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Statistical relative $\mathcal{A}-$summation process for sequences of monotone and sublinear operators on modular spaces

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Abstract. In this paper, we prove Korovkin theorems via statistical relative $\mathcal{A}-$ summation process for monotone and sublinear operators in the setting of modular spaces, which includes, in particular cases, L^p, Orlicz, and Musielak-Orlicz spaces. Furthermore, we introduce a new, more general version with results that bring a new perspective. Finally, we present an important example that satisfies our main theorem and shows that it is strong.

1. Introduction

Korovkin's theorem is of significant importance in the literature ([29], [30]). The power and applicability of this theorem is fundamentally linked to their role in approximating real-valued functions via positive linear operators. Moreover, this theorem has been studied in different convergence methods in different function spaces and many extensions have been obtained. In certain Korovkin theorem, when there is a lack of convergence, summation methods can be quite effective. The concept of the $\mathcal{A}-$summation process was introduced and analyzed by Nishishishiraho on a compact Hausdorff space ([37],[38]). Additionally, the $\mathcal{A}-$summation process has been studied in several function spaces and with different types of convergence ([1], [11], [15], [17], [42]). One of these is modular spaces, which are an important class of function spaces for our study ([3], [10], [12], [14], [19], [26], [27], [28], [33], [34], [35], [36], [39], [43]).

Recently, studies have been carried out on the concept of positivity and linearity, which are the properties of the operator. In the research, it has been seen that the positivity property cannot be dropped; however, the linearity property can be improved. In this framework, Gal-Niculescu extended Korovkin's theorem to monotone and sublinear sequences of operators, contributing significant results to the literature and has attracted considerable interest in the field ([16], [22],[23], [24],[25]).

The motivation of this study is to present a new proof of Korovkin's theorem via the statistical relative $\mathcal{A}-$ summation of monotone and sublinear operators on modular spaces while also providing a more general version of the theorem that brings a new perspective. Additionally, important results of this new approximation are expressed. Finally, a significant example is introduced that confirms our main theorem and illustrates its effectiveness.

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2. Preliminaries

This section reviews some essential definitions and notations related to our study, particularly those associated with statistical convergence, summation process, monotone and sublinear operators, and modular spaces, which will be used in the sequel.

2.1. Statistical convergence and summation process

Let K be a subset of $\mathbb{N}$, the set of natural numbers. Then the natural density of K, denoted by $\delta(K)$, is defined as

$$\delta(K) := \lim_k \frac{1}{k} |\{n \leq k : n \in K\}|,$$

where the vertical bars denote the cardinality of the set.

The real number sequence $x = (x_n)$ is said to be statistically convergent to the number L if for each $\varepsilon > 0$,

$$\lim_k \frac{1}{k} |\{n \leq k : \ |x_n - L| \geq \varepsilon\}| = 0.$$

In this case, we write $(st) - \lim x_n = L$ ([20], [40]).

The concepts of statistical limit superior and limit inferior have been introduced by Fridy and Orhan in [21]. Then, Demirci [13] has generalized these concepts to $A-$statistical limit superior and limit inferior. The real number sequence $x = (x_n)$, the statistical superior limit of x is

$$(st) - \limsup x_n = \begin{cases} \sup M_x^*, & \text{if } M_x^* \neq \emptyset, \\ -\infty, & \text{if } M_x^* = \emptyset, \end{cases}$$

where $M_x^* := \{m^* \in \mathbb{R} : \delta(n : x_n > m^*) \neq 0\}$ and $\emptyset$ denotes the empty set. We note that by $\delta(K) = 0$ we mean either $\delta(K) > 0$ or K fails to have the density. Similarly, the statistical inferior limit of x is

$$(st) - \liminf x_n = \begin{cases} \inf N_x^*, & \text{if } N_x^* \neq \emptyset, \\ \infty, & \text{if } N_x^* = \emptyset, \end{cases}$$

where $N_x^* := \{n^* \in \mathbb{R} : \delta(n : x_n < n^*) \neq 0\}$.

Let's recall some notations related to the summability theory.

Let $\mathcal{A} := \left(A^{(j)}\right) = \left(\left(a_{kn}^{(j)}\right)\right)$ be a sequence of infinite matrices with non-negative entries. For a sequence of real numbers, $x = (x_n)$, the double sequence

$$\mathcal{A}x := \left\{(Ax)_k^j : k, j \in \mathbb{N}\right\}$$

defined by $(Ax)_k^j := \sum_{n=1}^{\infty} a_{kn}^{(j)} x_n$ is called the $\mathcal{A}-$transform of x whenever the series converges for all k and j. A sequence x is said to be $\mathcal{A}-$summable to L if

$$\lim_k \sum_{n=1}^{\infty} a_{kn}^{(j)} x_n = L$$

uniformly in j ([8], [41]). If $A^{(j)} = A$ for a fixed matrix A, then $\mathcal{A}-$summability is the ordinary matrix summability by A. If $a_{kn}^{(j)} = \frac{1}{k}$ for $j \leq n \leq j + k$, $(j = 1, 2, ...)$ and $a_{kn}^{(j)} = 0$ otherwise, then $\mathcal{A}-$summability reduces to almost convergence ([32]).

2.2. Monoton and sublinear operators

$\mathcal{F}(I)$ will be denoted as the vector lattice of all real-valued functions defined on I and equipped with point ordering, where I be a Hausdorff topological space. Let the following be the vector sublattices of $\mathcal{F}(I)$:

$$\begin{aligned} C(I) &= \{f \in \mathcal{F}(I) : f \text{ continuous}\}, \\ C_b(I) &= \{f \in \mathcal{F}(I) : f \text{ continuous and bounded}\}, \end{aligned}$$

and

$$L^p(I) = \{f \in \mathcal{F}(I) : f \text{ is Borel measurable and Lebesgue integrable}\}$$

for $1 \le p < \infty$. On $C_b(I)$, we consider the uniform norm $\|f\| = \sup_{x\in I} |f(x)|$, while on $L^p(I)$, $1 \le p < \infty$, we consider the usual $p-$norm $\|f\|_p = \left(\int_I |f(x)|^p \, dx \right)^{\frac{1}{p}}$. Assume that I and I_0 are two Hausdorff topological spaces and E and F two respectively ordered vector subspaces (or the positive cones) of $C(I)$ and $C(I_0)$ that contain the unity.

An operator $L : E \to F$ is called a weakly nonlinear operator (respectively a weakly nonlinear functional when $F = \mathbb{R}$) if it satisfies the following two conditions:

1. (sublinearity) L is positively homogeneous and subadditive, that is

$$L(\alpha f) = \alpha L(f) \text{ and } f \in E \text{ and } L(f+g) \le L(f) + L(g)$$

for all $f, g \in E$ and $\alpha \ge 0$,

2. (translatability) $L(f + \alpha.1) \le L(f) + \alpha L(1)$ for all $f \in E$ and for all $\alpha \ge 0$, where that is the function $1(x) = 1$ for every x.

Also, in this article we are interested in the operator satisfying the conditions:

(monotonicity) $f \le g$ in E implies $L(f) \le L(g)$,

(subunital property) $L(1) \le 1$.

If E and F are closed vector sublattices of the Banach lattices $C(I)$ and $C(I_0)$, respectively, then every monotone and subadditive operator (functional when $F = \mathbb{R}$) $L : E \to F$ satisfies the inequality

$$|L(f) - L(g)| \le L(|f - g|) \text{ for all } f, g.$$

If an operator (functional when $F = \mathbb{R}$) L is monotone and positively homogeneous, then we necessarily have $L(0) = 0$.

