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A concept of $\alpha\beta$-statistical convergence of multiset sequence

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Abstract. In 2014, Aktuğlu [1] first introduced the notion of $\alpha\beta$-statistical convergence which is more general than the notion of statistical convergence and lacunary statistical convergence. In this paper, based on this notion, we introduce the notion of $\alpha\beta$-statistical convergence and $\alpha\beta$-strongly summability for multiset sequences which improves the results in [3, 7]. We investigated the relations between these two notions and presented several examples corresponding to our results. We also introduce the notion of $\alpha\beta$ statistically limit superior and limit inferior for multiset sequences by using the notion of $\alpha\beta$ density which improves the results in [3].

1. Introduction

According to classical set theory, a set is a well defined collection of distinct elements, i.e, every element(s) occur only once. But, in many situations multiple occurrence of a particular element plays an important role in our daily life. For example, in case of a telephone number, one number can occur multiple times. If instead of multiple times, if it occurs only one time then it becomes problematic. Similarly, in case of binary representation of a particular number one number occurs in multiple times. In such cases, the notion of multiset come into the picture. Let X be a non-empty set. A multiset M with elements from the set X contain elements $x \in X$ with the multiplicity $C(x)$ where $C : X \to \mathbb{N}_0$, where $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. $C(x)$ represents the multiplicity of the element x. Consider the set $\{5, 5, 2, 2, 2, 7\}$. This is an example of a multiset as 5 occurs 2 times, 2 occurs 3 times and 7 occurs 1 time. In this case we represent this set as $\{5|2, 2|3, 7|1\}$. Here, $C(5) = 2, C(2) = 3, C(7) = 1$. In 1980, Hickman [8] studied several notions and operations on multisets. In 1981, Knuth studied multisets related to the computer programming [9]. Bender [2] investigated the partitions of multisets. In 1976, Lake [10] discussed an axiomatization of the theory of multisets. In [11], Majumdar considered the notion of soft multisets and distance, similarity between two soft multisets. For detail study of multisets, interested readers can see the references [3, 12, 13] and the references therein. In 2021, Pachilangode and John [13] considered different notions of convergence, like, Wijsman convergence,

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Hausdorff convergence in case of multisets. As we already said if X is a non-empty set and M is a multiset over X then there is a function $C : X \to \mathbb{N}_0$. So, if (X, d) is a metric space then we cannot use the same metric d for multiset M. In [13], Pachilangode and John considered the metric $d_M(x, y) = d(x, y) + |C(x) - C(y)|$. A sequence where each term is a multiset is known as multiset sequence (see, [13] for details). For a sequence $\{x_n\} \subset X$, a multiset sequence is defined by $M = \{x_n|c_n : x_n \in X, c_n \in \mathbb{N}_0\}$. In the present work, we consider the metric as $d_M(x_n|c_n, y_n|d_n) = \sqrt{(x_n - y_n)^2 + (c_n - d_n)^2}$ on the multiset M.

On the other hand, the idea of convergence of a real sequence has been extended to statistical convergence by Fast [4] and Steinhaus [16] independently. Further it is re-introduced by Schoenberg [15] and it is based on the notion of asymptotic density of the subset of natural numbers. Let $K \subseteq \mathbb{N}$. Then asymptotic density of K is defined as $d(K) = \lim_{n\to\infty} \frac{1}{n} |\{k \le n : k \in K\}|$ provided the limit exists. A real sequence $\{x_n\}$ is said to be statistically convergent to $x \in \mathbb{R}$ if for each $\varepsilon > 0$, $\lim_{n\to\infty} \frac{1}{n} |\{k \le n : |x_k - x| \ge \varepsilon\}| = 0$. In 1980, Šalát [14] considered the set of all statistically convergent sequences in l_∞ over the sup norm and showed that the set is dense in l_∞. Fridy [5] defined the notion of statistically Cauchy sequence and investigated the relations between statistical convergence and statistically Cauchy sequence. In 1993, Fridy and Orhan [6] defined the concept of lacunary sequence and defined the notion of Lacunary statistical convergence for sequence of real numbers. A lacunary sequence $\{t_n\}_{n\ge 0}$ is an increasing integer sequence with $t_0 = 0$ and $T_r = (t_r - t_{r-1}) \to \infty$ as $r \to \infty$. A real sequence $\{x_n\}$ is said to be lacunary statistically convergent to $x \in \mathbb{R}$ if for each $\varepsilon > 0$, $\lim_{r\to\infty} \frac{1}{T_r} |\{k \in (t_{r-1}, t_r] : |x_k - x| \ge \varepsilon\}| = 0$. In 2021, Debnath and Debnath [3] introduced the notion of statistical convergence for multiset sequences and investigated various properties of this new convergence. They also defined the notion of statistically boundedness in case of multiset sequences and presented the relation between statistically boundedness and statistical convergence of multiset sequences. Also, very recently Gumus et al. [7] used the notion of lacunary sequence to introduce the concept of lacunary statistical convergence for multiset sequences. Motivated by paper [3, 7], in this present work, we introduce the notion of $\alpha\beta$-statistical convergence and $\alpha\beta$-strongly summability for multiset sequences and investigated the relations between these two concepts. Various examples are presented to discuss the results. If we take $\alpha_r = (t_{r-1} + 1)$ and $\beta_r = t_r$, then $\alpha\beta$-statistical convergence coincides with lacunary statistical convergence. As $\alpha\beta$-statistical convergence is more general than the notion of statistical convergence, lacunary statistical convergence, so our results discussed in the paper improve the results discussed in [3, 7].

2. Main results

Firstly, we recall the notion of $\alpha\beta$-statistical convergence from [1] as follows.

Definition 2.1. Let $\alpha = \{\alpha_n\}_{n\in\mathbb{N}}$ and $\beta = \{\beta_n\}_{n\in\mathbb{N}}$ be two sequences of positive numbers satisfying the following conditions:

1. α and β are both non-decreasing;
2. $\beta_n \ge \alpha_n$ for all $n \in \mathbb{N}$;
3. $(\beta_n - \alpha_n) \to \infty$ as $n \to \infty$.

This pair is denoted by (α, β). A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be $\alpha\beta$-statistically convergent to a point $x \in \mathbb{R}$ if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{(\beta_n - \alpha_n + 1)} \mid \{k \in [\alpha_n, \beta_n] : |x_k - x| \ge \varepsilon\} \mid = 0.$$

In the above definition, if we take $\alpha_n = 1$ and $\beta_n = n$ for all $n \in \mathbb{N}$, then $\alpha\beta$-statistical convergence coincides with the notion of statistical convergence. So, $\alpha\beta$-statistical convergence is more general than the notion of statistical convergence.

