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Theoretical analysis on the nonlinear fractional differential equations and generalized heat equation

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Abstract. Using Schauder's fixed-point theorem, we establish sufficient conditions for the existence and uniqueness of solutions to the nonlinear fractional boundary value problem:

$$\begin{cases} {}_CD^{\beta}\zeta(x) + f(x, \zeta(x), I^{\gamma}\zeta(x)) = 0, \quad x \in I = [0,1], \quad 1 < \beta \leq 2, \ \ \gamma > 0, \\ \zeta(0) = 0, \quad \zeta(1) = \phi(\zeta), \end{cases} \tag{0.1}$$

where ϕ is a functional defined on $C(I, \mathbb{R})$. By constructing an appropriate Green's function, we derive a Lyapunov-type inequality for a special case of the problem (0.1):

$$\begin{cases} {}_CD^{\beta}\zeta(x) + \lambda(x)I^{\gamma}\zeta(x) = \eta(x, \zeta(x)), \quad x \in I = [0,1], \quad 1 < \beta \leq 2, \ \ \gamma > 0, \\ \zeta(0) = 0, \quad \zeta(1) = \phi(\zeta). \end{cases} \tag{0.2}$$

We further make an analysis for equation (0.2) by applying the inverse operator method and the Mittag-Leffler function with illustrative examples demonstrating applications obtained. Finally, we construct an analytic solution to the following generalized fractional heat equation with an initial condition in n dimensions based on an inverse operator:

$$\begin{cases} {}_CD_t^{\alpha}u(t, x) = \triangle_{a_1(x_1), \cdots, a_n(x_n)}u(t, x) + f(t, x), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \ \ 0 < \alpha \leq 1, \\ u(0, x) = \psi(x), \end{cases} \tag{0.3}$$

where

$$\triangle_{a_1(x_1), \cdots, a_n(x_n)} = a_1(x_1)\frac{\partial^2}{\partial x_1^2} + \cdots + a_n(x_n)\frac{\partial^2}{\partial x_n^2}.$$

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1. Introduction and Preliminaries

We will begin with some basic definitions given in fractional calculus, the multivariate Mittag-Leffler function as well as several fixed-point theorems soon to be used. In addition, we present a survey on the research closely related to ours.

Definition 1.1. Let $\beta \geq 0$ and ζ be a real function defined on $[0,T]$ with $T > 0$. The Riemann-Liouville fractional integral of order β is defined by $(I^0\zeta)(t) = \zeta(t)$ and

$$(I^\beta \zeta)(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}\zeta(s)ds, \qquad \beta > 0, \qquad t \in [0,T],$$

where Γ is the Gamma function.

Definition 1.2. The Caputo fractional derivative of order $\beta \geq 0$ is given by $({}_C D^0\zeta)(t) = \zeta(t)$ and $({}_C D^\beta \zeta)(t) = (I^{n-\beta}D^n\zeta)(t)$, for $\beta > 0$, where $n \in \mathbb{N}$, $n-1 < \beta \leq n$, is defined by

$$({}_C D^\beta \zeta)(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1}\zeta^{(n)}(s)ds, \qquad t \in [0,T].$$

Lemma 1.3. *Let $\beta > 0$, then the differential equation (see [1])*

$${}_C D^\beta u(t) = 0$$

has solutions $u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^{n-1}$, $c_i \in \mathbb{R}$, $i = 0,1,\cdots,n$, $n = -\lfloor -\beta \rfloor$.

Moreover, it has been established that $I^\beta {}_C D^\beta u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^{n-1}$ for some $c_i \in \mathbb{R}$, $i = 0,1,\cdots,n$, $n = -\lfloor -\beta \rfloor$, (see Lemma 2.3 in [1]).

Let $\alpha_i, \beta > 0$ and $z_i \in \mathbb{C}$,

$$E_{(\alpha_1,\alpha_2,\cdots,\alpha_m),\,\beta}(z_1,\cdots,z_m) = \sum_{s=0}^{\infty} \sum_{s_1+s_2+\cdots+s_m=s} \binom{s}{s_1,s_2,\cdots,s_m} \frac{z_1^{s_1}\cdots z_m^{s_m}}{\Gamma(\alpha_1 s_1 + \cdots + \alpha_m s_m + \beta)}$$

is the well-known multivariate Mittag-Leffler function [2, 3], which is an entire function on complex plane $\mathbb{C}^m$.

Theorem 1.4. *(Schauder's fixed-point Theorem [4, Theorem 4.1.1]) Let U be a nonempty and convex subset of a normed space B. Let T be a continuous mapping of U into a compact set $K \subseteq U$. Then T has a fixed point.*

Theorem 1.5. *(Leray-Schauder's fixed-point theorem) Let T be a continuous and compact mapping of a Banach space X to itself, such that the set $\{x \in X : x = \delta T x$ for some $0 < \delta \leq 1\}$ is bounded. Then T has a fixed point.*

Boundary value problems of fractional differential equations, including PDEs, are dominant topics and have emerged as an important field of research due to their wide applications in various areas of engineering and science, such as control theory, mechanics, wave propagation and biology (see [5–11]. In 2014, Tariboon et al. [12] studied the existence and uniqueness of solutions for the following fractional differential equation:

$${}_C D^\alpha u(x) = g(x, u(x)), \quad 1 < \alpha \leq 2, \;\; x \in [0,T],$$

subject to nonlocal fractional integral boundary conditions:

$$\sum_{i=1}^{m} \lambda_i u(\eta_i) = \omega_1, \quad \sum_{j=1}^{n} \mu_j (I^{\beta_j} u(T) - I^{\beta_j} u(\zeta_j)) = \omega_2, \quad \eta_i, \zeta_j \in (0,T),$$

where $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, λ_i, $\mu_j \in \mathbb{R}$ for all $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$ and ω_1, $\omega_2 \in \mathbb{R}$, using Krasnoselskii's fixed point theorem, Banach's contractive principle and Leray-Schauder's nonlinear alternative.

