frac{\Re[x;\Pi]}{\langle (x^2+1)^{p^s}\rangle}.$$

Theorem 4.5. *All left ideals of $\mathcal{R}_2 = \frac{\Re[x;\Pi]}{\langle (x^2+1)^{p^s}\rangle}$ can be expressed in the following form:*

$$\mathcal{I} = \mathcal{R}_2\left(f_1(x)+ug(x)\right) + \mathcal{R}_2\left(uf_2(x)\right),$$

where $f_1(x)$ is either 0 or in $\mathcal{B}$, $g(x) \in \mathcal{R}_1'$, $f_2(x)$ is either 0 or in $\mathcal{B}$, $f_2(x) \mid_r f_1(x)$, and $\deg(f_2(x)) > \deg(g(x))$. Furthermore, $g(x)$ is unique under the above constraints.

Proof. As previously noted, the left ideal $\mathcal{I}$ of $\mathcal{R}_2$ is $\frac{\mathfrak{J}}{\langle (x^2+1)^{p^s}\rangle}$, where $\mathfrak{J}$ is a left ideal of $\mathfrak{R}[x;\Pi]$ containing $\langle (x^2+1)^{p^s}\rangle$. According to Note 2.18, since the map μ is a surjective homomorphism from $\mathfrak{R}[x;\Pi]$ to $\mathbb{F}_{p^m}[x;\pi]$, it follows that $\mu(\mathfrak{J})$ forms a left ideal in $\mathbb{F}_{p^m}[x;\pi]$. Thus, for some polynomial $s(x) \in \mathfrak{R}[x;\Pi]$, we have $\mu(\mathfrak{J}) = \mathbb{F}_{p^m}[x;\pi](\mu(s(x)))$.

If we restrict the map μ to $\mathfrak{J}$, denoted by $\mu|_{\mathfrak{J}} : \mathfrak{J} \to \mu(\mathfrak{J})$, this restricted map is also surjective. Therefore, there exists a polynomial $t(x) \in \mathfrak{J}$ such that $\mu(t(x)) = \mu(s(x))$.

It is evident that $\mathfrak{R}[x;\Pi](t(x)) + (\mathfrak{J} \cap u\mathfrak{R}[x;\Pi]) \subseteq \mathfrak{J}$. Consider an arbitrary skew polynomial $t_1(x) \in \mathfrak{J}$. We have $\mu(t_1(x)) \in \mu(\mathfrak{J}) = \mathbb{F}_{p^m}[x;\pi](\mu(t(x)))$, which implies there exists another skew polynomial $w(x) \in \mathfrak{R}[x;\Pi]$ such that $\mu(t_1(x)) = \mu(w(x))\mu(t(x)) = \mu(w(x)t(x))$. Hence, we can express $t_1(x)$ as $t_1(x) = w(x)t(x) + r(x)$ for some $r(x) \in u\mathfrak{R}[x;\Pi]$. Since $t_1(x) \in \mathfrak{J}$ and $r(x) = t_1(x) - w(x)t(x) \in \mathfrak{J}$, it follows that $r(x) \in \mathfrak{J} \cap u\mathfrak{R}[x;\Pi]$.

Therefore, $t_1(x) = w(x)t(x) + r(x) \in \mathfrak{R}[x;\Pi](t(x)) + (\mathfrak{J} \cap u\mathfrak{R}[x;\Pi])$, which leads to the conclusion that $\mathfrak{J} = \mathfrak{R}[x;\Pi](t(x)) + (\mathfrak{J} \cap u\mathfrak{R}[x;\Pi])$. Given that $\langle (x^2+1)^{p^s}\rangle \subseteq \mathfrak{J}$, the ideal $\mathcal{I}$ can be expressed as follows:

$$\begin{aligned}
\mathcal{I} &= \frac{\mathfrak{J}}{\langle (x^2+1)^{p^s}\rangle} \\
&= \frac{\Big(\mathfrak{R}[x;\Pi](t(x)) + (\mathfrak{J} \cap u\mathfrak{R}[x;\Pi])\Big) + \langle (x^2+1)^{p^s}\rangle}{\langle (x^2+1)^{p^s}\rangle} \\
&= \frac{\mathfrak{R}[x;\Pi](t(x)) + \langle (x^2+1)^{p^s}\rangle}{\langle (x^2+1)^{p^s}\rangle} + \left(\frac{\mathfrak{J}}{\langle (x^2+1)^{p^s}\rangle} \cap \frac{u\mathfrak{R}[x;\Pi] + \langle (x^2+1)^{p^s}\rangle}{\langle (x^2+1)^{p^s}\rangle}\right) \\
&= \mathcal{R}_2(t(x)) + (\mathcal{I} \cap \mathcal{R}_2(u)).
\end{aligned}$$

We have $\mathrm{Tor}(\mathcal{I}) = \mu(\mathcal{I} :_{\mathcal{R}_2} u) = \mathcal{R}_1'(f(x))$, where $f(x) \in \mathcal{B}$ or 0. It can be easily seen that

$$\mathcal{I} \cap \mathcal{R}_2(u) = \mu^{-1}\left(\mu(\mathcal{I} :_{\mathcal{R}_2} u)\right) = u\mu^{-1}\left(\mathrm{Tor}(\mathcal{I})\right) = u\mu^{-1}(\mathcal{R}_1'(f_2(x))) = \mathcal{R}_2(uf_2(x)).$$

Since $\mu(\mathcal{I})$ is a left ideal of $\mathcal{R}_1'$, by Proposition 3.9, we get

$$\mu(\mathcal{I}) = \mathcal{R}_1'\Big(\mu(t(x))\Big) = \mathcal{R}_1'\Big(f_1(x)\Big),$$

where $f_1(x)$ is either 0 or an element of $\mathcal{B}$. Thus, $f_1(x) = y(x)\mu(t(x))$ for some $y(x)$ in $\mathcal{R}_1'$. This implies that $\mu(t(x))$ is a factor of $(x^2+1)^{p^s}$.