Next, we introduce the notion of $\alpha\beta$-statistical convergence of real valued multiset sequence.

Definition 2.2. Let $H = \{x_n|c_n\}$ be a real valued multiset sequence. The sequence H is said to be $\alpha\beta$-statistically convergent to the multiset $\{x|c\}$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{(\beta_n - \alpha_n + 1)} \mid \left\{k \in [\alpha_n, \beta_n] : \sqrt{(x_k - x)^2 + (c_k - c)^2} \geq \varepsilon\right\} \mid = 0.$$

The set of all multiset sequences which are $\alpha\beta$-statistically convergent is denoted by $S^m_{\alpha\beta}$.

Theorem 2.3. *Let (α, β) be a pair of sequences of positive numbers satisfying the conditions of Definition 2.1. Let $H = \{x_n|c_n\}$ be a multiset sequence converging to a multiset $\{x|c\}$. Then the multiset $H = \{x_n|c_n\}$ is $\alpha\beta$-statistically convergent to $\{x|c\}$.*

Proof. Since the multiset sequence $H = \{x_n|c_n\}$ is convergent to the multiset $\{x|c\}$, for $\varepsilon > 0$ there exists a natural number $n_0 \in \mathbb{N}$ such that $d_M(x_n|c_n, x|c) < \varepsilon$ for all $n \geq n_0$. So, the cardinality $|\{k \in \mathbb{N} : d_M(x_n|c_n, x|c) \geq \varepsilon\}| = d$, is finite. So,

$$\frac{1}{(\beta_n - \alpha_n + 1)} |\{k \in [\alpha_n, \beta_n] : \sqrt{(x_k - x)^2 + (c_k - c)^2} \geq \varepsilon\}| \leq \frac{d}{(\beta_n - \alpha_n + 1)} \to 0$$

as $n \to \infty$. So, the multiset sequence $H = \{x_n|c_n\}$ is $\alpha\beta$-statistically convergent to $\{x|c\}$. □

The converse of Theorem 2.3 is not true which follows from the next example.

Example 2.4. Let us define a multiset sequence $H = \{x_n|c_n\}$ by

$$x_n = \begin{cases} n, \text{ if } n = p^2 \text{ for some } p \in \mathbb{N}; \\ 3, \textit{ otherwise} \end{cases} \tag{1}$$

and

$$c_n = \begin{cases} 4, \text{ if } n = p^2 \text{ for some } p \in \mathbb{N}; \\ 6, \textit{ otherwise.} \end{cases} \tag{2}$$

Now, Let $\alpha_n = 1$ and $\beta_n = n^2$ for all $n \in \mathbb{N}$. Then,

$$\frac{1}{(\beta_n - \alpha_n + 1)} |\{k \in [\alpha_n, \beta_n] : \sqrt{(x_k - 3)^2 + (c_k - 6)^2} \geq \varepsilon\}| \leq \frac{n}{n^2} \to 0$$

as $n \to \infty$. So, in this case, the multiset sequence $H = \{x_n|c_n\}$ is $\alpha\beta$-statistically convergent to $\{3|6\}$.

Now, we introduce the notion of $\alpha\beta$-strongly summability for multiset sequences.

Definition 2.5. Let $H = \{x_n|c_n\}$ be a real valued multiset sequence. The sequence H is said to be $\alpha\beta$-strongly summable to the multiset $\{x|c\}$ if

$$\lim_{n\to\infty} \frac{1}{(\beta_n - \alpha_n + 1)} \sum_{k \in [\alpha_n, \beta_n]} \sqrt{(x_k - x)^2 + (c_k - c)^2} = 0.$$

The set of all multiset sequences which are $\alpha\beta$-strongly summable is denoted by $W^m_{\alpha\beta}$.

Now, we present an example of a multiset sequence which is $\alpha\beta$-strongly summable to some multiset.

Example 2.6. Let us define a multiset sequence $H = \{x_n|c_n\}$ by

$$x_n = \begin{cases} 7, \text{ if } n = p^2 \text{ for some } p \in \mathbb{N}; \\ 3, \textit{ otherwise} \end{cases} \tag{3}$$

and

$$c_n = \begin{cases} 4, \text{ if } n = p^2 \text{ for some } p \in \mathbb{N}; \\ 6, \textit{ otherwise.} \end{cases} \tag{4}$$

Let $\alpha_n = 1$ and $\beta_n = n^2$ for all $n \in \mathbb{N}$. Now,

$$\frac{1}{(\beta_n - \alpha_n + 1)} \sum_{k \in [\alpha_n, \beta_n]} \sqrt{(x_k - 3)^2 + (c_k - 6)^2} = \frac{1}{n} \sqrt{20} \to 0$$

as $n \to \infty$. So, in this case the multiset sequence H is $\alpha\beta$-strongly summable to $\{3|6\}$.

Theorem 2.7. *A multiset sequence $H = \{x_n|c_n\}$ is $\alpha\beta$-statistically convergent to a unique limit.*

Proof. If possible, let the multiset sequence $H = \{x_n|c_n\}$ be $\alpha\beta$-statistically convergent to multisets $\{x|c\}$ and $\{x'|c'\}$ where $x \neq x'$. Let $|x - x'| = d > 0$. Now,

$$\begin{aligned} \sqrt{(x - x')^2 + (c - c')^2} &= \sqrt{\{(x - x_k) + (x_k - x')\}^2 + \{(c - c_k) + (c_k - c')\}^2} \\ &\leq \sqrt{(x_k - x)^2 + (c_k - c)^2} + \sqrt{(x_k - x')^2 + (c_k - c')^2}. \end{aligned}$$

So,

$$\begin{aligned} &\frac{1}{(\beta_n - \alpha_n + 1)} |\{k \in [\alpha_n, \beta_n] : \sqrt{(x - x')^2 + (c - c')^2} \geq d\}| \\ &\leq \frac{1}{(\beta_n - \alpha_n + 1)} |\{k \in [\alpha_n, \beta_n] : \sqrt{(x_k - x)^2 + (c_k - c)^2} \geq \frac{d}{2}\}| \\ &+ \frac{1}{(\beta_n - \alpha_n + 1)} |\{k \in [\alpha_n, \beta_n] : \sqrt{(x_k - x')^2 + (c_k - c')^2} \geq \frac{d}{2}\}|. \end{aligned}$$

