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On gradient Ricci-Bourguignon harmonic solitons in sequential warped products

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Abstract. Our investigation involves sequential warped product manifolds that contain gradient Ricci-Bourguignon harmonic solutions. We present the primary connections for a gradient Ricci-Bourguignon harmonic solution on sequential warped product manifolds. In practical applications, our research investigates gradient Ricci-Bourguignon harmonic solitons for sequential generalized Robertson-Walker spacetimes and sequential standard static spacetimes. Our finding generalizes all results proven in [23].

1. Introduction and motivations

The choice of the warping function has a significant impact on the properties of warped product manifolds, which have a broad and interesting geometry. Therefore, it is crucial to comprehend how this function behaves when studying these items. The study of warped product manifolds has attracted a lot of attention lately, partly because of its many applications and links to other branches of mathematics. Consequently, there are numerous significant applications in geometry and physics for the study of warped product manifolds. For instance, some black hole spacetimes are modelling using warped product manifolds in general relativity. They appear in the study of moduli spaces of vector bundles on algebraic varieties in algebraic geometry. In topology, they have been used to construct examples of exotic manifolds that do not admit a smooth structure [8].

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On the other hand, it has been noticed that a perturbation of the Ricci solitons equation was proposed by J. P. Bourguignon in [10] which is called the Ricci-Bourguignon soliton. After initiating these concepts, several authors studied them. For examples, S. Dwivedi [13] constructed some results for the solitons of the Ricci-Bourguignon flow and generalized corresponding results for Ricci solitons. The results he proved about Ricci-Bourguignon almost solitons were generalized by using Ricci-Bourguignon almost solitons, introduced from Ricci's almost solitons. He also developed the integral formulas for compact gradient Ricci-Bourguignon solitons and compact gradient Ricci-Bourguignon almost solitons. In the case of constant scalar curvature or conformal vector fields, a compact gradient Ricci-Bourguignon almost soliton is isometric to a Euclidean sphere. For similar studies (see [4, 5, 14, 26–28, 30–33]). In the next study, the concept of harmonic-Ricci solitons was introduced and provided some characterizations of rigidity, generalizing known results for Ricci solitons. In the complete case, the restriction to the steady and shrinking gradient soliton was imposed, and some rigidity results can be traced back to the vanishing of certain modified curvature tensors that take into account the geometry of a Riemannian manifold equipped with a smooth map η, called η-curvature, which is a natural generalization in the setting of harmonic-Ricci solitons of the standard curvature tensor [1]. Furthermore, almost all Ricci-harmonic solitons were defined as generalizations of Ricci-harmonic solitons and harmonic-Einstein metrics [2, 3].

It has been shown that a gradient shrinking almost Ricci-harmonic soliton on a compact domain can be almost harmonic Einstein under some necessary and sufficient conditions. Following the previous concept, the Ricci-Bourguignon harmonic solitons introduced in [29] used the idea of V-harmonic map to study for geometric properties of gradient Ricci-Bourguignon harmonic solitons. As a result, the relationship between gradient Ricci-harmonic solitons and sequential warped product manifolds was established in [22, 23] by considering sequential warped product manifolds consisting of gradient Ricci-harmonic solitons.

They also gave the physical applications of sequential standard static space-time and sequential generalized Robertson-Walker space-time. In the present paper, our main focus is on the studying gradient Ricci-Bourguignon harmonic solitons inspired by [29] in sequential warped product manifolds that are similar to the [23]. Taking motivation from Ricci-harmonic solitons in sequential warped product manifolds, we then introduce the notion of Ricci-Bourguignon harmonic solitons in sequential warped product manifolds and prove some results about them which generalize previous results for Ricci-harmonic solitons in sequential warped product manifolds. We also derive some significant applications for gradient Ricci-Bourguignon solitons in sequential standard static space-time and sequential generalized Robertson-Walker space-time.

2. Basic formulas and Notations

Müller [25] introduced Ricci-harmonic flow, which is defined as follows: for a closed manifold Σ, given a map η from Σ to some closed target manifold $\mathcal{N}$;

$$\frac{\partial}{\partial t}g = -2\mathrm{Ric} + 2\alpha\mathcal{D}\eta\otimes\mathcal{D}\eta, \quad \frac{\partial}{\partial t}\eta = \rho_g\eta,$$

where $g(t)$ is a time-dependent metric on Σ, Ric is the corresponding Ricci curvature, $\rho_g\eta$ is the tension field of η with respect to g and α is a positive constant (possibly time dependent). Moreover, $\mathcal{D}\eta$ stands for the gradient of the function η. In [4], S. Azami developed Ricci-Bourguignon harmonic flow, which is

$$\begin{cases} \frac{\partial}{\partial t}g = -2\mathrm{Ric} - 2\beta \mathrm{R}g + 2\alpha\mathcal{D}\eta\otimes\mathcal{D}\eta \\ \frac{\partial}{\partial t}\eta = \rho_g\eta. \end{cases}$$

Definition 2.1. Let $\eta : (\Sigma, g) \to (\mathcal{N}, h)$ a smooth map (not necessarily a harmonic map), where (Σ, g) and $(\mathcal{N}, h)$ are static Riemannian manifolds.

