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On n-nonlinear Caputo fractional q-differential systems

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Abstract. This paper discusses the significance of quantum calculus in some mathematical fields. It specifically investigates solutions' existence, uniqueness, and stability for a system of n-nonlinear fractional q-differential equations with initial conditions involving Caputo fractional q-derivatives. The paper utilizes Schauder's and Banach's fixed-point theorems and Ulam-Hyers' stability criteria to explore the analytical dynamics inherent in these solutions. Additionally, it provides two illustrative examples to demonstrate the practical applicability of the obtained results.

1. Introduction

Fractional calculus, which extends differentiation and integration to non-integer orders, has gained significant attention from researchers because of its broad applications in modeling a wide range of scientific and technical phenomena. Fractional equations are utilized across various fields, such as blood flow dynamics, electrical circuits, biology, chemistry, physics, control theory, wave propagation, and signal and image processing, among others. For a more in-depth examination of the practical applications of fractional calculus, readers can refer to works by Afshari *et al.* [2–4], Agrawal [5], Benchohra *et al.* [1, 6–9], Herrmann [10], Hilfer [11], Kilbas *et al.* [12], and Samko *et al.* [13].

In the 20th century, physicists sparked a scientific revolution in quantum mechanics, leading mathematicians to develop quantum calculus. Jackson was the first to introduce a new concept in quantum calculus, which later became influential in various disciplines, including mathematics, mechanics, and physics, see [14, 15]. Recognizing the substantial applications of fractional q-calculus in many physical, biological, and economic systems, Al-Salam and Agarwal proposed the concept of fractional q-calculus, see [16, 17]. The effective role of quantum fractional differential equations (q-FDEs) in physics and mechanics has resulted in numerous applications across different fields, as discussed in [18–22].

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In [23], Salim *et al.* investigated the following fractional q-difference problem:

$$\begin{cases} ({}^cD_q^\zeta \xi)(\varsigma) = \wp(\varsigma, \xi(\varsigma)); \ \varsigma \in \Psi := [0, \beta], \\ \\ \xi(0) = \xi_0 \in F, \end{cases}$$

where $q \in (0,1)$, $\zeta \in (0,1]$, $\beta > 0$, $\wp : \Psi \times F \to F$ is a given continuous function, F is a real (or complex) Banach space with norm $\|\cdot\|$, and ${}^cD_q^\zeta$ is the Caputo fractional q-difference derivative of order ζ.

In [24], the authors proved some existence of solutions for the following problem with implicit fractional q-difference equations in Banach algebras:

$$\begin{cases} {}^cD_q^\zeta \left(\frac{\xi(\varsigma)}{h(\varsigma,\xi(\varsigma))}\right) = \psi\left(\varsigma, \xi(\varsigma), {}^c D_q^\zeta \left(\frac{\xi(\varsigma)}{h(\varsigma,\xi(\varsigma))}\right)\right); \ \varsigma \in \Psi := [0, \beta], \\ \\ \xi(0) = \xi_0 \in \mathbb{R}, \end{cases}$$

where $q \in (0,1)$, $\zeta \in (0,1]$, $\beta > 0, h : \Psi \times \mathbb{R} \to \mathbb{R}^*$, $\psi : \Psi \times \mathbb{R}^2 \to \mathbb{R}$ are given functions.

In [25], Basti and Arioua investigated the existence and uniqueness of solutions for a system of n-nonlinear fractional differential equations with integral conditions of the form:

$$\begin{cases} {}^\rho\mathcal{D}_{0^+}^{\alpha_1} y_1(t) = f_1\left(t, y(t), {}^\rho\mathcal{D}_{0^+}^{\beta_1} y_1(t)\right), \ t \in [0,T], \\ {}^\rho\mathcal{D}_{0^+}^{\alpha_2} y_2(t) = f_2\left(t, y(t), {}^\rho\mathcal{D}_{0^+}^{\beta_2} y_2(t)\right), \ t \in [0,T], \\ \vdots \\ {}^\rho\mathcal{D}_{0^+}^{\alpha_n} y_n(t) = f_n\left(t, y(t), {}^\rho\mathcal{D}_{0^+}^{\beta_n} y_n(t)\right), \ t \in [0,T], \end{cases}$$

with the integral conditions

$${}^\rho\mathcal{I}_{0^+}^{1-\alpha_1} y(0^+) = {}^\rho\mathcal{I}_{0^+}^{1-\alpha_2} y(0^+) = \cdots = {}^\rho\mathcal{I}_{0^+}^{1-\alpha_n} y(0^+) = 0,$$

where $y = (y_1, y_2, \ldots, y_n)$, for $n \in \mathbb{N}$, also $T, \rho > 0$, $0 < \beta_i \le \alpha_i \le 1$ and $f_i : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ are continuous functions.

Based on previous research, we investigate the existence, uniqueness and stability of solutions for a system of n-nonlinear q-FDEs of the following form:

$$\begin{cases} {}^C\mathcal{D}_{q,0^+}^{\sigma_1} y_1(t) = f_1(t, y(t)), \ t \in [0,T], \\ {}^C\mathcal{D}_{q,0^+}^{\sigma_2} y_2(t) = f_2(t, y(t)), \ t \in [0,T], \\ \vdots \\ {}^C\mathcal{D}_{q,0^+}^{\sigma_n} y_n(t) = f_n(t, y(t)), \ t \in [0,T], \end{cases} \tag{1}$$

with the initial conditions:

$$y_i(0^+) = \zeta_i \in \mathbb{R}, \ \text{for each } i \in \{1, \ldots, n\}, \tag{2}$$

where $0 < \sigma_i \le 1$ and $y = (y_1, y_2, \ldots, y_n)$, for $n \in \mathbb{N}$. In addition $f_i : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ are continuous functions for each $i \in \{1, \ldots, n\}$. Here ${}^C\mathcal{D}_q^\sigma$ is the Caputo fractional q-derivative of order σ, with $T > 0$ and $0 < q < 1$.

The following are the primary novelties of the current paper:

- Considering the diverse conditions we applied to system (1), our work can be seen as a continuation of the studies mentioned above.
- Our work continues upon the obtained results in [23, 24] by extending the fractional equations to n-nonlinear q-FDEs. This will necessitate the use of more complex approaches with different spaces and norms.
- Our findings expand upon those in [25] by considering n-nonlinear with q-fractional derivatives.
- We also address the qualitative aspect by investigating the Ulam stability of our system.

