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Characterizations of C-parallel and C-proper θ_i-slant curves in homothetic s-th Sasakian manifolds

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Abstract. This paper explores C-parallel and C-proper curves within the tangent and normal bundles of homothetic s-th Sasakian manifolds. We establish the necessary and sufficient conditions for θ_i-slant curves to belong to these specific classes. To solidify our theoretical results, we present three concrete examples using parametric equations on a homothetic s-th Sasakian manifold, each satisfying the derived theorems.

1. Introduction

The study of curves using structures on almost contact metric manifolds has received significant attention because of their important geometric and physical properties. Curves on almost contact metric manifolds have been extensively studied, especially in the context of slant curves.

The classification and characterization of slant curves, particularly in Sasakian 3-manifolds were given in [5]. The theory of C-parallel and C-proper curves was introduced by Lee, Suh, and Lee [12], who examined these concepts along slant curves in Sasakian 3-manifolds, specifically in the tangent and normal bundles. The notion of a C-parallel integral submanifold in a Sasakian manifold was defined in [6].

In our previous works [7–9, 13], we investigated Legendre and slant curves in various manifolds, including S-manifolds, introducing new families of slant curves and exploring their geometric properties.

Homothetic s-th Sasakian manifolds, a more general class with a richer f-structure compared to Sasakian manifolds. Hasegawa et al. [11] introduced the the notion of an s-th Sasakian manifold. These manifolds are a special case of trans-S manifolds, where β_i are zero, and α_i are nonzero constants ($i = 1, 2, ..., s$). Furthermore, Alegre et al. [1] defined trans-S manifolds as a broader class of manifolds that includes s-th Sasakian manifolds, C-manifolds and S-manifolds, thus extending the understanding of related structures.

Our one of recent studies of magnetic curves in homothetic s-th Sasakian manifolds has provided that the magnetic curves are indeed certain θ_i-slant helices in these manifolds [10].

Despite these developments, the specific properties of C-parallel and C-proper curves in homothetic s-th Sasakian manifolds have yet to be fully explored. In this paper, we extend the theory by providing a detailed characterization of these curves as a generalization of our previous work in [8], with a focus on the

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necessary and sufficient conditions for θ_i-slant curves to belong to these distinct classes. We also present three examples satisfying the theorems by considering parametric equations in a homothetic s-th Sasakian manifold.

2. Preliminaries

A Riemannian manifold (N, g) is referred to as a *homothetic s-th Sasakian manifold* if it adheres to the following conditions for every $X, Y \in \chi(N)$:

$$f^2X = -X + \sum_{i=1}^{s} \eta_i(X)\,\xi_i, \tag{1}$$

$$\eta_i\left(\xi_j\right) = \delta_{ij},\ f\xi_i = 0,\ \eta_i(fX) = 0,\ \eta_i(X) = g(X, \xi_i),$$

$$g(fX, fY) = g(X, Y) - \sum_{i=1}^{s} \eta_i(X)\eta_i(Y), \tag{2}$$

$$d\eta_i(X, Y) = -d\eta_i(Y, X) = \alpha_i g(X, fY),$$

where f denotes a tensor field of type $(1,1)$, ξ_i $(i = 1, 2, ..., s)$ are characteristic vector fields that are Killing, η_i $(i = 1, 2, ..., s)$ represent 1-forms, and α_i $(i = 1, 2, ..., s)$ are positive constants. Additionally, this f-structure satisfies the normality condition (see [11] and [14]). The manifold is concisely expressed as $N = (N^{2n+s}, f, \xi_i, \eta_i, g)$. If $\alpha_i = 1$ $(i = 1, 2, ..., s)$, then N corresponds to an S-manifold (see [3] and [4]). It is noteworthy that these manifolds constitute a subclass of trans-S-manifolds [1]. In a homothetic s-th Sasakian manifold, the following properties hold:

$$(\nabla_X f)Y = \sum_{i=1}^{s} \alpha_i \left\{g(fX, fY)\xi_i + \eta_i(Y)f^2X\right\},$$

and

$$\nabla_X \xi_i = -\alpha_i fX, \quad i \in \{1, ..., s\},$$

where ∇ represents the Levi-Civita connection associated with g.