Then $((\Sigma, g), (\mathcal{N}, h), V, \eta, \beta, \lambda)$ is called Ricci-Bourguignon harmonic solitons if

$$\begin{cases} \mathrm{Ric} - \beta \mathrm{R} g - \alpha \mathcal{D}\eta \otimes \mathcal{D}\eta - \frac{1}{2}\mathcal{L}_V g = \lambda g, \\ \rho_g \eta + \langle \mathcal{D}\eta, V\rangle = 0 \end{cases} \tag{2.1}$$

where R is the scalar curvature function of metric$g$, and $\alpha > 0$ is a positive constant depending on real constants β and λ. On other hand, η is map between (Σ, g) and $(\mathcal{N}, h)$. In particular, when $V = \mathcal{D}f$, Then $((\Sigma, g), (\mathcal{N}, h), V, \eta, \beta, \lambda)$ is called a gradient Ricci-Bourguignon harmonic soliton if it satisfies the coupled system of elliptic partial differential equations

$$\begin{cases} \mathrm{Ric} - \beta \mathrm{R} g - \alpha \mathcal{D}\eta \otimes \mathcal{D}\eta + \mathcal{D}^2 f = \lambda g, \\ \rho_g \eta - \langle \mathcal{D}\eta, \mathcal{D}f\rangle = 0, \end{cases} \tag{2.2}$$

here $f : M \to \mathbb{R}$ be a smooth function and $D^2 f = Hess(f)$. The function f is called the potential function of the Ricci-Bourguignon harmonic soliton. It is obvious that Ricci-Bourguignon harmonic solitons $((\Sigma, g), (\mathcal{N}, h), V, \eta, \beta, \lambda)$ are Ricci harmonic solitons if $\beta = 0$. Azami et al.[5] gave the condition under which a complete shrinking Ricci-harmonic Bourguignon soliton must be compact. The gradient Ricci-harmonic soliton is said to be shrinking, steady, or expanding depending on whether $\lambda > 0, \lambda = 0$ or $\lambda < 0$.

Definition 2.2. The gradient Ricci-Bourguignon harmonic soliton is called trivial if the potential function f is constant.

It can be from (2.2) that when η and f are constants, (Σ, g) must be an Einstein manifold.

2.1. Sequential warped product manifolds

Let (Σ_i, g_i) be three Riemannian manifolds with the associated matric g_i for $i = 1, 2, 3$, then the sequential warped product of the form $\Sigma = (\Sigma_1 \times_{f_1} \Sigma_2) \times_{f_2} \Sigma_3$ is defined as the following metric,

$$g = (g_1 \oplus f_1^2 g_2) \oplus f_2^2 g_3, \tag{2.3}$$

where $f_1 : \Sigma_1 \longrightarrow \mathbb{R}$ and $f_2 : \Sigma_1 \times \Sigma_2 \longrightarrow \mathbb{R}$ are two smooth warping functions. Now we denote the Levi-Civita connections on Σ, Σ_1, Σ_2 and Σ_3 by $\bar{\mathcal{D}}, \mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_3$, respectively. Similarly, Ricci curvature is presented as $\bar{Ric}, Ric^1, Ric^2$, and Ric^3, respectively. We represent the gradient of f_1 on Σ_1 by $\mathcal{D}_1 f_1$ and $\|\mathcal{D}_1 f_1\|^2 = g_1(\mathcal{D}_1 f_1, \mathcal{D}_1 f_1)$. Similarly, the gradient of f_2 on Σ by $\bar{\mathcal{D}} f_2$ and $\|\bar{\mathcal{D}} f_2\|^2 = g(\bar{\mathcal{D}} f_2, \bar{\mathcal{D}} f_2)$.

Now we recall a lemma which will be important in the proof of our main theorem.

Lemma 2.1. [16] Assuming that $\Sigma = (\Sigma_1 \times_{f_1} \Sigma_2) \times_{f_2} \Sigma_3$ is a sequential warped product manifold with metric $g = (g_1 \oplus f_1^2 g_2) \oplus f_2^2 g_3$, then for any $\mathrm{B}_i, \mathrm{V}_i, \mathrm{Z}_i \in \Gamma(\Sigma_i)$ and $i = 1, 2, 3$, the following holds

1. $\bar{\mathcal{D}}_{\mathrm{B}_1} \mathrm{V}_1 = \mathcal{D}_{1\mathrm{B}_1} \mathrm{V}_1$.
2. $\bar{\mathcal{D}}_{\mathrm{B}_1} \mathrm{B}_2 = \bar{\mathcal{D}}_{\mathrm{B}_2} \mathrm{B}_1 = \mathrm{B}_1(\ln f_1)\mathrm{B}_2$.
3. $\bar{\mathcal{D}}_{\mathrm{B}_2} \mathrm{V}_2 = \mathcal{D}_{2\mathrm{B}_2} \mathrm{V}_2 - f_1 g_2(\mathrm{B}_2, \mathrm{V}_2)\mathcal{D}_1 f_1$.
4. $\bar{\mathcal{D}}_{\mathrm{B}_3} \mathrm{B}_1 = \bar{\mathcal{D}}_{\mathrm{B}_1} \mathrm{B}_3 = \mathrm{B}_1(\ln f_2)\mathrm{B}_3$.
5. $\bar{\mathcal{D}}_{\mathrm{B}_3} \mathrm{B}_2 = \bar{\mathcal{D}}_{\mathrm{B}_2} \mathrm{B}_3 = \mathrm{B}_2(\ln f_2)\mathrm{B}_3$.
6. $\bar{\mathcal{D}}_{\mathrm{B}_3} \mathrm{V}_3 = \mathcal{D}_{3\mathrm{B}_3} \mathrm{V}_3 - f_2 g_3(\mathrm{B}_3, \mathrm{V}_3)\bar{\mathcal{D}} f_2$.
7. $\bar{\mathrm{R}}(\mathrm{B}_1, \mathrm{V}_1)\mathrm{Z}_1 = \mathrm{R}_1(\mathrm{B}_1, \mathrm{V}_1)\mathrm{Z}_1$
8. $\bar{\mathrm{R}}(\mathrm{B}_2, \mathrm{V}_2)\mathrm{Z}_2 = \mathrm{R}_2(\mathrm{B}_2, \mathrm{V}_2)\mathrm{Z}_2 - \|\mathcal{D}_1 f_1\|^2\{g_2(\mathrm{B}_2, \mathrm{Z}_2)\mathrm{V}_2 - g_2(\mathrm{V}_2, \mathrm{Z}_2)\mathrm{B}_2\}$.
9. $\bar{\mathrm{R}}(\mathrm{B}_1, \mathrm{V}_2)\mathrm{Z}_1 = -\frac{1}{f_1}\mathcal{D}^2_{f_1}(\mathrm{B}_1, \mathrm{Z}_1)\mathrm{V}_2$.
10. $\bar{\mathrm{R}}(\mathrm{B}_1, \mathrm{V}_2)\mathrm{Z}_2 = f_1 g_2(\mathrm{V}_2, \mathrm{Z}_2)\mathcal{D}_{1\mathrm{B}_1}\mathcal{D}_1 f_1$.
11. $\bar{\mathrm{R}}(\mathrm{B}_1, \mathrm{V}_2)\mathrm{Z}_3 = 0$.
12. $\bar{\mathrm{R}}(\mathrm{B}_i, \mathrm{V}_i)\mathrm{Z}_j = 0, i \neq j$.
13. $\bar{\mathrm{R}}(\mathrm{B}_i, \mathrm{V}_3)\mathrm{Z}_j = -\frac{1}{f_2}\mathcal{D}^2_{f_2}(\mathrm{B}_i, \mathrm{Z}_j)\mathrm{V}_3, i, j = 1, 2$.