2. Preliminaries and Fundamental Concepts

This section presents concepts of q-calculus and provides some definitions and lemmas that will be used in this paper (see [26–29]).

The q-integer $[\sigma]_q$, for $\sigma \in \mathbb{R}$, given by

$$[\sigma]_q = \begin{cases} \frac{1-q^\sigma}{1-q}, & q \neq 1, \\ \sigma, & q = 1. \end{cases}$$

The definition of q-factorial $[m]_q!$ is

$$[m]_q! = \begin{cases} [m]_q [m-1]_q \cdots [1]_q, & m \in \mathbb{N}, \\ 1, & m = 0. \end{cases}$$

We introduce the q-shifted as follows

$$(t-s)_q^{(0)} = 1, \quad (t-s)_q^{(k)} = \prod_{j=0}^{k-1} \left(t - sq^j\right), \text{ for } k \in \mathbb{N}, \text{ and } 0 < s < t.$$

The concept of the q-shifted is also introduced for $\sigma \in \mathbb{R}$ and $\sigma \notin \mathbb{N}$

$$(t-s)_q^{(\sigma)} = t^\sigma \prod_{k=0}^{\infty} \frac{t - q^k s}{t - q^{\sigma+k} s}, \quad 0 \leq s \leq t.$$

The q-analogue of Gamma function for $\sigma \in \mathbb{R} \setminus \{-m,\ m \in \mathbb{N} \cup \{0\}\}$ is defined by

$$\Gamma_q(\sigma) = \frac{(1-q)^{(\sigma-1)}}{(1-q)^{\sigma-1}}, \quad 0 < q < 1.$$

Obviously

$$\Gamma_q(1) = 1, \quad \Gamma_q(m+1) = [m]_q!, \quad \text{and} \quad \Gamma_q(\sigma+1) = [\sigma]_q \Gamma_q(\sigma).$$

The q-derivative of a function u is defined by

$$D_q^0 u(t) = u(t), \quad \text{and} \quad D_q u(t) = \frac{u(t) - u(qt)}{(1-q)t}, \ t \neq 0,$$

and $D_q u(0) = \lim\limits_{t \to 0} D_q u(t)$. Also, the q-derivative of higher order is given by

$$\left(D_q^m u\right)(t) = D_q D_q^{m-1} u(t), \quad m \in \mathbb{N}.$$

We have the following q-derivatives with respect to t and s

$$\begin{aligned} {}_tD_q\left[(t-s)^{(\sigma)}\right] &= [\sigma]_q (t-s)^{(\sigma-1)}, \\ {}_sD_q\left[(t-s)^{(\sigma)}\right] &= -[\sigma]_q (t-qs)^{(\sigma-1)}. \end{aligned}$$

Definition 2.1 ([30]). *Let $\sigma > 0$ and u be a function defined on $[0, T]$, the fractional q-integral of the Riemann-Liouville is defined by*

$$\mathcal{I}_q^\sigma u(t) = \frac{1}{\Gamma_q(\sigma)} \int_0^t (t-qs)^{(\sigma-1)} u(s)\, d_q s,$$

and $\mathcal{I}_q^0 u(t) = u(t)$.

Definition 2.2 ([31]). *Let $\sigma > 0$ and u be a function defined on $[0, T]$, the fractional q-derivative of Riemann-Liouville is defined by*

$$\mathcal{D}_q^0 u(t) = u(t) \quad \text{and} \quad \mathcal{D}_q^\sigma u(t) = D_q^m \mathcal{I}_q^{m-\sigma} u(t),$$

where m is the smallest integer greater than or equal to σ.

Lemma 2.3 ([32]). *Let $\sigma, \beta > 0$, and u be a function defined on $[0, T]$. Then the next formula hold*

1. $\mathcal{I}_q^\sigma \mathcal{I}_q^\beta u(t) = \mathcal{I}_q^{\sigma+\beta} u(t)$,
2. $\mathcal{D}_q^\sigma \mathcal{I}_q^\sigma u(t) = u(t)$.

Definition 2.4 ([31]). *Let $\sigma > 0$, the fractional q-derivative of the Caputo type is defined by*

$${}^C\mathcal{D}_q^0 u(t) = u(t) \quad \text{and} \quad {}^C\mathcal{D}_q^\sigma u(t) = \mathcal{I}_q^{m-\sigma} D_q^m u(t), \ t \in [0, T],$$

where m is the smallest integer greater than or equal to σ.

Lemma 2.5 ([31]). *Let $\sigma \in \mathbb{R}^+$, then the following equality holds*

$$\left(\mathcal{I}_q^\sigma \ {}^C\mathcal{D}_q^\sigma u(t)\right) = u(t) - \sum_{k=0}^{m-1} \frac{D_q^k u(0^+)}{\Gamma_q(k+1)} t^k.$$

If $\sigma \in (0, 1)$, we have

$$\left(\mathcal{I}_q^\sigma \ {}^C\mathcal{D}_q^\sigma u(t)\right) = u(t) - u(0).$$

Lemma 2.6 ([27]). *Let $\sigma > 0$, assume that $u \in C([0, T], \mathbb{R})$, then*

$${}^C\mathcal{D}_q^\sigma u(t) = \mathcal{D}_q^\sigma \left(u(t) - \sum_{k=0}^{m-1} \frac{D_q^k u(0^+)}{\Gamma_q(k+1)} t^k \right),$$

where $C([0, T], \mathbb{R})$ is a Banach space of all continuous functions from $[0, T]$ into $\mathbb{R}$, with the norm

$$\|u\| = \sup_{0 \le t \le T} |u(t)|.$$

3. Main Results

Here, we provide several lemmas to demonstrate our main results.