Now suppose, and this does not affect the general case, $t(x) = f_1(x) + ug(x)$, where $g(x) \in \mathcal{R}_1'$. Hence,

$$\mathcal{I} = \mathcal{R}_2\Big(f_1(x) + ug(x)\Big) + \mathcal{R}_2(uf_2(x)).$$

Suppose $f_2(x) \neq 0$. Using the right division algorithm, we get

$$g(x) = h(x)f_2(x) + r'(x),$$

where $r'(x) = 0$ or $\deg(r'(x)) < \deg(f_2(x))$. Therefore,

$$\begin{aligned}
f_1(x) + ur'(x) &= f_1(x) + u\left(g(x) - h(x)f_2(x)\right) \\
&= f_1(x) + ug(x) - uh(x)f_2(x) \\
&\implies f_1(x) + ur'(x) \in \mathcal{I}.
\end{aligned}$$

Hence, we can write $\mathcal{I} = \mathcal{R}_2\Big(f_1(x) + ur'(x)\Big) + \mathcal{R}_2(uf_2(x))$, and $\deg(r'(x))$ is strictly less than $\deg(f_2(x))$. Furthermore, since $u(f_1(x) + ur'(x)) \in \mathcal{I}$, it implies that $f_1(x) + ur'(x) \in \mathrm{Tor}(\mathcal{I}) = \mathcal{R}_1'(f_2(x))$. Thus, we infer that $f_2(x) \mid_r f_1(x)$.

Now, suppose that $g'(x)$ is a polynomial satisfying

$$\mathcal{I} = \mathcal{R}_2\Big(f_1(x) + ug'(x)\Big) + \mathcal{R}_2(uf_2(x)).$$

Since $f_1(x) + ug(x)$ and $f_1(x) + ug'(x)$ are elements of $\mathcal{I}$, it follows that

$$(f_1(x) + ug(x)) - (f_1(x) + ug'(x)) = u(g(x) - g'(x)) \in \mathcal{I}.$$

This implies that $g(x) - g'(x) \in \mathrm{Tor}(\mathcal{I})$.

If $g(x) \neq g'(x)$, then $\deg(f_2(x)) < \deg(g(x) - g'(x))$. However, since $\deg(g(x))$ and $\deg(g'(x))$ are both strictly less than $\deg(f_2(x))$, it follows that $\deg(g(x) - g'(x)) < \deg(f_2(x))$. This leads to a contradiction because $\deg(g(x) - g'(x))$ should not be less than $\deg(f_2(x))$.

Therefore, we must have $g(x) = g'(x)$. This completes the proof. □

We explicitly present all possible types of generator polynomials of left ideals of $\mathcal{R}_2$.

Theorem 4.6. *All possible left ideals of the quotient ring $\mathcal{R}_2$ have generator polynomials of the following forms:*

- ***Type 1**: Trivial ideals: $\mathcal{R}_2(0)$ and $\mathcal{R}_2(1)$.*
- ***Type 2**: Principal left ideal generated by a non-monic skew polynomial: $\mathcal{R}_2\big(uf_2(x)\big)$, where $f_2(x) \in \mathcal{B}$ and $0 \leq \deg(f_2(x)) \leq 2(p^s - 1)$.*
- ***Type 3**: Principal left ideal generated by a monic skew polynomial: $\mathcal{R}_2\big(f_1(x) + ug(x)\big)$, where $f_1(x) \in \mathcal{B}$, $2 \leq \deg(f_1(x)) \leq 2(p^s - 1)$, and $\deg(f_1(x)) > \deg(g(x))$.*
- ***Type 4**: Non-principal left ideal: $\mathcal{R}_2(uf_2(x)) + \mathcal{R}_2\big(f_1(x) + ug(x)\big)$, where $f_1(x), f_2(x) \in \mathcal{B}$, $2 \leq \deg(f_1(x)) \leq 2(p^s - 1)$, $f_2(x) \mid_r f_1(x)$, and $\deg(g(x)) < \deg(f_2(x))$.*

Proof. From Theorem 4.5, let $\mathcal{I}$ be any left ideal of $\mathcal{R}_2$. Then

$$\mathcal{I} = \mathcal{R}_2\big(f_1(x) + ug(x)\big) + \mathcal{R}_2(uf_2(x)),$$

where $f_1(x)$ and $f_2(x)$ are either 0 or elements of $\mathcal{B}$, $g(x) \in \mathcal{R}_1'$, $f_2(x) \mid_r f_1(x)$, and $\deg(f_2(x)) > \deg(g(x))$.

Now, consider the following cases:

- **Type 1**: There are two cases for the trivial left ideal.

Case 1. If $f_1(x) = f_2(x) = g(x) = 0$, then $\mathcal{I}$ is the trivial left ideal $\mathcal{R}_2(0)$.

Case 2. If $f_1(x)$ and $f_2(x)$ are constant skew polynomials, then $g(x) = 0$ and $\mathcal{I}$ is the trivial left ideal $\mathcal{R}_2(1)$.

- **Type 2**: If $\deg(f_1(x)) = 2p^s$ and $f_2(x) \neq 0$, then $g(x) \in \mathrm{Tor}(\mathcal{I})$ and $\mathcal{I} = \mathcal{R}_2\big(uf_2(x)\big)$.
- **Type 3**: If $2 \leq \deg(f_1(x)) \leq 2(p^s - 1)$ and $\deg(f_2(x)) = \deg(f_1(x))$, then

$$\mathcal{I} = \mathcal{R}_2\big(f_1(x) + ug(x)\big).$$

- **Type 4**: If $2 \leq \deg(f_1(x)) \leq 2(p^s - 1)$ and $\deg(g(x)) < \deg(f_2(x)) < \deg(f_1(x))$, then

$$\mathcal{I} = \mathcal{R}_2\big(f_1(x) + ug(x)\big) + \mathcal{R}_2\big(uf_2(x)\big).$$

□

Example 4.7. *Consider the ring $\mathfrak{R} = \mathbb{F}_{3^3} + u\mathbb{F}_{3^3}$ with $u^2 = 0$, the quotient ring $\mathcal{R}_2 = \frac{\mathfrak{R}[x;\Pi]}{\langle (x^2+1)^3 \rangle}$, and the automorphism $\Pi \in \mathrm{Aut}(\mathfrak{R})$ defined by $\Pi(a) = a^3$ for any $a \in \mathfrak{R}$. Then $\mathrm{ord}(\Pi) = 3$. Let w be a $(3^3 - 1)$-primitive root of an irreducible polynomial of degree 3 in $\mathbb{F}_3[x]$.*