Given a function $u(x) \in L_1(\mathbb{R})$, the Fourier transform is given by

$$\hat{u}(k) = (\mathcal{F}_x u(x))(k) = \int_{-\infty}^{\infty} e^{ikx} u(x) dx,$$

and the inverse Fourier transform of $\mathcal{F}_x u(k)$ is defined as

$$u(x) = \mathcal{F}_k^{-1}(\mathcal{F}_x u(k))(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} (\mathcal{F}_x u(x))(k) dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \hat{u}(k) dk.$$

In 2019, Morales-Delgado et al. [13] studied the following fractional differential equation using the Laplace and Fourier transforms:

$$\begin{cases} {}_C D_t^\alpha u(t,x) = \mu \dfrac{\partial^2}{\partial x^2} u(t,x) \ \ (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \ \ 0 < \alpha \le 1, \\ u(0,x) = \psi(x), \end{cases} \tag{1.1}$$

where $\mu > 0$ is the diffusion coefficient, and derived that

$$u(t,x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{\alpha,1}(-\mu k^2 t^\alpha) \hat{\psi}(k) e^{-ikx} dk, \tag{1.2}$$

where

$$E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad z \in \mathbb{C}.$$

Very recently, Li [5] considered the following fractional differential equation by the inverse operator method:

$$\begin{cases} \dfrac{{}_c\partial^\alpha}{\partial t^\alpha} u(t,x) = \triangle_{\lambda_1,\cdots,\lambda_n} u(t,x) + g(t,x), \ \ 1 < \alpha \le 2, \\ u(0,x) = \phi_1(x), \ \ u_t'(0,x) = \phi_2(x), \ \ (t,x) \in \mathbb{R} \times R^n, \end{cases} \tag{1.3}$$

where

$$\triangle_{\lambda_1,\cdots,\lambda_n} = \lambda_1 \frac{\partial^2}{\partial x_1^2} + \cdots + \lambda_n \frac{\partial^2}{\partial x_n^2}, \ \ \text{all } \lambda_i \text{ are contants,}$$

and obtained

$$\begin{aligned} u(t,x) = &\sum_{k=0}^{\infty} I_t^{\alpha k + \alpha} \triangle_{\lambda_1,\cdots,\lambda_n}^k g(t,x) + \sum_{k=0}^{\infty} \frac{t^{\alpha k}}{\Gamma(\alpha k + 1)} \triangle_{\lambda_1,\cdots,\lambda_n}^k \phi_1(x) \\ &+ \sum_{k=0}^{\infty} \frac{t^{\alpha k + 1}}{\Gamma(\alpha k + 2)} \triangle_{\lambda_1,\cdots,\lambda_n} \phi_2(x). \end{aligned}$$

in a subspace of $C(\mathbb{R}^+, \mathbb{R}^n)$.

The rest of the paper is structured as follows: In Section 2, we present a Lyapunov-type inequality that provides a necessary condition for the existence of a non-trivial solution to equation (0.2) based on a Green's vector function. Further, we derive sufficient conditions for the existence and uniqueness of solutions to equation (0.1) by Schauder's fixed-point theorem in Section 3. Applying the inverse operator method and the Mittag-Leffler function, we study the uniqueness and existence to equation (0.2) by Banach's contractive principle and Leray-Schauder's fixed-point theorem in Section 4 with several illustrative examples. In Section 5, we construct an analytic solution for a generalized fractional heat equation with an initial condition using the inverse operator method and the multivariate Mittag-Leffler function. Finally, we summarize the entire work in Section 6.

2. A Lyapunov-type inequality

Proposition 2.1. *$\zeta \in C(I.\mathbb{R})$ is a solution of equation (0.2) if and only if ζ satisfies the integral equation*

$$\zeta(x) = \phi(\zeta)x + \int_0^1 G(x,\tau)\Lambda(\tau,\zeta)d\tau,$$

where

$$G(x,\tau) = \overline{G}(x,\tau)[1,1], \quad \overline{G}(x,\tau) = \begin{cases} \frac{-x(1-\tau)^{\beta-1}+(x-\tau)^{\beta-1}}{\Gamma(\beta)}, & 0 \leq \tau \leq x \leq 1, \\ \frac{-x(1-\tau)^{\beta-1}}{\Gamma(\beta)}, & 0 \leq x \leq \tau \leq 1, \end{cases}$$

$$\Lambda(\tau,\zeta) = [\eta(\tau,\zeta(\tau)), F(\tau,\zeta)]^T, \qquad F(\tau,\zeta) = \int_0^\tau G^*(\tau,s)\zeta(s)ds, \quad G^*(\tau,s) = -\frac{\lambda(\tau)}{\Gamma(\gamma)}(\tau-s)^{\gamma-1},\ 0 \leq s < \tau \leq 1.$$