Lemma 3.1. *Let $0 < \sigma_i \le 1$, for any $y_i \in C([0, T], \mathbb{R})$ and continuous functions $(f_i)_{i=\overline{1,n}} : [0, T] \times \mathbb{R}^n \to \mathbb{R}$, the solutions of problem (1)–(2) is equivalent to the n-fractional q-integral equations*

$$\begin{cases} y_1(t) = \zeta_1 + \int_0^t G_{\sigma_1}(t, qs) f_1(s, y(s)) \, d_q s, \\ y_2(t) = \zeta_2 + \int_0^t G_{\sigma_2}(t, qs) f_2(s, y(s)) \, d_q s, \\ \qquad \vdots \\ y_n(t) = \zeta_n + \int_0^t G_{\sigma_n}(t, qs) f_n(s, y(s)) \, d_q s, \end{cases} \tag{3}$$

where G_{σ_i} are continuous functions, also called the Green's functions written as follows

$$G_{\sigma_i}(t, qs) = \frac{(t - qs)^{(\sigma_i - 1)}}{\Gamma_q(\sigma_i)}, \tag{4}$$

for each $i \in \{1, \ldots, n\}$, $s \in (0, t]$ and $t \in [0, T]$.

Proof. Let $0 < \sigma_i \leq 1$, $0 < q < 1$, and $y_i \in C([0,T], \mathbb{R})$ for each $i \in \{1, \ldots, n\}$. Starting applying $\mathcal{I}_q^{\sigma_i}$ on both sides of system (1), we obtain

$$\left(\mathcal{I}_q^{\sigma_i}\, {}^C\mathcal{D}_q^{\sigma_i} y_i(t)\right) = \mathcal{I}_q^{\sigma_i} f_i(s, y(s)),$$

from Lemma 2.5, by using conditions (2), we get

$$y_i(t) = \zeta_i + \mathcal{I}_q^{\sigma_i} f_i(t, y(t)). \tag{5}$$

Otherwise, by applying $\mathcal{D}_q^{\sigma_i}$ on both sides of the equations (5), utilizing Lemma 2.6, we find that

$$\mathcal{D}_q^{\sigma_i}(y_i(t) - \zeta_i) = \mathcal{D}_q^{\sigma_i} \mathcal{I}_q^{\sigma_i} f_i(t, y(t)).$$

Then,

$${}^C\mathcal{D}_q^{\sigma_i} y_i(t) = f_i(t, y(t)).$$

Furthermore, from (5), we find

$$y_i(0^+) = \zeta_i, \text{ for each } i \in \{1, \ldots, n\}.$$

□

Let us define $E = X_1 \times X_2 \times \cdots \times X_n$, where $X_i = \{y_i, \quad y_i \in C([0,T], \mathbb{R})\}_{i=\overline{1,n}}$, with the norm

$$\left\|y\right\|_E = \max_{1\leq i\leq n} \left\|y_i\right\|.$$

Under this norm $(E, \|\cdot\|_E)$ is a Banach space.

Let us define the integral operators

$$\mathcal{T}_i y(t) = \zeta_i + \int_0^t G_{\sigma_i}(t, qs)\, f_i(s, y(s))\, d_q s. \tag{6}$$

Since $f_i(t, y)$ and $G_{\sigma_i}(t, qs)$ are continuous functions for each $i \in \{1, \ldots, n\}$, then $\mathcal{T}_i$ are continuous, i.e., $\mathcal{T}_i(E) \subset X_i$.

Let $\mathcal{T} : E \to E$ be a system of n-operators that defined by

$$\mathcal{T} y(t) = \begin{pmatrix} \mathcal{T}_1 y(t) \\ \mathcal{T}_2 y(t) \\ \vdots \\ \mathcal{T}_n y(t) \end{pmatrix}, \tag{7}$$

where $(\mathcal{T}_i)_{i=\overline{1,n}}$ are the integral operators given by (6), characterized by the norm

$$\left\|\mathcal{T} y\right\|_E = \max_{1\leq i\leq n} \left\|\mathcal{T}_i y\right\|,$$

with $\left\|\mathcal{T}_i y\right\| = \sup\limits_{0\leq t\leq T} \left|\mathcal{T}_i y(t)\right|$.

3.1. Existence and Uniqueness Results of Solutions

This part discusses the existence and uniqueness of solutions, along with illustrative examples. We impose the following hypotheses:

(H_1) For any $y, \tilde{y} \in \mathbb{R}^n$, there exist $n \times n$ nonnegative constants $(\lambda_{i,k})_{i,k=\overline{1,n}}$ such that the functions f_i satisfy

$$\left|f_i(t,y) - f_i(t,\tilde{y})\right| \leq \sum_{k=1}^{n} \lambda_{i,k} \left|y_k - \tilde{y}_k\right|, \quad \text{for all } t \in [0,T].$$

(H_2) For any $y \in \mathbb{R}^n$, there exist two families of nonnegative continuous functions $(\psi_i)_{i=\overline{1,n}}$ and $(\omega_{i,k})_{i,k=\overline{1,n}}$ such that

$$\left|f(t,y)\right| \leq \psi_i(t) + \sum_{k=1}^{n} \omega_{i,k}(t) \left|y_k\right|, \quad \text{for all } t \in [0,T].$$

For each $i \in \{1, \ldots, n\}$, we denote

$$\lambda_i = \max_{1 \leq k \leq n} \{\lambda_{i,k}\}, \quad \text{and} \quad \zeta = \max_{1 \leq i \leq n} \{|\zeta_i|\},$$

and

$$\psi_i^* = \sup_{0 \leq t \leq T} \psi_i(t), \quad \omega_i^* = \sup_{0 \leq t \leq T} \left\{\max_{1 \leq k \leq n} \{\omega_{i,k}(t)\}\right\}.$$

Theorem 3.2. *Assume the hypotheses* (H_1) *and* (H_2) *hold, if*

$$\frac{n\omega_i^* T^{\sigma_i}}{\Gamma_q(\sigma_i + 1)} < 1, \quad \text{for each } i \in \{1, \ldots, n\}, \tag{8}$$

then problem (1)–(2) has at least one solution on $[0,T]$.

