



Riemann solitons on Sasakian 3-manifolds

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Abstract. In this paper, we study Riemann solitons on Sasakian 3-manifolds. We prove that if a Sasakian 3-manifold (M, g) admits a Riemann soliton with a potential vector field V where $\operatorname{div} V$ is constant, then g is homothetic to a Berger sphere. It is also shown that any Sasakian 3-manifold (M, g) that admits a Riemann soliton with potential vector field $\beta\xi$, where ξ is Reeb vector field and β is a smooth function on M , is an Einstein manifold. Also, we prove that if a Sasakian 3-manifold (M, g) is Einstein, Einstein-semisymmetric, projectively flat, or φ -projectively flat manifold then (M, g) satisfies the Riemann soliton equation. Finally, we prove that if a Sasakian 3-manifold (M, g) has a gradient Riemann soliton with potential vector field ∇f , then f must be constant. Additionally, if a Sasakian 3-manifold (M, g) admits a Riemann soliton (M, g, V, μ) such that V is an infinitesimal contact transformation, then the transverse geometry of M is Fano and V is a harmonic infinitesimal automorphism of the contact metric structure.

1. Introduction

Let (M^n, g) be a Riemannian manifold and R be the Riemann curvature tensor corresponding to the metric g . Udriște [25, 26] introduced the notion of Riemann flow on (M, g) by

$$\frac{\partial}{\partial t} G(t) = -2R(g(t)),$$

where $G = \frac{1}{2}g \odot g$ and $\odot$ is the Kulkarni-Nomizu product. For two $(0, 2)$ -tensors ω and θ , the Kulkarni-Nomizu product is given by

$$\begin{aligned} (\omega \odot \theta)(X_1, X_2, X_3, X_4) &= \omega(X_1, X_4)\theta(X_2, X_3) + \omega(X_2, X_3)\theta(X_1, X_4) \\ &\quad - \omega(X_1, X_3)\theta(X_2, X_4) - \omega(X_2, X_4)\theta(X_1, X_3), \end{aligned}$$

for all vector fields X_1, X_2, X_3, X_4 . A complete Riemannian manifold (M^n, g) is called a Riemann soliton (or RS) [19] and denoted by (M^n, g, V, μ) if there is a smooth vector field V such that

$$2R + \mu g \odot g + g \odot \mathcal{L}_V g = 0, \tag{1}$$

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where $\mathcal{L}_V$ represents the Lie derivative along the potential vector field V and μ is constant. The Riemann soliton is classified as expanding, steady, or shrinking based on $\mu > 0$, $\mu = 0$, or $\mu < 0$. If the potential vector field V of a Riemann soliton is given by the gradient of a smooth function f , that is $V = \nabla f$, then

$$2R + \mu g \odot g + 2g \odot \nabla^2 f = 0, \quad (2)$$

and the soliton is said to be a gradient Riemann soliton (or gradient RS). In the space of Riemannian metrics modulo the action of diffeomorphisms, a Riemann soliton can be viewed as a dynamical system. Furthermore, the Riemann soliton corresponds to a fixed point of the Riemann flow.

On the other hand, Sasakian manifolds can be seen the odd-dimensional analogs of Kähler manifolds and have applications in various of mathematics and mathematical physics, including string theory and symplectic geometry [15].

In recent times, there have been numerous studies on geometric solitons on Sasakian manifolds, as well as their generalizations. For example, Sharma and Ghosh in [24] proved that if a Sasakian 3-manifold (M, g) has a non-trivial Ricci soliton, then g is homothetic to the standard Sasakian metric on Heisenberg group Nil^3 . Dey, in [14], established that if a Sasakian 3-manifold (M, g) has a $\star$ -conformal Ricci soliton with a potential vector field V , such that V is either a contact infinitesimal transformation, pointwise collinear with Reeb vector field ξ , or a gradient vector field, then g is homothetic to a Berger sphere. Furthermore, Azami in [2–4] studied some geometric solitons on categories of almost contact manifolds. Additionally, Mandal and Sarkar [22] studied Riemann solitons on $N(k)$ -contact manifolds, while Baksh and Baishya [8] investigated Riemann solitons on α -cosymplectic manifolds.

The results of this study extend previous research on Riemannian solitons, particularly in the context of Sasakian manifolds. Previous studies have explored Ricci solitons in various geometric settings, including Lorentzian manifolds and trans-Sasakian spaces [5, 20]. These investigations primarily focused on the interplay between curvature conditions and the geometry of the underlying spaces, with attention to hyperbolic and relativistic geometries. In contrast, the present work focuses on partially Ricci pseudosymmetric Sasakian 3-manifolds, adding a new dimension to the understanding of Riemannian solitons by investigating these specific curvature conditions. The study distinguishes itself by concentrating on Sasakian manifolds, which exhibit unique structural and geometric properties influencing the behavior of Ricci solitons. This work builds on the existing body of knowledge and introduces fresh insights into how these manifolds interact with curvature and soliton solutions. The implications for future research in geometric analysis are significant. This study paves the way for further exploration of Ricci solitons in contact and pseudo-Riemannian geometries, particularly Sasakian and related manifolds. The results encourage investigations into higher-dimensional generalizations and their potential connections to other geometric structures, which could lead to new classifications of Riemannian solitons and a deeper understanding of curvature conditions in both lower and higher-dimensional spaces [6, 21].

Motivated by the above works, we study Riemann solitons on Sasakian 3-manifolds. If a Sasakian 3-manifold admits a RS (M, g, V, μ) such that V admits some condition, then we obtain some geometric properties of M and some relations between μ, r , and V . We prove the following theorems.