Proof. Let us first rearrange equation (0.2) as follows:

$${}_C D^\beta \zeta(x) = \eta(x,\zeta(x)) + \int_0^x G^*(x,\tau)\zeta(\tau)d\tau, \qquad x \in I = [0,1], \qquad 1 < \beta \leq 2,$$

where $G^*(x,\tau) = -\frac{1}{\Gamma(\gamma)}\lambda(x)(x-\tau)^{\gamma-1}$, for $0 \leq \tau < x \leq 1$, $\gamma > 0$.

From Lemma 1.3, we come to

$$\zeta(x) = c_0 + c_1 x + I^\beta \eta(x,\zeta(x)) + I^\beta\Big(\int_0^x G^*(x,\tau)\zeta(\tau)d\tau\Big). \tag{2.1}$$

Using the boundary conditions, we can easily find that $c_0 = 0$ and

$$c_1 = \phi(\zeta) - \frac{1}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\eta(s,\zeta(s))ds - \frac{1}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\int_0^s G^*(s,\tau)\zeta(\tau)d\tau ds.$$

Consequently,

$$\begin{aligned}\zeta(x) &= \phi(\zeta)x - \frac{x}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\eta(s,\zeta(s))ds - \frac{x}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}F(s,\zeta)ds \\ &\quad + \frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}\eta(s,\zeta(s))ds + \frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}F(s,\zeta)ds \\ &= \phi(\zeta)x + \int_0^1 \overline{G}(x,s)\eta(s,\zeta(s))ds + \int_0^1 \overline{G}(x,s)F(s,\zeta)ds \\ &= \phi(\zeta)x + \int_0^1 G(x,s)\Lambda(s,\zeta)ds.\end{aligned}$$

Here, the vector Green's function $G(x,s)$ and vector function $\Lambda(s,\zeta)$ are defined in the proposition's hypotheses. This completes the proof. □

Lemma 2.2. *The function $\overline{G}(x,s)$ given in Proposition 2.1 satisfies the following inequality:*

$$|\overline{G}(x,s)| \leq \frac{1}{\Gamma(\beta)}\max\{g(\beta), h(\beta)\} \leq \frac{g(\beta)}{\Gamma(\beta)}, \qquad \text{on } I \times I$$

where $\beta \in (1,2]$,

$$h(\beta) = \begin{cases} (\beta-1)^{\frac{\beta-1}{2-\beta}} - (\beta-1)^{\frac{1}{2-\beta}}, & 1 < \beta < 2, \\ 0, & \beta = 2, \end{cases} \qquad g(\beta) = \frac{(\beta-1)^{\beta-1}}{\beta^\beta}, \quad \beta \in (1,2]. \tag{2.2}$$

Proof. Let us disregard the case $\beta = 2$, as the claim is easily proved in that scenario. Over $0 \le s \le x \le 1$,

$$|\overline{G}(x,s)| \le \frac{1}{\Gamma(\beta)} \max_{x\in I}\{x^{\beta-1} - x, x(1-x)^{\beta-1}\}.$$

Finding the extremum points of the two functions $x^{\beta-1} - x, x(1-x)^{\beta-1}$, one observes that they achieve their maximums at $x = (\beta-1)^{-\frac{1}{\beta-2}}$ and $x = \beta^{-1}$, respectively. This implies that

$$|\overline{G}(x,s)| \le \frac{1}{\Gamma(\beta)} \max\{g(\beta), h(\beta)\}, \qquad 0 \le s \le x \le 1,$$

where g, h are as given in the lemma's hypotheses. Additionally, numerical observations show that $h \le g$ over $(1,2)$ (see Figure 1). Similarly, for $0 \le x \le s \le 1$, we derive

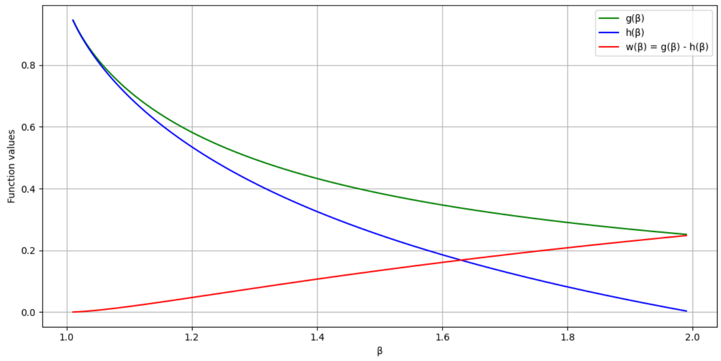

Figure 1: Graph of g, h and $w = g - h$ over $(1,2)$

$$|\overline{G}(x,s)| \le \frac{g(\beta)}{\Gamma(\beta)}, \qquad 0 \le x \le s \le 1.$$

We completes the proof. □

In the following theorem, we provide a necessary condition for the existence of a solution to equation (0.2). To do so, we need to impose some certain conditions to the coefficients as follow:

(C1) $\lambda \in C(I, \mathbb{R})$, which implies that $\|\lambda\|_\infty = \max_{x\in I} |\lambda(x)| < \infty$.

