Filomat 39:21 (2025), 7339–7346
https://doi.org/10.2298/FIL2521339M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

About G-topological groups

Osman Mucuk[a]

[a]*Department of Mathematics, Faculty of Science, Erciyes University, Kayseri 38039, Turkey*

Abstract. In this paper, we define a G-topological group on a group endowed with a method G, which generalises the notion of first countable topological groups in which any convergent sequence has a unique limit, give some counter-examples of G-topological groups and then extend the usual properties of topological groups to G-topological groups in this sense.

1. Introduction

Using the convergent sequences with unique limits in a topological space X one can define a function from convergent sequences to limits. Then sequential versions of some topological concepts can be stated in term of convergent sequences. If X is also first countable space, then sequential versions and standard forms agree.

Motivated by this, recently many mathematicians have been in afford to define some topological definitions associated with different convergences. Replacing the limit function defined by the convergent sequences with a function G is called a G-method [6]. As a result of this replacement some authors have been recently studied G-continuity [8] (see also [12] and [9] for other continuities), G-compactness [7] and the G-connectedness in [10] (see also [11]), G-open subsets and G-neighbourhoods [17].

G-methods have been extended to arbitrary sets not only to topological spaces and G-hulls, G-closures, G-kernels and G-interiors have been introduced [13] .

Recently, G-connectedness [18] and G-compactness [19] for the topological groups with operations extending the idea of topological groups [3] have been developed. We refer the readers to [20] some counter examples of G-methods and [2] for a variety of G-convergence. These ideas have been into account in neutrosophic topological spaces. [1]. The statistical convergence, statistically sequential spaces, statistically Frechet spaces with some applications in selection principles theory, function spaces and hyperspaces have been given in [16].

The notion of G-topological group on a group with operations which has a topology is defined in [21, Definition 5.4] and then some properties are given in [5]. In this paper we redefine G-topological group not on a topology but also on a set and characterise some properties of these setting together with some counter examples.

2020 *Mathematics Subject Classification*. Primary 40J05; Secondary 54A05, 22A05.
Keywords. Sequences, G-convegence, G-continuity, G-topological group.
Received: 29 January 2025; Revised: 19 April 2025; Accepted: 21 April 2025
Communicated by Ljubiša D. R. Kočinac
Email address: mucuk@erciyes.edu.tr (Osman Mucuk)
ORCID iD: https://orcid.org/0000-0001-7411-2871 (Osman Mucuk)

2. Preliminaries

We use the symbols $\mathbf{a}$, $\mathbf{b}$, ... for the indications of the sequences (a_n), (b_n), ... in a set or topological space X; and $s(X)$ and $c(X)$ respectively for the sets of all sequences and convergent sequences in X.

The convergent sequences with unique limits in a topological space X define a function $\lim\colon c(X) \to X$ which assigns each convergent sequence to its limit. Motivated by this idea a G-method depending on sequential convergence in X is defined to be a map from a subset $c_G(X)$ of $s(X)$ to X. We shortly write (X, G) for such a method. A sequence $\mathbf{a} = (a_n)$ is *G-convergent* to ℓ provided that $\mathbf{a} \in c_G(X)$ and $G(\mathbf{a}) = \ell$. In particular lim function from $c(X)$ to X yields a G-method.

If in a topological space X any convergent sequence $\mathbf{x} = (x_n)$ is in the domain $c_G(X)$ of the method G and $\lim \mathbf{x} = G(\mathbf{x})$, then the method is said to be *regular*. The regularity condition defined in a topological space is weakened and replaced with the following pointwise method in a set [14, p.279]: A method G in a set X is called *pointwise method* if for any $x \in X$, one has $x \in [\{x\}]^G$. which means the constant sequence $\mathbf{x} = (x, x, x, \cdots)$ is G-convergent to x.

Let X be a set, $A \subseteq X$ and $\ell \in X$. Then ℓ is said to be in *G-hull* of A denoted by $[A]^G$ whenever there exists a sequence $\mathbf{a} = (a_n)$ in A with $G(\mathbf{a}) = \ell$ and A is *G-closed* whenever $[A]^G \subseteq A$. Hence A is not G-closed whenever there exists a sequence $\mathbf{a} = (a_n)$ in A which is not G-convergent to a point in A. Eventually a subset A with $[A]^G = \emptyset$ is G-closed.

Whenever a method G is regular, then $A \subseteq [A]^G$ and therefore A is G-closed if and only if $[A]^G = A$. Even if a method G is regular, G-closure $[A]^G$ is not necessarily G-closed. The intersection of G-closed subsets is also G-closed but the union of G-closed subsets is not necessarily G-closed.