Let (N, g) be a Riemannian manifold and $\gamma : I \to N$ a smooth curve. The collection of vector fields $\{T = E_1, E_2, \ldots, E_r\}$ is referred to as the *Frenet frame field* of γ, and it satisfies the following relations:

$$\begin{aligned}
&T = E_1 = \gamma', \\
&\nabla_T T = \kappa_1 E_2, \\
&\nabla_T E_2 = -\kappa_1 T + \kappa_2 E_3, \\
&\vdots \\
&\nabla_T E_j = -\kappa_{j-1} E_{j-1} + \kappa_j E_{j+1}, \quad (2 < j < r), \\
&\vdots \\
&\nabla_T E_r = -\kappa_{r-1} E_{r-1},
\end{aligned}$$

where ∇ represents the Levi-Civita connection. The positive integer $r \leq n$ is called the *osculating order*, and the scalar functions $\kappa_1, \kappa_2, \ldots, \kappa_{r-1}$ are referred to as the *curvatures* of γ. In this context, γ is said to be a *Frenet curve of osculating order r*.

Curves are categorized based on their curvatures as follows: A Frenet curve with osculating order $r = 1$ is a *geodesic*. A Frenet curve with osculating order $r = 2$ and constant curvature κ_1 is a *circle*. A Frenet curve of osculating order $r \leq n$ with constant curvatures $\kappa_1, \kappa_2, \ldots, \kappa_{r-1}$ is referred to as a *helix of order r*. A helix of order $r = 3$ is simply called a *helix*.

Let $N = (N^{2n+s}, f, \xi_i, \eta_i, g)$ be a homothetic s-th Sasakian manifold, and let $\gamma : I \to N$ be a unit-speed curve. The functions $\theta_i = \theta_i(t)$ are known as the *contact angles* between T and ξ_i, and are defined by:

$$\cos\theta_i(t) = g(T, \xi_i).$$

We say that γ is a θ_i*-slant curve* if all θ_i are constant. If these constant contact angles are equal to the same value, we refer to γ as a *slant curve*. Furthermore, if all contact angles are equal to $\frac{\pi}{2}$, γ is called a *Legendre curve* (see [9]). These definitions generalize the definitions of Legendre and slant curves given in [4] and [5].

We start by introducing the following definition:

Definition 2.1. *Let $\gamma : I \to (N^{2n+s}, f, \xi_i, \eta_i, g)$ be parameterized by arc-length in a homothetic s-th Sasakian manifold. The curve γ is referred to as*

a) C-parallel (in the tangent bundle) if

$$\nabla_T H = \lambda \sum_{i=1}^{s} \xi_i,$$

b) C-parallel in the normal bundle if

$$\nabla_T^{\perp} H = \lambda \sum_{i=1}^{s} \xi_i,$$

c) C-proper (in the tangent bundle) if

$$\Delta H = \lambda \sum_{i=1}^{s} \xi_i,$$

d) C-proper in the normal bundle if

$$\Delta^{\perp} H = \lambda \sum_{i=1}^{s} \xi_i.$$

In all cases, ∇ is the Levi-Civita connection, Δ is the Laplacian, λ is a non-zero differentiable function and H is the mean curvature field of γ.

Let $\gamma : I \to N$ be a unit speed curve in an n-dimensional Riemannian manifold (N, g). It is known that (see [2])

$$\nabla_T H = -\kappa_1^2 E_1 + \kappa_1' E_2 + \kappa_1\kappa_2 E_3,$$

$$\nabla_T^{\perp} H = \kappa_1' E_2 + \kappa_1\kappa_2 E_3,$$

$$\begin{aligned} \Delta H &= -\nabla_T\nabla_T\nabla_T T \\ &= 3\kappa_1\kappa_1' E_1 + \left(\kappa_1^3 + \kappa_1\kappa_2^2 - \kappa_1''\right) E_2 \\ &\quad -(2\kappa_1'\kappa_2 + \kappa_1\kappa_2')E_3 - \kappa_1\kappa_2\kappa_3 E_4 \end{aligned}$$

and

$$\begin{aligned}\Delta^{\perp}H &= -\nabla_T^{\perp}\nabla_T^{\perp}\nabla_T^{\perp}T \\ &= \left(\kappa_1\kappa_2^2-\kappa_1''\right)E_2-\left(2\kappa_1'\kappa_2+\kappa_1\kappa_2'\right)E_3 \\ &\quad -\kappa_1\kappa_2\kappa_3E_4.\end{aligned}$$