This shows that

$$\frac{1}{(\beta_n - \alpha_n + 1)} |\{k \in [\alpha_n, \beta_n] : \sqrt{(x - x')^2 + (c - c')^2} \geq d\}| \to 0 \text{ as } n \to \infty,$$

a contradiction. Now, we suppose the case where $c \neq c'$. Let $|c - c'| = d' > 0$. So,

$$\begin{aligned} &\frac{1}{(\beta_n - \alpha_n + 1)} |\{k \in [\alpha_n, \beta_n] : \sqrt{(x - x')^2 + (c - c')^2} \geq d'\}| \\ &\leq \frac{1}{(\beta_n - \alpha_n + 1)} |\{k \in [\alpha_n, \beta_n] : \sqrt{(x_k - x)^2 + (c_k - c)^2} \geq \frac{d'}{2}\}| \\ &+ \frac{1}{(\beta_n - \alpha_n + 1)} |\{k \in [\alpha_n, \beta_n] : \sqrt{(x_k - x')^2 + (c_k - c')^2} \geq \frac{d'}{2}\}|. \end{aligned}$$

This shows that

$$\frac{1}{(\beta_n - \alpha_n + 1)} |\{k \in [\alpha_n, \beta_n] : \sqrt{(x - x')^2 + (c - c')^2} \geq d'\}| \to 0 \text{ as } n \to \infty,$$

a contradiction. So, the limit is unique. □

Theorem 2.8. *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences satisfying three conditions in Definition 2.1. Let $H = \{x_n|c_n\}$ be a multiset sequence. If $H \in W^m_{\alpha\beta}$ then $H \in S^m_{\alpha\beta}$.*

Proof. Let the multiset sequence $H = \{x_n|c_n\}$ be $\alpha\beta$-strongly summable to the multiset $\{x|c\}$. Then

$$\lim_{n\to\infty} \frac{1}{(\beta_n - \alpha_n + 1)} \sum_{k\in[\alpha_n,\beta_n]} \sqrt{(x_k - x)^2 + (c_k - c)^2} = 0.$$

Now,

$$\begin{aligned}
\sum_{k\in[\alpha_n,\beta_n]} \sqrt{(x_k - x)^2 + (c_k - c)^2} &\geq \sum_{k\in[\alpha_n,\beta_n],\ \sqrt{(x_k-x)^2+(c_k-c)^2}\geq\varepsilon} \sqrt{(x_k - x)^2 + (c_k - c)^2} \\
&\geq \varepsilon|\{k \in [\alpha_n, \beta_n] : \sqrt{(x_k - x)^2 + (c_k - c)^2} \geq \varepsilon\}| \\
\implies \frac{1}{(\beta_n - \alpha_n + 1)} &\sum_{k\in[\alpha_n,\beta_n]} \sqrt{(x_k - x)^2 + (c_k - c)^2} \geq \\
\frac{\varepsilon}{(\beta_n - \alpha_n + 1)}&|\{k \in [\alpha_n, \beta_n] : \sqrt{(x_k - x)^2 + (c_k - c)^2} \geq \varepsilon\}|.
\end{aligned}$$

This shows that

$$\lim_{n\to\infty} \frac{1}{(\beta_n - \alpha_n + 1)} \mid \left\{k \in [\alpha_n, \beta_n] : \sqrt{(x_k - x)^2 + (c_k - c)^2} \geq \varepsilon\right\} \mid = 0.$$

So, $H \in S^m_{\alpha\beta}$. □

Corollary 2.9. *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences satisfying three conditions in Definition 2.1. Let $H = \{x_n|c_n\}$ be a multiset sequence. If $H \in W^m_{\alpha\beta}$ then the limit of H is unique.*

Proof. Proof follows from Theorem 2.7 and Theorem 2.8. □

Now, we demonstrate an example of a multiset sequence which is statistically convergent to some multiset but is not $\alpha\beta$-statistically convergent for some pair (α, β).

Example 2.10. Let us define a multiset sequence $H = \{x_n|c_n\}$ by

$$x_j = \begin{cases} j^2, & \text{if } j \in [p!, p! + p] \cap \mathbb{N} \text{ for some } p \in \mathbb{N}; \\ 3, & \textit{otherwise} \end{cases} \tag{5}$$

and

$$c_j = \begin{cases} 4, & \text{if } j \in [p!, p! + p] \cap \mathbb{N} \text{ for some } p \in \mathbb{N}; \\ 6, & \textit{otherwise.} \end{cases} \tag{6}$$

Let $\alpha_n = n!$ and $\beta_n = (n! + n)$ for all $n \in \mathbb{N}$. Then the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences satisfying three conditions in Definition 2.1. Now, $\{k \in \mathbb{N} : \sqrt{(x_k - 3)^2 + (c_k - 6)^2} \geq \varepsilon\} = \bigcup_{n\geq1} A_n$ where $A_n = [n!, n! + n]$. Also, the natural density of the set $A = \bigcup_{n\geq1} A_n$ is zero since

$$\lim_{n\to\infty} \frac{|\{k \leq n : k \in A\}|}{n} \leq \limsup \frac{|\{k \leq n : k \in A\}|}{n} \leq \lim_{n\to\infty} \frac{1 + 2 + \cdots + n}{n! + n} = 0.$$

So,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : \sqrt{(x_k - 3)^2 + (c_k - 6)^2} \geq \varepsilon\}| = 0.$$

This shows that the multiset sequence is statistically convergent to $\{3|6\}$. But, for $0 < \varepsilon < 1$,

$$\lim_{n\to\infty} \frac{1}{(\beta_n - \alpha_n + 1)} \mid \left\{k \in [\alpha_n, \beta_n] : \sqrt{(x_k - 3)^2 + (c_k - 6)^2} \geq \varepsilon\right\} \mid = 1 \neq 0.$$

So, H is not $\alpha\beta$-statistically convergent to the multiset $\{3|6\}$ for $\alpha_n = n!$ and $\beta_n = (n! + n)$.

We recall the notion of boundedness and statistical boundededness of a multiset sequence from [3] as follows.

Definition 2.11. ([3]) A multiset sequence $H = \{x_n|c_n\}$ is said to be bounded if there exists a non-negetive number B such that $\sqrt{x_k^2 + (c_k - 1)^2} \leq B$ for all $k \in \mathbb{N}$.