A subset A with G-closed complement $X \setminus A$ is called *G-open*. Eventually unlike the intersection, the union of G-open subsets of X is G-open. We refer to [2] and [20] for various examples of G-methods, G-closed and G-open subsets; and some other evaluations.

We can define (G, H)-continuity of a function between two methods as follows.

Definition 2.1. Let (X, G) and (Y, H) be two methods and $f\colon (X, G) \to (Y, H)$ be a map between these methods. We call f, (G, H)-*continuous* if whenever $\mathbf{a} = (a_n) \in c_G(X)$, then $\mathbf{b} = (f(a_n)) \in c_H(Y)$ and $f(G(\mathbf{a})) = H(\mathbf{b})$.

3. G-topological groups

This section is assigned to the definition of G-topological groups, some examples, and some standard properties motivated by topological groups.

For the methods (X, G) and (Y, H) we have a product method $G \times H$ on $X \times Y$ defined by $(G \times H)(\mathbf{a}, \mathbf{b}) = (G(\mathbf{a}), H(\mathbf{b}))$ where $\mathbf{a} = (a_n) \in c_G(X)$ and $\mathbf{b} = (b_n) \in c_G(X)$. $G \times H$-open subsets are the subsets $A \times B$ whenever A is G-open in X and B is G-open in X. If the methods G and H are regular, so is $G \times H$.

Extending the definition of topological groups to the G-method setting, we define a G-topological group as follows. This definition is given in [21, Definition 5.4] on a group with operations and a topology, but our definition is stated only on a group with a G-method. As a convention, we use multiplication notation for groups, and it can be modified to the additive case when the group is commutative.

Definition 3.1. Let (X, G) be a method in which X is a group. Then X is called a *G-topological group* subject to that the multiplication map m and the inverse map n are G-continuous.

In this definition, when X is a first countable space in which any convergent sequence has a unique limit and G is the method lim, then a G-topological group X agrees with a standard topological group.

Theorem 3.2. *For a group X with a method G we have the following.*

(1) *The multiplicative map m is G-continuous iff for the sequences $\mathbf{a} = (a_n)$ and $\mathbf{b} = (b_n)$ in $c_G(X)$, one has that $\mathbf{ab} = (a_n b_n) \in c_G(X)$ and $G(\mathbf{ab}) = G(\mathbf{a})G(\mathbf{b})$.*

(2) *The inverse map n is G-continuous iff for any sequence $\mathbf{a} = (a_n)$ in $c_G(X)$ we have that $\mathbf{a}^{-1} = (a_n^{-1}) \in c_G(X)$ and $G(\mathbf{a})^{-1} = G(\mathbf{a}^{-1})$.*

Proof. (1) If m is G-continuous; and $\mathbf{a} = (a_n)$ and $\mathbf{b} = (b_n)$ are the sequence in $c_G(X)$, then G-continuity of m implies that $\mathbf{ab} \in c_G(X)$ and $G(m(a_n, b_n)) = m(G(\mathbf{a}), G(\mathbf{b}))$ which means that $G(\mathbf{ab}) = G(\mathbf{a})G(\mathbf{b})$.

Whenever for all the sequences $\mathbf{a}$ and $\mathbf{b}$ in $c_G(X)$ one has $\mathbf{ab} \in c_G(X)$ and the condition $G(\mathbf{ab}) = G(\mathbf{a})G(\mathbf{b})$ is satisfied, then $G(m(a_n, b_n)) = m(G(\mathbf{a}), G(\mathbf{b}))$ which implies that m is G-continuous.

(2) If the inverse map n is G-continuous, for any sequence $\mathbf{a} = (a_n)$ in $c_G(X)$ we have that $\mathbf{a}^{-1} =\in c_G$ and $n(G(\mathbf{a})) = G(n(\mathbf{a}))$ which means $G(\mathbf{a})^{-1} = G(\mathbf{a}^{-1})$.

Whenever for any sequence $\mathbf{a} \in c_G(X)$, one has $\mathbf{a}^{-1} \in c_G(X)$ and the equality $G(\mathbf{a})^{-1} = G(\mathbf{a}^{-1})$, then by $n(G(\mathbf{a})) = G(n(\mathbf{a}))$, the G-continuity of n follows. □

Theorem 3.3. *For a group X imposed with a pointwise method G, G-continuities of the multiplicative map m and the inverse map n are equivalent to G-continuity of the difference map δ.*

Proof. G-continuity of the map n gives G-continuity of the map $g\colon X \times X \to X \times X$ defined by $(x, y) \mapsto (x, y^{-1})$ and G-continuity of δ as a composite $m \circ g = \delta$ of G-continuous maps follows.