Thus, we can immediately state the following proposition:

Proposition 2.2. *Let $\gamma : I \to (N^{2n+s}, f, \xi_i, \eta_i, g)$ be a unit speed curve in a homothetic s-th Sasakian manifold. Then*

a) γ is C-parallel (in the tangent bundle) if and only if

$$-\kappa_1^2E_1+\kappa_1'E_2+\kappa_1\kappa_2E_3=\lambda\sum_{i=1}^{s}\xi_i, \tag{3}$$

b) γ is C-parallel in the normal bundle if and only if

$$\kappa_1'E_2+\kappa_1\kappa_2E_3=\lambda\sum_{i=1}^{s}\xi_i, \tag{4}$$

c) γ is C-proper (in the tangent bundle) if and only if

$$3\kappa_1\kappa_1'E_1+\left(\kappa_1^3+\kappa_1\kappa_2^2-\kappa_1''\right)E_2-(2\kappa_1'\kappa_2+\kappa_1\kappa_2')E_3-\kappa_1\kappa_2\kappa_3E_4=\lambda\sum_{i=1}^{s}\xi_i, \tag{5}$$

d) γ is C-proper in the normal bundle if and only if

$$\left(\kappa_1\kappa_2^2-\kappa_1''\right)E_2-\left(2\kappa_1'\kappa_2+\kappa_1\kappa_2'\right)E_3-\kappa_1\kappa_2\kappa_3E_4=\lambda\sum_{i=1}^{s}\xi_i. \tag{6}$$

Our goal now is to apply Proposition 2.2 to θ_i-slant curves in homothetic s-th Sasakian manifolds. Let $\gamma : I \to (N^{2n+s}, f, \xi_i, \eta_i, g)$ be a θ_i-slant curve. Then, differentiating

$$\eta_i(T)=\cos\theta_i,$$

we obtain

$$\eta_i(E_2)=0, \tag{7}$$

where θ_i represents the constant contact angles that satisfy

$$\sum_{i=1}^{s}\cos^2\theta_i\le 1.$$

Equality holds only for geodesics, which correspond to the integral curves of

$$T=\sum_{i=1}^{s}\cos\theta_i\xi_i$$

(see [10]).

For θ_i-slant curves where the sum of the contact angles vanish, we have an interesting Lemma:

Lemma 2.3. *Let $\gamma : I \to N$ be a non-geodesic unit-speed θ_i-slant curve with $\sum_{i=1}^{s} \cos\theta_i = 0$. Then the contact angles satisfy*

$$\frac{-1}{2} < \sum_{i<j} \cos\theta_i \cos\theta_j \leq 0.$$

Proof. The proof is straightforward and relies on the identity

$$\left(\sum_{i=1}^{s} \cos\theta_i\right)^2 = \sum_{i=1}^{s} \cos^2\theta_i + 2\sum_{i<j} \cos\theta_i \cos\theta_j,$$

and the fact that for non-geodesic θ_i-slant curves, the contact angles satisfy

$$\sum_{i=1}^{s} \cos^2\theta_i < 1.$$

□

3. C-parallel θ_i-slant Curves

The first theorem below is a consequence of Proposition 2.2 a).