2.3. Modular spaces

Assume that I will be considered as a locally compact Hausdorff topological space. We will denote by $\mathcal{H}(I)$ the space of all real-valued, measurable functions on I which are provided with equality almost everywhere. Also, $C_c(I)$ be the subspace of $C_b(I)$ of all functions with compact support on I.

The functional $\rho : \mathcal{H}(I) \to [0, \infty]$ is said to be modular if it satisfies the following conditions:

(i) $\rho(f) = 0$ iff $f = 0$ a.e. I,

(ii) $\rho(-f) = \rho(f)$ for every $f \in \mathcal{H}(I)$,

(iii) $\rho(\alpha f + \beta g) \le \rho(f) + \rho(g)$ for every $f, g \in \mathcal{H}(I)$ and for every $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$.

Recall that, a modular ρ is $N-$quasi convex if there is constant $N > 1$ such that $\rho(\alpha f + \beta g) \le N\alpha\rho(Nf) + N\beta\rho(Ng)$ for every $f, g \in \mathcal{H}(I)$, $\alpha, \beta \ge 0$, $\alpha + \beta = 1$. We say ρ is convex if $N = 1$. In addition, a modular ρ is $N-$quasi semi-convex if there exists a constant $N \ge 1$ such that $\rho(\alpha f) \le N\alpha\rho(Nf)$ holds for every $f \in \mathcal{H}(I)$ and $\alpha \in (0, 1]$.

Now let us recall I_ρ, which is a vector subspace of $\mathcal{H}(I)$, constructed with the modular ρ,

$$I_\rho := \left\{ f \in \mathcal{H}(I) : \lim_{\lambda \to 0^+} \rho(\lambda f) = 0 \right\}.$$

Also, the space of finite elements of the I_ρ is as follows,

$$I_\rho^* := \left\{ f \in I_\rho : \rho(\lambda f) < \infty \text{ for all } \lambda > 0 \right\}.$$

Modular functionals and modular spaces are discussed in detail in [4], [5], [6], [7], [33], [34], [35]. Now let us be given the following concepts recently expressed by Demirci and Kolay ([14]).

Let (f_n) be a function sequence whose terms belong to I_ρ. Then, (f_n) is said to be statistically modularly convergent relatively to a scale function σ, if there exists a function $\sigma(x)$, called a scale function $\sigma \in \mathcal{H}(I)$, $|\sigma(x)| \neq 0$ such that

$(st) - \lim \rho\left(\lambda_0\left(\frac{f_n - f}{\sigma}\right)\right) = 0$ for some $\lambda_0 > 0$ ([14]).

Also, (f_n) is statistically relatively $F-$norm convergent (or relatively strongly convergent) to f if

$(st) - \lim \rho\left(\lambda\left(\frac{f_n - f}{\sigma}\right)\right) = 0$ for every $\lambda > 0$ ([14]).

The two notations of convergence are equivalent if and only if the modular satisfies a Δ_2-condition, i.e. there exists a constant $M > 0$ such that $\rho(2f) \leq M\rho(f)$ for every $f \in \mathcal{H}(I)$ ([35]).

It will be observed that statistical modular convergence is the special case of statistical relative modular convergence in which the scale function is a non-zero constant.

Let us note the following properties about the modular concept that are necessary for this study:

ρ is monotone, i.e. for all $f, g \in \mathcal{H}(I)$ and $|f| \leq |g|$ then $\rho(f) \leq \rho(g)$.

ρ is finite, i.e. $\chi_B \in I_\rho$ where χ_B denotes the characteristic function of the set B.

ρ is absolutely finite, i.e. ρ is finite and for every $\varepsilon > 0$, $\lambda > 0$ there is $\delta > 0$ such that $\rho(\lambda\chi_{B_*}) < \varepsilon$ for any measurable subset $B_* \subset I$ with $\mu(B_*) < \infty$.

ρ is strongly finite, i.e. $\chi_B \in I_\rho^*$.

ρ is absolutely continuous, i.e. there exists $\alpha > 0$ such that for every $f \in \mathcal{H}(I)$, with $\rho(\lambda f) < +\infty$, the following condition is satisfied: for every $\varepsilon > 0$ there is $\delta > 0$ such that $\rho(\alpha f \chi_{B_*}) < \varepsilon$, for every measurable subset $B_* \subset I$ with $\mu(B_*) < \delta$.

If a modular ρ is monotone and finite, then $C(I) \subset I_\rho$. If ρ is monotone and strongly finite, then $C(I) \subset I_\rho^*$. Also, if ρ monotone, strongly finite and absolutely continuous, then $\overline{C_c(I)} = I_\rho$ with respect to the modular convergence in ordinary sense ([31, 33, 36]).

Now let us express the statistical relative $\mathcal{A}-$summation process for monotone and sublinear operators on modular spaces.

A sequence $L = (L_n)$ of monotone and sublinear operators from E into $\mathcal{H}(I)$ with $C_b(I) \subset E \subset \mathcal{H}(I)$ is called a statistical relative $\mathcal{A}-$summation process on E if $(L_n(f))$ is relatively $\mathcal{A}-$summable to f (with respect to modular ρ) for every $f \in E$, i.e.,

$$(st) - \lim_k \rho\left(\lambda\left(\frac{\sum_{n=1}^{\infty} a_{kn}^{(j)} L_n(f) - f}{\sigma}\right)\right) = 0 \text{ uniformly in } j, \text{ for some } \lambda > 0, \tag{1}$$

where for all $k, n, j \in \mathbb{N}$, $f \in E$ and it assume that the series in (1) is absolutely convergent almost everywhere with respect to Lebesgue measure.

Let (L_n) be a sequence of monotone and sublinear operators such that for each $k, j \in \mathbb{N}$

$$\sum_{n=1}^{\infty} a_{kn}^{(j)} |L_n(1)| < \infty. \tag{2}$$

For each $k, j \in \mathbb{N}$ and $f \in E \cap C_b(I)$,

$$\sum_{n=1}^{\infty} a_{kn}^{(j)} L_n(f;x),$$

which is well defined by (2).

Also, if each operator L_n is monotone, sublinear and translatable,then

$$\sum_{n=1}^{\infty} a_{kn}^{(j)} L_n(f;x),$$

is also monotone, sublinear and translatable for each k and j, respectively.

3. Main results

Recently, considering statistical convergence, Gal and Iancu ([25]) have studied some Korovkin Type results for a sequence of monotone and sublinear operators. By this motivation, using $\mathcal{A}$−summation process in statistical relative modular sense, we present new and expanded version the Korovkin theorems for monotone and sublinear operators.

Let $L = (L_n)$ be a sequence of monotone and sublinear operators from E into $\mathcal{H}(I)$ with $C_b(I) \subset E \subset I_\rho$. Let ρ be a monotone and finite modular on $\mathcal{H}(I)$. Assume further that the sequence L, together with modular ρ, satisfies the following property:

there exists a subsets $X_L \subset E \cap I_\rho$ with $C_b(I) \subset X_L$ and $\sigma \in \mathcal{H}(I)$ is an unbounded function satisfies $\sigma(x) \neq 0$ such that the inequality

$$(st) - \limsup_k \rho\left(\lambda\left(\frac{\sum_{n=1}^{\infty} a_{kn}^{(j)} L_n(f;x)}{\sigma(x)}\right)\right) \leq P\rho(\lambda f),\ \text{uniformly in } j, \tag{3}$$

holds for every $f \in X_L$, $\lambda > 0$ and for an absolute positive constant P.