Assuming G-continuity of the difference map δ we first prove G-continuity of $f\colon X \to X \times X, x \mapsto (e, x)$. Since G is a pointwise method for the constant sequence $\mathbf{e} = (e, e, \dots)$ we have $G(\mathbf{e}) = e$ and therefore a sequence $\mathbf{a} = (a_n)$ in $c_G(X)$ with $G(\mathbf{a}) = x$ implies

$$f(G(\mathbf{a})) = (e, G(\mathbf{a})) = (G(\mathbf{e}), G(\mathbf{a})) = G(\mathbf{e}, \mathbf{a}) = G(f(\mathbf{a}))$$

and therefore f is G-continuous. Hence n becomes G-continuous as a composite $n = \delta \circ f$ of G-continuous maps.

We next prove that the map $g\colon X \times X \to X \times X$ defined by $(x, y) \to (x, y^{-1})$ is G-continuous. If $\mathbf{a} = (a_n)$ and $\mathbf{b} = (b_n)$ are the sequences in $c_G(X)$, then we have

$$\begin{aligned} gG(\mathbf{a}, \mathbf{b}) &= g(G(\mathbf{a}), G(\mathbf{b})) \\ &= (G(\mathbf{a}), G(\mathbf{b})^{-1}) \\ &= (G(\mathbf{a}), G(\mathbf{b}^{-1})) && \text{(by G-continuity of n)} \\ &= G(\mathbf{a}, \mathbf{b}^{-1}) \\ &= G(g(\mathbf{a}, \mathbf{b})) \end{aligned}$$

Hence, the map g is G-continuous. Then, the multiplicative map m becomes G-continuous as the composite of G-continuous maps $m = \delta \circ g$. □

Hence, we can give the following corollary by Theorem 3.3.

Corollary 3.4. *A group X together with a pointwise method G is a G-topological group if and only if for the sequences $\mathbf{a} = (a_n)$ and $\mathbf{b} = b_n$ in $c_G(X)$, one has that $\mathbf{ab}^{-1} = (a_n b_n^{-1}) \in c_G(X)$ and $G(\mathbf{ab}^{-1}) = G(\mathbf{a})G(\mathbf{b})^{-1}$.*

Proof. The proof is a consequence that, the difference map δ is G-continuous if and only if for the sequences $\mathbf{a} = (a_n)$ and $\mathbf{b} = b_n$ in $c_G(X)$ one has $\mathbf{ab}^{-1} \in c_G(X)$ and $G(\delta(a_n, b_n)) = \delta(G((\mathbf{a}), G(\mathbf{b}))$ and the equivalently $G(\mathbf{ab}^{-1}) = G(\mathbf{a})G(\mathbf{b})^{-1}$. □

Below, we give an example that is not a topological group but is a G-topological group.

Example 3.5. Additive group $(\mathbb{R}, +)$ with co-countable topology is not a topological group. For example, $\mathbb{R} \setminus \mathbb{Z}$ is an open subset of $\mathbb{R}$ but $\delta^{-1}(\mathbb{R} \setminus \mathbb{Z})$ is not open in $\mathbb{R}^2$ concerning the product of co-countable topologies: Let l_n denote the set of the points (a, b)'s on the line $y = x - n$ for each $n \in \mathbb{Z}$. Then

$$\delta^{-1}(\mathbb{R} \setminus \mathbb{Z}) = \delta^{-1}(\bigcup_{n\in\mathbb{Z}}(n, n+1)) = \bigcup_{n\in\mathbb{Z}} \delta^{-1}(n, n+1) = \mathbb{R}^2 \setminus (\bigcup_{n\in\mathbb{Z}} l_n)$$

For a point $(a, b) \in \delta^{-1}(\mathbb{R} \setminus \mathbb{Z})$ there are no co-countable subsets U and V such that $(a, b) \in U \times V \subseteq \delta^{-1}(\mathbb{R} \setminus \mathbb{Z})$ since for such co-countable subsets U and V the product $U \times V$ includes some points on the lines l_n's. Hence the difference map $\delta\colon \mathbb{R}^2 \to \mathbb{R}, \delta(x,y) = x - y$ is not continuous and therefore $(\mathbb{R}, +)$ is not a topological group.

However, with co-countable topology, the terms of a sequence $\mathbf{a} = (a_n)$ converging to a are almost a, and the sequence has only one limit. Therefore, the function $\lim\colon c(\mathbb{R}) \to \mathbb{R}$ defines a G-method. Then $(\mathbb{R}, +)$ is a G-topological group with the method $G = \lim$. For if $\mathbf{a} = (a_n)$ and $\mathbf{b} = (b_n)$ are the sequences converging to x and y respectively, then the terms of the sequence $\delta(a_n, b_n) = (a_n - b_n)$ are almost $x - y$ and converges to $x - y$ which implies that the difference map δ is G-continuous.