14. $\bar{R}(B_i, V_3)Z_3 = f_2 g_3(V_3, Z_3)\mathcal{D}_{B_i}\bar{\mathcal{D}}f_2, i = 1, 2.$
15. $\bar{R}(B_3, V_3)Z_3 = R_3(B_3, V_3)Z_3 - \|\mathcal{D}f_2\|^2\{g_3(B_3, Z_3)V_3 - g_3(V_3, Z_3)B_3\}$

Lemma 2.2. [16] Assuming that $\Sigma = (\Sigma_1 \times_{f_1} \Sigma_2) \times_{f_2} \Sigma_3$ is a sequential warped product manifold with metric $g = (g_1 \oplus f_1^2 g_2) \oplus f_2^2 g_3$, for any $B_i, V_i, Z_i \in \Gamma(\Sigma_i)$ and $i = 1, 2, 3$, the following holds

1. $\bar{Ric}(B_1, V_1) = Ric_1(B_1, V_1) - \frac{n_2}{f_1}\mathcal{D}^2_{f_1}(B_1, V_1) - \frac{n_3}{f_2}\mathcal{D}^2_{f_2}(B_1, V_1).$
2. $\bar{Ric}(B_2, V_2) = Ric_2(B_2, V_2) - f_1^\star g_2(B_2, V_2) - \frac{n_3}{f_2}\mathcal{D}^2_{f_2}(B_2, V_2).$
3. $\bar{Ric}(B_3, V_3) = Ric_3(B_3, V_3) - f_2^\star g_3(B_3, B_3).$
4. $\bar{Ric}(B_i, V_j) = 0,\ i \neq j.$

where $f_1^\star = f_1\Delta_{\Sigma_1} f_1 + (n_2 - 1)\|\mathcal{D}_1 f_1\|^2$ and $f_2^\star = f_2\Delta f_2 + (n_3 - 1)\|\bar{\mathcal{D}} f_2\|^2$.

This paper will consider the harmonic map as a real function $\eta : \Sigma \to \mathbb{R}$. Our first result characterizes locally the harmonic map η_1 using the potential function $\eta_1 : \Sigma \to \mathbb{R}$. On the other hand, $\pi_i : \Sigma_1 \times \Sigma_2 \times \Sigma_3 \to \Sigma_i$ are projection maps for $i = 1, 2, 3$ and $\eta_{\Sigma_i} : \Sigma_i \to \mathbb{R}$ are partial non-constant harmonic maps. It has been shown that the potential function

$$\eta_1 = \eta_{1\Sigma_1} \circ \pi_1, \qquad \eta_{1\Sigma_1} \in C^\infty(\Sigma_1) \tag{2.4}$$

depends only on the base manifold Σ_1 [18, 24]. Precisely, we obtain the following: We now prove the key lemma as follows:

Lemma 2.3. Assuming that $\Sigma = ((\Sigma_1 \times_{f_1} \Sigma_2) \times_{f_2} \Sigma_3, \eta_1, \eta, \beta, \lambda)$ is a gradient Ricci-Bourguignon harmonic soliton on a sequential warped product manifold, including a non-constant harmonic map η, then the harmonic map η can be expressed in the form $\eta = \eta_{\Sigma_1} \circ \pi_1$, *or* $\eta = \eta_{\Sigma_2} \circ \pi_2$, *or* $\eta = \eta_{\Sigma_3} \circ \pi_3$ if and only if $\eta_1 = \eta_{1\Sigma_1} \circ \pi_1$ for a neighborhood v of a point $(p_1, p_2, p_3) \in \Gamma(\Sigma)$, where $\eta_1 \in C^\infty(\Sigma_1)$ is another potential function and $\pi_i : \Sigma_i \longrightarrow \mathbb{R}$ as projection maps for $i = 1, 2, 3$.

Proof. From Lemma 2.2, we can rewrite the following:

$$\bar{Ric}(B_1, V_1) = Ric_1(B_1, V_1) - \frac{n_2}{f_1}\mathcal{D}^2_{f_1}(B_1, V_1) - \frac{n_3}{f_2}\mathcal{D}^2_{f_2}(B_1, V_1) \tag{2.5}$$

$$\bar{Ric}(B_1, V_j) = 0, \qquad j = 2, 3 \tag{2.6}$$

$$\begin{aligned}\bar{Ric}(B_2, V_2) =& Ric_2(B_2, V_2) - \left(f_1\Delta_{\Sigma_1} f_1 + (n_2 - 1)\|\mathcal{D}_1 f_1\|^2\right)g_2(B_2, V_2)\\ &- \frac{n_3}{f_2}\mathcal{D}^2_{f_2}(B_2, V_2),\end{aligned} \tag{2.7}$$

$$\bar{Ric}(B_3, V_3) = Ric_3(B_3, V_3) - \left(f_2\Delta_{\Sigma_2} f_2 + (n_3 - 1)\|\mathcal{D}_2 f_2\|^2\right)g_3(B_3, B_3) \tag{2.8}$$

for all $B_1, V_1 \in \Gamma(\Sigma_1)$, $\quad B_2, V_2 \in \Gamma(\Sigma_2)$ and $B_3, B_3 \in \Gamma(\Sigma_3)$.