The factorization of $x^6 + 1$ in $\mathfrak{R}[x;\Pi]$ is given by

$$x^6 + 1 = (x^2 + w^2x + w^2)(x^2 + x + 2w^2 + 2w + 1)$$

$$\times (x^2 + (2w^2 + w + 1)x + 1).$$

Consider $C = \mathcal{R}_2\Big(f(x) + ug(x)\Big)$, *where* $g(x) = (x^2 + w^2x + w^2)$ *and*

$$\begin{aligned} f(x) &= (x^2 + x + 2w^2 + 2w + 1)(x^2 + (2w^2 + w + 1)x + 1) \\ &\implies f(x) = x^4 + (2w^2 + 2w + 2)x^3 + (w^2 + 2w + 1)x^2 \\ &\quad + (w^2 + 2w + 2)x + 2w^2 + 2w + 1. \end{aligned}$$

Therefore, C *has a generator matrix* G, *with its rows given as follows:*

- ***First row****:* $2w^2 + 2w + 1 + uw^2, w^2 + 2w + 2 + uw^2, w^2 + 2w + 1 + u, 2w^2 + 2w + 2, 1, 0.$
- ***Second row****:* $0, 2w^6 + 2w^3 + 1 + uw^6, w^6 + 2w^3 + 2 + uw^6, w^2 + 2w^3 + 1 + u, 2w^6 + 2w^3 + 2, 1.$

The generator matrix of $\phi(C)$ *is*

$$\phi(C) = \begin{pmatrix} \phi(G) \\ \phi(uG) \end{pmatrix},$$

and all four rows of the matrix $\phi(C)$ *can be obtained similarly as given in Example 4.3. Hence, using the MAGMA algebra system [11], we find that* $\phi(C)$ *is a near-MDS code with parameters* $[12, 4, 8]$.

Theorem 4.8. *Let* $\mathcal{I}$ *be any left ideal of* $\mathcal{R}_2$. *Then* $\mathcal{A}(\mathcal{I})^*$ *has generator polynomials of the following forms:*

- ***Type 1****: If* $\mathcal{I}$ *is a trivial ideal,* $\mathcal{R}_2(0)$ *or* $\mathcal{R}_2(1)$, *then* $\mathcal{A}(\mathcal{I})^* = \mathcal{R}_2(1)$ *or* $\mathcal{R}_2(0)$, *respectively.*
- ***Type 2****: If* $\mathcal{I} = \mathcal{R}_2(uf_2(x))$, *then*

$$\mathcal{A}(\mathcal{I})^* = \mathcal{R}_2\Big(\tilde{f}_2^*(x)\Big) + \mathcal{R}_2(u).$$

- ***Type 3****: If* $\mathcal{I} = \mathcal{R}_2(p(x))$, *where* $p(x) = f_1(x) + ug(x)$, *then*

$$\mathcal{A}(\mathcal{I})^* = \mathcal{R}_2\Big(\tilde{p}^*(x)\Big).$$

- ***Type 4****: If* $\mathcal{I} = \mathcal{R}_2\Big(f_1(x) + ug(x)\Big) + \mathcal{R}_2(uf_2(x))$, *then*

$$\mathcal{A}(\mathcal{I})^* = \mathcal{R}_2\Big((\tilde{f}_2(x) - uk(x))^*\Big) + \mathcal{R}_2\Big(u(\tilde{f}_1(x))^*\Big),$$

where $k(x) \in \mathcal{R}_1'$ *with* $\tilde{f}_1(x)g(x) = k(x)f_2(x)$ *and* $\tilde{a}(x) = \frac{(x^2+1)^{p^s}}{a(x)}$.

Proof. The proof directly follows from [25, Theorem 3.3]. □

By using a similar approach as in [23, Lemma 6.1], we can express any polynomial $Q(x) \in \mathcal{R}_2$ in the following form:

$$Q(x) = \sum_{i=0}^{p^s-1}(a_{0i}x + b_{0i})(x^2 + 1)^i + u\sum_{i=0}^{p^s-1}(a_{1i}x + b_{1i})(x^2 + 1)^i,$$

where $a_{ji}, b_{ji} \in \mathbb{F}_{p^m}$ for $j = 0, 1$.

We now have a different polynomial form in $\mathcal{R}_2$. Therefore, Theorem 4.6 can be rewritten as follows:

Theorem 4.9. *Let* $\mathcal{I}$ *be a left ideal of the quotient ring* $\mathcal{R}_2$. *Then,* $\mathcal{I}$ *has the following forms of generator polynomials based on the above assumptions:*

- ***Type 1***: *Trivial ideals:* $\mathcal{R}_2(0)$ *and* $\mathcal{R}_2(1)$.
- ***Type 2***: *Principal left ideal generated by a non-monic skew polynomial:* $\mathcal{R}_2\big(u(x^2+1)^i\big)$, *where* $0 \le i \le p^s - 1$.
- ***Type 3***: *Principal left ideal generated by a monic skew polynomial:*

  $$\mathcal{R}_2\big((x^2+1)^i + uh(x)(x^2+1)^t\big),$$

  where $1 \le i \le p^s - 1$, $0 \le t \le i$, *and either* $h(x) = 0$ *or* $h(x)$ *is a unit. Furthermore,* $h(x)$ *can be expressed as*

  $$h(x) = \sum_{j=0}^{i-t-1} (h_{0j}x + h_{1j})(x^2+1)^j,$$

  where $h_{0j}, h_{1j} \in \mathbb{F}_{p^m}$, $h_{00}x + h_{10} \neq 0$, *and* $\deg(h(x)) \le 2(T-t) - 1$, *with* T *being the smallest positive integer such that* $u(x^2+1)^T \in \mathcal{I}$.
- ***Type 4***: *Non-principal left ideal:*

  $$\mathcal{R}_2\left((x^2+1)^i + uh(x)(x^2+1)^t\right) + \mathcal{R}_2(u(x^2+1)^w),$$

  where $1 \le i \le p^s - 1$, $0 \le t \le i$, $w < T$, *and* $h(x)$ *and* T *are as described in Type 3, with* $deg(h(x)) \le w - t - 1$.