We now give a few more examples of G-topological groups.

Example 3.6. If X is a group with a G-method defined by $G(\mathbf{x}) = x_1$ for the sequences in X, which is a not a pointwise method, then for all sequences $\mathbf{a} = (a_n)$ and $\mathbf{b} = (b_n)$ in X it follows that $G(\mathbf{ab}) = a_1b_1$ and $G(\mathbf{a})G(\mathbf{b}) = a_1b_1$. Hence it follows that $G(\mathbf{ab}) = G(\mathbf{a})G(\mathbf{b})$ and therefore the multiplicative map is G-continuous. Moreover for any sequence $\mathbf{a} = (a_n)$ in X one has $G(\mathbf{a})^{-1} = G(\mathbf{a}^{-1})$ and therefore the inverse map n is continuous. Hence, X is a G-topological group.

Example 3.7. If the method G on the additive group $(\mathbb{R}, +)$ is defined by $G(\mathbf{x}) = \lim \dfrac{x_n + x_{n+1}}{2}$ which is regular and is not the same as the lim method. For example if $A = \{0, 1\}$, then $[A]^G = \{0, \frac{1}{2}, 1\}$ and therefore A is not G-closed, but A is a closed subset. For the sequences $\mathbf{a} = (a_n)$ and $\mathbf{b} = (b_n)$ in $c_G(\mathbb{R})$ one has $(\delta(a_n, b_n)) = \mathbf{a} - \mathbf{b} \in c_G(X)$; and

$$G(\mathbf{a} - \mathbf{b}) = \lim \frac{a_n - b_n + a_{n+1} - b_{n+1}}{2}.$$

and

$$\begin{aligned} G(\mathbf{a}) - G(\mathbf{b}) &= \lim \frac{a_n + a_{n+1}}{2} - \lim \frac{b_n + b_{n+1}}{2} \\ &= \lim \frac{a_n - b_n + a_{n+1} - b_{n+1}}{2} \end{aligned}$$

Hence, by the equability $G(\mathbf{a} - \mathbf{b}) = G(\mathbf{a}) - G(\mathbf{b})$, G-continuity of the difference map δ follows and therefore $(\mathbb{R}, +)$ becomes a G-topological group according to this method G.

Example 3.8. For a group X and a constant $x_0 \in X$; and the G-method defined by $G(\mathbf{a}) = x_0 \in X$ with domain all sequences, any sequences $\mathbf{a} = (a_n)$ and $\mathbf{b} = (b_n)$ in X yields $G(\mathbf{ab}) = x_0$ and $G(\mathbf{a})G(\mathbf{b})) = x_0^2$. Hence, if $x_0 = e$ is identity, then X becomes a G-topological group, and if $x_0 \neq e$, then X is not a G-topological group.

Theorem 3.9. *In a G-topological group X with a pointwise method G, the right and left translations are:*

(1) *G-continuous*

(2) *G-closed*

(3) *G-open.*

Proof. The proof for a right translation $R_a\colon X \to X$ defined by $R_a(x) = xa$ is sufficient since the proof for left translations is similar.

(1) Since the method G is pointwise by the G-continuity of the multiplicative map in the G-topological group X, for a sequence $\mathbf{x}$ with G-convergence to x and constant sequence $\mathbf{a} = (a, a, \dots,)$, the sequence $\mathbf{xa}$ has G-convergence to xa. Hence we have

$$G(R_a(\mathbf{x})) = G(\mathbf{xa}) = G(\mathbf{x})G(\mathbf{a}) = xa = R_a(x) = R_a(G(\mathbf{x}))$$

and therefore the right translation map $R_a\colon X \to X$ is G-continuous.

(2) For a G-closed subset F, if $\mathbf{x}a = \mathbf{xa}$ is a sequence in Fa with G-convergence to xa then $\mathbf{x}$ is G-convergent to x. Since F is G-closed we have $x \in F$ and $\mathbf{xa}$ is G-convergent to $xa \in Fa$ which means $R_a(F) = Fa$ is G-closed.