Definition 2.12. ([3]) A multiset sequence $H = \{x_n|c_n\}$ is said to be statistically bounded if there exists a non-negetive number B such that

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : \sqrt{x_k^2 + (c_k - 1)^2} \geq B\}| = 0.$$

Theorem 2.13. *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences satisfying three conditions in Definition 2.1. Let $H = \{x_n|c_n\}$ be a multiset sequence. If H is bounded and $H \in S^m_{\alpha\beta}$, then $H \in W^m_{\alpha\beta}$.*

Proof. As the multiset sequence $H = \{x_n|c_n\}$ is bounded so, there exists a non-negetive number M such that $\sqrt{x_k^2 + (c_k - 1)^2} \leq M$ for all $k \in \mathbb{N}$. If $M = 0$ then all the terms of the multiset sequence will be $0|1$ and in this case, the sequence is $\alpha\beta$-strongly summable to the multiset $\{0|1\}$. So, we assume that $M > 0$. Let $\varepsilon > 0$. Suppose that the multiset sequence is $\alpha\beta$-statistically convergent to $x|c$. First of all let $x > 0$. Then we have

$$\sqrt{(x_k - x)^2 + (c_k - c)^2} \leq \sqrt{x_k^2 + (c_k - 1)^2} \leq M \text{ for all } k \in \mathbb{N}.$$

Now,

$$\begin{aligned} \frac{1}{(\beta_n - \alpha_n + 1)} \sum_{k\in[\alpha_n,\beta_n]} d_M(x_k|c_k, x|c) &= \frac{1}{(\beta_n - \alpha_n + 1)} \sum_{k\in[\alpha_n,\beta_n], d_M(x_k|c_k,x|c)\geq\frac{\varepsilon}{2}} d_M(x_k|c_k, x|c) \\ &+ \frac{1}{(\beta_n - \alpha_n + 1)} \sum_{k\in[\alpha_n,\beta_n], d_M(x_k|c_k,x|c)<\frac{\varepsilon}{2}} d_M(x_k|c_k, x|c). \end{aligned} \tag{7}$$

So,

$$\frac{1}{(\beta_n - \alpha_n + 1)} \sum_{k\in[\alpha_n,\beta_n]} d_M(x_k|c_k, x|c) \leq \frac{M}{(\beta_n - \alpha_n + 1)} |\{k \in [\alpha_n, \beta_n] : d_M(x_k|c_k, x|c) \geq \frac{\varepsilon}{2}\}| + \frac{\varepsilon}{2}.$$

Now, as $H \in S^m_{\alpha\beta}$, so, the first sequence on the right hand side goes to zero as $n \to \infty$ and this implies that $H \in W^m_{\alpha\beta}$ as

$$\lim_{n\to\infty} \frac{1}{(\beta_n - \alpha_n + 1)} \sum_{k\in[\alpha_n,\beta_n]} d_M(x_k|c_k, x|c) = 0.$$

Now, if $x = -l$ where $l > 0$ then we have $\sqrt{(x_k - x)^2 + (c_k - c)^2} = \sqrt{(x_k + l)^2 + (c_k - c)^2} \leq \sqrt{x_k^2 + (c_k - 1)^2} + l \leq (M + l)$ for all $k \in \mathbb{N}$. So, from equation 7, we have

$$\frac{1}{(\beta_n - \alpha_n + 1)} \sum_{k\in[\alpha_n,\beta_n]} d_M(x_k|c_k, x|c) \leq \frac{(M + l)}{(\beta_n - \alpha_n + 1)} |\{k \in [\alpha_n, \beta_n] : d_M(x_k|c_k, x|c) \geq \frac{\varepsilon}{2}\}| + \frac{\varepsilon}{2}.$$

Similarly, as above we can show that

$$\lim_{n\to\infty}\frac{1}{(\beta_n-\alpha_n+1)}\sum_{k\in[\alpha_n,\beta_n]} d_M(x_k|c_k, x|c) = 0.$$

This shows that $H \in W^m_{\alpha\beta}$. $\square$

Now, we introduce the notion of $\alpha\beta$-statistical boundedness of a multiset sequence as follows.

Definition 2.14. Let (α, β) be a pair of sequences satisfying three conditions in Definition 2.1 and $H = \{x_n|c_n\}$ be a multiset sequence. Then H is said to be $\alpha\beta$-statistically bounded if there exists a non-negetive number M such that

$$\lim_{n\to\infty}\frac{1}{(\beta_n-\alpha_n+1)}|\{k \in [\alpha_n, \beta_n] : \sqrt{x_k^2 + (c_k-1)^2} \geq M\}| = 0.$$

From Definition 2.14, it is clear that $\alpha\beta$-statistical boundedness is more general than the statistical boundedness introduced in [3].

Example 2.15. Let us define a multiset sequence $H = \{x_n|c_n\}$ by

$$x_j = \begin{cases} \frac{1}{2}, \text{ if } j \in [p!, p!+p] \cap \mathbb{N} \text{ for some } p \in \mathbb{N}; \\ 3, \textit{ otherwise} \end{cases} \tag{8}$$

and

$$c_j = \begin{cases} 4, \text{ if } j \in [p!, p!+p] \text{ for some } p \in \mathbb{N}; \\ 6, \textit{ otherwise.} \end{cases} \tag{9}$$

Let $\alpha_n = n!$ and $\beta_n = n! + n$ for all $n \in \mathbb{N}$.

In this case, for some $\varepsilon > 0$,

$$\lim_{n\to\infty}\frac{1}{(\beta_n-\alpha_n+1)}|\{k \in [\alpha_n, \beta_n] : \sqrt{x_k^2 + (c_k-1)^2} \geq \varepsilon\}| = 1 \neq 0.$$

If we choose $M = 4$ then

$$\lim_{n\to\infty}\frac{1}{(\beta_n-\alpha_n+1)}|\{k \in [\alpha_n, \beta_n] : \sqrt{x_k^2 + (c_k-1)^2} \geq M\}| = 0.$$

So, the multiset sequence H is $\alpha\beta$-statistically bounded with $M > 0$.

Theorem 2.16. *Let (α, β) and (α', β') be two pairs of sequences satisfying the conditions in Definition 2.1. Let $\alpha_n \leq \alpha'_n \leq \beta'_n \leq \beta_n$ for all $n \in \mathbb{N}$ and there exists a sequence $\{\delta_n\}$ of positive numbers such that $\lim_{n\to\infty}\frac{1}{\delta_n}$ is non-zero finite and $(\beta'_n - \alpha'_n + 1)\delta_n \geq (\beta_n - \alpha_n + 1)$ for all n. Let $H = \{x_n|c_n\}$ be a multiset sequence. If $H \in S^m_{\alpha\beta}$, then $H \in S^m_{\alpha'\beta'}$.*