Operating equation (2.2) for B_i and B_j, then we have

$$\begin{aligned}&\bar{Ric}(B_i, B_j) + \bar{\mathcal{D}}^2\eta_1(B_i, B_j) - \alpha\bar{\mathcal{D}}\eta(B_i)\bar{\mathcal{D}}\eta(B_j) = (\beta R + \lambda)\bar{g}(B_i, B_j)\\ &\qquad \rho_{\bar{g}}\eta(B_i, B_j) - \bar{g}(\bar{\mathcal{D}}\eta(B_i), \bar{\mathcal{D}}\eta_1(B_j)) = 0,\end{aligned} \tag{2.9}$$

for $i \neq j$ and $2 \leq j \leq 3$. For the first part, let us assume that the harmonic map η can be expressed in the form $\eta = \eta_{\Sigma_1} \circ \pi_1$, $\quad$ or $\quad \eta = \eta_{\Sigma_2} \circ \pi_2$, $\quad$ or $\quad \eta = \eta_{\Sigma_3} \circ \pi_3$, then using (2.9), we find that

$$\begin{aligned}D^2_{\eta_1}(B_1, B_2) &= B_1(B_2(\eta_1)) - \mathcal{D}_{B_1}B_2(\eta_1)\\ &= \alpha\mathcal{D}_\eta(B_1)\mathcal{D}_\eta(B_2) = 0, \quad \forall \quad B_1 \in \Gamma(\Sigma_1) \quad \& \quad \forall \quad B_2 \in \Gamma(\Sigma_2).\end{aligned} \tag{2.10}$$

Also, we get easily

$$\bar{g}(B_1, B_2) = 0 \quad \text{and} \quad \bar{Ric}(B_1, B_2) = 0 \tag{2.11}$$

from (2.6). Similarly, we have

$$D^2_{\eta_1}(\mathrm{B}_1,\mathrm{B}_3) = \mathrm{B}_1(\mathrm{B}_3(\eta_1)) - \mathcal{D}_{\mathrm{B}_1}\mathrm{B}_3(\eta_1)$$
$$= \alpha\mathcal{D}_\eta(\mathrm{B}_1)\mathcal{D}_\eta(\mathrm{B}_3) = 0, \quad \forall \quad \mathrm{B}_1 \in \Gamma(\Sigma_1) \quad \& \quad \forall \quad \mathrm{B}_3 \in \Gamma(\Sigma_3). \tag{2.12}$$

Together, the following

$$\bar{g}(\mathrm{B}_1,\mathrm{B}_3) = 0 \quad \text{and} \quad \bar{\mathrm{Ric}}(\mathrm{B}_1,\mathrm{B}_3) = 0, \tag{2.13}$$

from (2.6). Now combining the (2.10), (2.11), (2.12) and (2.13), using in the first part of (2.9), we derive

$$\alpha\mathcal{D}\eta(\mathrm{B}_i)\mathcal{D}\eta(\mathrm{B}_j) = (\beta\mathrm{R} + \lambda)\bar{g}(\mathrm{B}_i,\mathrm{B}_j) = 0, \tag{2.14}$$

for $i = 1$ and $j = 2, 3$. Then it is easy to find that $\eta_1 = \eta_{1\Sigma_1} \circ \pi_1$ due to [24].

Conversely, we assume that η_1 can be written in the form $\eta_1 = \eta_{1\Sigma_1} \circ \pi_1 \in C^\infty(\Sigma_1)$, then using equation (2.1) and (2.3), we constructed

$$\alpha\mathcal{D}\eta(\mathrm{B}_i)\mathcal{D}\eta(\mathrm{B}_j) = 0 \tag{2.15}$$

for $\mathrm{B}_i \in \Gamma(\Sigma_i)$. From the hypothesis, η is a non-constant map, then there exist a neighborhood point $v = (p_1, p_2, p_3) \in \Gamma(\Sigma_1 \times \Sigma_2 \times \Sigma_3)$ and vector field $\mathrm{B}_i = \mathrm{B}_1 + \mathrm{B}_2 + \mathrm{B}_3$, such that

$$\mathcal{D}\eta(\mathrm{B}_1 + \mathrm{B}_2 + \mathrm{B}_3)\mathcal{D}\eta(\mathrm{B}_1 + \mathrm{B}_2 + \mathrm{B}_3) \neq 0. \tag{2.16}$$

Applying summing up to 3 in (2.16), we get

$$\sum_{i=1}^{3}(\mathcal{D}\eta(\mathrm{B}_i))^2 + \sum_{i=1}^{3}\sum_{j=1,i\neq j}^{3}\mathcal{D}\eta(\mathrm{B}_i)\mathcal{D}\eta(\mathrm{B}_j) \neq 0. \tag{2.17}$$

Now from (2.15) and (2.17), we reached that $\mathcal{D}\eta(\mathrm{B}_i) \neq 0$ for some $i = 1, 2, 3$. The proof is completed. □

2.2. *Main Theorems*

Theorem 2.4. A sequential warped product manifold of the type $\Sigma = ((\Sigma_1 \times_{f_1} \Sigma_2) \times_{f_2} \Sigma_3, \bar{g}, \eta_1, \eta, \beta, \lambda)$ is a gradient Ricci-Bourguignon harmonic soliton if and only if the functions f_i, η_1, η, β and λ satisfy one of the following conditions:

(a) If $\eta = \eta_{\Sigma_1} \circ \pi_1$, then

$$\begin{cases} \mathrm{Ric}_1 - \dfrac{n_2}{f_1}\mathcal{D}_1^2(f_1) - \dfrac{n_3}{f_2}\mathcal{D}^2(f_2) + \mathcal{D}^2(\eta_1) - \alpha\mathcal{D}_1\eta_{\Sigma_1} \otimes \mathcal{D}_1\eta_{\Sigma_1} = \big(\lambda + \beta\mathrm{R}\big)g_1, \\ \big\{\bar{\Delta}_1 - g_1(\mathcal{D}_1, \mathcal{D}_1(\eta_1 - n_2\log(f_1))\big\}\eta_{\Sigma_1} + n_3\mathcal{D}_1\eta_1(\log)(f_2)) = 0, \end{cases} \tag{2.18}$$

$$\mathrm{Ric}_2 - \frac{n_3}{f_2}\mathcal{D}^2(f_2) = \Big\{\big(\lambda + \beta\mathrm{R}\big)f_1 + f_1(\Delta_1 f_1) + (n_2 - 1)\|\mathcal{D}_1 f_1\|^2 - f_1(\mathcal{D}_1\eta_1(f_1)\Big\}g_2, \tag{2.19}$$

and together Σ_3 is Einstein with $\mathrm{Ric}_3 = \lambda_3 g_3$ such that

$$\lambda_3 = \big(\lambda + \beta\mathrm{R}\big)f_2^2 + f_2\Delta f_2 + (n_3 - 1)\|\mathcal{D}f_2\|^2 - f_2(\mathcal{D}_2\eta_1(f_2)). \tag{2.20}$$

(b) If $\eta = \eta_{\Sigma_2} \circ \pi_2$, then

$$\mathrm{Ric}_1 - \frac{n_2}{f_1}\mathcal{D}_1^2(f_1) - \frac{n_3}{f_2}\mathcal{D}^2(f_2) + \mathcal{D}^2(\eta_1) = \left(\lambda + \beta \mathrm{R}\right)g_1, \tag{2.21}$$

$$\begin{cases} \mathrm{Ric}_2 - \dfrac{n_3}{f_2}\mathcal{D}^2(f_2) - \dfrac{\alpha}{f_1^4}\mathcal{D}_2\eta_{\Sigma_2} \otimes \mathcal{D}_2\eta_{\Sigma_2} = \left\{\left(\lambda + \beta \mathrm{R}\right)f_1^2 + f_1\Delta_1 f_1 + (n_2 - 1)\|\mathcal{D}_1 f_1\|^2 - f_1(\mathcal{D}_1\eta_1(f_1))\right\}g_2, \\ \Delta_2\eta_{\Sigma_2} + n_3\mathcal{D}_2\eta_{\Sigma_2}(f_2) = 0, \end{cases} \tag{2.22}$$

together Σ_3 is Einstein with $\mathrm{Ric}_3 - \frac{\alpha}{f_2}\mathcal{D}_2\eta_{\Sigma_2} \otimes \mathcal{D}_2\eta_{\Sigma_2} = \lambda_3 g_3$ such that

$$\lambda_3 = \left(\lambda + \beta \mathrm{R}\right)f_2^2 + f_2\Delta f_2 + (n_3 - 1)\|\mathcal{D}f_2\|^2 - f_2(\mathcal{D}_1\eta_1(f_2)). \tag{2.23}$$

(c) If $\eta = \eta_{\Sigma_3} \circ \pi_3$, then

$$\mathrm{Ric}_1 - \frac{n_2}{f_1}\mathcal{D}_1^2(f_1) - \frac{n_3}{f_2}\mathcal{D}^2(f_2) + \mathcal{D}^2(\eta_1) = \left(\lambda + \beta \mathrm{R}\right)g_1, \tag{2.24}$$

$$\mathrm{Ric}_2 - \frac{n_3}{f_2}\mathcal{D}^2(f_2) = \left\{\left(\lambda + \beta \mathrm{R}\right)f_1^2 + f_1\Delta_1 f_1 + (n_2 - 1)\|\mathcal{D}_1 f_1\|^2 - f_1(\mathcal{D}_1\eta_1(f_1))\right\}g_2 \tag{2.25}$$

$$\begin{cases} \mathrm{Ric}_3 - \dfrac{\alpha}{f_2^4}\mathcal{D}_3\eta_{\Sigma_3} \otimes \mathcal{D}_3\eta_{\Sigma_3} = \lambda_3 g_3, \\ \Delta_3\eta_{\Sigma_3} = 0, \quad in \quad \Sigma_3 \end{cases} \tag{2.26}$$

together with the following,

$$\lambda_3 = \left(\lambda + \beta \mathrm{R}\right)f_2^2 + f_2\Delta f_2 + (n_3 - 1)\|\mathcal{D}f_2\|^2 - f_2(\mathcal{D}_1\eta_1(f_2)). \tag{2.27}$$

where $\mathcal{D}^2 f = Hess(f)$ and $\mathcal{D}f$ is the gradient of the function is f.

Proof. Let $\Sigma = ((\Sigma_1 \times_{f_1} \Sigma_2) \times_{f_2} \Sigma_3, \bar{g}, \eta_1, \eta, \beta, \lambda)$ be a gradient Ricci-Bourguignon harmonic soliton with the assumptions $\eta = \eta_{\Sigma_1} \circ \pi_1$. By applying Lemma 2.3 and Hessian equations from [20] in the main equation (2.1), we arrive at (2.18), Similar procedures, again using Lemma 2.3 and $\eta = \eta_{\Sigma_1} \circ \pi_1$ into the equation (2.1), from Lemma 2.2, we derive that

$$\mathrm{Ric}_2(\mathrm{B}_2, \mathrm{V}_2) - f_1^* g_2(\mathrm{B}_2, \mathrm{V}_2) - \frac{n_3}{f_2}\mathcal{D}^2(\mathrm{B}_2, \mathrm{V}_2) + \mathcal{D}^2\eta_1(\mathrm{B}_2, \mathrm{V}_2) = \left(\lambda + \beta \mathrm{R}\right)f_1^2 g_2(\mathrm{B}_2, \mathrm{V}_2) \tag{2.28}$$

for any $\mathrm{B}_2, \mathrm{V}_2 \in \Gamma(\Sigma_2)$ and $f_1^* = \left(f_1\Delta_1 f_1 + (n_2 - 1)\|\mathcal{D}_1 f_1\|^2\right)$. Including the results from Lemma 2.1 and the relation of the Hessian for any function gives the following

$$\mathcal{D}^2\eta_1(\mathrm{B}_2, \mathrm{V}_2) = f_1\mathcal{D}_1\eta_1(f_1)g_2(\mathrm{B}_2, \mathrm{V}_2). \tag{2.29}$$