Theorem 3.1. *Let $\gamma : I \to N$ be a unit-speed θ_i-slant curve. Then γ is C-parallel (in the tangent bundle) if and only if it is a θ_i-slant helix of order $r \geq 3$ satisfying*

$$\sum_{i=1}^{s} \cos\theta_i \neq 0, \ \sum_{i=1}^{s} \alpha_i \neq 0,$$

$$\sum_{i=1}^{s} \xi_i \in sp\{T, E_3\}, \tag{8}$$

$$fT \in sp\{E_2, E_4\},$$

$$\kappa_2 = \frac{\kappa_1 \sqrt{s - \left(\sum_{i=1}^{s} \cos\theta_i\right)^2}}{\left|\sum_{i=1}^{s} \cos\theta_i\right|}, \tag{9}$$

$$\lambda = \frac{-\kappa_1^2}{\sum_{i=1}^{s} \cos\theta_i} = constant, \tag{10}$$

and moreover if $\kappa_3 = 0$, then

$$\kappa_1 = \frac{1}{s} \sqrt{1 - \sum_{i=1}^{s} \cos^2\theta_i} \left|\sum_{i=1}^{s} \cos\theta_i\right| \left|\sum_{i=1}^{s} \alpha_i\right|, \tag{11}$$

$$\kappa_2 = \frac{1}{s} \sqrt{1 - \sum_{i=1}^{s} \cos^2\theta_i} \sqrt{s - \left(\sum_{i=1}^{s} \cos\theta_i\right)^2} \left|\sum_{i=1}^{s} \alpha_i\right|. \tag{12}$$

Proof. Let γ be C-parallel (in the tangent bundle). Then equation (3) must be satisfied. If we apply E_2, using (7), we directly find that $\kappa_1' = 0$. So κ_1 is a constant. Then we apply T and find (10). As a result, we can write

$$\sum_{i=1}^{s} \xi_i = \frac{-\kappa_1^2}{\lambda} T + \frac{\kappa_1 \kappa_2}{\lambda} E_3,$$

which is equivalent to

$$\sum_{i=1}^{s} \xi_i = \left(\sum_{i=1}^{s} \cos \theta_i \right) T - \frac{\kappa_2}{\kappa_1} \sum_{i=1}^{s} \cos \theta_i E_3. \tag{13}$$

So the condition (8) is valid. Notice that $r = 1$ leads to a contradiction, since $\lambda \neq 0$. Let us consider $r = 2$. Then $\kappa_2 = 0$, thus (13) becomes

$$\sum_{i=1}^{s} \xi_i = \left(\sum_{i=1}^{s} \cos \theta_i \right) T.$$

Since ξ_i $(i = 1, 2, ..., s)$ are Killing vector fields, we have $\nabla_T T = 0 = \kappa_1 E_2$. So again, $\kappa_1 = 0$ contradicts $\lambda \neq 0$. As a result $\kappa_2 \neq 0$ and $\sum\limits_{i=1}^{s} \cos \theta_i \neq 0$. Now let $r \geq 3$. From the norm of both sides in equation (13), we obtain

$$s = \left(\sum_{i=1}^{s} \cos \theta_i \right)^2 + \frac{\kappa_2^2}{\kappa_1^2} \left(\sum_{i=1}^{s} \cos \theta_i \right)^2,$$

which gives us (9). Then κ_2 is a constant. We can rewrite (13) as

$$\sum_{i=1}^{s} \xi_i = \left(\sum_{i=1}^{s} \cos \theta_i \right) T \mp \sqrt{s - \left(\sum_{i=1}^{s} \cos \theta_i \right)^2} E_3. \tag{14}$$

Differentiating (14), we find

$$\begin{aligned}
-\left(\sum_{i=1}^{s} \alpha_i \right) fT &= \left[\kappa_1 \left(\sum_{i=1}^{s} \cos \theta_i \right) \pm \kappa_2 \sqrt{s - \left(\sum_{i=1}^{s} \cos \theta_i \right)^2} \right] E_2 \mp \kappa_3 \sqrt{s - \left(\sum_{i=1}^{s} \cos \theta_i \right)^2} E_4 \\
&= \left[\kappa_1 \left(\sum_{i=1}^{s} \cos \theta_i \right) + \kappa_1 \frac{s - \left(\sum\limits_{i=1}^{s} \cos \theta_i \right)^2}{\sum\limits_{i=1}^{s} \cos \theta_i} \right] E_2 \mp \kappa_3 \sqrt{s - \left(\sum_{i=1}^{s} \cos \theta_i \right)^2} E_4 \\
&= \frac{\kappa_1 s}{\sum\limits_{i=1}^{s} \cos \theta_i} E_2 \mp \kappa_3 \sqrt{s - \left(\sum_{i=1}^{s} \cos \theta_i \right)^2} E_4.
\end{aligned} \tag{15}$$