Proof. Firstly, we transform problem (1)–(2) into a fixed point problem.
Let the operator $\mathcal{T} : E \to E$ defined by (7), with $[\mathcal{T}_i(y)(t)]_{i=\overline{1,n}}$ being the integral operators given by

$$\mathcal{T}_i(y)(t) = \zeta_i + \int_0^t G_{\sigma_i}(t,qs) f_i(s, y(s)) \, d_q s, \quad t \in [0,T],$$

where $G_{\sigma_i}(t,qs)$ defined in (4), for each $i \in \{1, \ldots, n\}$.
We demonstrate that $\mathcal{T}$ satisfies the assumptions of Schauder's fixed point theorem. This proof will be completed in three steps.
Step 1: $\mathcal{T}$ is a continuous operator.
Consider the sequences $(y_m)_{m \in \mathbb{N}} = \left(y_1^m, y_2^m, \ldots, y_n^m\right)$, such that $\lim_{m \to \infty} y_m \to y$ in E. For each $i \in \{1, \ldots, n\}$ and $t \in [0,T]$, we obtain

$$\begin{aligned}
\left|\mathcal{T}_i y_m(t) - \mathcal{T}_i y(t)\right| &\leq \int_0^t G_{\sigma_i}(t,qs) \left|f_i(s, y_m(s)) - f_i(s, y(s))\right| d_q s \\
&\leq \int_0^t G_{\sigma_i}(t,qs) \sum_{k=1}^{n} \lambda_{i,k} \left|y_k^m(s) - y_k(s)\right| d_q s \\
&\leq \frac{n\lambda_i T^{\sigma_i}}{\Gamma_q(\sigma_i + 1)} \left\|y_m - y\right\|_E.
\end{aligned}$$

Since $y_m \to y$, we get $\left|\mathcal{T}_i y_m(t) - \mathcal{T}_i y(t)\right| \to 0$, as $m \to \infty$. Hence $\lim_{m \to \infty} \left\|\mathcal{T} y_m - \mathcal{T} y\right\|_E = 0$, for all $t \in [0,T]$. Then $\mathcal{T}$ is continuous.

Step 2: $\mathcal{T}$ is defined from a bounded, closed and convex subset into itself.
For each $i \in \{1, \ldots, n\}$, using (8), we consider r satisfies

$$r \geq \left(\zeta + \frac{\psi_i^* T^{\sigma_i}}{\Gamma_q(\sigma_i + 1)}\right)\left(\frac{\Gamma_q(\sigma_i + 1)}{\Gamma_q(\sigma_i + 1) - n\omega_i^* T^{\sigma_i}}\right).$$

Thus the subset ϑ_r defined by

$$\vartheta_r = \left\{y \in E, \quad \|y\|_E \leq r\right\},$$

is clearly bounded, closed and convex in E.
Let $\mathcal{T} : \vartheta_r \to E$ be the integral operator defined by (7). Then $\mathcal{T}(\vartheta_r) \subset \vartheta_r$. Indeed, by using (H_2), for each $i \in \{1, \ldots, n\}$ and $t \in [0, T]$, we get

$$\begin{aligned} \left|f_i(t, y(t))\right| &\leq \psi_i(t) + \sum_{k=1}^{n} \omega_{i,k}(t)\left|y_k\right|, \\ &\leq \psi_i^* + n\omega_i^* \|y\|_E. \end{aligned}$$

Thus

$$\begin{aligned} \left|\mathcal{T}_i y(t)\right| &\leq \left|\zeta_i + \int_0^t G_{\sigma_i}(t, qs) f_i(s, y(s))\, d_q s\right|, \\ &\leq \max_{1\leq i\leq n}\{|\zeta_i|\} + \int_0^t G_{\sigma_i}(t, qs)\left|f_i(s, y(s))\right| d_q s, \\ &\leq \zeta + \frac{\left(\psi_i^* + n\omega_i^* \|y\|_E\right) T^{\sigma_i}}{\Gamma_q(\sigma_i + 1)}, \\ &\leq \frac{\left(\zeta + \frac{\psi_i^* T^{\sigma_i}}{\Gamma_q(\sigma_i+1)}\right)\left(\frac{\Gamma_q(\sigma_i+1)}{\Gamma_q(\sigma_i+1) - n\omega_i^* T^{\sigma_i}}\right)}{\left(\frac{\Gamma_q(\sigma_i+1)}{\Gamma_q(\sigma_i+1) - n\omega_i^* T^{\sigma_i}}\right)} + \frac{n\omega_i^* T^{\sigma_i}}{\Gamma_q(\sigma_i + 1)} r, \\ &\leq r. \end{aligned}$$

Hence, $\|\mathcal{T}_i y\| \leq r$ for each $i \in \{1, \ldots, n\}$, which implies that $\|\mathcal{T} y\|_E \leq r$. Consequently $\mathcal{T}(\vartheta_r) \subset \vartheta_r$.
Step 3: $\mathcal{T}(\vartheta_r)$ is an equicontinuous subset.
Let $t_1, t_2 \in [0, T]$, $t_1 < t_2$ and $y \in \vartheta_r$. Then, for each $i \in \{1, \ldots, n\}$, we have

$$\begin{aligned} &\left|\mathcal{T}_i y(t_2) - \mathcal{T}_i y(t_1)\right| \\ &= \left|\int_0^{t_2} G_{\sigma_i}(t_2, qs) f_i(s, y(s))\, d_q s - \int_0^{t_1} G_{\sigma_i}(t_1, qs) f_i(s, y(s))\, d_q s\right|, \\ &\leq \int_0^{t_1} \left|\left[G_{\sigma_i}(t_2, qs) - G_{\sigma_i}(t_1, qs)\right] f_i(s, y(s))\right| d_q s + \int_{t_1}^{t_2} G_{\sigma_i}(t_2, qs)\left|f_i(s, y(s))\right| d_q s, \\ &\leq \left(\psi_i^* + n\omega_i^* r\right)\left(\int_0^{t_1} \left|G_{\sigma_i}(t_2, qs) - G_{\sigma_i}(t_1, qs)\right| d_q s + \int_{t_1}^{t_2} G_{\sigma_i}(t_2, qs)\, d_q s\right). \end{aligned} \tag{9}$$

As $t_2 > t_1$, we obtain

$$\begin{aligned} \left|G_{\sigma_i}(t_2, qs) - G_{\sigma_i}(t_1, qs)\right| &= \frac{1}{\Gamma_q(\sigma_i)}\left[(t_1 - qs)^{(\sigma_i - 1)} - (t_2 - qs)^{(\sigma_i - 1)}\right], \\ &= \frac{1}{\Gamma_q(\sigma_i + 1)}\, {}_sD_q\left[(t_2 - s)^{(\sigma_i)} - (t_1 - s)^{(\sigma_i)}\right]. \end{aligned}$$