Throughout the paper assume that I is a bounded, locally compact subsets of $\mathbb{R}$ and E is a vector subset of $\mathcal{F}(I)$ containing the test function f_i for $i = 0, 1, 2, 3$ defined by $f_0(x) = 1$, $f_1(x) = x$, $f_2(x) = -x$, $f_3(x) = x^2$.

Theorem 3.1. *Let $\mathcal{A} = \left(a_{kn}^{(j)}\right)$ be a sequence of infinite matrices with non-negative real entries and let ρ be a monotone, strongly finite, absolutely continuous and N−quasi semi-convex modular on $\mathcal{H}(I)$. Let $L = (L_n)$ be a sequence of monotone, subunital and sublinear operators from E into $\mathcal{H}(I)$ satisfying (3). Moreover, suppose that σ and σ_i are unbounded function satisfying $|\sigma(x)| \geq b > 0$ and $|\sigma_i(x)| \geq b_i > 0$ $(i = 0, 1, 2, 3)$. If*

$$(st) - \lim_k \rho\left(\lambda\left(\frac{\sum_{n=1}^{\infty} a_{kn}^{(j)} L_n(f_i;x) - f_i(x)}{\sigma_i(x)}\right)\right) = 0,\ \textit{uniformly in } j, \tag{4}$$

for every $\lambda > 0$, $i = 0, 1, 2, 3$ in I_ρ, then for every $f \in E \cap C_b(I)$ such that $f - g \in X_L$ for every $g \in C_c(I)$

$$(st) - \lim_k \rho\left(\lambda_0\left(\frac{\sum_{n=1}^{\infty} a_{kn}^{(j)} L_n(f;x) - f(x)}{\sigma(x)}\right)\right) = 0,\ \textit{uniformly in } j, \tag{5}$$

for some $\lambda_0 > 0$ in I_ρ and E, X_L are before. Moreover, if L_n is translatable, then for all $f \in E \cap C_b(I)$, we have

$$(st) - \lim_k \rho\left(\lambda_0\left(\frac{\sum_{n=1}^{\infty} a_{kn}^{(j)} L_n(f;x) - f(x)}{\sigma(x)}\right)\right) = 0, \text{ uniformly in } j, \text{ for some } \lambda_0 > 0.$$

Proof. We first claim that

$$(st) - \lim_k \rho\left(\eta\left(\frac{\sum_{n=1}^{\infty} a_{kn}^{(j)} L_n(g;x) - g(x)}{\sigma(x)}\right)\right) = 0, \text{ uniformly in } j, \tag{6}$$

for every $g \in E \cap C_b(I)$ and $\eta > 0$ where $\sigma(x) = \max\{|\sigma_i(x)|;\ i = 0, 1, 2, 3\}$. To see this assume that g belong to $E \cap C_b(I)$ and $\eta > 0$ is any positive number. Then for $\varepsilon > 0$ arbitrary fixed, there is a $\delta > 0$ such that

$$\left|g(t) - g(x)\right| \leq \varepsilon \text{ for every } x \in I \text{ with } |t - x| \leq \delta.$$

If $|t - x| \geq \delta$, then

$$\left|g(t) - g(x)\right| \leq \frac{2\|g\|}{\delta^2}|t - x|^2,$$

so that

$$\left|g(t) - g(x)\right| \leq \varepsilon + \frac{2\|g\|}{\delta^2}|t - x|^2 \text{ for all } x \in I. \tag{7}$$

Letting $M := \max\{x, 0\}$. We can write (7) as

$$\left|g(t) - g(x)\right| \leq \varepsilon + \frac{2\|g\|}{\delta^2}\left[t^2 + 2t(M - x) + 2M(-t) + |x|^2\right].$$

Using this an the fact that the operators L_n are monotone, subunital and sublinear, we conclude that

$$\begin{aligned}
\left|\sum_{n=1}^{\infty} a_{kn}^{(j)} L_n(g;x) - g(x)\right| &\leq \left|\sum_{n=1}^{\infty} a_{kn}^{(j)} L_n(g - g(x);x) + g(x)\sum_{n=1}^{\infty} a_{kn}^{(j)} L_n(1;x) - 1\right| \\
&\leq \sum_{n=1}^{\infty} a_{kn}^{(j)} L_n\left(\left|g - g(x)\right|;x\right) + g(x)\left|\sum_{n=1}^{\infty} a_{kn}^{(j)} L_n(1;x) - 1\right| \\
&\leq \varepsilon + \frac{2\|g\|}{\delta^2}\sum_{n=1}^{\infty} a_{kn}^{(j)} L_n\left|t^2 + 2t(M - x) + 2M(-t) + |x|^2;x\right|
\end{aligned}$$

where $K = \max\left\{g(x) + \frac{2M\|g\|}{\delta^2}, \frac{2\|g\|}{\delta^2}, \frac{4M\|g\|}{\delta^2}\right\}$. Now, we multiply the both sides of the above inequality by $\frac{1}{\sigma(x)}$ and in view of the fact that for any $\eta > 0$, we have

$$\eta\left|\frac{\sum_{n=1}^{\infty} a_{kn}^{(j)} L_n(g;x) - g(x)}{\sigma(x)}\right| \leq \frac{\eta\varepsilon}{|\sigma(x)|} + \eta K\left\{\left|\frac{\sum_{n=1}^{\infty} a_{kn}^{(j)} L_n(f_3;x) - f_3(x)}{\sigma_3(x)}\right|\right.$$

$$+\left|\frac{\sum_{n=1}^{\infty}a_{kn}^{(j)}L_n\left(f_2;x\right)-f_2\left(x\right)}{\sigma_2\left(x\right)}\right|+\left|\frac{\sum_{n=1}^{\infty}a_{kn}^{(j)}L_n\left(f_1;x\right)-f_1\left(x\right)}{\sigma_1\left(x\right)}\right|$$

$$\left.+\left|\frac{\sum_{n=1}^{\infty}a_{kn}^{(j)}L_n\left(f_0;x\right)-f_0\left(x\right)}{\sigma_0\left(x\right)}\right|\right\}.$$

Applying the modular ρ on both sides of the above inequality, since ρ is monotone, we get

$$\rho\left(\eta\left(\frac{\sum_{n=1}^{\infty}a_{kn}^{(j)}L_n\left(g;x\right)-g\left(x\right)}{\sigma\left(x\right)}\right)\right)\leq\rho\left(\frac{\eta\varepsilon}{\sigma\left(x\right)}+\eta K\left(\frac{\sum_{n=1}^{\infty}a_{kn}^{(j)}L_n\left(f_3;x\right)-f_3\left(x\right)}{\sigma_3\left(x\right)}\right)\right.$$

$$+\eta K\left(\frac{\sum_{n=1}^{\infty}a_{kn}^{(j)}L_n\left(f_2;x\right)-f_2\left(x\right)}{\sigma_2\left(x\right)}\right)+\eta K\left(\frac{\sum_{n=1}^{\infty}a_{kn}^{(j)}L_n\left(f_1;x\right)-f_1\left(x\right)}{\sigma_1\left(x\right)}\right)$$

$$\left.+\eta K\left(\frac{\sum_{n=1}^{\infty}a_{kn}^{(j)}L_n\left(f_0;x\right)-f_0\left(x\right)}{\sigma_0\left(x\right)}\right)\right).$$