5. $\mathbb{F}_{p^m}\mathfrak{R}$-additive skew negacyclic codes

In this section, we discuss the algebraic structure of $\mathbb{F}_{p^m}\mathfrak{R}$-additive skew negacyclic codes of block length $(p^s, 2p^s)$. Moreover, we characterize all possible forms of generators of $\mathbb{F}_{p^m}\mathfrak{R}$-additive skew negacyclic codes of block length $(p^s, 2p^s)$. Subsequently, we study the separable $\mathbb{F}_{p^m}\mathfrak{R}$-additive skew negacyclic codes.

Consider the set

$$\mathbb{F}_{p^m}\mathfrak{R} = \{(a, b) \mid a \in \mathbb{F}_{p^m}, b \in \mathfrak{R}\}.$$

We can easily see that the set $\mathbb{F}_{p^m}\mathfrak{R}$ is not a $\mathfrak{R}$-module under usual ring multiplication. With the help of the ring epimorphism μ, we define a $\mathfrak{R}$-scalar multiplication " $*$ " on $\mathbb{F}_{p^m}\mathfrak{R}$ as $r * (a, b) = (\mu(r)a, b)$. Under this scalar multiplication and usual addition, the set $\mathbb{F}_{p^m}\mathfrak{R}$ forms a left $\mathfrak{R}$-module.

Further, we can define the $\mathfrak{R}$-module structure on $\mathbb{F}_{p^m}^{n_1}\mathfrak{R}^{n_2}$ by considering the $\mathfrak{R}$-scalar multiplication as follows:

$$r * (a_0, a_1, \ldots, a_{n_1-1}, b_0, \ldots, b_{n_2-1}) = (\mu(r)a_0, \mu(r)a_1, \ldots, \mu(r)a_{n_1-1}, b_0, b_1, \ldots, b_{n_2-1}),$$

where $r \in \mathfrak{R}$ and $(a_0, a_1, \ldots, a_{n_1-1}, b_0, \ldots, b_{n_2-1}) \in \mathbb{F}_{p^m}^{n_1}\mathfrak{R}^{n_2}$. From now on, we consider $n_1 = p^s$ and $n_2 = 2p^s$. We have a natural bijection between $\mathbb{F}_{p^m}^{p^s}\mathfrak{R}^{2p^s}$ and $\mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2$, which is given by

$$(a_0, a_1, \ldots, a_{p^s-1}, b_0, b_1, \ldots, b_{2p^s-1}) \mapsto \left(a_0 + a_1x + \cdots + a_{p^s-1}x^{p^s-1}, b_0 + b_1 + \cdots + b_{2p^s-1}x^{2p^s-1}\right).$$

Furthermore, the scalar multiplication $*$ induces a scalar multiplication on $\mathcal{R}$ as follows:

$$r(x) * (f(x), g(x)) = \big(\mu(r(x))f(x), r(x)g(x)\big),$$

where $r(x) \in \mathfrak{R}[x; \Pi]$ and $(f(x), g(x)) \in \mathcal{R}$. Under this $*$ operation, $\mathcal{R}$ forms a left $\mathfrak{R}[x; \Pi]$-module.

Let C be a non-empty subset of $\mathbb{F}_{p^m}^{p^s}\mathfrak{R}^{2p^s}$. If C forms a subgroup of $\mathbb{F}_{p^m}^{p^s}\mathfrak{R}^{2p^s}$, then C is called an $\mathbb{F}_{p^m}\mathfrak{R}$-additive code of block length $(p^s, 2p^s)$. For any $\mathbb{F}_{p^m}\mathfrak{R}$-additive code C, it is called an $\mathbb{F}_{p^m}\mathfrak{R}$-additive linear code of block length $(p^s, 2p^s)$ if and only if C forms an $\mathfrak{R}$-submodule of $\mathbb{F}_{p^m}^{p^s}\mathfrak{R}^{2p^s}$.

Any $\mathbb{F}_{p^m}\mathfrak{R}$-additive linear code is called an $\mathbb{F}_{p^m}\mathfrak{R}$-additive skew negacyclic code if, for any vector $(a_0, a_1, \ldots, a_{p^s-1}, b_0, b_1, \ldots, b_{2p^s-1}) \in C$, the vector $\left(-\pi(a_{p^s-1}), \pi(a_0), \ldots, \pi(a_{p^s-2}), -\Pi(b_{2p^s-1}), \Pi(b_0), \ldots, \Pi(b_{2p^s-2})\right)$ also belongs to C.

Proposition 5.1. *A code C is an $\mathbb{F}_{p^m}\mathfrak{R}$-additive skew negacyclic code of block length $(p^s, 2p^s)$ if and only if C is a left $\mathfrak{R}[x;\Pi]$-submodule of $\mathcal{R}$.*

Proof. We only prove the implication part, as the converse follows a similar argument. Let C be an $\mathbb{F}_{p^m}\mathfrak{R}$-additive skew negacyclic code of block length $(p^s, 2p^s)$, and let $\mathbf{c} = (\mathbf{a}, \mathbf{b}) \in C$, where $\mathbf{a} = (a_0, a_1, \ldots, a_{p^s-1}) \in \mathbb{F}_{p^m}^{p^s}$ and $\mathbf{b} = (b_0, b_1, \ldots, b_{2p^s-1}) \in \mathfrak{R}^{2p^s}$.

Since there is a bijection between $\mathbb{F}_{p^m}^{p^s}\mathfrak{R}^{2p^s}$ and $\mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2$, we have $c(x) = (a(x), b(x)) \in \mathcal{R}$. Consider the action of x on $c(x)$:

$$\begin{aligned} x * c(x) &= \left(xa_0 + xa_1x + \cdots + xa_{p^s-2}x^{p^s-2} + xa_{p^s-1}x^{p^s-1}, xb_0 + xb_1x + \cdots + xb_{2p^s-2}x^{2p^s-2} + xb_{2p^s-1}x^{2p^s-1}\right) \\ &= \left(-\pi(a_{p^s-1}) + \pi(a_0)x + \cdots + \pi(a_{p^s-2})x^{p^s-1}, -\Pi(b_{2p^s-1}) + \Pi(b_0)x + \Pi(b_1)x^2 + \cdots + \Pi(b_{2p^s-2})x^{2p^s-1}\right). \end{aligned}$$

This expression is the skew negacyclic shift of $\mathbf{c}$. Hence, $x * c(x) \in C$. Since C is linear, for any $f(x) \in \mathfrak{R}[x;\Pi]$, $f(x) * c(x) \in C$. Therefore, C is a left $\mathfrak{R}[x;\Pi]$-submodule of $\mathcal{R}$. □