(3) Since the right translation $R_{a^{-1}}$ is G-continuous, if U is G-open, then the inverse image $(R_{a^{-1}})^{-1}(U) = R_a(U) = Ua$ is G-open which proves that the map R_a is G-open. □

Theorem 3.10. *If X is a G-topological group and Y a H-topological group, then the product $X \times Y$ is a (G, H)-topological group.*

Proof. Let X and Y be respectively G and H-topological groups with the multiplicative operations. We prove that the multiplicative map $m\colon (X \times Y) \times (X \times Y) \to X \times Y$ defined by $m((x, y), (x', y')) = (xx', yy')$ is (G, H)-continuous. Let the sequences $\mathbf{a} = (a_n)$ and $\mathbf{a}' = (a'_n)$ be in $c_G(X)$; and $\mathbf{b} = (b_n)$ and $\mathbf{b}' = (b'_n)$ in $c_H(Y)$. By the G-continuity of the multiplicative map in X we have $G(\mathbf{a})G(\mathbf{a}') = G(\mathbf{aa}')$ and similarly $H(\mathbf{b})H(\mathbf{b}') = H(\mathbf{bb}')$. Then by Theorem 3.2 and the following evaluation, (G, H)-continuity of the multiplicative map m follows:

$$
\begin{aligned}
(G, H)((\mathbf{a}, \mathbf{b})(\mathbf{a}', \mathbf{b}')) &= (G, H)(\mathbf{aa}', \mathbf{bb}') \\
&= (G(\mathbf{aa}'), H(\mathbf{bb}')) \\
&= (G(\mathbf{a})G(\mathbf{a}'), H(\mathbf{b})G(\mathbf{b}')) && \text{(by G and H-continuities of m)} \\
&= (G(\mathbf{a}), H(\mathbf{b}))(G(\mathbf{a}'), H(\mathbf{b}')) \\
&= (G, H)(\mathbf{a}, \mathbf{b})(G, H)(\mathbf{a}', \mathbf{b}')
\end{aligned}
$$

By the G-continuity of the inverse map we have $G(\mathbf{a}^{-1}) = G(\mathbf{a})^{-1}$ and similarly $H(\mathbf{b}^{-1}) = H(\mathbf{b})^{-1}$. Hence we have

$$
\begin{aligned}
(G, H)((\mathbf{a}, \mathbf{b})^{-1}) &= (G, H)(\mathbf{a}^{-1}, \mathbf{b}^{-1}) \\
&= (G(\mathbf{a}^{-1}), H(\mathbf{b}^{-1})) \\
&= (G(\mathbf{a})^{-1}, H(\mathbf{b})^{-1}) && \text{(by G and H-continuities of n)} \\
&= (G(\mathbf{a}), H(\mathbf{b}))^{-1} \\
&= (G, H)(\mathbf{a}, \mathbf{b}))^{-1}
\end{aligned}
$$

which proves (G, H)-continuity of the inverse map for $X \times Y$. Hence by Theorem 3.2, $X \times Y$ is a (G, H)-topological group. □

Theorem 3.11. *In a G-topological group X for the subsets $A, B \subseteq X$, we have $[A]^G [B]^{G^{-1}} \subseteq [AB^{-1}]^G$ and $[A]^G [B]^G \subseteq [AB]^G$.*

Proof. Let X be a G-topological group and $A, B \subseteq X$. Then $[A]^G \times [B]^G = [A \times B]^G$ and the G-continuity of the difference map δ mean that

$$\delta([A]^G \times [B]^G) = \delta([A \times B]^G) \subseteq [\delta(A \times B)]^G$$

and therefore $[A]^G [B]^{G^{-1}} = \delta([A]^G \times [B]^G) \subseteq [\delta(A \times B)]^G = [AB^{-1}]^G$ □

The proof of $[A]^G [B]^G \subseteq [AB]^G$ can be similarly performed by replacing the difference map δ with the multiplicative m.

Subgroup and normal subgroup are rephrased as follows, and it is helpful for the proof of Theorem 3.13

Proposition 3.12. *Let X be a group, $Y \subseteq X$ a subset and δ the difference map. Then the following hold:*

(1) *Y is a subgroup of X if and only if $\delta(Y \times Y) \subseteq Y$.*

(2) *The subgroup Y is normal if for any $a \in X$ and the map $f_a\colon X \to X$ defined by $x \mapsto axa^{-1}$ we have $f_a(Y) \subseteq Y$.*

Theorem 3.13. *The following hold for a G-topological group X such that G is a pointwise method.*

(1) *If Y is a subgroup of X, then so also is $[Y]^G$.*

(2) *If Y is normal in X, then so also is $[Y]^G$.*

Proof. (1) Y is a subgroup of X, then by Proposition 3.12 (1) it is sufficient to prove that $\delta([Y]^G \times [Y]^G) \subseteq [Y]^G$. Here $[Y]^G \times [Y]^G = [Y \times Y]^G$ and by the G-continuity of δ we have $\delta([Y \times Y]^G) \subseteq [\delta(Y \times Y)]^G$ and as Y is a subgroup we have $\delta(Y \times Y) \subseteq Y$ which implies that $[\delta(Y \times Y)]^G \subseteq [Y]^G$. Assembling these, we conclude that

$$\delta([Y]^G \times [Y]^G) = \delta([Y \times Y]^G) \subseteq [\delta(Y \times Y)]^G \subseteq [Y]^G$$

which means $[Y]^G$ is a subgroup of X.