Combing the equations (2.28) and (2.29), we get our supposed result (2.19). Now for any $\mathrm{B}_3, \mathrm{V}_3 \in \Gamma(\Sigma_3)$ and using Lemma 2.2 with $\eta = \eta_{\Sigma_1} \circ \pi_1$, we get

$$\mathrm{Ric}_3(\mathrm{B}_3, \mathrm{V}_3) - \left(f_2\Delta_2 f_2 + (n_3 - 1)\|\mathcal{D}f_2\|^2\right)g_3(\mathrm{B}_3, \mathrm{V}_3) + \mathcal{D}^2\eta_1(\mathrm{B}_3, \mathrm{V}_3) = \left(\lambda + \beta \mathrm{R}\right)f_2^2 g_3(\mathrm{B}_3, \mathrm{V}_3). \tag{2.30}$$

the same property as in (2.29) is applied, ones have

$$\mathcal{D}^2\eta_1(\mathrm{B}_3, \mathrm{V}_3) = f_2\mathcal{D}_2\eta_1(f_2)g_3(\mathrm{B}_3, \mathrm{V}_3). \tag{2.31}$$

Inserting (2.31) into (2.30), we derive

$$\mathrm{Ric}_3(\mathrm{B}_3, \mathrm{V}_3) - \{f_2\Delta_2 f_2 + (n_3 - 1)\|\mathcal{D}f_2\|^2\}g_3(\mathrm{B}_3, \mathrm{V}_3) + f_2\mathcal{D}_2\eta_1(f_2)g_3(\mathrm{B}_3, \mathrm{V}_3) = \left(\lambda + \beta\mathrm{R}\right)f_2^2 g_3(\mathrm{B}_3, \mathrm{V}_3) \quad (2.32)$$

From the above equation, it is concluded that Σ_3 is an Einstein manifold. The same procedures will apply to another case, and then it completes the proof of our Theorem.
□

Theorem 2.5. Let a sequential warped product manifold of the type $\Sigma = ((\Sigma_1 \times_{f_1} \Sigma_2) \times_{f_2} \Sigma_3, \bar{g}, \eta_1, \eta, \beta, \lambda)$ is a gradient Ricci-Bourguignon harmonic soliton with non-constant harmonic map η. If $(\lambda + \beta\mathrm{R}) \geq 0$, η_1 tend to maximum or minimum in Σ_1 with the inequality,

$$\frac{n_1}{f_2} tr_{g_1}\mathcal{D}^2(f_2) + \frac{n_2}{f_1}\Delta_1(f_1) \geq \mathrm{R}_1, \quad (2.33)$$

then $\eta_1 = \eta_{1\Sigma_1} \circ \pi_1$ is constant functions, where R_1 is represent the scalar curvature on R_1.

Proof. From the first statement of the theorem and taking the trace in (2.18) for any $\mathrm{B}_1, \mathrm{V}_1 \in \Gamma(\Sigma_1)$

$$\Delta_1\eta_{1\Sigma_1} = n_1\left(\lambda + \beta\mathrm{R}\right) + \alpha\|d\pi_1(\eta)\|^2 - \mathrm{R}_1 + \frac{n_3}{f_2} tr_{g_1}\mathcal{D}^2(f_2) + \frac{n_2}{f_1}\Delta_1(f_1). \quad (2.34)$$

Now from (2.33) and $\left(\lambda + \beta\mathrm{R}\right) \geq 0$ together with η_1 tend to maximum or minimum in Σ_1, it easily conclude from (2.34), the map $\eta_1 = \eta_{1\Sigma_1} \circ \pi_1$ is a constant function. □

Theorem 2.6. Let a sequential warped product manifold of the type $\Sigma = ((\Sigma_1 \times_{f_1} \Sigma_2) \times_{f_2} \Sigma_3, \bar{g}, \eta_1, \eta, \beta, \lambda)$ is a gradient Ricci-Bourguignon harmonic soliton with non-constant harmonic map η such that f_2 tends to the maximum or minimum and the following inequalities hold

$$\left\{\left(\lambda + \beta\mathrm{R}\right) \leq \frac{\mu}{f_2^2} \quad or \quad \left(\lambda + \beta\mathrm{R}\right) \geq \frac{\mu}{f_2^2}\right\} \in \Sigma_1 \times \Sigma_2, \quad (2.35)$$

then f_2 is a constant function.

Proof. One of the most useful elliptic operators of 2^{th} order is defined by

$$\omega(\cdot) = \Delta(\cdot) - \mathcal{D}\eta_1(\cdot) + \frac{n_1 - 1}{f_2}\mathcal{D}f_2(\cdot). \quad (2.36)$$

Implementing (2.20), (2.23) (2.28) and (2.36), we get the following

$$\omega(\cdot) = \frac{\mu - \left(\lambda + \beta\mathrm{R}\right)f_2^2}{f_2}. \quad (2.37)$$

Applying our assumption (2.35) in Eq. (2.37) together, if f_2 tends to a maximum or minimum, then f_2 is a constant function. It completes the proof of the theorem. □

3. Applications in sequential standard static space-time

If we consider $\Sigma_3 = \mathcal{I}$ is an open interval associated with a subinterval of $\mathbb{R}$. In this case, dt^2 is the Euclidean metric tensor on $\mathcal{I}$, then a sequential warped product manifold of the form $\Sigma = ((\Sigma_1 \times_{f_1} \Sigma_2) \times_{f_2} \mathcal{I}, \bar{g})$ turns into *sequential standard static space-time* with metric tensor $\bar{g} = (g_1 \oplus f_1^2 g_2) \oplus f_2^2(-dt^2)$. This type of space-time is defined in [19]. If $\eta : \Sigma \longrightarrow \mathbb{R}$ is a harmonic map, then we have the following result:

Theorem 3.1. Assume that a sequential warped product manifold of the type $\Sigma = ((\Sigma_1 \times_{f_1} \Sigma_2) \times_{f_2} \mathcal{I}, \bar{g}, \eta_1, \eta, \beta, \lambda)$ is a gradient Ricci-Bourguignon harmonic soliton if and only if the functions f, η_1, η, β and λ satisfy one of the following conditions:

(a) If $\eta = \eta_{\Sigma_1} \circ \pi_1$, then

$$\begin{cases} \mathrm{Ric}_1 - \dfrac{n_2}{f_1}\mathcal{D}_1^2(f_1) - \dfrac{n_3}{f_2}\mathcal{D}^2(f_2) + \mathcal{D}^2(\eta_1) - \alpha \mathcal{D}_1\eta_{\Sigma_1} \otimes \mathcal{D}_1\eta_{\Sigma_1} = \Big(\lambda + \beta \mathrm{R}\Big) g_1 \\ \qquad \Big\{\Delta_1 - g_1(\mathcal{D}_1, \mathcal{D}_1(\eta_1 - n_2 \log(f_1))\Big\}\eta_{\Sigma_1} + n_3 \mathcal{D}_1\eta_1(\log)(f_2)) = 0, \end{cases} \tag{3.1}$$

$$\mathrm{Ric}_2 - \frac{n_3}{f_2}\mathcal{D}^2(f_2) = \Big\{\Big(\lambda + \beta\mathrm{R}\Big) f_1 + f_1(\Delta_1 f_1) + (n_2 - 1)\|\mathcal{D}_1 f_1\|^2 - f_1(\mathcal{D}_1\eta_1(f_1)\Big\} g_2, \tag{3.2}$$

and together with the following

$$\Big(\lambda + \beta\mathrm{R}\Big) f_2^2 + f_2\Delta f_2 - f_2(\mathcal{D}_1\eta_1(f_2)) = 0. \tag{3.3}$$

(b) If $\eta = \eta_{\Sigma_2} \circ \pi_2$, then

$$\mathrm{Ric}_1 - \frac{n_2}{f_1}\mathcal{D}_1^2(f_1) - \frac{n_3}{f_2}\mathcal{D}^2(f_2) + \mathcal{D}^2(\eta_1) = \Big(\lambda + \beta\mathrm{R}\Big) g_1, \tag{3.4}$$

$$\begin{cases} \mathrm{Ric}_2 - \dfrac{n_3}{f_2}\mathcal{D}^2(f_2) - \dfrac{\alpha}{f_1^4}\mathcal{D}_2\eta_{\Sigma_2} \otimes \mathcal{D}_2\eta_{\Sigma_2} = \Big\{\Big(\lambda + \beta\mathrm{R}\Big) f_1^2 + f_1\Delta_1 f_1 + (n_2 - 1)\|\mathcal{D}_1 f_1\|^2 \\ \qquad\qquad - f_1(\mathcal{D}_1\eta_1(f_1))\Big\} g_2, \\ \qquad \Delta_2\eta_{\Sigma_2} + n_3\mathcal{D}_2\eta_{\Sigma_2}(f_2) = 0, \end{cases} \tag{3.5}$$

and together with the following:

$$\Big(\lambda + \beta\mathrm{R}\Big) f_2^2 + f_2\Delta f_2 - f_2(\mathcal{D}_1\eta_1(f_2)) = 0. \tag{3.6}$$

(c) If $\eta = \eta_{\mathcal{I}} \circ \pi_{\mathcal{I}}$, then

$$\mathrm{Ric}_1 - \frac{n_2}{f_1}\mathcal{D}_1^2(f_1) - \frac{n_3}{f_2}\mathcal{D}^2(f_2) + \mathcal{D}^2(\eta_1) = \Big(\lambda + \beta\mathrm{R}\Big) g_1, \tag{3.7}$$

$$\mathrm{Ric}_2 - \frac{n_3}{f_2}\mathcal{D}^2(f_2) = \Big\{\Big(\lambda + \beta\mathrm{R}\Big) f_1^2 + f_1\Delta_1 f_1 + (n_2 - 1)\|\mathcal{D}_1 f_1\|^2 - f_1(\mathcal{D}_1\eta_1(f_1))\Big\} g_2, \tag{3.8}$$

$$\begin{cases} \alpha \mathcal{D}_{\mathcal{I}}\eta_{\mathcal{I}} \otimes \mathcal{D}_{\mathcal{I}}\eta_{\mathcal{I}} + f_2^4\Big\{\Big(\lambda + \beta\mathrm{R}\Big) f_2^2 + f_2\Delta f_2 - f_2(\mathcal{D}_1\eta_1(f_2))\Big\} = 0. \\ \qquad \Delta_{\mathcal{I}}\eta_{\mathcal{I}} = 0 \ \ in \ \ \mathcal{I}. \end{cases} \tag{3.9}$$

Proof. For the interval $\mathcal{I}$, the metric tensor is defined as $g_{\mathcal{I}}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = -1$ and Ricci curvature is given as $\mathrm{Ric}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = 0$ in Theorem 2.4, we desire result of theorem. The proof is completed. □

4. Applications in generalized Robertson-Walker space-time

If we consider $\eta : \Sigma \longrightarrow \mathbb{R}$ is a harmonic map through the sequential generalized Robertson-Walker space-time $\Sigma = ((\mathcal{I} \times_{f_1} \Sigma_2) \times_{f_2} \Sigma_3, \bar{g}, \eta_1, \eta, \beta, \lambda)$, then we have the following results.