In Section 2, we defined a Gray map ϕ from $\mathfrak{R}$ to $\mathbb{F}_{p^m}^2$, and subsequently extended this Gray map ϕ from $\mathfrak{R}^n$ to $\mathbb{F}_{p^m}^{2n}$. In this section, we further extend this Gray map from mixed alphabets $\mathbb{F}_{p^m}^{p^s}\mathfrak{R}^{2p^s}$ to $\mathbb{F}_{p^m}^{5p^s}$ as follows:

$$\begin{aligned} &\Phi : \mathbb{F}_{p^m}^{p^s}\mathfrak{R}^{2p^s} \longrightarrow \mathbb{F}_{p^m}^{5p^s}, \\ &\Phi((a_0, a_1, \ldots, a_{p^s-1}, b_0, b_1, \ldots, b_{2p^s-1})) = (a_0, a_1, \ldots, a_{p^s-1}, \phi(b_0), \phi(b_1), \ldots, \phi(b_{2p^s-1})). \end{aligned}$$

Since the map ϕ is distance-preserving, Φ is also a distance-preserving map, converting Lee distance to Hamming distance. Moreover, Φ is a bijective map, which implies that if C is an $\mathbb{F}_{p^m}\mathfrak{R}$-additive skew negacyclic code of block length $(p^s, 2p^s)$, then $\Phi(C)$ is a linear code of length $5p^s$ over $\mathbb{F}_{p^m}$.

Now, we present the key theorem of this section, which determines the generator polynomials of $\mathbb{F}_{p^m}\mathfrak{R}$-additive skew negacyclic codes of block length $(p^s, 2p^s)$ in the next result.

Theorem 5.2. *All possible $\mathbb{F}_{p^m}\mathfrak{R}$-additive skew negacyclic codes of block length $(p^s, 2p^s)$ are of the following types:*

- ***Type 1:*** $(0), \mathcal{R}$.
- ***Type 2:*** $\mathcal{R}_2((a(x), 0))$, *where* $0 \le \deg(a(x)) \le p^s - 1$.
- ***Type 3:*** $\mathcal{R}_2((k_1(x), f_1(x) + ug(x)))$, *where* $2 \le \deg(f_1(x)) \le 2(p^s - 1)$ *and* $\deg(f_1(x)) > \deg(g(x))$.
- ***Type 4:*** $\mathcal{R}_2((k_2(x), uf_2(x)))$, *where* $0 \le \deg(f_2(x)) \le 2(p^s - 1)$.
- ***Type 5:*** $\mathcal{R}_2((a(x), 0)) + \mathcal{R}_2((k_1(x), f_1(x) + ug(x)))$, *where* $0 \le \deg(a(x)) \le p^s - 1$, $2 \le \deg(f_1(x)) \le 2(p^s - 1)$, *and* $\deg(f_1(x)) > \deg(g(x))$.
- ***Type 6:*** $\mathcal{R}_2((a(x), 0)) + \mathcal{R}_2((k_2(x), uf_2(x)))$, *where* $0 \le \deg(a(x)) \le p^s - 1$ *and* $0 \le \deg(f_2(x)) \le 2(p^s - 1)$.
- ***Type 7:*** $\mathcal{R}_2((k_1(x), f_1(x) + ug(x))) + \mathcal{R}_2((k_2(x), uf_2(x)))$, *where* $0 \le \deg(f_2(x)) \le 2(p^s - 1)$, $2 \le \deg(f_1(x)) \le 2(p^s - 1)$, $f_2(x) \mid_r f_1(x)$, *and* $\deg(f_2(x)) > \deg(g(x))$.
- ***Type 8:*** $\mathcal{R}_2((a(x), 0)) + \mathcal{R}_2((k_1(x), f_1(x) + ug(x))) + \mathcal{R}_2((k_2(x), uf_2(x)))$, *where* $\deg(f_2(x)) > \deg(g(x))$, $f_2(x) \mid_r f_1(x)$, $0 \le \deg(a(x)) \le p^s - 1$, $0 \le \deg(f_2(x)) \le 2(p^s - 1)$, $2 \le \deg(f_1(x)) \le 2(p^s - 1)$, *and* $\max\{\deg(k_1(x)), \deg(k_2(x))\} < \deg(a(x))$.

Moreover, in all these types, $a(x) \in \mathcal{B}$, $f_1(x), f_2(x) \in \mathcal{B}$, $k_1(x), k_2(x) \in \mathcal{R}_1$, $g(x) \in \mathcal{R}_1'$, and the polynomial $g(x)$ is unique under the conditions mentioned. Here, $\mathcal{B}$ is the set of all monic factors of $(x+1)^{p^s}$ over $\mathbb{F}_{p^m}[x;\pi]$.

Proof. Let Θ be the canonical projection on the $\Re[x;\Pi]$-module defined by

$$\Theta : C \to \mathcal{R}_2,$$

$$\Big(a(x), b(x)\Big) \mapsto b(x).$$

The kernel of Θ is given by

$$\mathrm{Ker}(\Theta) = \{(v(x), 0) \in C \mid v(x) \in \mathcal{R}_1\}.$$

Thus, $J = \{v(x) \in \mathcal{R}_1 \mid (v(x), 0) \in \mathrm{Ker}(\Theta)\}$ is a left ideal of $\mathcal{R}_1$. By [25, Proposition 3.1], there exists $a(x) \in \mathfrak{B}$ such that $J = \mathcal{R}_1(a(x))$. Therefore, $\mathrm{Ker}(\Theta)$ is a left $\Re[x;\Pi]$-submodule of C generated by a single element of the form $(a(x), 0)$, i.e.,

$$\mathrm{Ker}(\Theta) = \mathcal{R}_2(a(x), 0).$$

By the first isomorphism theorem, we have

$$\frac{C}{\mathrm{Ker}(\Theta)} \cong \Theta(C),$$

which is a left ideal of $\mathcal{R}_2$. From Theorem 4.5, we get

$$\frac{C}{\mathrm{Ker}(\Theta)} \cong \Theta(C) = \mathcal{R}_2\left(f_1(x) + ug(x)\right) + \mathcal{R}_2\left(uf_2(x)\right),$$

where $f_1(x), f_2(x) \in \mathcal{B}$, $g(x) \in \mathcal{R}_1'$, $f_2(x) \mid_r f_1(x)$, and $\deg(f_2(x)) > \deg(g(x))$. This implies that $\Theta(C)$ is generated by $f_1(x)+ug(x)$ and $uf_2(x)$. For $k_1(x), k_2(x) \in \mathcal{R}_1$, there exist two elements $(k_1(x), f_1(x)+ug(x))$ and $(k_2(x), uf_2(x))$ such that

$$\Theta((k_1(x), f_1(x) + ug(x))) = f_1(x) + ug(x),$$
$$\Theta((k_2(x), uf_2(x))) = uf_2(x).$$

