



Some corrections to uniqueness results for the solutions of singular fractional boundary value problems with p-Laplacian

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Abstract. In the published articles “Positive Solutions of Singular Fractional Boundary Value Problem with p-Laplacian” by Dehong and “Existence of positive solutions for singular fractional boundary value problems with p-Laplacian” by Hamal and Erkan, we identified an inconsistency in the formulations of the boundary conditions. This inconsistency has significant implications for the results presented in Lemma 2.3 in [3] and Lemma 2.5 in [6]. Specifically, while the lemmas assert the existence of a unique solution under the stated boundary conditions, our analysis shows that the chosen boundary conditions do not guarantee uniqueness.

The purpose of this erratum is to clarify the errors, examine their impact on the results, and provide the corrected boundary conditions along with a revised statement regarding the uniqueness of the solution.

1. Introduction and Preliminary

Fractional differential equations, with derivatives of non-integer order, have emerged as a powerful mathematical framework for modeling intricate phenomena with applications spanning across various scientific and engineering disciplines. These equations provide a more accurate representation of systems that exhibit memory effects, non-local interactions, and anomalous diffusion. Some examples are seen in biology, economics, control theory, chemistry, physics and biophysics, just to mention but a few [1, 2, 4, 5, 7–9, 11].

Definition 1.1. [10] The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $y : (0, +\infty) \rightarrow \mathbb{R}$ is defined by

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds$$

provided that the right-hand side is pointwise defined on $(0, +\infty)$.

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Definition 1.2. [10] The Caputo derivative of order $\alpha > 0$ for a function $y : (0, +\infty) \rightarrow \mathbb{R}$ is defined by

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{y^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds$$

where $n = [\alpha] + 1$ and $[\alpha]$ means the integer part of α .

Lemma 1.3. [12] Let $\alpha > 0$ then

$$I_{0+}^{\alpha} D_{0+}^{\alpha} y(t) = y(t) + c_1 + c_2 t + c_3 t^2 + \dots + c_n t^{n-1},$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n$ and $n = [\alpha] + 1$.

In [3], Dehong obtained the existence results of solutions for the following problem.

$$(\phi_p(D_{0+}^{\alpha} u(t)))' + f(t, u(t)) = 0, \quad t \in (0, 1) \quad (1)$$

$$u'(0) = 0, \quad u(1) - \gamma u(\eta) = 0, \quad (2)$$

where $1 < \alpha \leq 2$, $\phi_p(s) = |s|^{p-2}s$, with $p > 1$, $\gamma, \eta \in (0, 1)$, D_{0+}^{α} is the Caputo fractional derivative and $f(t, u)$ may be singular at $u = 0$.

Although the author claims in Lemma 2.3 of [3] that the solution is unique, it has been observed that the given boundary conditions are not sufficient to guarantee uniqueness.

In [6], motivated by the problem (1)-(2), the authors investigated the existence of at least three positive solutions for the following boundary value problem with p-Laplacian

$$(\phi_p(D_{0+}^{\alpha} u(t)))' + f(t, u(t), u'(t)) = 0, \quad t \in (0, 1) \quad (3)$$

$$u(0) = u'(0) = 0, \quad D_{0+}^{\alpha-1} u(1) = \sum_{i=1}^{m-2} a_i D_{0+}^{\alpha-1} u(\xi_i), \quad (4)$$

where $2 < \alpha \leq 3$, $\xi_i \in \mathbb{R}$ for all $i = 1, 2, \dots, m-2$ such that $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$, $\phi_p(s) = |s|^{p-2}s$, with $p > 1$, D_{0+}^{α} is the Caputo fractional derivative and $f(t, x, y)$ may be singular at $t = 0$.

By making the same mistake, the authors claimed in Lemma 2.5 of [6] that the solution is unique, but the given boundary conditions are lacking for the uniqueness result.

In this paper, since the proof of Lemma 2.5 in [6] can be done in a very similar way, we first aim to give the correct proof for Lemma 2.3 in [3] and then we will edit these lemmas by expressing the problem correctly.

2. Corrected Result

We shall first give the corrected form of the proof for Lemma 2.3 in [3]. Let $F : [0, 1] \times \mathbb{R} \rightarrow (0, +\infty)$ be continuous.

Lemma 2.1. For any $u \in C[0, 1]$, the following boundary value problem

$$(\phi_p(D_{0+}^{\alpha} u(t)))' + F(t, u(t)) = 0, \quad t \in (0, 1) \quad (5)$$

$$u'(0) = a, \quad u(1) - \gamma u(\eta) = a, \quad (6)$$

has a solution which is given by

$$\begin{aligned} u(t) = & \int_0^1 G(t, s) \phi_q \left(\int_0^s F(\tau, u(\tau)) d\tau - \phi_p(D_{0+}^{\alpha} u(0)) \right) ds \\ & + \frac{\gamma}{1-\gamma} \int_0^1 G(\eta, s) \phi_q \left(\int_0^s F(\tau, u(\tau)) d\tau - \phi_p(D_{0+}^{\alpha} u(0)) \right) ds + \frac{a\eta\gamma}{1-\gamma} + at, \end{aligned}$$

where a is a fixed positive constant and

$$G(t, s) = \begin{cases} \frac{(1-s)^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases} \quad (7)$$

Proof. Integrating both sides of (5), we have

$$\phi_p(D_{0+}^\alpha u(t)) - \phi_p(D_{0+}^\alpha u(0)) = - \int_0^t F(s, u(s)) ds$$

and

$$\phi_p(D_{0+}^\alpha u(t)) = - \int_0^t F(s, u(s)) ds + \phi_p(D_{0+}^\alpha u(0))$$

so

$$D_{0+}^\alpha u(t) = -\phi_q \left(\int_0^t F(s, u(s)) ds - \phi_p(D_{0+}^\alpha u(0)) \right).$$