Then,

$$\int_0^{t_1} \left| G_{\sigma_i}(t_2, qs) - G_{\sigma_i}(t_1, qs) \right| d_q s \leq \frac{1}{\Gamma_q(\sigma_i + 1)} \left[(t_2 - t_1)^{(\sigma_i)} + \left(t_2^{\sigma_i} - t_1^{\sigma_i} \right) \right],$$

and

$$\int_{t_1}^{t_2} G_{\sigma_i}(t_2, qs) d_q s = \frac{-1}{\Gamma_q(\sigma_i + 1)} \left[(t_2 - s)^{(\sigma_i)} \right]_{t_1}^{t_2} = \frac{1}{\Gamma_q(\sigma_i + 1)} (t_2 - t_1)^{(\sigma_i)}.$$

Thus, (9) gives

$$\left| \mathcal{T}_i y(t_2) - \mathcal{T}_i y(t_1) \right| \leq \frac{\left(\psi_i^* + n\omega_i^* r \right)}{\Gamma_q(\sigma_i + 1)} \left[2(t_2 - t_1)^{(\sigma_i)} + \left(t_2^{\sigma_i} - t_1^{\sigma_i} \right) \right].$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero for each $i \in \{1, \ldots, n\}$. Combining steps 1 to 3 and applying the Ascoli-Arzelà theorem, we conclude that $\mathcal{T} : \vartheta_r \to \vartheta_r$ is continuous and compact, satisfies the assumptions of Schauder's fixed point theorem [33]. Therefore, $\mathcal{T}$ has at least a fixed point which is the solution of problem (1)–(2) on $[0, T]$. □

Example 3.3. *Consider the following problem:*

$$\begin{cases} {}^{C}\mathcal{D}^{\frac{1}{2}}_{\frac{1}{4},0^+}(y_1)(t) = \frac{\pi t}{2\sqrt{e}} + \frac{\sin(t)\left(|y_1(t)| + |y_2(t)|\right)}{\sqrt{1+e}\left(1 + q + |y_1(t)| + |y_2(t)|\right)}, & t \in \left[0, \frac{\pi}{6}\right], \\ {}^{C}\mathcal{D}^{\frac{2}{3}}_{\frac{1}{4},0^+}(y_2)(t) = \exp\left(-\pi^2 t\right)\left(\frac{1}{100\sqrt{e}} + \frac{\tan(t)}{\pi}\left(|y_1(t)| + |y_2(t)|\right)\right), & t \in \left[0, \frac{\pi}{6}\right], \end{cases} \tag{10}$$

and

$$y_i(0) = \zeta_i \in \mathbb{R}, \quad \textit{for each } i \in \{1, 2\}.$$

We set

$$\begin{aligned} f_1(t, y_1, y_2) &= \frac{\pi t}{2\sqrt{e}} + \frac{\sin(t)\left(|y_1| + |y_2|\right)}{\sqrt{1+e}\left(1 + q + |y_1| + |y_2|\right)}, \\ f_2(t, y_1, y_2) &= \exp\left(-\pi^2 t\right)\left(\frac{1}{100\sqrt{e}} + \frac{\tan(t)}{\pi}\left(|y_1| + |y_2|\right)\right). \end{aligned}$$

The functions f_i are continuous for any $y \in \mathbb{R}^2$. Also

$$\begin{aligned} \left| f_1(t, y_1, y_2) - f_1(t, \tilde{y}_1, \tilde{y}_2) \right| &\leq \sum_{k=1}^{2} \frac{1+q}{2\sqrt{1+e}} \left| y_k - \tilde{y}_k \right|, \\ \left| f_2(t, y_1, y_2) - f_2(t, \tilde{y}_1, \tilde{y}_2) \right| &\leq \sum_{k=1}^{2} \frac{\sqrt{3}}{3\pi} \left| y_k - \tilde{y}_k \right|, \end{aligned}$$

hence, (H_1) is satisfied for each $i \in \{1, 2\}$, with

$$\lambda_1 = \max_{1 \leq k \leq 2} \{\lambda_{1,k}\} = \frac{5}{8\sqrt{1+e}} \quad \textit{and} \quad \lambda_2 = \max_{1 \leq k \leq 2} \{\lambda_{2,k}\} = \frac{\sqrt{3}}{3\pi}.$$

For each $i \in \{1, 2\}$ and $t \in \left[0, \frac{\pi}{6}\right]$, we have

$$\left| f_1(t, y_1, y_2) \right| \leq \frac{\pi t}{2\sqrt{e}} + \frac{\sin(t)}{\sqrt{1+e}} \left(|y_1| + |y_2| \right),$$

$$\left|f_2(t,y_1,y_2)\right| \leq \frac{\exp\left(-\pi^2 t\right)}{100\sqrt{e}} + \frac{\exp\left(-\pi^2 t\right)\tan(t)}{\pi}\left(\left|y_1\right|+\left|y_2\right|\right).$$

Thus, hypothesis (H_2) *is satisfied, with*

$$\psi_1(t) = \frac{\pi t}{2\sqrt{e}}, \quad \omega_{1,1}(t) = \omega_{1,2}(t) = \frac{\sin(t)}{\sqrt{1+e}},$$

and

$$\psi_2(t) = \frac{\exp\left(-\pi^2 t\right)}{100\sqrt{e}}, \quad \omega_{2,1}(t) = \omega_{2,2}(t) = \frac{\exp\left(-\pi^2 t\right)\tan(t)}{\pi}.$$

For each $i \in \{1,2\}$, *we obtain*

$$\psi_1^* = \frac{\pi^2}{12\sqrt{e}}, \quad \omega_1^* = \frac{1}{2\sqrt{1+e}} \quad \text{and} \quad \psi_2^* = \frac{1}{100\sqrt{e}}, \quad \omega_2^* = \frac{\sqrt{3}}{3\pi}.$$

Now, we show that

$$\frac{n\omega_1^* T^{\sigma_1}}{\Gamma_q(\sigma_1+1)} = \frac{2\times\frac{1}{2\sqrt{1+e}}\times\left(\frac{\pi}{6}\right)^{\frac{1}{2}}}{\Gamma_{\frac{1}{4}}\left(\frac{1}{2}+1\right)} \simeq 0,395925 < 1,$$

$$\frac{n\omega_2^* T^{\sigma_2}}{\Gamma_q(\sigma_2+1)} = \frac{2\times\frac{\sqrt{3}}{3\pi}\times\left(\frac{\pi}{6}\right)^{\frac{2}{3}}}{\Gamma_{\frac{1}{4}}\left(\frac{2}{3}+1\right)} \simeq 0,249391 < 1.$$

It follows from Theorem 3.2, that problem (10) has at least one solution.