Therefore, we may write that

$$\rho\left(\eta\left(\frac{\sum_{n=1}^{\infty}a_{kn}^{(j)}L_n\left(g\right)-g}{\sigma}\right)\right)\leq\rho\left(\frac{4\eta\varepsilon}{\sigma}\right)+\rho\left(4\eta K\left(\frac{\sum_{n=1}^{\infty}a_{kn}^{(j)}L_n\left(f_3\right)-f_3}{\sigma_3}\right)\right)$$

$$+\rho\left(4\eta K\left(\frac{\sum_{n=1}^{\infty}a_{kn}^{(j)}L_n\left(f_2\right)-f_2}{\sigma_2}\right)\right)+\rho\left(4\eta K\left(\frac{\sum_{n=1}^{\infty}a_{kn}^{(j)}L_n\left(f_1\right)-f_1}{\sigma_1}\right)\right)$$

$$+\rho\left(4\eta K\left(\frac{\sum_{n=1}^{\infty}a_{kn}^{(j)}L_n\left(f_0\right)-f_0}{\sigma_0}\right)\right).$$

Since ρ is $N-$quasi semi-convex and strongly finite. We have, assuming $0<\varepsilon\leq1$,

$$\rho\left(\eta\left(\frac{\sum_{n=1}^{\infty}a_{kn}^{(j)}L_n\left(g\right)-g}{\sigma}\right)\right)\leq N\varepsilon\rho\left(\frac{4\eta N}{\sigma}\right)+\rho\left(4\eta K\left(\frac{\sum_{n=1}^{\infty}a_{kn}^{(j)}L_n\left(f_3\right)-f_3}{\sigma_3}\right)\right)$$

$$+\rho\left(4\eta K\left(\frac{\sum_{n=1}^{\infty}a_{kn}^{(j)}L_n\left(f_2\right)-f_2}{\sigma_2}\right)\right)+\rho\left(4\eta K\left(\frac{\sum_{n=1}^{\infty}a_{kn}^{(j)}L_n\left(f_1\right)-f_1}{\sigma_1}\right)\right)$$

$$+\rho\left(4\eta K\left(\frac{\sum_{n=1}^{\infty}a_{kn}^{(j)}L_n\left(f_0\right)-f_0}{\sigma_0}\right)\right).$$

For a given $r>0$, choose on $\varepsilon\in(0,1]$ such that $N\varepsilon\rho\left(\frac{4\eta N}{\sigma}\right)<r$. Now we define the following sets:

$$\begin{aligned} G_\eta &= \left\{k:\ \rho\left(\eta\left(\frac{\sum_{n=1}^{\infty}a_{kn}^{(j)}L_n\left(g\right)-g}{\sigma}\right)\right)\geq r\right\}, \\ G_{\eta,i} &= \left\{k:\ \rho\left(\eta\left(\frac{\sum_{n=1}^{\infty}a_{kn}^{(j)}L_n\left(f_i\right)-f_i}{\sigma}\right)\right)\geq \frac{r-N\varepsilon\rho\left(\frac{4\eta N}{\sigma}\right)}{4}\right\},\ i=0,1,2,3. \end{aligned}$$

Then it is easy to see that $G_\eta\subset\bigcup_{i=0}^{3}G_{\eta,i}$. So we can write $\delta\left(G_\eta\right)\leq\sum_{i=0}^{3}\delta\left(G_{\eta,i}\right)$. Then using (4), we obtain that $\delta\left(G_\eta\right)=0$. We get

$$(st)-\lim_k\rho\left(\eta\left(\frac{\sum_{n=1}^{\infty}a_{kn}^{(j)}L_n\left(g\right)-g}{\sigma}\right)\right)=0,\ \text{uniformly in } j,$$

which proves our claim (6).

Observe that (6) also holds for every $g\in C_c(I)$. Now let $f\in E\cap C_b(I)$ satisfying $f-g\in X_L$ for every $g\in C_c(I)$. Since ρ *is* strongly finite and absolutely continuous, we can see that ρ is also absolutely finite on $\mathcal{H}(I)$ (see for details, [3]). It is known from ([4, 33]) that there exists a sequence $(g_n)\subset C_c(I)$ such that $\lim_n\rho\left(3\lambda_0^*\left(g_n-f\right)\right)=0$ for some $\lambda_0^*>0$. This means that, for every $\varepsilon>0$, there is a positive number $n_0=n_0(\varepsilon)$ with

$$\rho\left(3\lambda_0^*\left(g_n-f\right)\right)<\varepsilon \text{ for every } n\geq n_0. \tag{8}$$

On the other hand, monotone and sublinear of the operators L_n, we may write that

$$\begin{aligned} &\lambda_0^*\left|\sum_{n=1}^{\infty}a_{kn}^{(j)}L_n\left(f;x\right)-f\left(x\right)\right|\leq\lambda_0^*\left|\sum_{n=1}^{\infty}a_{kn}^{(j)}L_n\left(f-g_{n_0};x\right)\right| \\ &+\lambda_0^*\left|\sum_{n=1}^{\infty}a_{kn}^{(j)}L_n\left(g_{n_0};x\right)-g_{n_0}\left(x\right)\right|+\lambda_0^*\left|g_{n_0}\left(x\right)-f\left(x\right)\right| \end{aligned}$$

holds for every $x\in I$ and $j\in\mathbb{N}$.

Now, applying modular ρ in the last inequality and using the monotonicity of ρ and moreover multiplying both sides of the above inequality by $\frac{1}{\sigma(x)}$, we get

$$\rho\left(\lambda_0^*\left(\frac{\sum_{n=1}^{\infty}a_{kn}^{(j)}L_n\left(f\right)-f}{\sigma}\right)\right)\leq\rho\left(3\lambda_0^*\left(\frac{\sum_{n=1}^{\infty}a_{kn}^{(j)}L_n\left(f-g_{n_0}\right)}{\sigma}\right)\right) \tag{9}$$

$$+\rho\left(3\lambda_0^*\left(\frac{\sum_{n=1}^{\infty} a_{kn}^{(j)} L_n\left(g_{n_0}\right)-g_{n_0}}{\sigma}\right)\right)+\rho\left(3\lambda_0^*\left(g_{n_0}-f\right)\right).$$

Then using the (8) in (9), we have

$$\rho\left(\lambda_0^*\left(\frac{\sum_{n=1}^{\infty} a_{kn}^{(j)} L_n(f)-f}{\sigma}\right)\right) \leq \varepsilon+\rho\left(3\lambda_0^*\left(\frac{\sum_{n=1}^{\infty} a_{kn}^{(j)} L_n\left(f-g_{n_0}\right)}{\sigma}\right)\right)$$

$$+\rho\left(3\lambda_0^*\left(\frac{\sum_{n=1}^{\infty} a_{kn}^{(j)} L_n\left(g_{n_0}\right)-g_{n_0}}{\sigma}\right)\right).$$