Since $r \geq 3$, we have $\sum\limits_{i=1}^{s} \alpha_i \neq 0$ and $fT \in sp\{E_2, E_4\}$ .We also know that

$$g(fT, fT) = 1 - \sum_{i=1}^{s} \cos^2 \theta_i$$

is a constant. Hence, the norm of both sides in (15) results κ_3 =constant. Moreover, if $\kappa_3 = 0$, we conclude

$$fT = \frac{-\kappa_1 s}{\sum\limits_{i=1}^{s}\alpha_i \sum\limits_{i=1}^{s}\cos\theta_i} E_2.$$

The norm of this last equation gives (11). Then, (12) follows from (9) and (11). Finally, notice that differentiating (14) and (15) repeatedly, we calculate κ_i $(i = 1, 2, ..., r-1)$ are constants, so that γ is a helix of order r.

Conversely, let γ be a θ_i-slant helix of order $r \geq 3$ satisfying the given conditions. Since γ is a θ_i-slant curve and $\sum\limits_{i=1}^{s}\xi_i \in sp\{T, E_3\}$, one can calculate (14). Then, for the given λ, equation (3) is validated. Thus γ is C-parallel (in the tangent bundle). □

Corollary 3.2. *There does not exist a C-parallel Legendre curve in a homothetic s-th Sasakian manifold in the tangent bundle.*

The following theorem holds for C-parallel θ_i-slant curves in the normal bundle:

Theorem 3.3. *Let $\gamma : I \to N$ be a unit-speed θ_i-slant curve. Then γ is C-parallel in the normal bundle if and only if it is a θ_i-slant helix of order $r \geq 3$ satisfying*

$$\sum_{i=1}^{s}\cos\theta_i = 0,\ \sum_{i=1}^{s}\alpha_i \neq 0,$$

$$\sum_{i=1}^{s}\xi_i = \pm\sqrt{s}E_3,$$

$$fT = \frac{\pm\kappa_2\sqrt{s}}{\sum\limits_{i=1}^{s}\alpha_i}E_2 \mp \frac{\kappa_3\sqrt{s}}{\sum\limits_{i=1}^{s}\alpha_i}E_4, \tag{16}$$

$$\lambda = \frac{\pm\kappa_1\kappa_2}{\sqrt{s}} \tag{17}$$

and moreover if $\kappa_3 = 0$, then

$$\kappa_2 = \frac{1}{\sqrt{s}}\sqrt{1 - \sum_{i=1}^{s}\cos^2\theta_i}\left|\sum_{i=1}^{s}\alpha_i\right|,\ fT = \pm\sqrt{1 - \sum_{i=1}^{s}\cos^2\theta_i}E_2. \tag{18}$$

Proof. Let γ be C-parallel in the normal bundle. If we apply T in equation (4), we find $\sum\limits_{i=1}^{s}\cos\theta_i = 0$ because $\lambda \neq 0$. Secondly, from the fact that $\eta_i(E_2) = 0$, $(i = 1, 2, ..., s)$, equation (4) gives us $\kappa_1' = 0$, that is, κ_1 is a constant. We can rewrite (4) as

$$\kappa_1\kappa_2 E_3 = \lambda\sum_{i=1}^{s}\xi_i. \tag{19}$$

This last equation will provide us the rest of the conditions of the Theorem. From the norm of (19), we have

$$\kappa_1\kappa_2 = |\lambda|\sqrt{s},$$

which is the same as (17). Then (19) becomes

$$E_3 = \frac{\pm 1}{\sqrt{s}} \sum_{i=1}^{s} \xi_i. \tag{20}$$

Differentiating E_3 along the curve, we obtain

$$-\kappa_2 E_2 + \kappa_3 E_4 = \frac{\mp 1}{\sqrt{s}} \left(\sum_{i=1}^{s} \alpha_i \right) fT. \tag{21}$$