Proof. Since $[\alpha'_n, \beta'_n] \subseteq [\alpha_n, \beta_n]$, so,

$$|\{k \in [\alpha_n, \beta_n] : \sqrt{(x_k-x)^2 + (c_k-c)^2} \geq \varepsilon\}| \geq |\{k \in [\alpha'_n, \beta'_n] : \sqrt{(x_k-x)^2 + (c_k-c)^2} \geq \varepsilon\}|$$

$$\Longrightarrow \frac{1}{(\beta_n-\alpha_n+1)}|\{k \in [\alpha_n, \beta_n] : \sqrt{(x_k-x)^2 + (c_k-c)^2} \geq \varepsilon\}| \geq$$

$$\frac{1}{(\beta_n - \alpha_n + 1)}|\{k \in [\alpha'_n, \beta'_n] : \sqrt{(x_k - x)^2 + (c_k - c)^2} \geq \varepsilon\}|$$

$$\implies \frac{1}{(\beta_n - \alpha_n + 1)}|\{k \in [\alpha_n, \beta_n] : \sqrt{(x_k - x)^2 + (c_k - c)^2} \geq \varepsilon\}| \geq$$

$$\frac{1}{\delta_n(\beta'_n - \alpha'_n + 1)}|\{k \in [\alpha'_n, \beta'_n] : \sqrt{(x_k - x)^2 + (c_k - c)^2} \geq \varepsilon\}|.$$

Now, taking the limit both sides, we have $H \in S^m_{\alpha'\beta'}$. □

In our next result, we investigate the relation between $\alpha\beta$-statistical convergence and statistical convergence of sequence of multisets.

Theorem 2.17. *Let (α, β) be a pair of sequences satisfying three conditions in Definition 2.1 and $H = \{x_n|c_n\}$ be a multiset sequence. Let $\liminf \frac{\beta_n}{\alpha_n} > 1$. If $H \in S^m$, then $H \in S^m_{\alpha\beta}$. Here S^m denotes the collection of all multiset sequences which are statistically convergent.*

Proof. Let $\liminf \frac{\beta_n}{\alpha_n} = c > 1$ and $\varepsilon' > 0$ such that $(c - \varepsilon') > 1$. So, there exists $n_0 \in \mathbb{N}$ such that $\frac{\beta_n}{\alpha_n} > (c - \varepsilon')$ for all $n \geq n_0$. Let us choose $n \in \mathbb{N}$ with $n \geq n_0$. let $H = \{x_n|c_n\}$ is statistically convergent to the multiset $\{x|c\}$. Let $t_n = (\beta_n - \alpha_n + 1)$. Now,

$$\begin{aligned}
\frac{1}{\beta_n}|\{k \leq \beta_n : d_M(x_k|c_k, x|c) \geq \varepsilon\}| &\geq \frac{1}{\beta_n}|\{k \in [\alpha_n, \beta_n] : d_M(x_k|c_k, x|c) \geq \varepsilon\}| \\
&= \frac{t_n}{\beta_n t_n}|\{k \in [\alpha_n, \beta_n] : d_M(x_k|c_k, x|c) \geq \varepsilon\}| \\
&> \frac{\delta}{t_n}|\{k \in [\alpha_n, \beta_n] : d_M(x_k|c_k, x|c) \geq \varepsilon\}|
\end{aligned}$$

where $\delta = \frac{c - \varepsilon' - 1}{c - \varepsilon'} > 0$. Now taking the limits at both sides as $n \to \infty$ we have,

$$\lim_{n\to\infty} \frac{1}{(\beta_n - \alpha_n + 1)}|\{k \in [\alpha_n, \beta_n] : \sqrt{(x_k - x)^2 + (c_k - c)^2} \geq \varepsilon\}| = 0.$$

So, $H \in S^m_{\alpha\beta}$. □

Note 2.18. In Example 2.4, we see that the multiset sequence H is $\alpha\beta$-statistically convergent to the multiset $\{3|6\}$ for $\alpha_n = 1$ and $\beta_n = n^2$ for all $n \in \mathbb{N}$. Also, this multiset sequence H is statistically convergent to the multiset $\{3|6\}$. In this case $\liminf \frac{\beta_n}{\alpha_n} > 1$.

Theorem 2.19. *Let (α, β) be a pair of sequences satisfying three conditions in Definition 2.1 and $H = \{x_n|c_n\}$ be a multiset sequence. Let $\liminf \frac{\beta_n}{\alpha_n} > 1$. If $H \in W^m$, then $H \in W^m_{\alpha\beta}$. Here W^m denotes the collection of all multiset sequences which are strongly Cesáro summable.*

Proof. Since, $\liminf \frac{\beta_n}{\alpha_n} > 1$, so, for each $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that $\frac{\beta_n}{\alpha_n} > (1 + \delta)$ for all $n \geq n_0$. Let

$H \in W^m$ and the multiset sequence H is strongly Cesáro summable to $x|c$. Now,

$$\begin{aligned}
\frac{1}{\beta_n}\sum_{i=1}^{\beta_n}\sqrt{(x_i-x)^2+(c_i-c)^2} &\geq \frac{1}{\beta_n}\sum_{i\in[\alpha_n,\beta_n]}\sqrt{(x_i-x)^2+(c_i-c)^2}\\
&= \frac{(\beta_n-\alpha_n)}{\beta_n}.\frac{1}{(\beta_n-\alpha_n)}\sum_{i\in[\alpha_n,\beta_n]}\sqrt{(x_i-x)^2+(c_i-c)^2}\\
&\geq \frac{(\beta_n-\alpha_n)}{\alpha_n}.\frac{\alpha_n}{\beta_n}\frac{1}{(\beta_n-\alpha_n+1)}\sum_{i\in[\alpha_n,\beta_n]}\sqrt{(x_i-x)^2+(c_i-c)^2}\\
&> \frac{\delta}{1+\delta}.\frac{1}{(\beta_n-\alpha_n+1)}\sum_{i\in[\alpha_n,\beta_n]}\sqrt{(x_i-x)^2+(c_i-c)^2}.
\end{aligned}$$

Since $H \in W^m$, so, $\frac{1}{\beta_n}\sum_{i=1}^{\beta_n}\sqrt{(x_i-x)^2+(c_i-c)^2} \to 0$ as $n \to \infty$. We have, $\frac{1}{(\beta_n-\alpha_n+1)}\sum_{i\in[\alpha_n,\beta_n]}\sqrt{(x_i-x)^2+(c_i-c)^2} \to 0$ as $n \to \infty$. This shows that $H \in W^m_{\alpha\beta}$. □

Open problem: Under what conditions, a sequence of multisets which is $\alpha\beta$-statistically convergent to a multiset, is statistically convergent?

We recall from [3], the definition of the set $m\mathbb{R}$ as

$$m\mathbb{R} = \{x|c : x \in \mathbb{R} \text{ and } c \in \mathbb{N}\}.$$

$m\mathbb{R}$ is a multi-subset of the set of all real numbers $\mathbb{R}$. On the other hand, if $K \subset \mathbb{N}$, then, the $\alpha\beta$ density of K is defined as (see [1])

$$\delta_{\alpha\beta}(K) = \lim_{n\to\infty}\frac{1}{(\beta_n-\alpha_n+1)}|\{k \in [\alpha_n,\beta_n] : k \in K\}|.$$

If we take $\alpha_n = 1$ and $\beta_n = n$ for all $n \in \mathbb{N}$ then $\delta_{\alpha\beta}(K)$ coincides with the natural density of K.