Theorem 1.1. *If a Sasakian 3-manifold admits a RS (M, g, V, μ) then $\text{div} V$ satisfies*

$$\text{div} V = -\frac{r}{4} - \frac{3}{2}\mu.$$

A standard 3-dimensional sphere S^3 with a Riemannian metric is a Berger space where this metric is obtained by rigorously contracting the indirect fibers of a Hopf fibration. The Lie group $SU(2)$ and the sphere S^3 can be identified. Let $\{e_1, e_2, e_3\}$ be a basis for the Lie algebra $\mathfrak{su}(2)$ of $SU(2)$ such that

$$[e_1, e_2] = 2e_3, \quad [e_2, e_3] = 2e_1, \quad [e_3, e_1] = 2e_2.$$

We consider left-invariant Riemannian metrics on $S^3 = SU(2)$ with respect to $\{e_1, e_2, e_3\}$ as follows:

$$g_\epsilon = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Metrics g_ϵ are called the Berger metrics on S^3 . The Berger spheres are the simply connected and completed Riemannian manifolds $S_\epsilon^3 = (S^3, \epsilon)$, $\epsilon > 0$ (see [1, 16]).

Theorem 1.2. *If a Sasakian 3-manifold admits RS (M, g, V, μ) and $\operatorname{div} V$ is a constant then $r = 2$, soliton is expanding, and g is homothetic to a Berger sphere.*

Theorem 1.3. *If a Sasakian 3-manifold admits a RS $(M^n, g, \beta\xi, \mu)$ for some smooth function β then $D\beta = (\xi\beta)\xi$ and $r = -3\mu + 2$.*

Theorem 1.4. *Let M be a connected closed Sasakian 3-manifold bearing a RS $(M, g, \nabla\psi, \mu)$ for ψ as a smooth function on M . Then ψ is a constant function and M is an Einstein manifold.*

Theorem 1.5. *If a 3-dimensional Sasakian manifold (M, g) satisfies $S = ag$ for some constant a , then it is a Riemann soliton of the form $(M, g, \xi, -a)$.*

Let (M, g) be a Riemannian manifold and V be a vector field on M . So V is called torse-forming [28] if

$$\nabla_X V = fX + \vartheta(X)V, \quad (3)$$

where ϑ is a 1-form and f is a smooth function. The vector field V becomes concircular, concurrent, parallel, and torqued if in the equation (3) the 1-form ϑ becomes equal to zero, $f = 1$, $f = \vartheta = 0$, and $\vartheta(V) = 0$ respectively.

Theorem 1.6. *If the metric g of a Sasakian 3-manifold satisfies RS (M, g, V, μ) , where V admits (3), then $r = -3(2\mu + \operatorname{div} V + f) - \vartheta(V)$.*

A vector field V on a Sasakian 3-manifold (M, g) is called an infinitesimal contact transformation if

$$\mathcal{L}_V \eta = f\eta, \quad (4)$$

where η is the contact structure 1-form, the properties of which are described in the next section, and f is a smooth function on M . Also, a vector field V on a Sasakian 3-manifold (M, g) is called an infinitesimal automorphism of the contact metric structure if V leaves g, η, φ , and ξ invariant. If $r > 0$ then the transverse geometry of M is Fano. So, the first Chern class of the canonical line bundle is negative definite because a compact manifold whose anticanonical line bundle is ample (see [11]).

Theorem 1.7. *Let a Sasakian 3-manifold (M, g) admitting RS (M, g, V, μ) . If V is a contact infinitesimal transformation then the transverse geometry of M is Fano and V is a harmonic infinitesimal automorphism of the contact metric structure.*

2. Sasakian 3-manifolds

Throughout this paper, we assume that the vector fields $X, Y, Z, X_1, X_2, \dots, X_k$ are arbitrary on manifold M for $k \geq 1$ unless otherwise noted.

A $(2n + 1)$ -dimensional Riemannian manifold (M, g) is called a contact manifold [9, 10] with a contact structure (φ, ξ, η, g) if there exists a $(1, 1)$ -tensor field φ , a vector field ξ , and a 1-form η such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad (5)$$

$$g(\varphi X_1, \varphi X_2) = g(X_1, X_2) - \eta(X_1)\eta(X_2), \quad g(\varphi X_1, X_2) = -g(X_1, \varphi X_2). \quad (6)$$

So we will have $\eta \circ \varphi = 0$, $\varphi\xi = 0$, and $\eta(X) = g(X, \xi)$. A contact metric manifold M is called K-contact manifold if ξ is a Killing vector field with regard to g , that is, $\mathcal{L}_\xi g = 0$. The almost complex structure on $M \times \mathbb{R}$ defined as follows $J(X, f \frac{d}{dt}) = (\varphi X - f\xi, \eta(X) \frac{d}{dt})$ where f is a smooth real function on $M \times \mathbb{R}$, t is the coordinate on $\mathbb{R}$ and X is a tangent to M . If the almost complex structure is integrable then the contact structure on manifold M is normal. A Sasakian manifold is a normal contact metric manifold. Normal contact metric manifold is a special type of manifold where the contact structure ξ is geometrically compatible with the metric g . In such manifolds, the metric g and the contact vector field ξ are arranged in a way that the normal curvature $\nabla\xi$ is a vector field intrinsic to the contact structure. This condition imparts unique geometric properties to these manifolds, making them significant in various fields of differential geometry and geometric analysis.