By property (3) and also using the facts that $g_{n_0} \in C_c(I)$ and $f-g \in X_L$, we obtain

$$(st)-\limsup_k \rho\left(\lambda_0^*\left(\frac{\sum_{n=1}^{\infty} a_{kn}^{(j)} L_n(f)-f}{\sigma}\right)\right) \leq \varepsilon+R\rho\left(3\lambda_0^*\left(f-g_{n_0}\right)\right) \tag{10}$$

$$+\limsup_k \rho\left(3\lambda_0^*\left(\frac{\sum_{n=1}^{\infty} a_{kn}^{(j)} L_n\left(g_{n_0}\right)-g_{n_0}}{\sigma}\right)\right)$$

$$+\limsup_k \rho\left(3\lambda_0^*\left(\frac{\sum_{n=1}^{\infty} a_{kn}^{(j)} L_n\left(g_{n_0}\right)-g_{n_0}}{\sigma}\right)\right)$$

$$\leq \varepsilon(1+R)+\limsup_k \rho\left(3\lambda_0^*\left(\frac{\sum_{n=1}^{\infty} a_{kn}^{(j)} L_n\left(g_{n_0}\right)-g_{n_0}}{\sigma}\right)\right).$$

By (6), since

$$(st)-\lim_k \rho\left(3\lambda_0^*\left(\frac{\sum_{n=1}^{\infty} a_{kn}^{(j)} L_n\left(g_{n_0}\right)-g_{n_0}}{\sigma}\right)\right)=0,$$

we get

$$\limsup_k \rho\left(3\lambda_0^*\left(\frac{\sum_{n=1}^{\infty} a_{kn}^{(j)} L_n\left(g_{n_0}\right)-g_{n_0}}{\sigma}\right)\right)=0. \tag{11}$$

Combining (10) and (11), we conclude that

$$(st) - \lim_k \sup \rho\left(\lambda_0^*\left(\frac{\sum\limits_{n=1}^{\infty} a_{kn}^{(j)} L_n(f) - f}{\sigma}\right)\right) \leq \varepsilon (R+1).$$

Since $\varepsilon > 0$ was arbitrary, we find

$$(st) - \lim_k \sup \rho\left(\lambda_0^*\left(\frac{\sum\limits_{n=1}^{\infty} a_{kn}^{(j)} L_n(f) - f}{\sigma}\right)\right) = 0.$$

Furthermore, since $\rho\left(\lambda_0^*\left(\frac{\sum_{n=1}^{\infty} a_{kn}^{(j)} L_n(f) - f}{\sigma}\right)\right)$ is non-negative for all $k, j \in \mathbb{N}$, we can easily show that

$$(st) - \lim_k \rho\left(\lambda_0^*\left(\frac{\sum\limits_{n=1}^{\infty} a_{kn}^{(j)} L_n(f) - f}{\sigma}\right)\right) = 0, \text{ uniformly in } j.$$

Now suppose that in addition that L_n is translatable for $n \in \mathbb{N}$. Since $f + \|f\| \geq 0$, in view of (5), we get, for some $\lambda_0 > 0$,

$$(st) - \lim_k \rho\left(\lambda_0\left(\frac{\sum\limits_{n=1}^{\infty} a_{kn}^{(j)} L_n\left(f + \|f\|\right) - \left(f + \|f\|\right)}{\sigma}\right)\right) = 0, \text{ uniformly in } j,$$

and since L_n is also translatable, we can write

$$L_n\left(f + \|f\|\right) = L_n(f) + \|f\| L_n(1) = L_n(f) + \|f\| L_n(f_0).$$

Thanks to our hypotheses (4), we get $f \in E \cap C_b(I)$, for some $\lambda_0 > 0$,

$$(st) - \lim_k \rho\left(\lambda_0\left(\frac{\sum\limits_{n=1}^{\infty} a_{kn}^{(j)} L_n(f;x) - f(x)}{\sigma(x)}\right)\right) = 0, \text{ uniformly in } j.$$

□

Remark 3.2. Note that, in Theorem 3.1, in general it is not possible to obtain statistical relative $F-$norm convergence unless the modular ρ satisfies Δ_2-condition.

If one replaces the scale function by non-zero constant, then the condition (3) reduces to

$$(st) - \lim_k \sup \rho\left(\lambda\left(\sum_{n=1}^{\infty} a_{kn}^{(j)} L_n(f;x)\right)\right) \leq P\rho(\lambda f), \text{ uniformly in } j, \tag{12}$$

for every $f \in X_L$, $\lambda > 0$ and for an absolute positive constant P. In this case, the next result immediately follows from our Theorem 3.1.

Corollary 3.3. *Let $\mathcal{A} = \left(a_{kn}^{(j)}\right)$ be a sequence of infinite matrices with non-negative real entries and let ρ be a monotone, strongly finite, absolutely continuous and $N-$quasi semi-convex modular on $\mathcal{H}(I)$. Assume that $L = (L_n)$ be a sequence of monotone, subunital and sublinear operators from E into $\mathcal{H}(I)$ satisfying (12). If*

$$(st) - \lim_k \rho\left(\lambda\left(\sum_{n=1}^{\infty} a_{kn}^{(j)} L_n(f_i; x) - f_i(x)\right)\right) = 0, \text{ uniformly in } j,$$

for every $\lambda > 0$, $i = 0,1,2,3$ in I_ρ, then for every $f \in E \cap C_b(I)$ such that $f - g \in X_L$ for every $g \in C_c(I)$

$$(st) - \lim_k \rho\left(\lambda_0\left(\sum_{n=1}^{\infty} a_{kn}^{(j)} L_n(f; x) - f(x)\right)\right) = 0, \text{ uniformly in } j,$$

for some $\lambda_0 > 0$ in I_ρ where E, X_L are before.

If one replaces the matrices $\mathcal{A} := \left(A^{(j)}\right)$ by the identity matrix and take the scale function as a non-zero constant, then the condition (3) reduces to

$$(st) - \lim_n \sup \rho(\lambda(L_n(f; x))) \leq P\rho(\lambda f) \tag{13}$$

for every $f \in X_L$, $\lambda > 0$ and for an absolute positive constant P. In this case, the following result immediately follows from our Theorem 3.1.

Corollary 3.4. *Let ρ be a monotone, strongly finite, absolutely continuous and $N-$quasi semi-convex modular on $\mathcal{H}(I)$. Let $L = (L_n)$ be a sequence of monotone, subunital and sublinear operators from E into $\mathcal{H}(I)$ satisfying (13). If $(L_n(f_i))$ is statistically $F-$norm convergent to f_i for $i = 0,1,2,3$ in I_ρ then $(L_n(f))$ is statistically modularly convergent to f in I_ρ provided that f is any function belonging to I_ρ such that $f - g \in X_L$ for every $g \in C_c(I)$.*

4. Application

In this section we will present an example that proves our main theorem. Firstly, we will calculate the Bernstein-Kantorovich-Choquet operator necessary for our example. Before calculating this operator, let us recall some notations:

Suppose that (I, Λ) is a measurable space with $I \neq \emptyset$ and Λ is a $\sigma-$algebra of subsets of I. The function $m : \Lambda \to [0, +\infty]$ is said to be a capacity (or monotone set function) iff $m(\emptyset) = 0$ and $m(E) \leq m(F)$ for all $E, F \in \Lambda$ with $E \subset F$ (monotonicity). A capacity m is submodular if

$$m(E \cup F) + m(E \cap F) \leq m(E) + m(F), \text{ for all } E, F \in \Lambda.$$

If $m(I) = 1$, then a capacity m is said to be normailzed. The capacities provide a non additive generalization of probability measures, that is, of capacities m having the property of $\sigma-$additivity,

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} m(E_n)$$

for every sequence (E_n) of disjoint sets with $\bigcup_{n=1}^{\infty} E_n \in \Lambda$.