Let $H = \{x_n|c_n\}$ be a multiset sequence. Let us introduce two sets corresponding to the sequences $\alpha = (\alpha_n)$ and $\beta = (\beta_n)$ as follows.

$$B_{\alpha\beta,H} = \{x|c \in m\mathbb{R} : \delta_{\alpha\beta}(\{k \in \mathbb{N} : \sqrt{x_k^2+(c_k-1)^2} > \sqrt{x^2+(c-1)^2}\}) \neq 0\}.$$

$$A_{\alpha\beta,H} = \{x|c \in m\mathbb{R} : \delta_{\alpha\beta}(\{k \in \mathbb{N} : \sqrt{x_k^2+(c_k-1)^2} < \sqrt{x^2+(c-1)^2}\}) \neq 0\}.$$

We define the supremum of the set $B_{\alpha\beta,H}$ as $x|c$ where c is the largest multiplicity in $B_{\alpha\beta,H}$ satisfying the condition $c \leq \max\{c' : c' \in H\}$ and x is the supremum of different sets of real numbers having multiplicity c. Similarly, we define the infimum of the set $A_{\alpha\beta,H}$ as $x|c$ where c is the smallest multiplicity in $A_{\alpha\beta,H}$ satisfying the condition $c \geq \max\{c' : c' \in H\}$ and x is the infimum of different sets of real numbers having multiplicity c.

Definition 2.20. Let $H = \{x_n|c_n\}$ be a multiset sequence and $\alpha = (\alpha_n), \beta = (\beta_n)$ be two sequences satisfying the conditions in Definition 2.1. Then we define,

$$\alpha\beta - st - limsupH = \begin{cases}\sup B_{\alpha\beta,H} \text{ if } B_{\alpha\beta,H} \neq \emptyset;\\ -\infty \text{ if } B_{\alpha\beta,H} = \emptyset\end{cases} \tag{10}$$

and

$$\alpha\beta - st - liminfH = \begin{cases}\inf A_{\alpha\beta,H} \text{ if } A_{\alpha\beta,H} \neq \emptyset;\\ \infty \text{ if } A_{\alpha\beta,H} = \emptyset.\end{cases} \tag{11}$$

In our next example, we use the notation $(a, b)|c$ to denote that for every real number $p \in (a, b)$, the multiplicity of p is c.

Example 2.21. Let us define a multiset sequence $H = \{x_n|c_n\}$ by

$$x_n = \begin{cases} 7, \text{ if } n = p^2 \text{ for some } p \in \mathbb{N}; \\ 3, \textit{ otherwise} \end{cases} \tag{12}$$

and

$$c_n = \begin{cases} 4, \text{ if } n = p^2 \text{ for some } p \in \mathbb{N}; \\ 6, \textit{ otherwise.} \end{cases} \tag{13}$$

Let $\alpha_n = 1$ and $\beta_n = n^2$ for all $n \in \mathbb{N}$. For this sequence, firstly we calculate the set $B_{\alpha\beta,H}$. Let $x|c \in B_{\alpha\beta,H}$. So, $\delta_{\alpha\beta}(\{k \in \mathbb{N} : \sqrt{x_k^2 + (c_k - 1)^2} > \sqrt{x^2 + (c-1)^2}\}) \neq 0\}$. Here, in this case $\{k \in \mathbb{N} : \sqrt{x_k^2 + (c_k - 1)^2} > \sqrt{x^2 + (c-1)^2}\} \neq \{k \in \mathbb{N} : k = p^2$ for some $p\}$ because in that case the $\alpha\beta$-density of the set become zero. So, $k \neq p^2$ for any $p \in \mathbb{N}$. In this case, $\sqrt{x^2 + (c-1)^2} < \sqrt{x_k^2 + (c_k - 1)^2}$ will imply $\sqrt{x^2 + (c-1)^2} < \sqrt{34}$. So, $x^2 < 17$ or $(c-1)^2 < 17$. In the first case if $x^2 < 17$ and $(c-1)^2 \geq 17$ then $x \in (-4.12, 4.12)$ and $c \geq 5.12$. Also, as c is the multiplicity so, c cannot be negetive and in this case c cannot be greater than 6. In this case $\{x|c\} = (-4.12, 4.12)|6$. In the second case, suppose $x^2 < 17$ and $(c-1)^2 < 17$. In this case, $x \in (-4.12, 4.12)$ and $1 \leq c \leq 5$. In the third case, let $x^2 \geq 17$ and $(c-1)^2 < 17$. So, $1 \leq c \leq 5$. Also, in this case, $x \in [4.12, 5.83)$ or $x \leq -4.12$. Now, we can see that

$$B_{\alpha\beta,H} = \{(-4.12, 4.12)|1, (-4.12, 4.12)|2, \ldots, (-4.12, 4.12)|6,$$

$$[4.12, 5.83)|1, \ldots, [4.12, 5.83)|5, (-\infty, -4.12]|1, (-\infty, -4.12]|2, \ldots, (-\infty, -4.12]|5\}.$$

So, $\alpha\beta - st - limsupH = 4.12|6$.

Now, if the set $\{k \in \mathbb{N} : \sqrt{x_k^2 + (c_k - 1)^2} < \sqrt{x^2 + (c-1)^2}\} = \{k \in \mathbb{N} : k \neq p^2$ for some $p\}$, we consider the relation, $\sqrt{x^2 + (c-1)^2} > \sqrt{34}$ for the set $A_{\alpha\beta,H}$. So, $x^2 > 17$ or $(c-1)^2 > 17$. In our first case, let $x^2 > 17$ and $(c-1)^2 > 17$. In this case, $x > 4.12$ or $x < -4.12$ and $c > 5.12$. Here we ignore the case where c is negetive. In our second case, let $x^2 > 17$ and $(c-1)^2 \leq 17$. So, $x > 4.12$ or $x < -4.12$ and $1 \leq c \leq 5$. In our third case, let $x^2 \leq 17$ and $(c-1)^2 > 17$. So, in this case, $x \in [-4.12, 4.12]$ and $c \geq 6$. So,

$$A_{\alpha\beta,H} = \{(4.12, \infty)|1, (4.12, \infty)|2, \ldots, (-\infty, -4.12)|1, (-\infty, -4.12)|2, \ldots,$$

$$[-4.12, 4.12]|6, \ldots\}.$$

In this case 6 is the smallest such that $6 \geq \max\{c' : c' \in H\}$. Also, the infimum is $-\infty$ of all real numbers having multiplicity 6 So, $\alpha\beta - st - liminfH = -\infty$.