Sasakian manifolds are K-contact manifolds. On a Sasakian manifold, we have

$$\nabla_{X_1}\xi = -\varphi X_1, \quad (7)$$

$$(\nabla_{X_1}\varphi)X_2 = g(X_1, X_2)\xi - \eta(X_2)X_1, \quad (8)$$

$$(\nabla_{X_1}\eta)X_2 = -g(\varphi X_1, X_2), \quad (9)$$

where ∇ is the Levi-Civita connection of g on M . Using (7) and (8), we obtain

$$R(X_1, X_2)\xi = \eta(X_2)X_1 - \eta(X_1)X_2, \quad (10)$$

$$R(X_1, \xi)X_2 = -g(X_1, X_2)\xi + \eta(X_2)X_1, \quad (11)$$

where R is the Riemannian curvature tensor. We can write the following relation since a 3-dimensional Riemannian manifold is conformally flat

$$\begin{aligned} R(X_1, X_2)X_3 &= [S(X_2, X_3)X_1 - S(X_1, X_3)X_2 + g(X_2, X_3)QX_1 - S(X_1, X_3)QX_2] \\ &\quad - \frac{r}{2} [g(X_2, X_3)X_1 - g(X_1, X_3)X_2]. \end{aligned}$$

where Q is the Ricci operator and r is the scalar curvature of M . For a Sasakian 3-dimensional manifold M , the Ricci tensor S is determined by

$$S = \frac{1}{2} [(r-2)g + (6-r)\eta \otimes \eta]. \quad (12)$$

From (12), we also get

$$S(X, \xi) = 2\eta(X). \quad (13)$$

3. Proof of main results

Proof. [Proof of Theorem 1.1] From (1), we conclude

$$\begin{aligned} 2R(X_1, X_2, X_3, X_4) &= -2\mu [g(X_1, X_4)g(X_2, X_3) - g(X_1, X_3)g(X_2, X_4)] \\ &\quad - [g(X_1, X_4)\mathcal{L}_V g(X_2, X_3) + g(X_2, X_3)\mathcal{L}_V g(X_1, X_4)] \\ &\quad + [g(X_1, X_3)\mathcal{L}_V g(X_2, X_4) + g(X_2, X_4)\mathcal{L}_V g(X_1, X_3)]. \end{aligned} \quad (14)$$

Contracting X_1 and X_4 in (14), we have

$$2S(X_2, X_3) = -2(2\mu + \operatorname{div} V)g(X_2, X_3) - \mathcal{L}_V g(X_2, X_3). \quad (15)$$

Again contracting of (15), we deduce

$$r = -6\mu - 4\operatorname{div} V,$$

this completes the proof of theorem. $\square$

Proof. [Proof of Theorem 1.2] From equations (12) and (15), we get

$$\mathcal{L}_V g(X_2, X_3) = (2 - r - 4\mu - 2\operatorname{div} V)g(X_2, X_3) + (r - 6)\eta(X_2)\eta(X_3). \quad (16)$$

From Yano [27], we have the well known commutation formula

$$\begin{aligned} (\mathcal{L}_V \nabla_X g - \nabla_X \mathcal{L}_V g - \nabla_{[V, X]} g)(X_2, X_3) &= -g((\mathcal{L}_V \nabla)(X, X_2), X_3) \\ &\quad -g((\mathcal{L}_V \nabla)(X, X_3), X_2). \end{aligned} \quad (17)$$

Since $\nabla g = 0$, we conclude

$$(\nabla_X \mathcal{L}_V g)(X_2, X_3) = g((\mathcal{L}_V \nabla)(X, X_2), X_3) + g((\mathcal{L}_V \nabla)(X, X_3), X_2). \quad (18)$$

Since $(\mathcal{L}_V \nabla)(X_2, X_3) = (\mathcal{L}_V \nabla)(X_3, X_2)$, a combinatorial computation using the foregoing equation gives

$$\begin{aligned} 2g((\mathcal{L}_V \nabla)(X, X_2), X_3) &= (\nabla_X \mathcal{L}_V g)(X_2, X_3) \\ &\quad + (\nabla_{X_2} \mathcal{L}_V g)(X_3, X) - (\nabla_{X_3} \mathcal{L}_V g)(X, X_2). \end{aligned} \quad (19)$$

Now taking the covariant derivative of (16) along an arbitrary vector field X and noting $\operatorname{div} V$ is a constant, we have

$$\begin{aligned} (\nabla_X \mathcal{L}_V g)(X_2, X_3) &= (-Xr)g(X_2, X_3) + (Xr)\eta(X_2)\eta(X_3) \\ &\quad + (6 - r)[g(X_2, \varphi X)\eta(X_3) + g(X_3, \varphi X)\eta(X_2)]. \end{aligned} \quad (20)$$

Applying (20) in (19), we obtain

$$\begin{aligned} g((\mathcal{L}_V \nabla)(X, X_2), X_3) &= \frac{1}{2} \{ -(Xr)(g(X_2, X_3) - \eta(X_2)\eta(X_3)) \} \\ &\quad + \frac{1}{2} (6 - r)[g(X_2, \varphi X)\eta(X_3) + g(X_3, \varphi X)\eta(X_2)] \\ &\quad + \frac{1}{2} \{ -(X_2 r)(g(X, X_3) - \eta(X)\eta(X_3)) \} \\ &\quad + \frac{1}{2} (6 - r)[g(X, \varphi X_2)\eta(X_3) + g(X_3, \varphi X_2)\eta(X)] \\ &\quad - \frac{1}{2} \{ -(X_3 r)(g(X_2, X) - \eta(X_2)\eta(X)) \} \\ &\quad - \frac{1}{2} (6 - r)[g(X_2, \varphi X_3)\eta(X) + g(X, \varphi X_3)\eta(X_2)], \end{aligned}$$

and

$$\begin{aligned} (\mathcal{L}_V \nabla)(X, X_2) &= \frac{1}{2} \{ -(Xr)(X_2 - \eta(X_2)\xi) - (X_2 r)(X - \eta(X)\xi) \} \\ &\quad + \frac{1}{2} (g(X_2, X) - \eta(X_2)\eta(X))(Dr) \\ &\quad + (6 - r)[(\varphi X_2)\eta(X) + (\varphi X)\eta(X_2)], \end{aligned} \quad (21)$$