Theorem 3.4. *Assume that the hypothesis* (H_1) *holds. If*

$$\max_{1\le i\le n}\left\{\frac{n\lambda_i T^{\sigma_i}}{\Gamma_q(\sigma_i+1)}\right\} < 1, \tag{11}$$

then problem (1)–(2) admits a unique solution on $[0,T]$.

Proof. Let $y,\tilde{y} \in E$. By using (H_1), we obtain for all $t\in[0,T]$ that

$$\left|\mathcal{T}_i y(t) - \mathcal{T}_i\tilde{y}(t)\right| \le \frac{n\lambda_i T^{\sigma_i}}{\Gamma_q(\sigma_i+1)}\left\|y-\tilde{y}\right\|_E, \quad \text{for each } i\in\{1,\dots,n\}.$$

Consequently

$$\left\|\mathcal{T}y - \mathcal{T}\tilde{y}\right\|_E \le \max_{1\le i\le n}\left\{\frac{n\lambda_i T^{\sigma_i}}{\Gamma_q(\sigma_i+1)}\right\}\left\|y-\tilde{y}\right\|_E.$$

Based on statement (11), it follows that $\mathcal{T}$ is a contraction operator. The Banach contraction principle [33] leads us to conclude that $\mathcal{T}$ has a unique fixed point. This fixed point is the unique solution of problem (1)–(2) on $[0,T]$. □

Example 3.5. *Consider the following problem:*

$$\begin{cases} {}^{C}\mathcal{D}^{\frac{1}{2}}_{\frac{1}{2},0^+}(y_1)(t) = \frac{\arcsin(t)\left(|y_1(t)|+|y_2(t)|+|y_3(t)|\right)}{\pi\sqrt{1+qt}}, & t\in\left[0,\frac{1}{2}\right], \\ {}^{C}\mathcal{D}^{\frac{5}{6}}_{\frac{1}{2},0^+}(y_2)(t) = \frac{\exp^{-\pi^2 t}}{2\pi(1-q)\left(1+|y_1(t)|+|y_2(t)|+|y_3(t)|\right)}, & t\in\left[0,\frac{1}{2}\right], \\ {}^{C}\mathcal{D}^{\frac{6}{7}}_{\frac{1}{2},0^+}(y_3)(t) = \frac{\arccos(t)\left(2+|y_1(t)|+|y_2(t)|+|y_3(t)|\right)}{\pi(t^2+1)\left(1+|y_1(t)|+|y_2(t)|+|y_3(t)|\right)}, & t\in\left[0,\frac{1}{2}\right], \end{cases} \tag{12}$$

and

$$y_i(0) = \zeta_i \in \mathbb{R}, \quad \textit{for each } i \in \{1,2,3\}.$$

We set

$$\begin{aligned}
f_1(t, y_1, y_2, y_3) &= \frac{\arcsin(t)\left(\left|y_1\right| + \left|y_2\right| + \left|y_3\right|\right)}{\pi\sqrt{1+qt}}, \\
f_2(t, y_1, y_2, y_3) &= \frac{\exp^{-\pi^2 t}}{2\pi(1-q)\left(1 + \left|y_1\right| + \left|y_2\right| + \left|y_3\right|\right)}, \\
f_3(t, y_1, y_2, y_3) &= \frac{\arccos(t)\left(2 + \left|y_1\right| + \left|y_2\right| + \left|y_3\right|\right)}{\pi(t^2+1)\left(1 + \left|y_1\right| + \left|y_2\right| + \left|y_3\right|\right)}.
\end{aligned}$$

The functions f_i are continuous for any $y, \tilde{y} \in \mathbb{R}^3$. Also

$$\begin{aligned}
\left|f_1(t, y_1, y_2, y_3) - f_1(t, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3)\right| &\leq \sum_{k=1}^{3} \frac{1}{6}\left|y_k - \tilde{y}_k\right|, \\
\left|f_2(t, y_1, y_2, y_3) - f_2(t, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3)\right| &\leq \sum_{k=1}^{3} \frac{1}{2\pi(1-q)}\left|y_k - \tilde{y}_k\right|, \\
\left|f_3(t, y_1, y_2, y_3) - f_3(t, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3)\right| &\leq \sum_{k=1}^{3} \frac{1}{3}\left|y_k - \tilde{y}_k\right|.
\end{aligned}$$

Hence, the hypothesis (H_1) is satisfied with $\lambda_1 = \frac{1}{6}$, $\lambda_2 = \frac{1}{\pi}$, and $\lambda_3 = \frac{1}{3}$.
For each $i \in \{1,2,3\}$ we obtain

$$\begin{aligned}
\frac{n\lambda_1 T^{\sigma_1}}{\Gamma_q(\sigma_1+1)} &= \frac{3 \times \frac{1}{6} \times \left(\frac{1}{2}\right)^{\frac{1}{2}}}{\Gamma_{\frac{1}{2}}\left(\frac{1}{2}+1\right)} \simeq 0,383932 < 1, \\
\frac{n\lambda_2 T^{\sigma_2}}{\Gamma_q(\sigma_2+1)} &= \frac{3 \times \frac{1}{\pi} \times \left(\frac{1}{2}\right)^{\frac{5}{6}}}{\Gamma_{\frac{1}{2}}\left(\frac{5}{6}+1\right)} \simeq 0,557904 < 1, \\
\frac{n\lambda_3 T^{\sigma 3}}{\Gamma_q(\sigma_3+1)} &= \frac{3 \times \frac{1}{3} \times \left(\frac{1}{2}\right)^{\frac{6}{7}}}{\Gamma_{\frac{1}{2}}\left(\frac{6}{7}+1\right)} \simeq 0,571772 < 1.
\end{aligned}$$

It remains to show that the condition (11)

$$\max_{1 \leq i \leq 3}\left\{\frac{n\lambda_i T^{\sigma_i}}{\Gamma_q(\sigma_i+1)}\right\} \simeq 0,571772 < 1,$$

is satisfied. It follows from Theorem 3.4, that problem (12) admits a unique solution.

3.2. ***Ulam-Hyers Stability***

Next, we use Definition 3.6 and 3.7 to study the Ulam-Hyers stability for our system of n-nonlinear q-FDEs (1).