(C2) Let $\eta \in C(I \times \mathbb{R}, \mathbb{R})$ and ϕ be a functional defined on $C(I, \mathbb{R})$. We assume either

$$\left\|\eta\right\|_\infty = \sup_{(x,y)\in I\times\mathbb{R}} |\eta(x,y)| < \infty, \ \ A_\phi = \sup_{\zeta\in C(I,\mathbb{R})} |\phi(\zeta)| < \infty,$$

or satisfy the following conditions:

$$\begin{aligned} &\|\eta\| = \inf\{C_\eta > 0,\ |\eta(t,x)| \le C_\eta |x|,\ \text{for all } (t,x) \in I \times \mathbb{R}\} < \infty, \\ &\|\phi\| = \inf\{C_\phi > 0,\ |\phi(\zeta)| \le C_\phi \|\zeta\|_\infty,\ \text{for all } \zeta \in C(I,\mathbb{R})\} < \infty. \end{aligned} \tag{2.3}$$

Theorem 2.3. *Suppose there exists a nontrivial continuous solution to the nonlinear fractional boundary value problem* (0.2)*. Then*

(a) *If ϕ and η satisfy* (2.3), *then*

$$\|\phi\| + \frac{1}{\Gamma(\beta+1)}\Big(\frac{\beta-1}{\beta}\Big)^{\beta-1}\Big(\|\eta\| + \frac{\|\lambda\|_\infty}{\Gamma(\gamma+2)}\Big) \geq 1,$$

equivalently,

$$\|\phi\| + \frac{g(\beta)}{\Gamma(\beta)}\Big(\|\eta\| + \frac{\|\lambda\|_\infty}{\Gamma(\gamma+2)}\Big) \geq 1.$$

(b) *If ϕ is uniformly bounded, that is $A_\phi := \sup_{\zeta\in C(I,\mathbb{R})}|\phi(\zeta)| < \infty$, and $\|\eta\|_\infty$ is bounded with*

$$\|\lambda\|_\infty < \frac{\Gamma(\beta)\Gamma(\gamma+2)}{g(\beta)},$$

then an upper bound for the solution ζ is given by

$$\|\zeta\|_\infty \leq \Gamma(\gamma+2)\cdot\frac{\Gamma(\beta)A_\phi + g(\beta)\|\eta\|_\infty}{\Gamma(\beta)\Gamma(\gamma+2) - g(\beta)\|\lambda\|_\infty}.$$

Proof. Suppose $\zeta \in C(I,\mathbb{R})$ is a solution of equation (0.2), then using Proposition 2.1,

$$\zeta(x) = \phi(\zeta)x + \int_0^1 G(x,\tau)\Lambda(\tau,\zeta)d\tau.$$

Case 1. For the case ϕ and η satisfy (2.3), by employing Lemma 2.2 we have

$$|\zeta(x)| \leq |\phi(\zeta)|\cdot|x| + \frac{g(\beta)}{\Gamma(\beta)}\cdot\int_0^1(\|\eta\|\cdot\|\zeta\|_\infty + |F(\tau,\zeta)|)d\tau. \tag{2.4}$$

On the other hand,

$$|F(\tau,\zeta)| = \left|\int_0^\tau G^*(\tau,s)\zeta(s)ds\right| \leq \|\lambda\|_\infty\cdot\|\zeta\|_\infty\cdot\frac{\tau^\gamma}{\Gamma(\gamma+1)}. \tag{2.5}$$

Combining (2.4) and (2.5) yields that

$$\|\zeta\|_\infty \leq \|\phi\|\cdot\|\zeta\|_\infty + \frac{g(\beta)}{\Gamma(\beta)}\Big(\|\eta\|\cdot\|\zeta\|_\infty + \frac{\|\lambda\|_\infty\cdot\|\zeta\|_\infty}{\Gamma(\gamma+2)}\Big),$$

which as ζ is a nontrivial solution we derive

$$1 \leq \|\phi\| + \frac{g(\beta)}{\Gamma(\beta)}\Big(\|\eta\| + \frac{\|\lambda\|_\infty}{\Gamma(\gamma+2)}\Big),$$

as the desired consequence.

Case 2. Now let us consider $\|\eta\|_\infty < \infty$ and ϕ is uniformly bounded, that is, $A_\phi := \sup_{\zeta\in C(I,\mathbb{R})}|\phi(\zeta)| < \infty$, then following the same argument as mentioned in previous case we get

$$\|\zeta\|_\infty \leq A_\phi + \frac{g(\beta)}{\Gamma(\beta)}\Big(\|\eta\|_\infty + \frac{\|\lambda\|_\infty\cdot\|\zeta\|_\infty}{\Gamma(\gamma+2)}\Big).$$

This implies

$$\|\zeta\|_\infty \leq \Gamma(\gamma+2)\cdot\frac{\Gamma(\beta)A_\phi + g(\beta)\|\eta\|_\infty}{\Gamma(\beta)\Gamma(\gamma+2) - g(\beta)\|\lambda\|_\infty},$$

which completes the proof. □

3. Existence and Uniqueness to Equation (0.1)

In this section, we focus on the fractional boundary value problem (0.1), which is a generalization of the fractional differential equation (0.2). The analysis of solutions is carried out using the integral representation that forms the basis of the subsequent existence results.