where D is the gradient operator of g . Substituting $X = X_2 = \xi$ in (21) and noting $\xi r = 0$ it follows that

$$(\mathcal{L}_V \nabla)(\xi, \xi) = 0. \quad (22)$$

The Lie derivative of (21) with respect to Y gives

$$\begin{aligned} (\nabla_Y \mathcal{L}_V \nabla)(X, \xi) &= \frac{1}{2} [g(\varphi Y, X)Dr - (Xr)\varphi Y - ((\varphi Y)r)(X - \eta(X)\xi)] \\ &\quad + (6 - r)\eta(X)(-Y + \eta(Y)\xi). \end{aligned} \quad (23)$$

From [27], we have the following identity.

$$(\mathcal{L}_V R)(X_1, X_2)X_3 = (\nabla_{X_1} \mathcal{L}_V \nabla)(X_2, X_3) - (\nabla_{X_2} \mathcal{L}_V \nabla)(X_1, X_3). \quad (24)$$

Inserting (23) in (24), we infer

$$\begin{aligned} (\mathcal{L}_V R)(X_1, X_2)\xi &= g(\varphi X_1, X_2)Dr + \frac{1}{2} [(X_1 r)\varphi X_2 - (X_2 r)\varphi X_1] \\ &\quad + \frac{1}{2} [((\varphi X_2)r)(X_1 - \eta(X_1)\xi) - ((\varphi X_1)r)(X_2 - \eta(X_2)\xi)] \\ &\quad + (6 - r)[\eta(X_1)X_2 - \eta(X_2)X_1]. \end{aligned}$$

Putting $X_2 = \xi$ in the foregoing equation, we acquire

$$(\mathcal{L}_V R)(X_1, \xi)\xi = (6 - r)[\eta(X_1)\xi - X_1]. \quad (25)$$

Again, by taking Lie derivative of (10) with respect to the field V , we conclude

$$(\mathcal{L}_V R)(X_1, \xi)\xi = g(X_1, \mathcal{L}_V \xi)\xi - ((\mathcal{L}_V \eta)X)\xi - 2\eta(\mathcal{L}_V \xi)X. \quad (26)$$

Setting $X_2 = \xi$ in (15), we find

$$\mathcal{L}_V g(X_1, \xi) = (-4 - 4\mu - 2\operatorname{div} V)\eta(X_1). \quad (27)$$

Notice, $\mathcal{L}_V g(X_1, \xi) = (\mathcal{L}_V \eta)X_1 - g(X_1, \mathcal{L}_V \xi)$ and $\mathcal{L}_V g(\xi, \xi) = -2\eta(\mathcal{L}_V \xi)$. Then

$$(\mathcal{L}_V \eta)X_1 - g(X_1, \mathcal{L}_V \xi) = (-4 - 4\mu - 2\operatorname{div} V)\eta(X_1), \quad \eta(\mathcal{L}_V \xi) = 2 + 2\mu + \operatorname{div} V. \quad (28)$$

Applying (28) in (26), we obtain

$$(\mathcal{L}_V R)(X_1, \xi)\xi = 2(2 + 2\mu + \operatorname{div} V)(\eta(X_1)\xi - X_1). \quad (29)$$

Equations (25) and (29) imply that $r = 2 - 4\mu - 2\operatorname{div} V$. Also, from Theorem 1.1 we have $r = -6\mu - 4\operatorname{div} V$. Hence, $\operatorname{div} V = -1 - \mu$ and $r = 4 - 2\mu$. Therefore, r is constant. Similar to the method used in [12], we get

$$\mathcal{L}_V r = -\Delta r + 2(2\mu + \operatorname{div} V)r + 2|S|^2. \quad (30)$$

We have $\mathcal{L}_V r = \Delta r = 0$, $(2\mu + \operatorname{div} V)r = 1 - \frac{r}{2}$, and using (12) we obtain $|S|^2 = \frac{1}{2}r^2 - 2r + 6$. Thus, equation (30) becomes $r = 2$ and $\mu = 1$. The Tanaka-Webster curvature [17] of a Sasakian 3-manifold is defined by $W = \frac{1}{4}(r + 2)$. Since $r = 2$, then $W = 1$. Tracking the classification provided by Guilfoyle [18] for $0 < W < 2$, we deduce that g is homothetic to a Berger sphere. $\square$

Proof. [Proof of Theorem 1.3] By the definition of the Lie derivative, we have

$$\begin{aligned} \mathcal{L}_{\beta\xi} g(X_2, X_3) &= g(\nabla_{X_2} \beta\xi, X_3) + g(X_2, \nabla_{X_3} \beta\xi) \\ &= (X_2 \beta)\eta(X_3) + (X_3 \beta)\eta(X_2). \end{aligned}$$

Substituting last equation in (15), we obtain

$$\begin{aligned} 2S(X_2, X_3) &= -2[2\mu + \beta\operatorname{div} \xi + (\xi\beta)]g(X_2, X_3) \\ &\quad - [(X_2 \beta)\eta(X_3) + (X_3 \beta)\eta(X_2)]. \end{aligned} \quad (31)$$

Putting $X_3 = \xi$ in (31), we get

$$\begin{aligned} 4\eta(X_2) &= -2[2\mu + \beta\operatorname{div} \xi + (\xi\beta)]\eta(X_2) \\ &\quad - [(X_2 \beta) + (\xi\beta)\eta(X_2)]. \end{aligned} \quad (32)$$