Definition 3.6. *The system (1) is Ulam-Hyers stable if there exists a real number $\eta > 0$, such that for all $\varepsilon = \max\limits_{1\leq i\leq n}\{\varepsilon_i\} > 0$ and for any solution $z \in E$ satisfying the inequality*

$$\left|{}^{C}\mathcal{D}_{q,0^+}^{\sigma_i} z_i(t) - f_i(t, z(t))\right| \leq \varepsilon_i,\ t \in [0, T],\ i \in \{1, \ldots, n\}, \tag{13}$$

there exists a solution $y \in E$ of system (1) satisfying

$$\left\|y - z\right\|_E \leq \eta\varepsilon.$$

Definition 3.7. *The system (1) is generalized Ulam-Hyers stable if there exists a function $\varphi \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfying $\varphi(0) = 0$, such that for any solution $z \in E$ of inequality (13), there exists a solution $y \in E$ of (13) satisfying the inequality*

$$\left\|y - z\right\|_E \leq \varphi(\varepsilon).$$

Remark 3.8. *A function $z \in E$ is a solution of inequality (13) if and only if there exists a function $g \in E$ and $(\varepsilon_i)_{i=\overline{1,n}} > 0$, such that for all $t \in [0, T]$ and $i \in \{1, \ldots, n\}$,*

1. $\left|g_i(t)\right| \leq \varepsilon_i$,
2. ${}^{C}\mathcal{D}_{q,0^+}^{\sigma_i} z_i(t) = f_i(t, z(t)) + g_i(t)$.

Lemma 3.9. *If a function $z \in E$ is the solution of inequality (13), then there exists $(\varepsilon_i)_{i=\overline{1,n}} > 0$, ensuring that z remains the solution of the integral inequality*

$$\left|z_i(t) - z_i(0) - \frac{1}{\Gamma_q(\sigma_i)}\int_0^t G_{\sigma_i}(t, qs) f_i(s, z(s))\, d_q s\right| \leq \frac{\varepsilon_i T^{\sigma_i}}{\Gamma_q(\sigma_i + 1)}.$$

Proof. If z is a solution of (13), by Remark 3.8, we have

$$z_i(t) = z_i(0) + \frac{1}{\Gamma_q(\sigma_i)}\int_0^t G_{\sigma_i}(t, qs)\left[f_i(s, z(s)) + g_i(s)\right] d_q s,$$

and

$$\begin{aligned}
&\left|z_i(t) - z_i(0) - \frac{1}{\Gamma_q(\sigma_i)}\int_0^t G_{\sigma_i}(t, qs) f_i(s, z(s))\, d_q s\right|, \\
&= \left|z_i(0) + \frac{1}{\Gamma_q(\sigma_i)}\int_0^t G_{\sigma_i}(t, qs)\left[f_i(s, z(s)) + g_i(s)\right] d_q s\right. \\
&\quad \left. - z_i(0) - \frac{1}{\Gamma_q(\sigma_i)}\int_0^t G_{\sigma_i}(t, qs) f_i(s, z(s))\, d_q s\right|, \\
&\leq \frac{1}{\Gamma_q(\sigma_i)}\int_0^t G_{\sigma_i}(t, qs)\left|g_i(s)\right| d_q s, \\
&\leq \frac{\varepsilon_i T^{\sigma_i}}{\Gamma_q(\sigma_i + 1)}, \quad \text{for each } i \in \{1, \ldots, n\}.
\end{aligned}$$

□

Theorem 3.10. *Assume that (H_1) and (11) hold, then system (1) is Ulam-Hyers stable and consequently generalized Ulam-Hyers stable.*

Proof. Let $\varepsilon = \max\limits_{1\leq i\leq n}\{\varepsilon_i\} > 0$, assume that $z \in E$ be a solution of inequality (13), and $y \in E$ is the unique solution of system (1), with the conditions

$$y_i(0) = z_i(0), \quad \text{for each } i \in \{1, \ldots, n\},$$

then

$$y_i(t) = z_i(0) + \frac{1}{\Gamma_q(\sigma_i)} \int_0^t G_{\sigma_i}(t, qs) f_i(s, y(s))\, d_q s,$$

and

$$\begin{aligned} \left|z_i(t) - y_i(t)\right| &= \left|z_i(t) - z_i(0) - \frac{1}{\Gamma_q(\sigma_i)} \int_0^t G_{\sigma_i}(t, qs) f_i(s, y(s))\, d_q s\right|, \\ &\leq \left|z_i(t) - z_i(0) - \frac{1}{\Gamma_q(\sigma_i)} \int_0^t G_{\sigma_i}(t, qs) f_i(s, z(s))\, d_q s\right| \\ &\quad + \frac{1}{\Gamma_q(\sigma_i)} \int_0^t G_{\sigma_i}(t, qs) \left|f_i(s, z(s)) - f_i(s, y(s))\right| d_q s. \end{aligned}$$

Using (H_1) and Lemma 3.9, for all $t \in [0, T]$ and $i \in \{1, \ldots, n\}$, we get

$$\left\|z_i - y_i\right\| \leq \frac{T^{\sigma_i}}{\Gamma_q(\sigma_i + 1)} \left(\varepsilon_i + n\lambda_i \left\|y - z\right\|_E\right),$$

then

$$\left\|y - z\right\|_E \leq \max_{1\leq i\leq n} \left\{\frac{T^{\sigma_i}}{\Gamma_q(\sigma_i + 1)} \left(\varepsilon_i + n\lambda_i \left\|y - z\right\|_E\right)\right\}.$$

Since $\varepsilon = \max\limits_{1\leq i\leq n}\{\varepsilon_i\} > 0$, we have

$$\left\|y - z\right\|_E \leq \max_{1\leq i\leq n} \left\{\frac{T^{\sigma_i}}{\Gamma_q(\sigma_i + 1)}\right\} \varepsilon + \Lambda \left\|y - z\right\|_E,$$

with $\Lambda = \max\limits_{1\leq i\leq n} \left\{\frac{n\lambda_i T^{\sigma_i}}{\Gamma_q(\sigma_i+1)}\right\} < 1$. Then

$$\left\|y - z\right\|_E \leq \eta\varepsilon,$$

where $\eta = \max\limits_{1\leq i\leq n} \left\{\frac{T^{\sigma_i}}{(1-\Lambda)\Gamma_q(\sigma_i+1)}\right\}$.
We deduce that system (1) is Ulam-Hyers stable and is consequently generalized Ulam-Hyers stable if we set $\varphi(t) = \eta t$. □