Using Lemma 1.1, we get

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q \left(\int_0^s F(\tau, u(\tau)) d\tau - \phi_p(D_{0+}^\alpha u(0)) \right) ds + A + Bt. \quad (8)$$

Using the boundary conditions (6), we have $B = a$ and

$$\begin{aligned} A &= \frac{1}{(1-\gamma)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi_q \left(\int_0^s F(\tau, u(\tau)) d\tau - \phi_p(D_{0+}^\alpha u(0)) \right) ds \\ &\quad - \frac{\gamma}{(1-\gamma)\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} \phi_q \left(\int_0^s F(\tau, u(\tau)) d\tau - \phi_p(D_{0+}^\alpha u(0)) \right) ds + \frac{a\eta\gamma}{1-\gamma}. \end{aligned}$$

Substituting A and B in (8), we have

$$\begin{aligned} u(t) &= \int_0^t \left(\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \right) \phi_q \left(\int_0^s F(\tau, u(\tau)) d\tau - \phi_p(D_{0+}^\alpha u(0)) \right) ds \\ &\quad + \int_t^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\int_0^s F(\tau, u(\tau)) d\tau - \phi_p(D_{0+}^\alpha u(0)) \right) ds \\ &\quad + \frac{\gamma}{(1-\gamma)\Gamma(\alpha)} \int_0^\eta ((1-s)^{\alpha-1} - (\eta-s)^{\alpha-1}) \phi_q \left(\int_0^s F(\tau, u(\tau)) d\tau - \phi_p(D_{0+}^\alpha u(0)) \right) ds \\ &\quad + \frac{\gamma}{(1-\gamma)\Gamma(\alpha)} \int_\eta^1 (1-s)^{\alpha-1} \phi_q \left(\int_0^s F(\tau, u(\tau)) d\tau - \phi_p(D_{0+}^\alpha u(0)) \right) ds + \frac{a\eta\gamma}{1-\gamma} + at, \end{aligned}$$

which depends on the value of $\phi_p(D_{0+}^\alpha u(0))$. So, the solution of problem (5)-(6) is not unique.

If we consider the boundary value problem as follows

$$(\phi_p(D_{0+}^\alpha u(t)))' + F(t, u(t)) = 0, \quad t \in (0, 1) \quad (9)$$

$$u'(0) = a, \quad \phi_p(D_{0+}^\alpha u(0)) = b \quad u(1) - \gamma u(\eta) = a. \quad (10)$$

The following result is obtained, which gives the unique solution.

Lemma 2.2. For any $u \in C[0, 1]$, the boundary value problem (9)-(10) has a unique solution which is given by

$$\begin{aligned} u(t) &= \int_0^1 G(t, s) \phi_q \left(\int_0^s F(\tau, u(\tau)) d\tau - b \right) ds \\ &\quad + \frac{\gamma}{1-\gamma} \int_0^1 G(\eta, s) \phi_q \left(\int_0^s F(\tau, u(\tau)) d\tau - b \right) ds + \frac{a\eta\gamma}{1-\gamma} + at, \end{aligned}$$

where a, b are the fixed positive constants and $G(t, s)$ is given in (7).

Now, we will give the corrected form of Lemma 2.5 in [6].

Lemma 2.3. *We consider the following fractional boundary value problem*

$$(\phi_p(D_{0+}^\alpha u(t)))' + h(t) = 0, \quad t \in (0, 1) \quad (11)$$

$$u(0) = u'(0) = 0, \quad D_{0+}^{\alpha-1} u(1) = \sum_{i=1}^{m-2} a_i D_{0+}^{\alpha-1} u(\xi_i), \quad (12)$$

where $h \in C(0, 1) \cup L^1(0, 1)$. We denote by $\Delta = \Gamma(\alpha)(1 - \sum_{i=1}^{m-2} a_i)$.

Then the solution of problem (11)-(12) is given by

$$u(t) = \frac{t^{\alpha-1}}{\Delta} \int_0^1 w(s) ds - \frac{t^{\alpha-1}}{\Delta} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} w(s) ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} w(s) ds,$$

where $w(s) = \phi_q \left(\int_0^s h(\tau) d\tau - \phi_p(D_{0+}^\alpha u(0)) \right)$, $\phi_q(u)$ is the inverse function of $\phi_p(u)$, such that $\frac{1}{p} + \frac{1}{q} = 1$.

From this we can see that the solution depends on the value of $\phi_p(D_{0+}^\alpha u(0))$. So, the solution of problem (11)-(12) is not unique.

3. Conclusion

If

$$\begin{aligned} &(\phi_p(D_{0+}^\alpha u(t)))' + f(t, u(t)) = 0, \quad t \in (0, 1) \\ &u'(0) = 0, \quad D_{0+}^\alpha u(0) = 0, \quad u(1) - \gamma u(\eta) = 0, \end{aligned}$$

had been considered instead of the boundary value problem (1)-(2) in the reference [3], the given results would have been correct.

Similarly, if

$$\begin{aligned} &(\phi_p(D_{0+}^\alpha u(t)))' + f(t, u(t), u'(t)) = 0, \quad t \in (0, 1) \\ &u(0) = u'(0) = D_{0+}^\alpha u(0) = 0, \quad D_{0+}^{\alpha-1} u(1) = \sum_{i=1}^{m-2} a_i D_{0+}^{\alpha-1} u(\xi_i), \end{aligned}$$

had been considered instead of the boundary value problem (3)-(4) in the reference [6], the given results would have been correct.

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