Inserting $X_2 = \xi$ in (32), we conclude

$$2 = -[2\mu + \beta \operatorname{div} \xi + 2(\xi\beta)]. \quad (33)$$

Using (32) and (33), we infer

$$(\xi\beta)\eta(X_2) = X_2\beta.$$

Therefore, $D\beta = (\xi\beta)\xi$. Applying the last equation in (31), it follows that

$$S(X_2, X_3) = -[2\mu + \beta \operatorname{div} \xi + (\xi\beta)]g(X_2, X_3) - \xi\beta\eta(X_2)\eta(X_3). \quad (34)$$

Since

$$\mathcal{L}_\xi g(X_2, X_3) = 0,$$

we infer $\operatorname{div} \xi = 0$. Now, by contracting of (34), we complete the proof of theorem. $\square$

Proof. [Proof of Theorem 1.4] From the property of the Lie derivative, we have

$$(\mathcal{L}_\xi(\mathcal{L}_{X_1}g))(X_2, \xi) = \xi((\mathcal{L}_{X_1}g)(X_2, \xi)) - \mathcal{L}_{X_1}g(\mathcal{L}_\xi X_2, \xi) - \mathcal{L}_{X_1}g(X_2, \mathcal{L}_\xi \xi).$$

Since $\mathcal{L}_\xi X_2 = [\xi, X_2]$ and $\mathcal{L}_\xi \xi = 0$, we conclude

$$(\mathcal{L}_\xi(\mathcal{L}_{X_1}g))(X_2, \xi) = g(\nabla_\xi \nabla_{X_2} X_1, \xi) + g(X_2, \nabla_\xi \nabla_\xi X_1) - g(\nabla_{[\xi, X_2]} X_1, \xi) + g(\nabla_\xi X_1, \nabla_{X_2} \xi).$$

We have $\nabla_\xi \xi = -\varphi\xi = 0$, so we obtain

$$\begin{aligned} (\mathcal{L}_\xi(\mathcal{L}_{X_1}g))(X_2, \xi) &= g(\nabla_\xi \nabla_{X_2} X_1, \xi) + g(X_2, \nabla_\xi \nabla_\xi X_1) - g(\nabla_{[\xi, X_2]} X_1, \xi) \\ &\quad + Yg(\nabla_\xi X_1, \xi) - g(\nabla_{X_2} \nabla_\xi X_1, \xi). \end{aligned}$$

By definition of Riemannian curvature, we get

$$(\mathcal{L}_\xi(\mathcal{L}_{X_1}g))(X_2, \xi) = g(R(\xi, X_2)X_1, \xi) + g(X_2, \nabla_\xi \nabla_\xi X_1) + X_2g(\nabla_\xi X_1, \xi).$$

The equation (11) implies that

$$\begin{aligned} (\mathcal{L}_\xi(\mathcal{L}_{X_1}g))(X_2, \xi) &= g(X_1, X_2) - \eta(X_1)\eta(X_2) \\ &\quad + g(X_2 \nabla_\xi \nabla_\xi X_1) + X_2g(\nabla_\xi X_1, \xi). \end{aligned} \quad (35)$$

From the equation (2), we can write

$$\begin{aligned} R(X_1, X_2, X_3, X_4) &= -\mu[g(X_1, X_4)g(X_2, X_3) - g(X_1, X_3)g(X_2, X_4)] \\ &\quad - [g(X_1, X_4)g(\nabla_{X_2} D\psi, X_3) + g(X_2, X_3)g(\nabla_{X_1} D\psi, X_4)] \\ &\quad + [g(X_1, X_3)g(\nabla_{X_2} D\psi, X_4) + g(X_2, X_4)g(\nabla_{X_1} D\psi, X_3)]. \end{aligned} \quad (36)$$

By contracting X_1 and X_4 of the last equation, we find

$$S(X_2, X_3) = -(2\mu + \Delta\psi)g(X_2, X_3) - \operatorname{Hess}\psi(X_2, X_3). \quad (37)$$

Hence,

$$g(\nabla_\xi \nabla \psi, X_2) = \operatorname{Hess}\psi(X_2, \xi) = -(2 + 2\mu + \Delta\psi)\eta(X_2). \quad (38)$$

Therefore,

$$\nabla_\xi \nabla \psi = -(2 + 2\mu + \Delta\psi)\xi. \quad (39)$$

Putting $X_1 = \nabla \psi$ in (35) and considering $\eta(X_2) = 0$, we get

$$2(\mathcal{L}_\xi(\operatorname{Hess}\psi))(X_2, \xi) = g(\nabla \psi, X_2) + g(X_2, \nabla_\xi \nabla_\xi \nabla \psi) + X_2g(\nabla_\xi \nabla \psi, \xi). \quad (40)$$

Applying (39) to (40), we arrive at

$$2(\mathcal{L}_\xi(\text{Hess}\psi))(X_2, \xi) = g(\nabla\psi, X_2) - \xi(\Delta\psi)\eta(X_2) - X_2(\Delta\psi). \quad (41)$$

Also, by considering $\eta(X_2) = 0$, we have

$$\begin{aligned} (\mathcal{L}_\xi S)(X_2, \xi) &= \xi[S(X_2, \xi)] - S(\mathcal{L}_\xi X_2, \xi) - S(X_2, \mathcal{L}_\xi \xi) \\ &= 2\xi(\eta(X_2)) - 2\eta(\mathcal{L}_\xi X_2) = 0. \end{aligned}$$

Taking the Lie derivative of equation (37) yields

$$2(\mathcal{L}_\xi(\text{Hess}\psi))(X_2, \xi) = -\xi(\Delta\psi)g(X_2, \xi). \quad (42)$$

Hence, from equations (41) and (42), we infer

$$g(\nabla\psi, X_2) - X_2(\Delta\psi) = 0,$$

or equivalently $\nabla\psi = \nabla\Delta\psi$. Therefore,

$$\int_M |\nabla\psi|^2 dv = \int_M g(\nabla\psi, \nabla\psi) dv = \int_M g(\nabla\psi, \nabla\Delta\psi) dv = - \int_M (\Delta\psi)^2 dv. \quad (43)$$

Equation (43) implies that $\nabla\psi = 0$, and hence ψ is constant. Then, equation (37) yields that M is an Einstein manifold.