So $\sum_{i=1}^{s} \alpha_i \neq 0$, because otherwise $\kappa_2 = 0$ gives us $\lambda = 0$, which is a contradiction. As a result, from equation (21), we deduce (16). Moreover, let $\kappa_3 = 0$. In this case, from the norm of

$$fT = \frac{\pm \kappa_2 \sqrt{s}}{\sum_{i=1}^{s} \alpha_i} E_2,$$

we obtain κ_2 as given in (18). Then we write κ_2 in the above equation and calculate fT as desired. If we differentiate (20) and (21) repeatedly, we obtain κ_i $(i = 1, 2, ..., r-1)$ are constants, i.e., γ is a helix of order r. The converse statement is easily proved, by observing that if γ satisfies the given conditions of the Theorem, then equation (4) is satisfied. □

Corollary 3.4. *In a homothetic s-th Sasakian manifold, if a Legendre curve of osculating order $r = 3$ is C-parallel in the normal bundle, then*

$$\kappa_2 = \frac{1}{\sqrt{s}} \left| \sum_{i=1}^{s} \alpha_i \right|, \; fT = \pm E_2.$$

If we take $\alpha_i = 1$, $(i = 1, 2, ..., s)$ in Corollary 3.4, we have the results of Theorem 3.2 in [8].

4. C-proper θ_i-slant Curves

The following theorem can be stated for C-proper θ_i-slant curves in the tangent bundle:

Theorem 4.1. *Let $\gamma : I \to N$ be a unit-speed θ_i-slant curve. Then γ is C-proper (in the tangent bundle) if and only if*

$$r \geq 3, \; \sum_{i=1}^{s} \cos\theta_i \neq 0,$$

$$\sum_{i=1}^{s} \xi_i \in sp\{T, E_3, E_4\},$$

$$\kappa_1 \neq constant, \; \kappa_2 \neq 0,$$

$$\lambda = \frac{3\kappa_1 \kappa_1'}{\sum_{i=1}^{s} \cos\theta_i}, \tag{22}$$

$$\kappa_1^2 + \kappa_2^2 = \frac{\kappa_1''}{\kappa_1}, \tag{23}$$

$$\lambda \sum_{i=1}^{s} \eta_i(E_3) = -(2\kappa_1'\kappa_2 + \kappa_1\kappa_2'), \tag{24}$$

$$\lambda \sum_{i=1}^{s} \eta_i(E_4) = -\kappa_1\kappa_2\kappa_3, \tag{25}$$

$$\left[\sum_{i=1}^{s} \eta_i(E_3)\right]^2 + \left[\sum_{i=1}^{s} \eta_i(E_4)\right]^2 = s - \left(\sum_{i=1}^{s} \cos\theta_i\right)^2. \tag{26}$$

Additionally if $\sum_{i=1}^{s} \alpha_i \neq 0$, *then*

$$fT \in sp\{E_2, E_3, E_4, E_5\}.$$

Moreover if $\kappa_3 = 0$, *then*

$$E_3 = \frac{\pm 1}{\sqrt{s - \left(\sum_{i=1}^{s} \cos\theta_i\right)^2}} \left[-\left(\sum_{i=1}^{s} \cos\theta_i\right) T + \sum_{i=1}^{s} \xi_i\right], \tag{27}$$

$$\kappa_2 = \frac{1}{\sqrt{s - \left(\sum_{i=1}^{s} \cos\theta_i\right)^2}} \left[\left(\sum_{i=1}^{s} \alpha_i\right) \sqrt{1 - \sum_{i=1}^{s} \cos^2\theta_i} \pm \kappa_1 \sum_{i=1}^{s} \cos\theta_i\right], \tag{28}$$

and in this case if $\sum_{i=1}^{s} \alpha_i \neq 0$, *then*

$$fT = \pm\sqrt{1 - \sum_{i=1}^{s} \cos^2\theta_i E_2}. \tag{29}$$

Proof. Let γ be C-proper (in the tangent bundle). If we apply T to equation (5), we obtain