Lemma 3.1. *$\zeta \in C(I, \mathbb{R})$ is a solution of equation (0.1) if and only if ζ satisfies the integral equation*

$$\zeta(x) = \phi(\zeta)x + \int_0^1 \overline{G_0}(x,s) f(s, \zeta(s), I^\gamma \zeta(s))ds, \tag{3.1}$$

where

$$\overline{G_0}(x,s) = \begin{cases} \frac{x(1-s)^{\beta-1} - (x-s)^{\beta-1}}{\Gamma(\beta)}, & 0 \le s \le x \le 1, \\ \\ \frac{x(1-s)^{\beta-1}}{\Gamma(\beta)}, & 0 \le x \le s \le 1. \end{cases} \tag{3.2}$$

Proof. From Lemma 1.3, we convert equation (0.1) to an equivalent integral equation

$$\zeta(x) = c_0 + c_1 x - \int_0^x \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} f(s, \zeta(s), I^\gamma \zeta(s))ds. \tag{3.3}$$

Using the boundary conditions, we can easily find that $c_0 = 0$ and

$$c_1 = \phi(\zeta) + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s, \zeta(s), I^\gamma \zeta(s))ds.$$

Consequently,

$$\begin{aligned} \zeta(x) &= \phi(\zeta)x + \int_0^1 \frac{x(1-s)^{\beta-1}}{\Gamma(\beta)} f(s, \zeta(s), I^\gamma \zeta(s))ds - \int_0^x \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} f(s, \zeta(s), I^\gamma \zeta(s))ds \\ &= \phi(\zeta)x + \int_0^1 \overline{G_0}(x,s) f(s, \zeta(s), I^\gamma \zeta(s))ds, \end{aligned}$$

where the Green's function $\overline{G}_0(x,s)$ is defined in (3.2). □

Applying Schauder's fixed point theorem, we are ready to establish sufficient conditions for the existence of solutions to Eq. (0.1) in the space $C(I, \mathbb{R})$.

Theorem 3.2. *Assume that $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and $\phi : C(I, \mathbb{R}) \to \mathbb{R}$ is a continuous functional such that the norm $\|\phi\|$ as defined in (2.3), satisfies the inequality $\|\phi\| < 1$. In addition, suppose that one of the following conditions is satisfied*

(H_1) *There exist a non-negative function $m(t) \in L^1(I)$, with $0 < p, q < 1$ and constants $C_p, C_q \ge 0$ such that*

$$|f(t,x,y)| \le m(t) + C_p|x|^p + C_q|y|^q, \quad \forall t \in I, \forall x, y \in \mathbb{R};$$

(H_2) *There exist constants $p, q > 1$ and $C_p, C_q \ge 0$ such that*

$$|f(t,x,y)| \le C_p|x|^p + C_q|y|^q, \quad \forall t \in I, \forall x, y \in \mathbb{R};$$

(H_3) *There exist constants $C, \overline{C} \ge 0$ such that*

$$\|\phi\| + \frac{g(\beta)}{\Gamma(\beta)}(C + \overline{C}) \le 1, \tag{3.4}$$

and

$$|f(t,x,y)| \le C|x| + \overline{C}|y|, \quad \forall t \in I, \forall x, y \in \mathbb{R}.$$

Then Eq. (0.1) *has a solution in the ball* $B_R \subset C(I,\mathbb{R})$, *for an appropriate* $R > 0$.

Proof. Suppose that $T : C(I,\mathbb{R}) \to C(I,\mathbb{R})$ is the operator given by

$$T\zeta(x) = \phi(\zeta)x + \int_0^1 \overline{G_0}(x,s) f(s,\zeta(s),I^\gamma\zeta(s))ds. \tag{3.5}$$

First, assume that (H_1) holds. Define the following ball in $C(I,\mathbb{R})$,

$$B_R = \{\zeta(x) \in C(I,\mathbb{R}) \mid \|\zeta\|_\infty \le R\},$$

where

$$R \ge \max\left\{\frac{g(\beta)}{\Gamma(\beta)\alpha_1}\|m\|_{L^1(I)}, \left(\frac{g(\beta)C_p}{\Gamma(\beta)\alpha_2}\right)^{\frac{1}{1-p}}, \left(\frac{g(\beta)C_q}{\Gamma(\beta)\alpha_3}\right)^{\frac{1}{1-q}}\right\},$$

where $0 < p,q < 1$, and the positive constants α_i, $i = 1,2,3$, are sufficiently small to satisfy

$$\alpha_1 + \alpha_2 + \alpha_3 = 1 - \|\phi\|.$$

Now we aim to prove that $T : B_R \to B_R$ is a self-map. For any $\zeta \in B_R$, utilizing Lemma 2.2, we derive

$$\begin{aligned}|T\zeta(x)| &= \left|\phi(\zeta)x + \int_0^1 \overline{G_0}(x,s) f(s,\zeta(s),I^\gamma\zeta(s))ds\right| \\ &\le \|\phi\|R + \frac{g(\beta)}{\Gamma(\beta)}\Big(\|m\|_{L^1(I)} + C_pR^p + C_qR^q\Big),\ x \in I,\end{aligned}$$

by noting that $\overline{G_0}(x,s) = -\overline{G}(x,s)$ previously defined in Proposition 2.1.