Hence,

$$\Theta\left(\mathcal{R}_2(k_1(x), f_1(x) + ug(x)) + \mathcal{R}_2(k_2(x), uf_2(x))\right) = \mathcal{R}_2\left(f_1(x) + ug(x)\right) + \mathcal{R}_2\left(uf_2(x)\right).$$

Thus, any codeword of C is generated by these three elements, namely, $(a(x), 0)$, $(k_1(x), f_1(x) + ug(x))$, and $(k_2(x), uf_2(x))$. Therefore,

$$C = \mathcal{R}_2(a(x), 0) + \mathcal{R}_2(k_1(x), f_1(x) + ug(x)) + \mathcal{R}_2(k_2(x), uf_2(x)).$$

We then have the following cases:

- **Type 1:** We have two cases for the trivial left ideal.
  1. **Case 1.** If $\deg(a(x)) = \deg(f_1(x)) = \deg(f_2(x)) = 0$, then $k_1(x) = k_2(x) = g(x) = 0$ and $C = \mathcal{R}(1)$.
  2. **Case 2.** If $\deg(a(x)) = p^s$, $\deg(f_1(x)) = \deg(f_2(x)) = 2p^s$, and $k_1(x) = k_2(x) = g(x) = 0$, then $C = \mathcal{R}(0)$.
- **Type 2:** If $\deg(f_1(x)) = \deg(f_2(x)) = 2p^s$, $0 \leq \deg(a(x)) \leq p^s - 1$, and $k_1(x) = k_2(x) = g(x) = 0$, then $C = \mathcal{R}_2\left(a(x), 0\right)$.
- **Type 3:** If $\deg(a(x)) = p^s$, $0 \leq \deg(k_1(x)) \leq p^s - 1$, $f_1(x) \in \mathcal{B}$, $2 \leq \deg(f_1(x)) \leq 2(p^s - 1)$, $\deg(f_2(x)) = \deg(f_1(x))$, $\deg(f_1(x)) > \deg(g(x))$, and $k_2(x) = 0$, then $C = \mathcal{R}_2\left(k_1(x), f_1(x) + ug(x)\right)$.
- **Type 4:** If $\deg(a(x)) = p^s$, $\deg(f_1(x)) = 2p^s$, $0 \leq \deg(k_2(x)) \leq p^s - 1$, $f_2(x) \in \mathcal{B}$, $0 \leq \deg(f_2(x)) \leq 2(p^s - 1)$, $\deg(f_2(x)) > \deg(g(x))$, and $k_1(x) = 0$, then $C = \mathcal{R}_2\left(k_2(x), uf_2(x)\right)$.

- **Type 5:** If $0 \leq \deg(a(x)) \leq p^s-1$, $0 \leq \deg(k_1(x)) \leq p^s-1$, $f_1(x) \in \mathcal{B}$, $2 \leq \deg(f_1(x)) \leq 2(p^s-1)$, $\deg(f_2(x)) = \deg(f_1(x))$, $\deg(f_1(x)) > \deg(g(x))$, and $k_2(x) = 0$, then $C = \mathcal{R}_2(a(x), 0) + \mathcal{R}_2(k_1(x), f_1(x) + ug(x))$.
- **Type 6:** If $\deg(f_1(x)) = 2p^s$, $0 \leq \deg(a(x)) \leq p^s - 1$, $0 \leq \deg(k_2(x)) \leq p^s - 1$, $f_2(x) \in \mathcal{B}$, $0 \leq \deg(f_2(x)) \leq 2(p^s - 1)$, $\deg(f_2(x)) > \deg(g(x))$, and $k_1(x) = 0$, then $C = \mathcal{R}_2(a(x), 0) + \mathcal{R}_2(k_2(x), uf_2(x))$.
- **Type 7:** If $\deg(a(x)) = p^s$, $0 \leq \deg(k_1(x)) \leq p^s - 1$, $0 \leq \deg(k_2(x)) \leq p^s - 1$, $f_1(x), f_2(x) \in \mathcal{B}$, $0 \leq \deg(f_2(x)) \leq 2(p^s - 1)$, $2 \leq \deg(f_1(x)) \leq 2(p^s - 1)$, $f_2(x) \mid_r f_1(x)$, and $\deg(f_2(x)) > \deg(g(x))$, then $C = \mathcal{R}_2(k_1(x), f_1(x) + ug(x)) + \mathcal{R}_2(k_2(x), uf_2(x))$.
- **Type 8:** If $0 \leq \deg(a(x)) \leq p^s - 1$, $0 \leq \deg(k_1(x)) \leq p^s - 1$, $0 \leq \deg(k_2(x)) \leq p^s - 1$, $f_1(x), f_2(x) \in \mathcal{B}$, $0 \leq \deg(f_2(x)) \leq 2(p^s - 1)$, $2 \leq \deg(f_1(x)) \leq 2(p^s - 1)$, $f_2(x) \mid_r f_1(x)$, and $\deg(f_2(x)) > \deg(g(x))$, then $C = \mathcal{R}_2(a(x), 0) + \mathcal{R}_2(k_1(x), f_1(x) + ug(x)) + \mathcal{R}_2(k_2(x), uf_2(x))$.

□

By considering our above discussion, we present the algebraic structure of separable $\mathbb{F}_{p^m}\mathfrak{R}$-additive skew negacyclic codes. A $\mathbb{F}_{p^m}\mathfrak{R}$-additive skew negacyclic code C is called a separable $\mathbb{F}_{p^m}\mathfrak{R}$-additive skew negacyclic code if it can be written as $C = C_1 \otimes C_2$, where C_1 and C_2 are the canonical projections of C on the first p^s and the last $2p^s$ coordinates, respectively.