$$\lambda \sum_{i=1}^{s} \cos\theta_i = 3\kappa_1\kappa_1'. \tag{30}$$

If we apply E_2, we find $\kappa_1^3 + \kappa_1\kappa_2^2 - \kappa_1'' = 0$. If κ_1 is a constant, then this last equation becomes $\kappa_1\left(\kappa_1^2 + \kappa_2^2\right) = 0$. This contradicts with $\lambda \neq 0$. Thus $\kappa_1 \neq$constant, so (30) gives us (22). Equation (23) is the same with $\kappa_1^3 + \kappa_1\kappa_2^2 - \kappa_1'' = 0$, divided by κ_1. Equations (24) and (25) are directly calculated by applying E_3 and E_4 to (5), respectively. Then, from the norm of $\sum_{i=1}^{s} \xi_i$ being $\sqrt{s}$, we find equation (26). Differentiating $\sum_{i=1}^{s} \xi_i$ along the curve, if $\sum_{i=1}^{s} \alpha_i \neq 0$, we observe that fT can be written in terms of E_2, E_3, E_4 and E_5. For $\kappa_3 = 0$, the proof is obtained as the previous cases. Conversely, if γ satisfies the given conditions, we can show that equation (5) is satisfied. Thus γ becomes C-proper (in the tangent bundle). □

Corollary 4.2. *There does not exist a C-proper Legendre curve in a homothetic s-th Sasakian manifold in the tangent bundle.*

Finally, we present the following theorem for C-proper θ_i-slant curves in the normal bundle:

Theorem 4.3. *Let $\gamma : I \to N$ be a unit-speed θ_i-slant curve. Then γ is C-proper in the normal bundle if and only if*

$$\sum_{i=1}^{s} \cos\theta_i = 0,$$

$$\sum_{i=1}^{s} \xi_i \in sp\{E_3, E_4\},$$

$$\kappa_1 \neq constant,\ \kappa_2 \neq 0,$$

$$\kappa_1\kappa_2^2 - \kappa_1'' = 0,$$

$$\lambda \sum_{i=1}^{s} \eta_i(E_3) = -(2\kappa_1'\kappa_2 + \kappa_1\kappa_2'),$$

$$\lambda \sum_{i=1}^{s} \eta_i(E_4) = -\kappa_1\kappa_2\kappa_3,$$

$$\left[\sum_{i=1}^{s} \eta_i(E_3)\right]^2 + \left[\sum_{i=1}^{s} \eta_i(E_4)\right]^2 = s$$

Additionally if $\sum_{i=1}^{s} \alpha_i \neq 0$, then

$$fT \in sp\{E_2, E_3, E_4, E_5\}.$$

Moreover if $\kappa_3 = 0$, then

$$\sum_{i=1}^{s} \xi_i = \pm\sqrt{s}E_3,$$

$$\kappa_2 = \frac{1}{\sqrt{s}}\sqrt{1 - \sum_{i=1}^{s} \cos^2\theta_i}\left|\sum_{i=1}^{s} \alpha_i\right|,\ fT = \pm\sqrt{1 - \sum_{i=1}^{s} \cos^2\theta_i}E_2.$$

Proof. The proof is done similar to the proof of Theorem 4.1, using equation (6). □

5. Examples

Let $N = (\mathbb{R}^{2n+s}, f, \xi_i, \eta_i, g)$ be the homothetic s-the Sasakian manifold given in [10]. Let us consider $N = (\mathbb{R}^{2n+s}, f, \xi_i, \eta_i, g)$ with $n = 1$, $s = 2$, $a = 1/2$ and $b = 1/4$. Then N is a homothetic s-th Sasakian manifold with

$$\alpha_1 = \alpha_2 = \frac{a}{b} = 2,\ \sum_{i=1}^{2} \alpha_i = 4.$$

We give the following three examples in N as follows:

Example 5.1. *The curve $\gamma : I \to N$ given by*

$$\gamma(t) = \left(\frac{\sqrt{6}}{6}\sin(6t), \frac{\sqrt{6}}{6}\cos(6t), \frac{7}{2}t + \frac{1}{24}\sin(12t), \frac{3}{2}t + \frac{1}{24}\sin(12t)\right)$$

is C-parallel (in the tangent bundle) with $\lambda = -3/2$. It satisfies Theorem 3.1 with

$$\kappa_1 = \kappa_2 = \frac{\sqrt{6}}{2},\ \kappa_3 = 0,$$

$$\theta_1 = \arccos\left(\frac{3}{4}\right),\ \theta_2 = \arccos\left(\frac{1}{4}\right),\ \sum_{i=1}^{2} \cos\theta_i = 1,$$

$$\sum_{i=1}^{2} \cos^2\theta_i = \frac{5}{8},\ \sqrt{1 - \sum_{i=1}^{2} \cos^2\theta_i} = \frac{\sqrt{6}}{4}.$$

It has the Frenet frame field

$$\left\{T, \frac{2\sqrt{6}}{3} fT, \frac{1}{\sqrt{2}} \sum_{i=1}^{2} \xi_i\right\}.$$

Example 5.2. *The curve $\gamma : I \to N$ given by*

$$\gamma(t) = \left(2\sin\left(\sqrt{2}t\right), 2\cos\left(\sqrt{2}t\right), 2\left(\sqrt{2}+1\right)t + \sin\left(2\sqrt{2}t\right), 2\left(\sqrt{2}-1\right)t + \sin\left(2\sqrt{2}t\right)\right)$$

is C-parallel in the normal bundle with $\lambda = \sqrt{2}$. It satisfies Theorem 3.3 with

$$\kappa_1 = 1,\ \kappa_2 = 2,\ \kappa_3 = 0,$$

$$\theta_1 = \frac{\pi}{3},\ \theta_2 = \frac{2\pi}{3},\ \sum_{i=1}^{2} \cos\theta_i = 0,$$

$$\sum_{i=1}^{2} \cos^2\theta_i = \frac{1}{2},\ \sqrt{1 - \sum_{i=1}^{2} \cos^2\theta_i} = \frac{1}{\sqrt{2}}.$$

It has the Frenet frame field

$$\left\{T, \sqrt{2} fT, \frac{1}{\sqrt{2}} \sum_{i=1}^{2} \xi_i\right\}.$$

Example 5.3. *The curve $\gamma : I \to N$, $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t), \gamma_4(t))$ is C-proper in the normal bundle with $\lambda = -\frac{56\sqrt{2}}{9} e^{\frac{2\sqrt{14}}{3}t}$, where*

$$\gamma_1(t) = \frac{4\sqrt{7}}{3} \int_0^t \cos\left(\frac{-9\sqrt{2}}{28} e^{\frac{2\sqrt{14}}{3}u}\right) du,$$

$$\gamma_2(t) = \frac{4\sqrt{7}}{3} \int_0^t \sin\left(\frac{-9\sqrt{2}}{28} e^{\frac{2\sqrt{14}}{3}u}\right) du,$$

$$\begin{aligned} \gamma_3(t) &= \frac{8t}{3} + \gamma_4(t) \\ &= \frac{4t}{3} + \frac{112}{9} \int_0^t \cos\left(\frac{-9\sqrt{2}}{28} e^{\frac{2\sqrt{14}}{3}u}\right) \left(\int_0^u \sin\left(\frac{-9\sqrt{2}}{28} e^{\frac{2\sqrt{14}}{3}v}\right) dv\right) du. \end{aligned}$$

It satisfies Theorem 4.3 with

$$\kappa_1 = e^{\frac{2\sqrt{14}}{3}t},\ \kappa_2 = \frac{2\sqrt{14}}{3},\ \kappa_3 = 0,$$

$$\theta_1 = \arccos\left(\frac{1}{3}\right),\ \theta_2 = \arccos\left(\frac{-1}{3}\right),\ \sum_{i=1}^{2} \cos\theta_i = 0,$$

$$\sum_{i=1}^{2} \cos^2\theta_i = \frac{2}{9},\ \sqrt{1 - \sum_{i=1}^{2} \cos^2\theta_i} = \frac{\sqrt{7}}{3}.$$

It has the Frenet frame field

$$\left\{T, \frac{3\sqrt{7}}{7} fT, \frac{1}{\sqrt{2}} \sum_{i=1}^{2} \xi_i\right\}.$$

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