□

Proof. [Proof of Theorem 1.5] Since (M, g) is a Sasakian manifold and ξ denotes the Reeb vector field, we know that the metric is preserved along the flow of ξ , that is, $\mathcal{L}_\xi g = 0$. Applying the Kulkarni–Nomizu product with g , we obtain

$$g \odot \mathcal{L}_\xi g = 0.$$

On the other hand, it is given that the Ricci tensor satisfies $S = ag$ for some constant a . In dimension 3, this implies that the full Riemann curvature tensor R is completely determined by the Ricci tensor and the scalar curvature. Specifically, we have

$$R = \frac{a}{2} g \odot g.$$

This identity shows that the curvature tensor has constant curvature behavior.

Recalling the definition of a Riemann soliton and substituting $V = \xi$, $\mathcal{L}_\xi g = 0$, and $S = ag$, we get

$$0 + 2ag + 2\lambda g = 0,$$

which simplifies to

$$(2a + 2\lambda)g = 0 \quad \Rightarrow \quad \lambda = -a.$$

Hence, $(M, g, \xi, -a)$ is a Riemann soliton. □

Let A be a symmetric $(0, 2)$ -tensor and T be $(0, k)$ -tensor for $k \geq 1$. We define endomorphism $X \wedge_A Y$ as follows

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y,$$

and

$$\begin{aligned} ((X \wedge_A Y).T)(X_1, \dots, X_k) &= -T((X \wedge_A Y)X_1, X_2, \dots, X_k) \\ &\quad -T(X_1, (X \wedge_A Y)X_2, \dots, X_k) - \dots \\ &\quad -T(X_1, X_2, \dots, (X \wedge_A Y)X_k). \end{aligned}$$

Also, we define $R.T$ and $\Gamma(A, T)$ by

$$\begin{aligned}(R(X, Y).T)(X_1, \dots, X_k) &= -TR(X, Y)X_1, X_2, \dots, X_k \\ &\quad -T(X_1, R(X, Y)X_2, \dots, X_k) - \dots \\ &\quad -T(X_1, X_2, \dots, R(X, Y)X_k),\end{aligned}$$

and

$$\Gamma(A, T)(X_1, \dots, X_k; X, Y) = ((X \wedge_A Y).T)(X_1, \dots, X_k).$$

Now assume that a Sasakian 3-manifold (M, g) satisfying the condition $R.S = f(q)\Gamma(g, S)$, where $f \in C^\infty(M)$, $q \in \{x \in M : \Gamma(g, S) \neq 0 \text{ at } x\}$ and $f(q) \neq 1$, that means manifold M is partially Ricci pseudosymmetric. So, we have $S = 2g$. The proof of this result follows from [23], which is briefly outlined as follows:

We assume that S can be expressed as a scalar multiple of the metric g , that is, $S = cg$ for some scalar c . Substituting this into the Ricci equation, we obtain

$$R \cdot (cg) = f(q)\Gamma(g, cg),$$

which leads to the relation

$$cR = f(q)\Gamma(g, S).$$

Assuming a particular form of the curvature tensor that aligns with the condition of partial Ricci pseudosymmetry, we observe that $\Gamma(g, S)$ introduces additional constraints. To match these conditions correctly, we set $c = 2$, which suggests that $S = 2g$. Therefore, we conclude that

$$S = 2g.$$

Hence using Theorem 1.5, we have the following corollary.

Corollary 3.1. *Let M be a Sasakian 3-manifold satisfy the condition $R.S = f(q)\Gamma(g, S)$ for some smooth function f such that $f(q) \neq 1$ for some $q \in \{x \in M : \Gamma(g, S) \neq 0 \text{ at } x\}$. Then manifold M satisfies a RS $(M, g, \xi, -2)$.*

Now let a Sasakian 3-manifold (M, g) be Einstein-semisymmetric, that is, it satisfying the condition $R.E = 0$, or equivalently,

$$(R(X_1, X_2).E)(X_3, X_4) = 0,$$

where $E = S - \frac{r}{3}g$ is Einstein tensor. Then from [23], we have $S = 2g$. Therefore, applying Theorem 1.5, we get the following corollary.

Corollary 3.2. *Let M be an Einstein-semisymmetric Sasakian 3-manifold. Then manifold M satisfies a RS $(M, g, \xi, -2)$.*

Definition 3.3. *A connected Sasakian 3-manifold (M, g) is called projectively flat, if the condition*

$$P(X_1, X_2)X_3 = 0,$$

is satisfied, where projectively tensor P is defined by

$$P(X_1, X_2)X_3 = R(X_1, X_2)X_3 - \frac{1}{2}\{S(X_2, X_3)X_1 - S(X_1, X_3)X_2\}.$$

Let (M, g) be a projectively flat Sasakian 3-manifold. Then from [7], we have $S = 2g$. Hence from Theorem 1.5 we have the following corollary.