On the other hand, if the set $\{k \in \mathbb{N} : \sqrt{x_k^2 + (c_k - 1)^2} < \sqrt{x^2 + (c-1)^2}\}$ contains natural numbers $k = p^2$ as well as $k \neq p^2$ then we consider the relation, $\sqrt{x^2 + (c-1)^2} > \sqrt{58}$ for the set $A_{\alpha\beta,H}$. Similarly, as above we can see that in this case

$$A_{\alpha\beta,H} = \{(5.39, \infty)|1, (5.39, \infty)|2, \ldots, (5.39, \infty)|5, (5.39, \infty)|7, \ldots,$$

$$(-\infty, -5.39)|1, (-\infty, -5.39)|2, \ldots, (-\infty, -5.39)|5, (-\infty, -5.39)|7, \ldots, [-5.39, 5.39]|7, \ldots\}.$$

In this case, $\alpha\beta - st - liminfH = -\infty$.

Lemma 2.22. *Let $H = \{x_n|c_n\}$ be a multiset sequence and $\alpha = (\alpha_n), \beta = (\beta_n)$ be two sequences satisfying the conditions in Definition 2.1. If $x|c = \alpha\beta - st - limsupH$ then for all $\varepsilon > 0$,*

$$\delta_{\alpha\beta}(\{k \in \mathbb{N} : \sqrt{x_k^2 + (c_k - 1)^2} > \sqrt{(x - \varepsilon)^2 + (c - 1)^2}\}) \neq 0$$

and

$$\delta_{\alpha\beta}(\{k \in \mathbb{N} : \sqrt{x_k^2 + (c_k - 1)^2} > \sqrt{(x + \varepsilon)^2 + (c - 1)^2}\}) = 0.$$

Proof. Let $\varepsilon > 0$. So, there exists $x^* \in \mathbb{R}$ such that $x - \varepsilon < x^*$ and $x^*|c \in B_{\alpha\beta,H}$. Since $x^*|c \in B_{\alpha\beta,H}$, so,

$$\delta_{\alpha\beta}(\{k \in \mathbb{N} : \sqrt{x_k^2 + (c_k - 1)^2} > \sqrt{(x^*)^2 + (c - 1)^2}\}) \neq 0.$$

Since $\{k \in \mathbb{N} : \sqrt{x_k^2 + (c_k - 1)^2} > \sqrt{(x^*)^2 + (c - 1)^2}\} \subset \{k \in \mathbb{N} : \sqrt{x_k^2 + (c_k - 1)^2} > \sqrt{(x - \varepsilon)^2 + (c - 1)^2}\}$, so,

$$\delta_{\alpha\beta}(\{k \in \mathbb{N} : \sqrt{x_k^2 + (c_k - 1)^2} > \sqrt{(x - \varepsilon)^2 + (c - 1)^2}\}) \neq 0.$$

The other part is straightforward as x is the supremum of different sets of real numbers having the multiplicity c. So for any positive ε, $(x + \varepsilon)|c \notin B_{\alpha\beta,H}$. □

Lemma 2.23. *Let $H = \{x_n|c_n\}$ be a multiset sequence and $\alpha = (\alpha_n), \beta = (\beta_n)$ be two sequences satisfying the conditions in Definition 2.1. If $x|c = \alpha\beta - st - liminfH$ then for all $\varepsilon > 0$,*

$$\delta_{\alpha\beta}(\{k \in \mathbb{N} : \sqrt{x_k^2 + (c_k - 1)^2} < \sqrt{(x + \varepsilon)^2 + (c - 1)^2}\}) \neq 0$$

and

$$\delta_{\alpha\beta}(\{k \in \mathbb{N} : \sqrt{x_k^2 + (c_k - 1)^2} < \sqrt{(x - \varepsilon)^2 + (c - 1)^2}\}) = 0.$$

Proof. Proof is straightforward, so omitted. □

For, $x_1|c_1, x_2|c_2 \in m\mathbb{R}$ by $x_1|c_1 \leq x_2|c_2$ will mean $\sqrt{x_1^2 + (c_1 - 1)^2} \leq \sqrt{x_2^2 + (c_2 - 1)^2}$. In our next results we mention $\alpha = (\alpha_n), \beta = (\beta_n)$ be two sequences to denote the sequences satisfying the conditions in Definition 2.1.

Theorem 2.24. *Let $H = \{x_n|c_n\}$ be a multiset sequence and $\alpha = (\alpha_n), \beta = (\beta_n)$ be two sequences. Then $\alpha\beta - st - liminfH \leq \alpha\beta - st - limsupH$.*

Proof. If $\alpha\beta - st - limsupH = \infty$ then there is nothing to prove. Suppose $\alpha\beta - st - limsupH = -\infty$. Then $B_{\alpha\beta,H} = \emptyset$. So, for all $x|c \in m\mathbb{R}$ we have

$$\delta_{\alpha\beta}(\{k \in \mathbb{N} : \sqrt{x_k^2 + (c_k - 1)^2} > \sqrt{x^2 + (c - 1)^2}\}) = 0,$$

$$\Rightarrow \delta_{\alpha\beta}(\{k \in \mathbb{N} : \sqrt{x_k^2 + (c_k - 1)^2} < \sqrt{x^2 + (c - 1)^2}\}) = 1.$$

This shows that for all $x|c \in m\mathbb{R}$ we have $x|c \in A_{\alpha\beta,H}$. So, $\alpha\beta - st - liminfH = -\infty$. Now, we assume that $\alpha\beta - st - limsupH = x|c$ and $\alpha\beta - st - liminfH = x'|c'$. We have for any $\varepsilon > 0$,

$$\delta_{\alpha\beta}(\{k \in \mathbb{N} : \sqrt{x_k^2 + (c_k - 1)^2} > \sqrt{(x + \varepsilon)^2 + (c - 1)^2}\}) = 0,$$

$$\Rightarrow \delta_{\alpha\beta}(\{k \in \mathbb{N} : \sqrt{x_k^2 + (c_k - 1)^2} < \sqrt{(x + \varepsilon)^2 + (c - 1)^2}\}) = 0.$$

So, $x'|c' \leq x + \varepsilon|c$. This implies $\sqrt{x'^2 + (c' - 1)^2} \leq \sqrt{(x + \varepsilon)^2 + (c - 1)^2}$. Since this is true for any $\varepsilon > 0$, we have, $\sqrt{x'^2 + (c' - 1)^2} \leq \sqrt{x^2 + (c - 1)^2}$. So, $\alpha\beta - st - liminfH \leq \alpha\beta - st - limsupH$. □