Theorem 5.3. *Let $C = C_1 \otimes C_2$ be a separable $\mathbb{F}_{p^m}\mathfrak{R}$-additive linear code of block length $(p^s, 2p^s)$, where C_1 and C_2 are linear codes of length p^s and $2p^s$ over $\mathbb{F}_{p^m}$ and $\mathfrak{R}$, respectively. Then C is an $\mathbb{F}_{p^m}\mathfrak{R}$-additive skew negacyclic code if and only if C_1 and C_2 are skew negacyclic codes of length p^s and $2p^s$ over $\mathbb{F}_{p^m}$ and $\mathfrak{R}$, respectively.*

Proof. Let C be an $\mathbb{F}_{p^m}\mathfrak{R}$-additive skew negacyclic code and $(a_0, a_1, \ldots, a_{p^s-1}, b_0, \ldots, b_{2p^s-1}) \in C$ such that $(a_0, a_1, \ldots, a_{p^s-1}) \in C_1$ and $(b_0, b_1, \ldots, b_{2p^s-1}) \in C_2$. Since C is closed under a skew negacyclic shift, we get

$$\Big(-\pi(a_{p^s-1}), \pi(a_0), \pi(a_1), \ldots, \pi(a_{p^s-2}), -\Pi(b_{2p^s-1}), \Pi(b_0), \Pi(b_1), \ldots, \Pi(b_{2p^s-2})\Big) \in C,$$

which implies that

$$\Big(-\pi(a_{p^s-1}), \pi(a_0), \ldots, \pi(a_{p^s-2})\Big) \in C_1 \text{ and } \Big(-\Pi(b_{2p^s-1}), \Pi(b_0), \ldots, \Pi(b_{2p^s-2})\Big) \in C_2.$$

Therefore, C_1 is a skew negacyclic code of length p^s over $\mathbb{F}_{p^m}$ and C_2 is a skew negacyclic code of length $2p^s$ over $\mathfrak{R}$.

Conversely, let C_1 and C_2 be skew negacyclic codes of length p^s and $2p^s$ over $\mathbb{F}_{p^m}$ and $\mathfrak{R}$, respectively, and let $\big(a_0, a_1, \ldots, a_{p^s-1}, b_0, b_1, \ldots, b_{2p^s-1}\big) \in C$. Then we have $\Big(-\pi(a_{p^s-1}), \pi(a_0), \ldots, \pi(a_{p^s-2})\Big) \in C_1$ and $\Big(-\Pi(b_{2p^s-1}), \Pi(b_0), \ldots, \Pi(b_{2p^s-2})\Big) \in C_2$. Thus,

$$\Big(-\pi(a_{p^s-1}), \pi(a_0), \ldots, \pi(a_{p^s-2}), -\Pi(b_{2p^s-1}), \Pi(b_0), \ldots, \Pi(b_{2p^s-2})\Big) \in C,$$

that is, C is closed under skew negacyclic shift. Hence, C is an $\mathbb{F}_{p^m}\mathfrak{R}$-additive skew negacyclic code of block length $(p^s, 2p^s)$. □

Theorem 5.3 states that any separable $\mathbb{F}_{p^m}\mathfrak{R}$-additive skew negacyclic code of block length $(p^s, 2p^s)$ is precisely the direct product of left ideals of $\mathcal{R}_1 = \frac{\mathbb{F}_{p^m}[x;\pi]}{\langle x^{p^s}+1\rangle}$ and left ideals of $\mathcal{R}_2 = \frac{\mathfrak{R}[x;\Pi]}{\langle x^{2p^s}+1\rangle}$. In Section 3, we discussed the complete structure of left ideals of $\mathcal{R}_1'$, and in Section 4, a similar study was conducted for the left ideals of $\mathcal{R}_2$. We now present the generator polynomials of separable codes in the following result:

Theorem 5.4. *Let C be a separable $\mathbb{F}_{p^m}\mathfrak{R}$-additive skew negacyclic code of block length $(p^s, 2p^s)$. Then the generator polynomials of C are given as follows:*

- ***Type 1:*** *$\langle((x-1)^k, 0)\rangle$ and $\langle((x-1)^k, 1)\rangle$, where $0 \leq k \leq p^s - 1$.*
- ***Type 2:*** *$\langle((x-1)^k, 0), (0, u(x^2+1)^i)\rangle$, where $0 \leq k \leq p^s - 1$ and $0 \leq i \leq p^s - 1$.*

- ***Type 3:*** $\langle((x-1)^k,0),(0,(x^2+1)^i+uh(x)(x^2+1)^t)\rangle$, *where* $0 \le k \le p^s-1$, $1 \le i \le p^s-1$, $0 \le t \le i$, *and either* $h(x)=0$ *or* $h(x)$ *is a unit. Furthermore,* $h(x)$ *can be expressed as* $h(x)=\sum_{j=0}^{i-t-1}(h_{0j}x+h_{1j})(x^2+1)^j$, *where* $h_{0j},h_{1j} \in \mathbb{F}_{p^m}$, $h_{00}x+h_{10} \neq 0$, *and* $\deg(h(x)) \le 2(T-t)-1$, *where* T *is the smallest positive integer such that* $u(x^2+1)^T \in \langle(x^2+1)^i+uh(x)(x^2+1)^t\rangle$.

- ***Type 4:*** $\langle((x-1)^k,0),(0,(x^2+1)^i+uh(x)(x^2+1)^t),(0,u(x^2+1)^w)\rangle$, *where* $0 \le k \le p^s-1$, $1 \le i \le p^s-1$, $0 \le t \le i$, $w < T$, *and* $h(x)$ *and* T *are as in Type 3. Additionally,* $\deg(h(x)) \le w-t-1$.