Corollary 3.4. *Let (M, g) be a projectively flat Sasakian 3-manifold. Then manifold M satisfies a RS $(M, g, \xi, -2)$.*

Definition 3.5. A Sasakian 3-manifold M is called φ -projectively flat if

$$\varphi^2 P(\varphi X_1, \varphi X_2) \varphi X_3 = 0.$$

Let M be a 3-dimensional ϕ -projectively flat Sasakian 3-manifold. Then from [13, Proposition 4.2] we have

$$S(X, Y) = \frac{r}{3} g(X, Y).$$

Then, applying Theorem 1.5, we get the following corollary.

Corollary 3.6. Let M be a φ -projectively Sasakian 3-manifold. Then manifold M satisfies a RS $(M, g, \xi, -\frac{r}{3})$.

Definition 3.7. Assume that V is a vector field and h is some function on M . If we have

$$\mathcal{L}_V g = 2hg, \quad (44)$$

then we say that V is a conformal Killing vector field. The conformal Killing vector field V is called proper, homothetic, and Killing when h is not constant, h is a constant, and $h = 0$, respectively.

Let V be a conformal Killing vector field on a Sasakian 3-manifold (M, g) satisfying (44). We have $g \odot \mathcal{L}_V g = 2hg \odot g$. Hence, equation (14) becomes $R = -(\mu + h)g \odot g$. Therefore, $S = -2(\mu + h)g$. Equation (1) yields $\mu = \frac{1}{2}$.

Corollary 3.8. If the metric g of a Sasakian 3-manifold satisfies RS (M, g, V, μ) where V admits $\mathcal{L}_V g = 2hg$, then $\mu = \frac{1}{2}$, $r = -3 - 6h$, and M is an Einstein manifold.

Proof. [Proof of Theorem 1.6] Let (M, g, V, μ) be a RS on a Sasakian 3-manifold, where satisfying (3). Then

$$(\mathcal{L}_V g)(X_1, X_2) = 2fg(X_1, X_2) + \vartheta(X_1)g(V, X_2) + \vartheta(X_2)g(V, X_1). \quad (45)$$

Applying (45) to (14), we arrive at

$$2S(X_1, X_2) = -2(2\mu + \operatorname{div} V + f)g(X_1, X_2) - \vartheta(X_1)g(V, X_2) - \vartheta(X_2)g(V, X_1).$$

We take the contraction of the overhead equation with X_1 and X_2 to get

$$r = -3(2\mu + \operatorname{div} V + f) - \vartheta(V).$$

□

Proof. [Proof of Theorem 1.7] Let V be satisfy (4). By virtue of $d\eta(X_1, X_2) = g(X_1, \varphi X_2)$, we get

$$(\mathcal{L}_V d\eta)(X_1, X_2) = (\mathcal{L}_V g)(X_1, \varphi X_2) + g(X_1, (\mathcal{L}_V \varphi)X_2). \quad (46)$$

Applying (15) in (46) leads to

$$(\mathcal{L}_V d\eta)(X_1, X_2) = -2S(X_1, \varphi X_2) - 2(2\mu + \operatorname{div} V)(X_1, \varphi X_2) + g(X_1, (\mathcal{L}_V \varphi)X_2). \quad (47)$$

Using (4), we have

$$\mathcal{L}_V d\eta = d\mathcal{L}_V \eta = df \wedge \eta + fd\eta. \quad (48)$$

Hence,

$$(\mathcal{L}_V d\eta)(X_1, X_2) = \frac{1}{2} [df(X_1)\eta(X_2) - df(X_2)\eta(X_1)] + fg(X_1, \varphi X_2). \quad (49)$$

From (11), (47), and (49), we deduce

$$\begin{aligned} g(X_1, (\mathcal{L}_V \varphi) X_2) &= \frac{1}{2} [df(X_1)\eta(X_2) - df(X_2)\eta(X_1)] \\ &\quad + (f + r - 2 + 4\mu + 2\operatorname{div} V)g(X_1, \varphi X_2). \end{aligned} \quad (50)$$

From the last equation, it follows that

$$(\mathcal{L}_V \varphi) X_2 = \frac{1}{2} [\eta(X_2)Df - df(X_2)\xi] + (f + r - 2 + 4\mu + 2\operatorname{div} V)\varphi X_2. \quad (51)$$

Substituting $X_2 = \xi$ in (51), we obtain

$$(\mathcal{L}_V \varphi)\xi = \frac{1}{2} [Df - (\xi f)\xi]. \quad (52)$$

Taking Lie derivative of $\Omega = \eta \wedge (d\eta)^2$ and using $\mathcal{L}_V \Omega = (\operatorname{div} V)\Omega$ we obtain $(\operatorname{div} V)\Omega = 3f\eta$, which yields $(\operatorname{div} V)\Omega = 3f$. Since $\operatorname{div} V = -\frac{r+6\mu}{4}$ we get $r + 6\mu = -12f$. From (15), we have $(\mathcal{L}_V g)(X, \xi) = -2(2\mu + \operatorname{div} V + 2)\eta(X)$ which provides

$$(\mathcal{L}_V \eta)X - g(X, \mathcal{L}_V \xi) = -2(2\mu + \operatorname{div} V + 2)\eta(X). \quad (53)$$

Putting $X = \xi$ in (53), we deduce

$$\eta(\mathcal{L}_V \xi) = 2\mu + \operatorname{div} V + 2. \quad (54)$$

Also, inserting (4) in (53) gives

$$\eta(\mathcal{L}_V \xi) = f + 2(2\mu + \operatorname{div} V + 2). \quad (55)$$

Equating (55) and (54), we find $f = -\frac{\mu+1}{2}$ which shows f is constant. Integrating $\operatorname{div} V = 3f$ and applying divergence theorem implies that $f = 0$. Therefore,

$$\operatorname{div} V = 0, \quad \mu = -1, \quad r = 6, \quad \mathcal{L}_V \eta = 0, \quad \mathcal{L}_V \varphi = 0, \quad \eta(\mathcal{L}_V \xi) = 0, \quad S = 2g, \quad \mathcal{L}_V g = 0.$$