Theorem 2.25. *Let $H = \{x_n|c_n\}$ be a multiset sequence and $\alpha = (\alpha_n), \beta = (\beta_n)$ be two sequences. Then H is $\alpha\beta$-statistically convergent if and only if $\alpha\beta - st - liminf H = \alpha\beta - st - limsup H$.*

Proof. Let $\alpha\beta - st - limsup H = l_1|c_1$ and $\alpha\beta - st - liminf H = l_2|c_2$. First of all, let the multisequence H be $\alpha\beta$-statistically convergent to $x|c$. Then for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{(\beta_n - \alpha_n + 1)} \left| \left\{ k \in [\alpha_n, \beta_n] : \sqrt{(x_k - x)^2 + (c_k - c)^2} \geq \varepsilon \right\} \right| = 0;$$

$$\Rightarrow \delta_{\alpha\beta}(\{k \in \mathbb{N} : |x_k - x| \leq \varepsilon, |c_k - c| \leq \varepsilon\}) = 1;$$

$$\Rightarrow \delta_{\alpha\beta}(\{k \in \mathbb{N} : |x_k - x| > \varepsilon, |c_k - c| > \varepsilon\}) = 0;$$

$$\Rightarrow \delta_{\alpha\beta}(\{k \in \mathbb{N} : x_k > x + \varepsilon, (c_k - 1) > c - 1 + \varepsilon\}) = 0;$$

$$\Rightarrow \delta_{\alpha\beta}(\{k \in \mathbb{N} : \sqrt{x_k^2 + (c_k - 1)^2} > \sqrt{(x + \varepsilon)^2 + (c + \varepsilon - 1)^2}\}) = 0.$$

So, $x + \varepsilon|c + \varepsilon \notin B_{\alpha\beta,H}$. We have,

$$\sqrt{l_1^2 + (c_1 - 1)^2} < \sqrt{(x + \varepsilon)^2 + (c + \varepsilon - 1)^2}.$$

Since this is true for all $\varepsilon > 0$, so, $\alpha\beta - st - limsup H \leq x|c$. Similarly, we can show that $x - \varepsilon|c - \varepsilon \notin A_{\alpha\beta,H}$ and this will imply

$$\sqrt{(x - \varepsilon)^2 + (c - \varepsilon - 1)^2} < \sqrt{l_2^2 + (c_2 - 1)^2}.$$

So, $x|c \leq \alpha\beta - st - liminf H$. From Theorem 2.24, we have $\alpha\beta - st - liminf H = \alpha\beta - st - limsup H$. Now, on the other hand, let $\alpha\beta - st - liminf H = \alpha\beta - st - limsup H = x|c$. Let $\varepsilon > 0$. So, from Lemma 2.22 and Lemma 2.23 we have,

$$\delta_{\alpha\beta}(\{k \in \mathbb{N} : \sqrt{x_k^2 + (c_k - 1)^2} > \sqrt{(x + \varepsilon)^2 + (c - 1)^2}\}) = 0$$

and

$$\delta_{\alpha\beta}(\{k \in \mathbb{N} : \sqrt{x_k^2 + (c_k - 1)^2} < \sqrt{(x - \varepsilon)^2 + (c - 1)^2}\}) = 0.$$

Now, $\sqrt{(x + \varepsilon)^2 + (c - 1)^2} \leq \sqrt{x^2 + (c - 1)^2} + \varepsilon$ and $\sqrt{(x - \varepsilon)^2 + (c - 1)^2} \geq \sqrt{x^2 + (c - 1)^2} - \varepsilon$. So, $\{k \in \mathbb{N} : \sqrt{x_k^2 + (c_k - 1)^2} \geq \sqrt{x^2 + (c - 1)^2} + \varepsilon\} \subseteq \{k \in \mathbb{N} : \sqrt{x_k^2 + (c_k - 1)^2} \geq \sqrt{(x + \varepsilon)^2 + (c - 1)^2}\}$ and $\{k \in \mathbb{N} : \sqrt{x_k^2 + (c_k - 1)^2} \leq \sqrt{x^2 + (c - 1)^2} - \varepsilon\} \subseteq \{k \in \mathbb{N} : \sqrt{x_k^2 + (c_k - 1)^2} \leq \sqrt{(x - \varepsilon)^2 + (c - 1)^2}\}$. This implies that

$$\delta_{\alpha\beta}(\{k \in \mathbb{N} : \sqrt{x_k^2 + (c_k - 1)^2} \geq \sqrt{x^2 + (c - 1)^2} + \varepsilon\}) = 0$$

and

$$\delta_{\alpha\beta}(\{k \in \mathbb{N} : \sqrt{x_k^2 + (c_k - 1)^2} \leq \sqrt{x^2 + (c - 1)^2} - \varepsilon\}) = 0.$$

So,

$$\delta_{\alpha\beta}(\{k \in \mathbb{N} : \sqrt{(x_k - x)^2 + (c_k - c)^2} \geq \varepsilon\}) = 0.$$

So, the multiset sequence is $\alpha\beta$-statistically convergent to $x|c$. □

Theorem 2.26. *$H = \{x_n|c_n\}$ be a multiset sequence which is $\alpha\beta$-statistically convergent to a multiset $\{x|c\}$. Then the multiset H is $\alpha\beta$-statistically bounded.*

Proof. Let the multiset sequence $H = \{x_n|c_n\}$ be $\alpha\beta$-statistically convergent to a multiset $\{x|c\}$. Then from Theorem 2.25, we have $\alpha\beta - st - liminf H = \alpha\beta - st - limsup H = x|c$. So, for $\varepsilon > 0$, we have from Lemma 2.22,

$$\delta_{\alpha\beta}(\{k \in \mathbb{N} : \sqrt{x_k^2 + (c_k - 1)^2} > \sqrt{(x + \varepsilon)^2 + (c - 1)^2}\}) = 0.$$

So, the sequence H is $\alpha\beta$-statistically bounded with $M = \sqrt{(x + \varepsilon)^2 + (c - 1)^2}$. □

3. Conclusion

In the present paper, we introduce on the notion of $\alpha\beta$-statistical convergence for multiset sequences and discussed the relation between $\alpha\beta$-statistical convergence and $\alpha\beta$-strongly summability for multiset sequences. We present several examples to discuss our results. Also, we have defined the concept of $\alpha\beta - st - liminf$ and $\alpha\beta - st - limsup$ for a multiset sequences and present some results on this concept. We pose an open problem in our article for finding conditions under which $\alpha\beta$-statistical convergence (for given two sequences α and β) implies statistical convergence for multiset sequence.

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