Hence,

$$\begin{aligned}\|T\zeta\|_\infty &\le \|\phi\|R + \frac{g(\beta)}{\Gamma(\beta)}\Big(\|m\|_{L^1(I)} + C_pR^p + C_qR^q\Big) \\ &\le \|\phi\|R + \alpha_1R + \alpha_2R + \alpha_3R = R.\end{aligned}$$

Note that $T\zeta$ is continuous over $C(I,\mathbb{R})$ because f and ϕ are continuous. Therefore, we claim that $T : B_R \to B_R$ is continuous.

Under hypothesis (H_2), taking

$$R \ge \max\left\{\left(\frac{g(\beta)C_p}{\Gamma(\beta)\hat{\alpha}_1}\right)^{\frac{1}{1-p}}, \left(\frac{g(\beta)C_q}{\Gamma(\beta)\hat{\alpha}_2}\right)^{\frac{1}{1-q}}\right\} > 0,$$

where $p,q > 1$, and the positive constants $\hat{\alpha}_1$ and $\hat{\alpha}_2$ satisfy

$$\hat{\alpha}_1 + \hat{\alpha}_2 = 1 - \|\phi\|,$$

and repeating the same arguments similar to that above we can arrive at

$$\begin{aligned}\|T\zeta\|_\infty &\le \|\phi\|R + \frac{g(\beta)}{\Gamma(\beta)}\Big(C_pR^p + C_qR^q\Big) \\ &\le \|\phi\|R + \hat{\alpha}_1R + \hat{\alpha}_2R = R.\end{aligned}$$

Finally, if (H_3) holds, using (3.4) we easily find that $\|T\zeta\|_\infty \le R$ and consequently, $T : B_R \to B_R$.

In summary, we imply that $T : B_R \to B_R$ is a continuous operator.

In what follows we prove that T is equicontinuous. Let us set

$$L := \max_{\substack{\zeta\in B_R\\ x\in I}}\{f(x,\zeta(x),I^\gamma\zeta(x))\} < \infty,$$

and assume that $x_1, x_2 \in I$ and $x_1 < x_2$, then for any $\zeta \in B_R$ we have

$$\begin{aligned}|T\zeta(x_2) - T\zeta(x_1)| &= \left|\phi(\zeta)(x_2 - x_1) + \int_0^1 \left(\overline{G_0}(x_2,t) - \overline{G_0}(x_1,t)\right) f(t,\zeta(t),I^\gamma\zeta(t))dt\right| \\ &\le \|\phi\|\,\|\zeta\|_\infty(x_2 - x_1) + L\int_0^1 \left|\overline{G_0}(x_2,t) - \overline{G_0}(x_1,t)\right|dt \\ &\le \|\phi\|R(x_2 - x_1) + \frac{L}{\Gamma(\beta+1)}\left[x_2^\beta - x_1^\beta + x_2 - x_1\right],\end{aligned}$$

which infers the set TB_R is equicontinuous. Additionally, TB_R is uniformly bounded and closed. By Ascoli's theorem, TB_R is compact. Using Schauder's fixed-point theorem, we conclude that there exists a solution within B_R for the fractional boundary value problem given in (0.1). This completes the proof. □

Theorem 3.3. *Assume that $f : C(I \times \mathbb{R} \times \mathbb{R}) \to \mathbb{R}$ is a Lipschitz continuous functional and satisfies*

$$|f(t,x_2,y_2) - f(t,x_1,y_1)| \le C_1|x_2 - x_1| + C_2|y_2 - y_1|, \quad \forall t \in I, \forall x_i, y_i \in \mathbb{R}, \textit{ for } i = 1,2,$$

where $C_1, C_2 > 0$ are constants, and ϕ is a functional defined on $C(I,\mathbb{R})$ and satisfies

$$|\phi(\zeta_1) - \phi(\zeta_2)| \le C_0\,\|\zeta_1 - \zeta_2\|_\infty\,, \quad \zeta,\ \zeta_2 \in C(I,\mathbb{R}). \tag{3.6}$$

In addition,

$$C_0 + \frac{g(\beta)}{\Gamma(\beta)}\left(C_1 + \frac{C_2}{\Gamma(\gamma+1)}\right) < 1. \tag{3.7}$$

Then Eq. (0.1) *has a unique solution.*

Proof. To prove the uniqueness, we suppose the converse is true, that means equation (0.1) has two distinct solutions ζ_1, ζ_2 in $C(I,\mathbb{R})$. Then

$$\begin{aligned}|\zeta_2(x) - \zeta_1(x)| &= \left|(\phi(\zeta_2) - \phi(\zeta_1))x + \int_0^1 \overline{G}(x,s)(f(s,\zeta_2(s),I^\gamma\zeta_2(s)) - f(s,\zeta_1(s),I^\gamma\zeta_1(s)))ds\right| \\ &\le C_0\,\|\zeta_2 - \zeta_1\|_\infty + \frac{g(\beta)}{\Gamma(\beta)}\left(C_1\|\zeta_2 - \zeta_1\|_\infty + C_2\|I^\gamma\zeta_2 - I^\gamma\zeta_1\|_\infty\right) \\ &\le \left[C_0 + \frac{g(\beta)}{\Gamma(\beta)}\left(C_1 + \frac{C_2}{\Gamma(\gamma+1)}\right)\right]\|\zeta_2 - \zeta_1\|_\infty,\end{aligned}$$

which implies that

$$1 \le C_0 + \frac{g(\beta)}{\Gamma(\beta)}\left(C_1 + \frac{C_2}{\Gamma(\gamma+1)}\right)$$

as ζ_1, ζ_2 are distinct functions on I, hence it contradicts with (3.7). Therefore, there is a unique solution. This completes the proof. □