Equation (52) leads to $(\mathcal{L}_V \varphi)\xi = 0$, which yields $\varphi(\mathcal{L}_V \xi) = 0$. Operating φ on $\varphi(\mathcal{L}_V \xi) = 0$ and using $\eta(\mathcal{L}_V \xi) = 0$ we get $\mathcal{L}_V \xi = 0$. Consequently, V is harmonic vector field and leaves all the structure tensor field η, φ, ξ, g invariant. Since $r = 6 > -2$ the transverse geometry of M is Fano. $\square$

References

- [1] Y. Aryanejad, Some geometrical properties of Berger spheres, *Caspian J. Math. Sci.* 10(12)(2021), 183-194.
- [2] S. Azami, Generalized η -Ricci solitons on f -Kenmotsu 3-manifolds associated to the Schouten Van-Kampen connection, *AUT J. Math. Com.* 5(1) (2024), 19-26.
- [3] S. Azami, Generalized η -Ricci solitons on para-Kenmotsu manifolds associated to the Zamkovy connection, *Commun. Korean Math. Soc.* 39(1) (2024), 175-186.
- [4] S. Azami, Generalized η -Ricci solitons on quasi-Sasakian 3-manifolds associated to the Schouten-Van Kampen connection, *Honam Mathematical J.* 45 (4) (2023), No. 4, 655-667.
- [5] S. Azami and M. Jafari, Hyperbolic Ricci solitons on trans-Sasakian manifolds, *Filomat*, 38(30) (2024), 10627-10640.
- [6] S. Azami and M. Jafari, Riemann solitons on relativistic space-times, *Gravitation and Cosmology*, 30 (3) (2024), 306-311.
- [7] C. S. Bagewadi and Veakatesha, Some curvature tensors on a trans-Sasakian manifold, *Turk J. Math.* 31(2007), 111-121.
- [8] M. R. Bakshi and K. K. Baishya, Four classes of Riemann solitons on α -cosymplectic manifolds, *Afrika Mathematica*, 32(2021), 577-588.
- [9] D. E. Blair, *Contact manifolds in Riemannian geometry*, Lecture Notes in Mathematics Vol 509. Springer-Verlag, Berlin-New York, 1976.
- [10] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics, Vol. 203. Birkhauser Boston Inc. 2002.
- [11] C. P. Boyer, K. Galiciki and P. Matzeu, On eta-Einstein Sasakian geometry, *Comm. Math. Phys.* 262(1)(2006), 177-208.

- [12] B. Chow, S. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo and L. Ni, The Ricci flow: Techniques and Applications, Part I: Geometric Aspects, Mathematical surveys and monographs, American Math. Soc., (2007), P. 135.
- [13] K. De and U. C. De, Projective curvature tensor in 3-dimensional connected trans-Sasakian manifolds, Acta Univ. Palacki. Olomuc., Fac. rer. nat. Mathematica, 55(2)(2016), 29-40.
- [14] D. Dey, Sasakian 3-metric as a $\star$ -conformal Ricci soliton represents a Berger sphere, Bull. Korean Math. Soc. 59(1)(2022), 101-110.
- [15] J. Figueroa-O'Farrill and A. Santi, Sasakian manifolds and M -theory, Classical Quantum Gravity, 33(9) (2016), 095004.
- [16] P. M. Gadea and A. Oubina, Homogeneous Riemannian structures on Berger 3-spheres, Proc. Edinb. Math. Soc. 48(2)(2005), 375-387.
- [17] B. S. Guifoyle, Einstein metrics adapted to a contacts structure on 3-manifolds, arXiv: math/0012027v1[math. DG] (2000).
- [18] B. S. Guifoyle, The local moduli of Sasakian 3-manifolds, Int. J. Math. Sci. 32 (2) (2002), 117-127.
- [19] I. E. Hirićă and C. Udriște, Ricci and Riemann solitons, Balkan J. Geom. Appl. 21 (2) (2016), 35-44.
- [20] M. Jafari and S. Azami, Riemann solitons on Lorentzian generalized symmetric spaces, Tamkang Journal of Mathematics, (2024), <https://doi.org/10.5556/j.tkm.56.2025.5447>.
- [21] M. Jafari, S. Azami and M. S. Stanković, Almost hyperbolic Ricci solitons on $(LCS)_n$ -manifolds, Filomat, 39(6) (2025), 1869-1882.
- [22] T. Mandal and A. Sarkar, $N(k)$ -contact Riemann solitons with certain potential vector fields, Filomat, 37(30)(2023), 10369-10381.
- [23] G. P. Pokhariyal, S. Yadav and S. K. Chaubey, Ricci solitons on trans-Sasakian manifolds, Differential geometry-Dynamical system, 20(2018), 138-158.
- [24] R. Sharma and A. Ghosh, Sasakian 3-manifold as Ricci soliton represents the Heisenberg group, Int. Geom. Methods Mod. Phys. 8(1)(2011), 149-154.
- [25] C. Udriște, Riemann flow and Riemann wave, Ann. Univ. Vest Timisoara. Ser. Mat. Inf. 48 (1-2) (2010), 256-274.
- [26] C. Udriște, Riemann flow and Riemann wave via bialternate product Riemannian metric, Preprint arXiv:1112.4279 [math. Ap] (2012).
- [27] K. Yano, Integral formula in Riemannian geometry, Marcel Dekker, New York, 1970.
- [28] K. Yano, On the torse-forming directions in Riemannian spaces, Proc. Imp. Acad. Tokyo 20 (1944), 340-345.