(2) If Y is a normal subgroup of X, then by (1), $[Y]^G$ is also a subgroup, and therefore, we need to prove that $[Y]^G$ is normal. By the normality of Y, for any $a \in X$ and the map $f_a\colon X \to X, x \mapsto axa^{-1}$ we have $f_a(Y) \subseteq Y$ which implies that $[f_a(Y)]^G \subseteq [Y]^G$ and since by Theorem 3.9 the translation maps $L_a\colon X \to X, x \mapsto ax$ and $R_{a^{-1}}\colon X \to X, x \mapsto xa^{-1}$ are G-continuous, as a composite of G-continuous maps $f_a = L_a \circ R_{a^{-1}}$ is G-continuous. Hence $f_a([Y]^G) \subseteq [f_a(Y)]^G \subseteq [Y]^G$ and therefore $[Y]^G$ is a normal subgroup. □

We define G-topological subgroups as follows:

Theorem 3.14. *If X is a G-topological group and Y a subgroup of X, then Y becomes a G-topological group with the method G_Y defined by $G_Y(\mathbf{a}) = G(\mathbf{a})$ on the domain*

$$c_G(Y) = \{\mathbf{a} = (a_n) \in c_G(X) \cap s(Y)\colon G(\mathbf{a}) \in Y\}$$

whenever it is non-empty.

Proof. Let $\mathbf{a} = (a_n) \in c_G(Y)$ and $\mathbf{b} = (b_n) \in c_G(Y)$. Since Y is a subgroup $\mathbf{ab} \in s(Y)$ and by G-continuity of the multiplicative map m in the G-topological group X, we have $\mathbf{ab} \in c_G(Y)$ and $G_Y(\mathbf{ab}) = G_Y(\mathbf{a})G_Y(\mathbf{b})$; and $G_Y(\mathbf{a}^{-1}) = G_Y(\mathbf{a})^{-1}$. Hence, by Theorem 3.2, Y becomes a G-topological group. □

Theorem [17, Theorem 27] can be revised for G-topological groups as follows.

Theorem 3.15. *Let X be a G-topological group and Y an H-topological group provided that the methods G and H are pointwise methods. Then a map $f\colon (X,G) \to (Y,H)$ is (G,H)-continuous if and only if f is so at the identity $e \in X$.*

Proof. If the map $f\colon (X,G) \to (Y,H)$ is (G,H)-continuous at the identity $e \in X$, then $f(\mathbf{x}) \in c_H(Y)$ and $H(f(\mathbf{x})) = e$ whenever $\mathbf{x} \in c_G(X)$ and $G(\mathbf{x}) = e$. Hence for a sequence $\mathbf{x}$ in $c_G(X)$ with $G(\mathbf{x}) = a$ and the constant sequence $\mathbf{a} = (a, a, \dots)$, we have $G(\mathbf{xa}^{-1}) = G(\mathbf{x})G(\mathbf{a})^{-1} = e$ since G is a pointwise method and X is a G-topological group. By assumption $H(f(\mathbf{xa}^{-1})) = e$ and by the multiplicative of f and H we have $H(f(\mathbf{x}))H(f(\mathbf{a}))^{-1} = e$, where H is pointwise method which implies $H(f(\mathbf{a})) = f(a)$ and therefore $H(f(\mathbf{x})) = f(a) = f(G(\mathbf{x}))$. Hence the (G,H)-continuity of f at any point $a \in X$ follows. □

Theorem 3.16. *For a G-topological group X provided with a pointwise method G, G-openness of one of the sets A and B implies that of the product AB.*

Proof. By Theorem 3.9 (3) G-openness of A means that Ab for $b \in B$ is G-open. Then, as the union of G-open subsets

$$AB = \bigcup_{b \in B} Ab$$

becomes G-open. □

4. G-topological quotient groups

It is well known that the quotient group of a topological group is also a topological group. Below, we prove a similar result for G-topological groups.