In the following, we introduce a class of functions that play a crucial role in constructing $f(t,x,y)$ as defined in Eq. (0.1). This function class is essential for the subsequent example and is formally defined as

$$\mathcal{H} = \left\{ h \in C\left(\mathbb{R}_0^+, \mathbb{R}\right) \;\middle|\; \frac{h(t)}{t} \text{ is decreasing on } \mathbb{R}^+, h(t) \le ct^{p_0} \text{ over } \mathbb{R}^+, \text{ for some } c, p_0 > 0 \right\}$$

where $\mathbb{R}_0^+ = [0,\infty)$ and $\mathbb{R}^+ = (0,\infty)$.

Clearly, $\mathcal{H}$ is non-empty, as it includes a variety of commonly used functions. For instance, the functions

$$h_1(t) = at^p, \quad h_2(t) = 1 - e^{-at}, \quad h_3(t) = \ln(1+at), \quad h_4(t) = \tan^{-1}(at),$$

all belong to $\mathcal{H}$ for any $a > 0$ and $0 < p < 1$.

Remark 3.4. A key property of any function h in $\mathcal{H}$ is subadditivity, as shown below

$$h(t_1+t_2) = \frac{t_1 h(t_1+t_2)}{t_1+t_2} + \frac{t_2 h(t_1+t_2)}{t_1+t_2} \le \frac{t_1 h(t_1)}{t_1} + \frac{t_2 h(t_2)}{t_2} = h(t_1) + h(t_2), \quad t_1, t_2 \ge 0.$$

Example 3.5. Consider the following fractional differential equation:

$$\begin{cases} {}_C D^{1.7}\zeta(x) + \lambda(x)\tan^{-1} a(|\zeta(x)| + |I^{1.4}\zeta(x)|) = 0, \quad x \in I, \ a > 0, \\ \zeta(0) = 0, \quad \zeta(1) = \int_0^1 w(t)\zeta(t)dt. \end{cases} \tag{3.8}$$

Here, $w \in L^1(I)$ is either non-negative or non-positive, and $\lambda : I \to \mathbb{R}$ is a bounded, non-trivial function satisfying the condition

$$\mathcal{A} := \|w\|_{L^1(I)} + \frac{2ag(1.7)\|\lambda\|_\infty}{\Gamma(1.7)} \le 1.$$

Define the functional

$$\phi(\zeta) := \int_0^1 w(t)\zeta(t)dt, \quad \text{for } \zeta \in C(I,\mathbb{R}),$$

where $C_0 = \|w\|_{L^1(I)} < 1$, ensuring that the condition (3.6) holds.

To verify the required assumptions, we observe that

$$\begin{aligned} |f(t,x,y)| &:= |\lambda(t) \cdot \tan^{-1} a(|x| + |y|)| \\ &\le \|\lambda\|_\infty \left(\tan^{-1} a|x| + \tan^{-1} a|y|\right) \\ &\le a\|\lambda\|_\infty (|x| + |y|). \end{aligned}$$

Thus, by Theorem 3.2, the fractional differential equation (3.8) admits at least one solution.

Furthermore, since

$$\|w\|_1 + \frac{ag(1.7)\|\lambda\|_\infty}{\Gamma(1.7)}\left(1 + \frac{1}{\Gamma(2.4)}\right) < A \le 1$$

the uniqueness of the solution follows from Theorem 3.3.

Example 3.6. Consider the fractional differential equation

$$\begin{cases} {}_CD^{1.7}\zeta(x) + \lambda(x) + \sin\left(\dfrac{|\zeta(x)|^p}{1+a|\zeta(x)|^q} + \dfrac{|I^{1.4}\zeta(x)|^p}{1+b|I^{1.4}\zeta(x)|^q}\right) = 0, \quad x \in I,\ a,b>0, \\ \zeta(0) = 0, \quad \zeta(1) = \int_0^1 w(t)\zeta(t)dt. \end{cases} \tag{3.9}$$

Here, λ and w are defined as in the previous example. The exponents satisfy the constraint $\max\{1,q\} < p < q+1$, while the constants a,b,p,q and the function w are chosen to ensure that the condition

$$\mathcal{A} := \|w\|_{L^1(I)} + \frac{g(1.7)}{\Gamma(1.7)}\left(A_a + \frac{A_b}{\Gamma(2.4)}\right) < 1, \tag{3.10}$$

is satisfied, where

$$A_a := \max_{x\in\mathbb{R}_0^+} \frac{px^{p-1} + a(p-q)x^{p+q-1}}{(1+ax^q)^2} < \infty,$$

$$A_b := \max_{x\in\mathbb{R}_0^+} \frac{px^{p-1} + b(p-q)x^{p+q-1}}{(1+bx^q)^2} < \infty.$$