Let m be normailzed capacity on Λ. If $f : I \to \mathbb{R}$ is $\Lambda-$measurable, then for any $E \in \Lambda$, the Choquet integral is given by

$$(C)\int_E f dm = \int_0^{+\infty} m\left(F_\gamma(f) \cap E\right) d\gamma + \int_{-\infty}^0 \left[m\left(F_\gamma(f) \cap E\right) - m(E)\right] d\gamma,$$

where $F_\gamma(f) = \{x \in I : f(x) \geq \gamma\}$. If $(C)\int_E f dm \in \mathbb{R}$, then f is called Choquet integrable on E. Note that if $f \geq 0$, then $\int_{-\infty}^{0} . = 0$. Also, if m is $\sigma-$additive measure, then the Choquet integral coincides with Lebesgue integral. We now conclude by mentioning the following assumptions concerning Choquet integrals:

· if $f \geq 0$, then $(C)\int_I f dm \geq 0$,

· $f \leq g$ implies $(C)\int_I f dm \leq (C)\int_I g dm$,

· for all $c \geq 0$, we have $(C)\int_I c f dm = c(C)\int_I f dm$,

· $(C)\int_I 1 dm = m(I)$,

· if m is submodular, then $(C)\int_I (f+g)\, dm \leq (C)\int_I f dm + (C)\int_I g dm$.

For some other notions about capacity and Choquet integral, we refer the readers to [2, 9, 18, 44, 45] (see also [23] and [25], as well as the references there in).

Let us consider Bernstein-Kantorovich-Choquet operators. Then, $C^{(1)}_{n,m}(f;x) : C([0,1]) \to C([0,1])$ by

$$C^{(1)}_{n,m}(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \frac{(C)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\, dm(t)}{m\left(\left[\frac{k}{n+1}, \frac{k+1}{n+1}\right]\right)}$$

where $m := \sqrt{\mathcal{L}}$ with Lebesgue measure $\mathcal{L}$. It is known from ([24]) that the operators monotone, translatable and sublinear.

Clearly, $C^{(1)}_{n,m}(f_0;x) = f_0(x)$. In order to find $C^{(1)}_{n,m}(f_1;x)$, we will first calculate the integrals $(C)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} t dm(t)$.

We get

$$\begin{aligned}
(C)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} t dm(t) &= \int_0^{\infty} m\left(\left\{t \in \left[\frac{k}{n+1}, \frac{k+1}{n+1}\right] : t \geq \gamma\right\}\right) d\gamma \\
&= \int_0^{\frac{k}{n+1}} m\left(\left\{t \in \left[\frac{k}{n+1}, \frac{k+1}{n+1}\right] : t \geq \gamma\right\}\right) d\gamma \\
&\quad + \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} m\left(\left\{t \in \left[\frac{k}{n+1}, \frac{k+1}{n+1}\right] : t \geq \gamma\right\}\right) d\gamma \\
&= \frac{1}{\sqrt{n+1}}\left(\frac{k}{n+1}\right) + \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \sqrt{\frac{k+1}{n+1} - \gamma d\gamma} \\
&= \frac{1}{\sqrt{n+1}}\left(\frac{k}{n+1}\right) + \int_0^{\frac{1}{n+1}} \sqrt{\omega} d\omega = \frac{1}{\sqrt{n+1}}\left(\frac{3k+2}{3(n+1)}\right).
\end{aligned}$$

Using the above equation in $C_{n,m}^{(1)}(f_1;x)$, we have

$$C_{n,m}^{(1)}(f_1;x)=\frac{n}{n+1}x+\frac{2}{3(n+1)}.$$

In order to find $C_{n,m}^{(1)}(f_2;x)$, we will first calculate the integrals $(C)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(-t)\,dm(t)$. We get

$$(C)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(-t)\,dm(t)=\int_{-\infty}^{0}\left\{m\left(\left\{t\in\left[\frac{k}{n+1},\frac{k+1}{n+1}\right]:-t\geq\gamma\right\}\right)-\frac{1}{\sqrt{n+1}}\right\}d\gamma$$

$$\begin{aligned}
&= \int_{-\frac{k}{n+1}}^{0}\left\{m\left(\left\{t\in\left[\frac{k}{n+1},\frac{k+1}{n+1}\right]:t\leq-\gamma\right\}\right)-\frac{1}{\sqrt{n+1}}\right\}d\gamma \\
&\quad+\int_{-\frac{k+1}{n+1}}^{-\frac{k}{n+1}}\left\{m\left(\left\{t\in\left[\frac{k}{n+1},\frac{k+1}{n+1}\right]:t\leq-\gamma\right\}\right)-\frac{1}{\sqrt{n+1}}\right\}d\gamma
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{n+1}}\left(-\frac{k}{n+1}\right)+\int_{-\frac{k+1}{n+1}}^{-\frac{k}{n+1}}\sqrt{-\gamma-\frac{k}{n+1}}\,d\gamma-\frac{1}{\sqrt{n+1}}\left(\frac{1}{n+1}\right) \\
&= \frac{1}{\sqrt{n+1}}\left(-\frac{k}{n+1}\right)+\int_{0}^{\frac{1}{n+1}}\sqrt{\omega}\,d\omega-\frac{1}{\sqrt{n+1}}\left(\frac{1}{n+1}\right)=\frac{-3k-1}{3\sqrt{n+1}\,(n+1)}
\end{aligned}$$

which gives

$$C_{n,m}^{(1)}(f_2;x)=\frac{n}{n+1}(-x)-\frac{1}{3(n+1)}.$$

Finally, in order to find $C_{n,m}^{(1)}(f_3;x)$, we will first calculate the integrals $(C)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}t^2dm(t)$. We get

$$\begin{aligned}
(C)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}t^2dm(t) &= \int_{0}^{\infty}m\left(\left\{t\in\left[\frac{k}{n+1},\frac{k+1}{n+1}\right]:t\geq\sqrt{\gamma}\right\}\right)d\gamma \\
&= \int_{0}^{\left(\frac{k}{n+1}\right)^2}m\left(\left\{t\in\left[\frac{k}{n+1},\frac{k+1}{n+1}\right]:t\geq\sqrt{\gamma}\right\}\right)d\gamma \\
&\quad+\int_{\left(\frac{k}{n+1}\right)^2}^{\left(\frac{k+1}{n+1}\right)^2}m\left(\left\{t\in\left[\frac{k}{n+1},\frac{k+1}{n+1}\right]:t\geq\sqrt{\gamma}\right\}\right)d\gamma
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{n+1}}\left(\frac{k}{n+1}\right)^2 + \int_{\left(\frac{k}{n+1}\right)^2}^{\left(\frac{k+1}{n+1}\right)^2} \sqrt{\frac{k+1}{n+1} - \sqrt{\gamma}}\,d\gamma \\
&= \frac{1}{\sqrt{n+1}}\left(\frac{k}{n+1}\right)^2 + 2\int_{0}^{\left(\frac{k+1}{n+1}\right)^2} \sqrt{\omega}\left(\frac{k+1}{n+1} - \omega\right)d\omega \\
&= \frac{1}{\sqrt{n+1}}\frac{1}{(n+1)^2}\left(k^2 + \frac{4}{3}(k+1) - \frac{4}{5}\right)
\end{aligned}
$$

which gives

$$
C^{(1)}_{n,m}(f_3;x) = \frac{n(n-1)}{(n+1)^2}x^2 + \frac{7}{3}\frac{n}{(n+1)^2}x + \frac{8}{15(n+1)^2}.
$$