Proof. Let $C = C_1 \otimes C_2$ be a separable $\mathbb{F}_{p^m}\mathfrak{R}$-additive skew negacyclic code of block length $(p^s, 2p^s)$. By Theorem 5.3, C_1 is a skew negacyclic code of length p^s over $\mathbb{F}_{p^m}$. This implies that there exists a skew polynomial $a(x)$ such that $C_1 = \langle a(x)\rangle$, where $a(x)$ is a monic divisor of $(x+1)^{p^s}$. Similarly, C_2 is a skew negacyclic code of length $2p^s$ over $\mathfrak{R}$. By Theorem 4.9, $C_2 = \langle b(x)\rangle$, where $b(x)$ is one of the 4 types of polynomials. Therefore, by the definition of a separable code, $C = \langle(a(x),0),(0,b(x))\rangle$. □

Remark 5.5. *The obtained additive skew negacyclic codes over* $\mathbb{F}_{p^m}\mathfrak{R}$ *contribute significantly to the literature by offering a novel extension of the existing framework of skew constacyclic and negacyclic codes over finite chain rings and mixed alphabets. These codes generalize the classical additive codes by incorporating both the algebraic structure of finite fields and the underlying non-commutative nature of skew polynomial rings.*

In the current literature, additive codes have been extensively studied over various algebraic structures, including finite fields, finite chain rings, and mixed alphabets such as $\mathbb{Z}_2\mathbb{Z}_4$ *and* $\mathbb{Z}_2\mathbb{Z}_2[u]$. *The present work builds upon these foundations by introducing and classifying skew negacyclic codes over the mixed alphabet* $\mathbb{F}_{p^m}\mathfrak{R}$, *where* $\mathfrak{R} = \mathbb{F}_{p^m}+u\mathbb{F}_{p^m}$ *with* $u^2=0$. *This extension is motivated by the algebraic properties of the chain ring* $\mathfrak{R}$, *which allows the construction of new classes of additive codes with potentially improved error-correcting capabilities.*

A key contribution of this work is the explicit characterization of the algebraic structure of these codes through their generator polynomials and separability conditions. The relationship between the obtained codes and classical constacyclic codes is established via the decomposition of skew negacyclic codes into direct sums of skew constacyclic codes of shorter lengths, as demonstrated in Theorem 3.2, Proposition 3.9, Theorems 4.2, 4.5, 5.2 and 5.3. This decomposition not only provides insight into the structure of the codes but also facilitates the construction of Maximum Distance Separable (MDS) and near-MDS codes, which are of great importance in coding theory.

Moreover, the separability property of the additive codes over $\mathbb{F}_{p^m}\mathfrak{R}$ *plays a crucial role in linking these codes to their torsion codes over* $\mathbb{F}_{p^m}$, *as shown through the Gray map and the annihilator structure of left ideals. This connection establishes a bridge between additive codes over mixed alphabets and their classical counterparts, further enriching the existing body of knowledge on skew cyclic and additive codes.*

Example 5.6. *Consider the ring* $\mathfrak{R} = \mathbb{F}_{3^3}+u\mathbb{F}_{3^3}$ *with* $u^2=0$, *the quotient ring* $\mathcal{R}_2 = \frac{\mathfrak{R}[x;\Pi]}{\langle(x^2+1)^3\rangle}$, *and the automorphism* $\Pi \in \mathrm{Aut}(\mathfrak{R})$ *defined by* $\Pi(a)=a^3$ *for any* $a \in \mathfrak{R}$. *Then* $\mathrm{ord}(\Pi)=3$. *Let* w *be a* (3^3-1)*-primitive root of an irreducible polynomial of degree* 3 *in* $\mathbb{F}_3[x]$.

The factorization of x^6+1 *in* $\mathfrak{R}[x;\Pi]$ *is given by*

$$x^6+1 = (x^2+(2w^2+2)x+1)(x^2+(2w^2+2w)x+2w^2+w+1)(x^2+(2w^2+w+2)x+w^2+w+1).$$

Consider $C = \mathcal{R}_2\big(a(x),0\big)+\mathcal{R}_2\big(k_2(x)+uf_2(x)\big)$, *where* $a(x)=k_2(x)=x^3+1$ *and*

$$\begin{aligned} & f(x) = (x^2+(2w^2+2w)x+2w^2+w+1)(x^2+(2w^2+w+2)x+w^2+w+1) \\ \implies & f(x) = x^4+(w^2+w+2)x^3+(w^2+w)x^2+(w^2+1)x+1. \end{aligned}$$

Therefore, C *has a generator matrix* G, *with its rows given as follows:*

- ***First row****:* $0,0,0,u,u(w^2+1),u(w^2+w),u(w^2+w+2),u,0$.
- ***Second row****:* $0,0,0,0,u,u(w^6+1),u(w^6+w^3),u(w^6+w^3+2),u$.

The rows of the generator matrix $\Phi(C)$ *are*

- ***First row:*** $0, 0, 0, 1, 1, w^2+1, w^2+1, w^2+w, w^2+w, w^2+w+2, w^2+w+2, 1, 1, 0, 0.$
- ***Second row:*** $0, 0, 0, 0, 0, 1, 1, w^6+1, w^6+1, w^6+w^3, w^6+w^3, w^6+w^3+2, w^6+w^3+2, 1, 1.$

Hence, using the MAGMA algebra system [11], $\Phi(C)$ *having parameters* $[15, 2, 10]$.

6. Conclusion

This study delves into the algebraic properties of skew negacyclic codes of length $2p^s$ over $\mathbb{F}_{p^m}$ and $\mathfrak{R} = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ with $u^2 = 0$ for an odd prime p. These codes can be represented as left ideals of $\frac{\mathbb{F}_{p^m}[x;\pi]}{\langle x^{2p^s}+1\rangle}$ and $\frac{\mathfrak{R}[x;\Pi]}{\langle x^{2p^s}+1\rangle}$, respectively, where $\mathrm{ord}(\pi)$ and $\mathrm{ord}(\Pi)$ divide p^s. We examine two cases based on $p \equiv a \pmod 4$, where $a = 1$ and 2. For each case, we analyze the algebraic structure and generator polynomials of these skew negacyclic codes. Similarly, we study the structural properties and the generator polynomials of skew negacyclic codes of length $2p^s$ over $\mathfrak{R}$ for both congruence cases. After establishing the code structures for a single alphabet, we extend our investigation to $\mathbb{F}_{p^m}\mathfrak{R}$-additive skew negacyclic codes of block length $(p^s, 2p^s)$. Moreover, we determine generator polynomials for $\mathbb{F}_{p^m}\mathfrak{R}$-additive skew negacyclic codes and separable skew negacyclic codes of block length $(p^s, 2p^s)$. To illustrate our findings, we present some examples and obtain MDS and near-MDS codes. Investigating skew negacyclic codes of block length (np^s, kp^s) over the mixed alphabets $\mathbb{F}_{p^m}\mathfrak{R}$ in the future might be an interesting field of investigation.

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