Theorem 4.1. *For a G-topological group X and a normal subgroup N of X, which is G-closed, there exists a method denoted by G_N on the quotient group X/N such that X/N is a G_N-topological group and the quotient map $p\colon X \to X/N$ is (G, G_N)-continuous.*

Proof. When X is a G-topological group and N a normal subgroup N, we define a method on the quotient group X/N by

$$G_N\colon c_G(X/N) \to X/N, G_N(\mathbf{a}N) = G(\mathbf{a})N$$

where the domain $c_G(X/N)$ of the method G_N is the set of the sequences $\mathbf{a}N = (a_nN)$ in X/N for the sequences $\mathbf{a} = (a_n) \in c_G(X)$. Prove that the method G_N is well defined. If $\mathbf{a}, \mathbf{b} \in c_G(X)$ and $\mathbf{a}N = \mathbf{b}N$, then $a_n{b_n}^{-1} \in N$ for any $n \in \mathbb{N}$ which means $\mathbf{ab}^{-1}$ is a sequence in N and by the G-continuity of the difference map $X\times X \to X$ with $(x, y) \to xy^{-1}$ we have $\mathbf{ab}^{-1} \in c_G(X)$ and $G(\mathbf{ab}^{-1}) = G(\mathbf{a})G(\mathbf{b})^{-1}$. Since N is G-closed one has $G(\mathbf{a})G(\mathbf{b})^{-1} \in N$ which means $G(\mathbf{a})N = G(\mathbf{b})N$. Hence, the method G_N is well-defined.

Now, prove that for the sequences $\mathbf{a}N, \mathbf{b}N \in c_G(X/N)$ we have

$$G(\mathbf{a}N\mathbf{b}N) = G(\mathbf{a}N)G(\mathbf{a}N) = G(\mathbf{a})G(\mathbf{b})\mathbf{N}$$

Let $\mathbf{a}N, \mathbf{b}N \in c_G(X/N)$. Hence $\mathbf{a} = (a_n) \in c_G(X)$ and $\mathbf{b} = (b_n) \in c_G(X)$; and the G-continuity of the multiplicative map $m\colon X \times X \to X, (x, y) \mapsto xy$ implies that $\mathbf{ab} \in c_G(X)$ and $G(\mathbf{ab}) = G(\mathbf{a})G(\mathbf{b})$. Since N is a normal subgroup $a_nN{b_n}^{-1}N = (a_n{b_n}^{-1})N$ which implies that $\mathbf{a}N\mathbf{b}N = (\mathbf{ab})N$. Hence, by the following evaluation, we have G_N-continuity of the multiplicative map for X/N

$$\begin{aligned} G_N(\mathbf{a}N)G_N(\mathbf{b}N) &= G(\mathbf{a})NG(\mathbf{b})N \\ &= (G(\mathbf{a})G(\mathbf{b}))N \\ &= G(\mathbf{ab})N \\ &= G_N(\mathbf{a}N\mathbf{b}N) \end{aligned}$$

For $\mathbf{a} \in c_G(X/N)$, we have $G_N(\mathbf{a}^{-1}N) = G(\mathbf{a}^{-1})N$ and since X is a G-topological group $G(\mathbf{a}^{-1}) = G(\mathbf{a})^{-1}$; and therefore we have $G_N(\mathbf{a}^{-1}N) = (G_N(\mathbf{a}N))^{-1}$ which guarantees G_N continuity of the inverse map.

By Theorem 3.2, these complete the proof that the quotient group X/N is a G_N-topological group.

We now prove that the quotient map $p\colon X \to X/N$ is (G, G_N)-continuous. If $\mathbf{a} = (a_n)$ is a sequence in $c_G(X)$, then the sequence $\mathbf{b} = (p(a_n)) = \mathbf{a}N$ is in the domain $c_G(X/N)$ of G_N and

$$p(G(\mathbf{a})) = G(\mathbf{a})N = G_N(\mathbf{a}N) = G_N(\mathbf{b})$$

which proves that the map p is G-continuous or equivalently (G, G_N)-continuous. □

Below, we give an isomorphism theorem for G-topological groupoids.

Theorem 4.2. *Let X a G-topological group, N a G-closed normal subgroup and $p\colon X \to X/G$ the quotient map. Then for another H-topological group Y and a (G, H)-continuous group homomorphism $f\colon (X, G) \to (Y, H)$ with $N \subseteq \mathsf{Ker} f$, there exists a unique (G_N, H) homomorphism $\widetilde{f}\colon X/N \to Y$ such that $\widetilde{f} \circ p = f$ and $\widetilde{f}$ is (G_N, H)-continuous.*