Let us consider $I=[0,1]$ and let $\varphi:[0,\infty)\to[0,\infty)$ be a continuous function with φ is convex function, $\varphi(0)=0$, $\varphi(u)>0$ for any $u>0$ and $\lim\limits_{u\to\infty}\varphi(u)=\infty$. Then, the functional defined by

$$
\rho^{\varphi}(f) := \int_0^1 \varphi\left(\left|f(x)\right|\right)dx, \text{ for } f\in\mathcal{H}(I) \tag{14}
$$

is a convex modular on $\mathcal{H}(I)$ and

$$
I_{\rho^{\varphi}} = \{f\in\mathcal{H}(I) : \rho^{\varphi}(\lambda f)<\infty \text{ for some } \lambda>0\}
$$

is the Orlicz space generated by φ. Using the Bernstein-Kantorovich-Choquet operators that we have shown and calculated with the $C=\left(C^{(1)}_{n,m}\right)$ we have previously expressed, we define the sequence of monotone, translatable and sublinear $L:=(L_{n,m})$ on $I_{\rho^{\varphi}}$ as follows:

$$
L_{n,m}(f;x) = (1+g_n(x))\,C^{(1)}_{n,m}(f;x) \text{ for } f\in I_{\rho^{\varphi}},
$$

and $x\in I$, $n\in\mathbb{N}$ and $m:=\sqrt{\mathcal{L}}$ with Lebesgue measure $\mathcal{L}$, where

$$
g_n(x) = \begin{cases} 1, & n=k^2, \\ n^3x, & 0<x<\frac{1}{n};\ n\neq k^2, \\ 0, & x=0 \text{ or } \frac{1}{n}\le x\le 1;\ n\neq k^2. \end{cases}
$$

If $\varphi(x)=x^p$ for $1\le p<\infty$, $x\ge 0$ then $I_{\rho^{\varphi}}=L_p(I)$ and we have $f\in I_{\rho^{\varphi}}$, $\rho^{\varphi}(f)=\left\|f\right\|_p$. Let us choose $p=1$. It is clear that

$$
\begin{aligned}
\rho\left(\lambda_0\left(\frac{g_n-g}{\sigma}\right)\right) &= \left\|\lambda_0\left(\frac{g_n-g}{\sigma}\right)\right\|_1 \\
&= \begin{cases} \lambda_0, & n=k^2, \\ \frac{\lambda_0}{4n}, & n\neq k^2, \end{cases}
\end{aligned}
$$

where $g=0$ and $\sigma(x)=\begin{cases} 1, & x=0 \\ \frac{1}{x^2}, & 0<x\le 1 \end{cases}$, then (g_n) converges to statistical relative modular to $g=0$. That is,

$$
(st)-\lim_n \left\|\lambda_0\left(\frac{g_n-g}{\sigma}\right)\right\|_1 = 0. \tag{15}
$$

However, since $(st)-\lim_n \rho(\lambda_0(g_n-g)) = (st)-\lim_n \|\lambda_0(g_n-g)\|_1 \neq 0$, does not converge to $g=0$ statistical modular. Now, we choose $\sigma_i(x)=\sigma(x)$ $(i=0,1,2,3)$ where $\sigma(x)=\begin{cases} 1, & x=0 \\ \frac{1}{x^2}, & 0<x\leq 1 \end{cases}$ on $L_1(I)$. Also, assume that $\mathcal{A} := \left(A^{(j)}\right) = \left(a_{kn}^{(j)}\right)$ is a sequence of infinite non-negative real matrices defined by $a_{kn}^{(j)} = \frac{1}{k+1}$ if $j \leq n \leq j+k$, $(j=1,2,...)$ and $a_{kn}^{(j)}=0$ otherwise. It can be seen that, for every $f\in L_1(I)$, $\lambda>0$ and for positive constant P_0 that

$$(st)-\limsup_k \left\| \lambda \left(\frac{\sum_{n=1}^{\infty} a_{kn}^{(j)} L_{n,m}(f)}{\sigma} \right) \right\|_1 \leq P_0 \|\lambda f\|_1, \text{ uniformly in } j.$$

Hence, we can see, for any $\lambda>0$, that

$$(st)-\lim_k \left\| \lambda \left(\frac{\sum_{n=1}^{\infty} a_{kn}^{(j)} L_{n,m}(f_i) - f_i}{\sigma} \right) \right\|_1 = 0, \text{uniformly in } j, \ (i=0,1,2,3) \tag{16}$$

where $\sigma_i(x)=\sigma(x)$ for $i=0,1,2,3$.

Now, observe that the Bernstein-Kantorovich-Choquet operators we calculated

$$\begin{aligned} C_{n,m}^{(1)}(f_0;x) &= 1, \\ C_{n,m}^{(1)}(f_2;x) &= \frac{n}{n+1}x + \frac{2}{3(n+1)}, \\ C_{n,m}^{(1)}(f_3;x) &= \frac{n}{n+1}(-x) - \frac{1}{3(n+1)}, \\ C_{n,m}^{(1)}(f_3;x) &= \frac{n(n-1)}{(n+1)^2}x^2 + \frac{7}{3}\frac{n}{(n+1)^2}x + \frac{8}{15(n+1)^2}. \end{aligned}$$

So, we can see,

$$\begin{aligned} & \left\| \lambda \left(\frac{\sum_{n=1}^{\infty} a_{kn}^{(j)} (1+g_n) C_{n,m}^{(1)}(f_0) - f_0}{\sigma} \right) \right\|_1 \\ = & \left\| \lambda \left(\frac{\sum_{n=j}^{j+k} \frac{1}{k+1}(1+g_n) - 1}{\sigma} \right) \right\|_1 \leq \frac{1}{k+1} \sum_{n=j}^{j+k} \left\| \lambda \frac{g_n}{\sigma} \right\|_1 . \end{aligned}$$

As is known if a sequence is convergent then the arithemetic mean of the sequence converges to the same value. Thus, by virtue of statistical convergence and from (15) it can easily seen that

$$(st)-\lim_k \left(\sup_j \left(\frac{1}{k+1} \sum_{n=j}^{j+k} \left\| \lambda \frac{g_n}{\sigma} \right\|_1 \right) \right) = 0, \tag{17}$$

so we get

$$(st)-\lim_k \left\| \lambda \left(\frac{\sum_{n=1}^{\infty} a_{kn}^{(j)} L_{n,m}(f_0) - f_0}{\sigma} \right) \right\|_1 = 0, \text{ uniformly in } j,$$

which guarantees that (16) holds true for $i = 0$. In addition, we have

$$\begin{aligned}
&\left\| \lambda \left(\frac{\sum_{n=1}^{\infty} a_{kn}^{(j)} (1+g_n) C_{n,m}^{(1)}(f_1) - f_1}{\sigma} \right) \right\|_1 \\
&= \left\| \lambda \left(\sum_{n=j}^{j+k} \frac{1}{k+1} \frac{(1+g_n)}{\sigma} C_{n,m}^{(1)}(f_1) - \frac{f_1}{\sigma} \right) \right\|_1 \\
&\leq \left\| \lambda x^3 \left(\frac{1}{k+1} \sum_{n=j}^{j+k} \frac{n}{n+1} - 1 \right) \right\|_1 + \left\| \lambda x^2 \left(\frac{1}{k+1} \sum_{n=j}^{j+k} \frac{2}{3(n+1)} \right) \right\|_1 \\
&\quad + \frac{1}{k+1} \sum_{n=j}^{j+k} \left(\left\| \lambda \frac{g_n}{\sigma} \frac{nx}{n+1} \right\|_1 + \left\| \lambda \frac{g_n}{\sigma} \frac{2}{3(n+1)} \right\|_1 \right) \\
&< \left(\frac{1}{k+1} \sum_{n=j}^{j+k} \frac{n}{n+1} - 1 \right) \left\| \lambda x^3 \right\|_1 + \left(\frac{1}{k+1} \sum_{n=j}^{j+k} \frac{2}{3(n+1)} \right) \left\| \lambda x^2 \right\|_1 \\
&\quad + 2 \frac{1}{k+1} \sum_{n=j}^{j+k} \left\| \lambda \frac{g_n}{\sigma} \right\|_1 .
\end{aligned}$$