Declarations

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References

[1] S. Abbas, B. Ahmad, M. Benchohra and A. Salim, *Fractional Difference, Differential Equations and Inclusions: Analysis and Stability*, Morgan Kaufmann, Cambridge, 2024. https://doi.org/10.1016/C2023-0-00030-9

[2] H. Afshari and A. Ahmadkhanlu, The existence of positive solutions for a CaputoHadamard boundary value problem with an integral boundary condition. *Adv. Theory Nonlinear Anal. Appl.* **7** (5), 155164 (2023). https://doi.org/10. 17762/atnaa.v7.i5.332

[3] H. Afshari and M. N. Sahlan, The existence of solutions for some new boundary value problems involving the q-derivative operator in quasib-metric and b-metric-like spaces, *Lett. Nonlinear Anal. Appl.* **2** (1) (2024), 1622.

[4] H. Afshari and A. Ahmadkhanlu, A discussion on some new boundary value problems involving the q-derivative operator via the existence of positive solutions. *Opt. Quantum Electron.* **56**, 1312 (2024). https://doi.org/10.1007/ s11082-024-07189-7

[5] O. P. Agrawal, Some generalized fractional calculus operators and their applications in integral equations, *Frac. Cal. Appl. Anal.* **15** (4) (2012), 700-711.

[6] M. Benchohra, S. Bouriah, A. Salim and Y. Zhou, *Fractional Differential Equations: A Coincidence Degree Approach*, Berlin, Boston: De Gruyter, 2024. https://doi.org/10.1515/9783111334387

[7] M. Benchohra, E. Karapınar, J. E. Lazreg and A. Salim, *Advanced Topics in Fractional Differential Equations: A Fixed Point Approach*, Springer, Cham, 2023. https://doi.org/10.1007/978-3-031-26928-8

[8] M. Benchohra, E. Karapınar, J. E. Lazreg and A. Salim, *Fractional Differential Equations: New Advancements for Generalized Fractional Derivatives*, Springer, Cham, 2023. https://doi.org/10.1007/978-3-031-34877-8

[9] S. Bouriah, A. Salim, M. Benchohra and E. Karapınar, On Periodic solutions for implicit nonlinear Caputo tempered fractional differential problems, *Demonstr. Math.* **57** (1) (2024), pp. 20230154. https://doi.org/10.1515/dema-2023-0154

[10] R. Herrmann, *Fractional Calculus: An Introduction for Physicists.* Singapore: World Scientific Publishing Company, 2011.

[11] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[12] A. A. Kilbas, Hari M. Srivastava and Juan J. Trujillo, *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[13] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[14] F. H. Jackson, On q-definite integrals, *Quart. J. Pure Appl. Math.* **41** (1910), 193–203.

[15] F. H. Jackson, On q-functions and a certain difference operator, *Trans. Roy. Soc. Edinburgh.* **46** (1908), 253–281.

[16] W. A. Al-Salam, Some fractional q-integrals and q-derivatives. *Proc. Edinburgh. Math. Soc.* **15**(1966), 135–140.

[17] R. P. Agarwal, Certain fractional q-integrals and q-derivatives. *Proc. Camb. Philol. Soc.* **66** (1969), 365–370.

[18] C. Yu, S. Wang, J. Wang and J. Li, Solvability criterion for fractional q-integro-difference system with Riemann-Stieltjes integrals conditions. *Fractal Fract.* **6** (10) (2022), 554.

[19] Y. Li, J. Liu, D. O'Regan and J. Xu, Nontrivial solutions for a system of fractional q-difference equations involving q-integral boundary conditions. *Mathematics.* **8** (2020), 828.

[20] R. Luca, Positive solutions for a system of fractional q-difference equations with multi-point boundary conditions, *Fractal Fract.* **8** (1) (2024), p70.

[21] M. E. Sami, Existence of solutions for a system of singular sum fractional q-differential equations via quantum calculus, *Adv. Differ. Equ.* **2020** (2020), 23.

[22] A. Benchaib, A. Salim, S. Abbas and M. Benchohra, Qualitative analysis of neutral implicit fractional q-difference equations with delay, *Differ. Equ. Appl.* **16** (1) (2024), 19-38. http://dx.doi.org/10.7153/dea-2024-16-02

[23] A. Salim, S. Abbas, M. Benchohra and J. E. Lazreg, Caputo fractional q-difference equations in Banach spaces. *J. Innov. Appl. Math. Comput. Sci.* **3** (1) (2023), 1-14. https://doi.org/10.58205/jiamcs.v3i1.67

[24] N. Laledj, A. Salim, J. E. Lazreg, S. Abbas, B. Ahmad and M. Benchohra, On implicit fractional q-difference equations: Analysis and stability. *Math. Methods Appl. Sci.* **45** (17) (2022), 10775-10797. https://doi.org/10.1002/mma.8417

[25] B. Basti and Y. Arioua, Existence study of solutions for a system of n-nonlinear fractional differential equations, *J. Math. Phys. Anal. Geom.* **18** (2) (2022), 350–367.

[26] V. Kac and P. Cheung, *Quantum Calculus*, Springer, New York, NY, USA, 2002.

[27] M. H. Annaby and Z. S. Mansour, *q-fractional calculus and Equation*, Springer, London, 2012.

[28] T. Ernst, *A comprehensive treatment of q-calculus*, Birkhauser/ Springer, Basel AG, Basel, 2012.

[29] T. Ernst, *A new method of q-calculus*, Doctoral thesis, Uppsal university, 2000.

[30] R. P. Agarwal, Certain fractional q-integrals and q-derivatives, *Proc. Camb. Philos. Soc.* **66** (1) (1969), 365–370.

[31] P. M. Rajkovic, S. Marinkovic, and M. Stankovic, On q–analogues of Caputo derivative and Mittag-Leffler function, *Fract. Cal. Appl. Anal.* **10** (4) (2007), 359–373.

[32] W. A. AL-Salam, *Some fractional q-integrals and q-derivatives*. Proc. Edinb. Math. Soc. **15** (1966), 135–140.

[33] A. Granas and J. Dugundji, *Fixed point theory*, (Vol. 14), New York: Springer, 2003.