Theorem 4.1. A sequential generalized Robertson-Walker space-time $\Sigma = ((\mathcal{I} \times_{f_1} \Sigma_2) \times_{f_2} \Sigma_3, \bar{g}, \eta_1, \eta, \beta, \lambda)$ is a gradient Ricci-Bourguignon harmonic soliton if and only if the following differential equations satisfied

(a) If $\eta = \eta_{\Sigma_1} \circ \pi_1$, then

$$\begin{cases} \dfrac{n_2 f_1''}{f_1} + \dfrac{n_3 \mathcal{D}^2(f_2)}{f_2} - \eta_1'' + \alpha \eta_{\mathcal{I}}'' = \lambda + \beta \mathrm{R}, \\ \eta_{\mathcal{I}}'' - \eta_{\mathcal{I}}' \eta_1' + \dfrac{n_2 f_1'}{f_1} \eta_{\mathcal{I}}' + \dfrac{n_3 \mathcal{D} f_2}{f_2} \eta_{\mathcal{I}}' = 0, \end{cases}$$

$$\mathrm{Ric}_2 - \frac{n_3}{f_2} \mathcal{D}^2(f_2) = \Big\{ \Big(\lambda + \beta \mathrm{R} \Big) f_1^2 + f_1 f_1'' + (n_2 - 1)(f_1')^2 - f_1 f_1' \eta_1' \Big\} g_2,$$

and together Σ_3 is Einstein with $\mathrm{Ric}_3 = \lambda_3 g_3$ such that

$$\lambda_3 = \Big(\lambda + \beta \mathrm{R} \Big) f_2^2 + f_2 \Delta f_2 + (n_3 - 1) \|\mathcal{D} f_2\|^2 - (\mathcal{D} f_2) f_2 \eta_1.$$

(b) If $\eta = \eta_{\Sigma_2} \circ \pi_2$, then

$$\frac{n_2 f_1''}{f_1} + \frac{n_3 \mathcal{D}^2(f_2)}{f_2} - \eta_1'' = \lambda + \beta \mathrm{R},$$

$$\begin{cases} \mathrm{Ric}_2 - \dfrac{n_3}{f_2} \mathcal{D}^2(f_2) - \dfrac{\alpha}{f_1^4} \mathcal{D}_2 \eta_{\Sigma_2} \otimes \mathcal{D}_2 \eta_{\Sigma_2} = \Big\{ \Big(\lambda + \beta \mathrm{R} \Big) f_1^2 + f_1 f_1'' + (n_2 - 1)(f_1')^2 - f_1 f_1' \eta_1' \Big\} g_2, \\ \Delta_2 \eta_{\Sigma_2} + n_3 \mathcal{D}_2 \eta_{\Sigma_2}(f_2) = 0, \end{cases}$$

and together Σ_3 is Eintein with $\mathrm{Ric}_3 - \frac{\alpha}{f_2} \mathcal{D}_2 \eta_{\Sigma_2} \otimes \mathcal{D}_2 \eta_{\Sigma_2} = \lambda_3 g_3$ such that

$$\lambda_3 = \Big(\lambda + \beta \mathrm{R} \Big) f_2^2 + f_2 \Delta f_2 + (n_3 - 1) \|\mathcal{D} f_2\|^2 - (\mathcal{D} f_2) f_2 \eta_1.$$

(c) If $\eta = \eta_{\Sigma_3} \circ \pi_3$, then

$$\frac{n_2 f_1''}{f_1} + \frac{n_3 \mathcal{D}^2(f_2)}{f_2} - \eta_1'' = \lambda + \beta \mathrm{R},$$

$$\mathrm{Ric}_2 - \frac{n_3}{f_2} \mathcal{D}^2(f_2) = \Big\{ \Big(\lambda + \beta \mathrm{R} \Big) f_1^2 + f_1 f_1'' + (n_2 - 1)(f_1')^2 - f_1 f_1' \eta_1' \Big\} g_2,$$

$$\begin{cases} \mathrm{Ric}_3 - \dfrac{\alpha}{f_2^4} \mathcal{D}_3 \eta_{\Sigma_3} \otimes \mathcal{D}_3 \eta_{\Sigma_3} = \lambda_3 g_3, \\ \Delta_3 \eta_{\Sigma_3} = 0, \ \ in \ \ \Sigma_3, \end{cases}$$

and together with the following

$$\lambda_3 = \Big(\lambda + \beta \mathrm{R} \Big) f_2^2 + f_2 \Delta f_2 + (n_3 - 1) \|\mathcal{D} f_2\|^2 - (\mathcal{D} f_2) f_2 \eta_1.$$

Proof. Now we define the following for first factor $\mathcal{I}$

$$\begin{aligned} \mathcal{D}_1 f_1 &= -f_1', \\ \mathcal{D}_1^2 f_1 \Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\Big) &= f_1'', \\ \Delta_1 f_1 &= -f_1'', \\ g_{\mathcal{I}} \Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\Big) &= -1, \\ g_{\mathcal{I}}(\mathcal{D}_1 f_1, \mathcal{D}_1 f_1) &= -(f_1')^2 \end{aligned}$$

All the above equations substitute in Theorem 2.4, and we get our desired results. it completes the proof of our Theorem. □

Remark 4.2. As we know if $\beta = 0$ in (2.2), then a gradient Ricci-Bourguignon harmonic soliton is generalized to gradient Ricci-Harmonic soliton which is given in [2, 4]. Now Substitute $\beta = 0$ in Theorem 2.4, Theorem 2.5, Theorem 2.6, Theorem 3.1 and Theorem 3.1. Then Theorem 2.4, Theorem 2.5, Theorem 2.6, Theorem 3.1 and Theorem 3.1 coincide with Theorem 2.1, Theorem 2.2, Theorem 3.1 and Theorem 3.2 in [23]. As a result, our results are the natural generalization of gradient Ricci-Harmonic solitons on sequential warped product manifolds.

Declarations

Ethical Approval

Not applicable.

Competing interest

The authors have no competing interests to declare relevant to this article's content.

Authors' contributions

All authors contributed equally to this work.

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