Since $(st) - \lim_k \left(\sup_j \left(\frac{1}{k+1} \sum_{n=j}^{j+k} \frac{n}{n+1} - 1 \right) \right) = 0$, $(st) - \lim_k \left(\sup_j \left(\frac{1}{k+1} \sum_{n=j}^{j+k} \frac{2}{3(n+1)} \right) \right) = 0$ and from (17) we have,

$$(st) - \lim_k \left(\sup_j \left\| \lambda \left(\frac{\sum_{n=1}^{\infty} a_{kn}^{(j)} L_{n,m}(f_1) - f_1}{\sigma} \right) \right\|_1 \right) = 0$$

which gives $(st) - \lim_k \left\| \lambda \left(\frac{\sum_{n=1}^{\infty} a_{kn}^{(j)} L_{n,m}(f_1) - f_1}{\sigma} \right) \right\|_1 = 0$, uniformly in j. So (16) holds true for $i = 1$.

For the case $i = 2$, the following result is obtained by applying the similar technique with $i = 1$

$$(st) - \lim \left\| \lambda \left(\frac{\sum_{n=1}^{\infty} a_{kn}^{(j)} L_{n,m}(f_2) - f_2}{\sigma} \right) \right\|_1 = 0, \text{ uniformly in } j.$$

So (16) holds true for $i = 2$.

Finally, since

$$\begin{aligned}
&\left\| \lambda \left(\frac{\sum_{n=1}^{\infty} a_{kn}^{(j)} (1+g_n) C_{n,m}^{(1)}(f_3) - f_3}{\sigma} \right) \right\|_1 \\
&= \left\| \lambda \left(\sum_{n=j}^{j+k} \frac{1}{k+1} \frac{(1+g_n)}{\sigma} C_{n,m}^{(1)}(f_3) - \frac{f_3}{\sigma} \right) \right\|_1
\end{aligned}$$

$$\leq \left\|\lambda x^4\left(\frac{1}{k+1}\sum_{n=j}^{j+k}\frac{n(n-1)}{(n+1)^2}-1\right)\right\|_1+\left\|\lambda x^3\left(\frac{1}{k+1}\sum_{n=j}^{j+k}\frac{n}{(n+1)^2}\right)\right\|_1$$
$$+\left\|\lambda x^2\left(\frac{1}{k+1}\sum_{n=j}^{j+k}\frac{8}{15(n+1)^2}\right)\right\|_1+\frac{1}{k+1}\sum_{n=j}^{j+k}\left(\left\|\lambda\frac{g_n}{\sigma}\frac{n(n-1)x^2}{(n+1)^2}\right\|_1\right.$$
$$\left.+\left\|\lambda\left(\frac{g_n}{\sigma}\frac{7nx}{3(n+1)^2}\right)\right\|_1+\left\|\lambda\left(\frac{g_n}{\sigma}\frac{8}{15(n+1)^2}\right)\right\|_1\right)$$
$$< \left(\frac{1}{k+1}\sum_{n=j}^{j+k}\frac{n(n-1)}{(n+1)^2}-1\right)\left\|\lambda x^4\right\|_1+\left(\frac{1}{k+1}\sum_{n=j}^{j+k}\frac{n}{(n+1)^2}\right)\left\|\lambda x^3\right\|_1$$
$$+\left(\frac{1}{k+1}\sum_{n=j}^{j+k}\frac{8}{15(n+1)^2}\right)\left\|\lambda x^2\right\|_1+3\left(\frac{1}{k+1}\sum_{n=j}^{j+k}\left\|\lambda\frac{g_n}{\sigma}\right\|_1\right).$$

Since $(st)-\lim_k\left(\sup_j\left(\frac{1}{k+1}\sum_{n=j}^{j+k}\frac{n(n-1)}{(n+1)^2}-1\right)\right)=0$, $(st)-\lim_k\left(\sup_j\left(\frac{1}{k+1}\sum_{n=j}^{j+k}\frac{n}{(n+1)^2}\right)\right)=0$, $(st)-\lim_k\left(\sup_j\left(\frac{1}{k+1}\sum_{n=j}^{j+k}\frac{8}{15(n+1)^2}\right)\right)=0$ and from (17) we have,

$$(st)-\lim_k\left(\sup_j\left\|\lambda\left(\frac{\sum_{n=1}^{\infty}a_{kn}^{(j)}L_{n,m}(f_3)-f_3}{\sigma}\right)\right\|_1\right)=0$$

which gives $(st)-\lim_k\left\|\lambda\left(\frac{\sum_{n=1}^{\infty}a_{kn}^{(j)}L_{n,m}(f_3)-f_3}{\sigma}\right)\right\|_1=0$, uniformly in j. So, our claim (16) holds true for each $i=0,1,2,3$.

So, our new operator $L=(L_{n,m})$ satisfies all conditions of Theorem 3.1 and therefore we obtain

$$(st)-\lim_k\left\|\lambda_0\left(\frac{\sum_{n=1}^{\infty}a_{kn}^{(j)}L_{n,m}(f)-f}{\sigma}\right)\right\|_1=0,\ \text{uniformly in } j,$$

for some $\lambda_0>0$, for any $f\in L_1(I)$. However, $(L_{n,m}(f_0))$ is neither statistical modular $\mathcal{A}$−summable nor statistical modularly convegent to f_0. Thus $(L_{n,m})$ does not fulfil the Corollary 3.3 and Corollary 3.4.

5. Concluding remark

In this study, we investigated the approximation properties of monotone and sublinear operators in modular spaces using the statistical relative $\mathcal{A}-$ summation method. This approach enabled us to obtain new and significant results concerning such operators within the modular framework. Compared to classical approximation techniques, our method proved to be more effective in terms of convergence behavior and applicability. The theoretical contributions were further supported by a concrete example, emphasizing the practical relevance and strength of the proposed approach.

It is well known that if $\varphi(x)=x^p$ for $1\leq p<\infty$, $x\geq 0$ then $I_{\rho^{\varphi}}=L_p(I)$ and we have $f\in I_{\rho^{\varphi}}$, $\rho^{\varphi}(f)=\|f\|_p$. Therefore, in Theorem 3.1, if the identity matrix is taken instead of $\mathcal{A}:=\left(A^{(j)}\right)$ and the scale function, σ, is chosen to be a non-zero constant, Theorem 3 given by Sorin and Iancu in [25] is obtained.

Future research may extend this framework to explore other types of convergence, specific selections of the matrix sequence $\mathcal{A}$, and more extensive families of function spaces, potentially leading to enhanced comprehension and more general results.

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