Proof. By the algebraic version of the theorem, we have the existence of the group homomorphism $\widetilde{f}\colon X/N \to Y$ defined by $\widetilde{f}(xN) = f(x)$ for $xN \in X/N$. Hence, we must prove that $\widetilde{f}$ is (G_N, H)-continuous. If $\mathbf{a}N = (a_nN)$ is a sequence with $\mathbf{a} = (a_n) \in c_G(X)$, then by (G, H)-continuity of f we have $\mathbf{b} = (f(a_n)) \in c_H(Y)$ and $f(G(\mathbf{a})) = H(\mathbf{b})$; and then

$$\widetilde{f}(G_N(\mathbf{a}N)) = \widetilde{f}(G(\mathbf{a})N) = f(G(\mathbf{a})) = H(\mathbf{b}) = H(\widetilde{f}(\mathbf{a}N))$$

and therefore $\widetilde{f}$ is (G_N, H)-continuous. □

5. Conclusion

In this paper, we define G-topological group and give some characterisations of standard properties. A groupoid is a category such that any morphism has an inverse and generalises a group (one can be referred to [4] for more discussion about groupoids). That means a groupoid with one object is just a group; equivalently, a group can be considered a groupoid with one object. A topological groupoid is a groupoid with topologies on both the sets of morphisms and objects such that constructional maps are continuous. A topological group is a particular topological groupoid with one object. The primary reference on topological groupoids might be [15].

The paper's results can be extended to the groupoid setting by defining G-topological groupoid and giving some characterisations.

Acknowledgement

We thank the referee for valuable and helpful comments, which enabled us to improve the paper much more.

References

[1] A. Açıkgöz, H. Çakallı, F. Esenbel, Lj. D. R. Kočinac, *A quest of G-continuity in neutrosophic spaces*, Math. Methods Appl. Sci. **44** (2021), 7834–7844.
[2] S. Behram, O. Mucuk, *About varieties of G-sequential methods, G-hulls and G-closures*, Proc. Internat. Math. Sci. **5** (2023) 81–86.
[3] R. Brown, O. Mucuk, *Covering groups of non-connected topological groups revisited*, Math. Proc. Cambridge Phil. Soc. **115** (1994), 97–110.
[4] R. Brown, *Topology and Groupoids*, BookSurge, LLC, Charleston, SC, 2006.
[5] J. Chen, J. Zhang, *On G-mappings defined by G-methods and G-topological groups*, Filomat **35** (2021), 2245–2256.
[6] J. Connor, K.-G. Grosse-Erdmann, *Sequential definitions of continuity for real functions*, Rocky Mountain J. Math. **33** (2003), 93–121.
[7] H. Çakallı, *Sequential definitions of compactness*, Appl. Math. Lett. **21** (2008), 594–598
[8] H. Çakallı, *On G-continuity*, Comput. Math. Appl. **61** (2011), 313–318.
[9] H. Çakallı, *New kinds of continuities*, Comput. Math. Appl. **61** (2011), 960–965.
[10] H. Çakallı, *Sequential definitions of connectedness*, Appl. Math. Lett. **25** (2012), 461–465.
[11] H. Çakallı, O. Mucuk, *On connectedness via a sequential method*, Rev. Un. Mat. Argentina **54** (2013), 101–109.
[12] İ. Çanak, M. Dik, *New types of continuities*, Abstr. Appl. Anal. 2010, Art. ID 258980, 6 pp.
[13] S. Lin, L. Liu, *G-methods, G-sequential spaces and G-continuity in topological spaces.* Topol. Appl. **212** (2016), 29–48.
[14] L. Liu, *G-kernel-open sets, G-kernel-neighbourhoods and G-kernal-derived sets*, J. Math. Res. Appl. **38** (2018), 276–286.
[15] K. Mackenzie, *Lie Groupoids and Lie Algebroids in Differential Geometry*, London Mathematical Society Lecture Note Series, 124. Cambridge University Press, Cambridge, 1987.
[16] G. Di Maio, Lj. D. R. Kočinac, *Statistical convergence in topology*, Topol. Appl. **156** (2008), 28–45.
[17] O. Mucuk, T. Şahan, *On G-sequential continuity*, Filomat **28** (2014), 1181–1189
[18] O. Mucuk, H. Çakallı, *G-connectedness for topological groups with operations*, Filomat **32** (2018) 1079–1089.
[19] O. Mucuk, H. Çakallı,*On G-compactness of topological groups with operations*, Filomat **36** (2022) 7113–7121.
[20] O. Mucuk, S. Behram, *Some counter examples of G-convergence*, Sixth International Conference of Mathematical Sciences (ICMS 2022), AIP Conf. Proc. 2879, 1–4.
[21] Y. Wu, F. Lin, *The G-connected property and G-topological groups*, Filomat **33** (2019), 4441–4450