To estimate the nonlinear term, we observe that for any $t \in I$, and $x, y \in \mathbb{R}$, the function $f(t,x,y)$ satisfies the bound

$$\begin{aligned} |f(t,x,y)| &:= \left|\lambda(t) + \sin\left(\frac{|x|^p}{1+a|x|^q} + \frac{|y|^p}{1+b|y|^q}\right)\right| \\ &\le |\lambda(t)| + \frac{|x|^p}{1+a|x|^q} + \frac{|y|^p}{1+b|y|^q} \\ &\le |\lambda(t)| + \frac{1}{a}|x|^{p-q} + \frac{1}{b}|y|^{p-q}, \end{aligned}$$

Moreover, given $t \in I$ and any $x_1, x_2, y_1, y_2 \in \mathbb{R}$, we establish the Lipschitz continuity property

$$\begin{aligned} \left|f(t,x_2,y_2) - f(t,x_1,y_1)\right| &= \left|\sin\left(\frac{|x_2|^p}{1+a|x_2|^q} + \frac{|y_2|^p}{1+b|y_2|^q}\right) - \sin\left(\frac{|x_1|^p}{1+a|x_1|^q} + \frac{|y_1|^p}{1+b|y_1|^q}\right)\right| \\ &\le \left|\frac{|x_2|^p}{1+a|x_2|^q} - \frac{|x_1|^p}{1+a|x_1|^q}\right| + \left|\frac{|y_2|^p}{1+b|y_2|^q} - \frac{|y_1|^p}{1+b|y_1|^q}\right| \\ &\le A_a|x_2 - x_1| + A_b|y_2 - y_1|, \end{aligned}$$

By applying Theorems 3.2 and 3.3, we conclude that equation (3.9) admits a unique solution.

As a concrete illustration, let us consider the parameter values $a = b = p = 2, q = 1.5$, and define $\lambda(x) = w(x) = \tan x$. Under these settings, we obtain

$$A_a = A_b \approx 0.4079, \qquad \mathcal{A} = \ln\sec 1 + \frac{0.3161}{0.9086}\left(0.4079 + \frac{0.4079}{1.2422}\right) \approx 0.8717 < 1.$$

This confirms the existence and uniqueness of a solution to equation (3.9).

4. Uniqueness and Existence for Equation (0.2)

In this section, we are going to use the inverse operator method to study the uniqueness and existence to equation (0.2) based on the Mittag-Leffler function and several notable fixed-point theorems.

Theorem 4.1. *Let η be a continuous and bounded function over $[0,1]\times\mathbb{R}$, $\lambda(x)\in C[0,1]$, $\phi : C(I,\mathbb{R})\to\mathbb{R}$ be a functional, $1<\beta\le 2$ and $\gamma\ge 0$. Then ζ is a solution of equation (0.2) if and only if it is equivalent to the following implicit integral equation in $C(I,\mathbb{R})$:*

$$\zeta(x)=\sum_{k=0}^{\infty}(-1)^k\left(I^{\beta}\lambda(x)I^{\gamma}\right)^k I^{\beta}\eta(x,\zeta(x))+\phi(\zeta)\sum_{k=0}^{\infty}(-1)^k\left(I^{\beta}\lambda(x)I^{\gamma}\right)^k x$$
$$+I^{\beta}_{x=1}\lambda(x)\,I^{\gamma}\zeta(x)\sum_{k=0}^{\infty}(-1)^k\left(I^{\beta}\lambda(x)I^{\gamma}\right)^k x-I^{\beta}_{x=1}\eta(x,\zeta(x))\sum_{k=0}^{\infty}(-1)^k\left(I^{\beta}\lambda(x)I^{\gamma}\right)^k x. \tag{4.1}$$

In addition, if

$$\mathcal{W}=1-\frac{\|\lambda\|_\infty}{\Gamma(\beta+\gamma+1)}E_{\beta+\gamma,\,1}(\|\lambda\|_\infty)>0,$$

then

$$\|\zeta\|_\infty\le\frac{1}{\mathcal{W}}\Big(E_{\beta+\gamma,\,\beta+1}(\|\lambda\|_\infty)\|\eta\|_\infty+|\phi(\zeta)|\,E_{\beta+\gamma,\,1}(\|\lambda\|_\infty)$$
$$+\frac{1}{\Gamma(\beta+1)}E_{\beta+\gamma,\,1}(\|\lambda\|_\infty)\|\eta\|_\infty\Big)<+\infty.$$

Proof. For $1<\beta\le 2$,

$$I^{\beta}({}_CD^{\beta})\zeta(x)=\zeta(x)-\zeta(0)-\zeta'(0)x=\zeta(x)-\zeta'(0)x,$$

since $\zeta(0)=0$. Applying the integral operator I^{β} to both sides of the equation,

$${}_CD^{\beta}\zeta(x)+\lambda(x)\,I^{\gamma}\zeta(x)=\eta(x,\zeta(x)),$$

we get

$$\zeta(x)-\zeta'(0)x+I^{\beta}\lambda(x)I^{\gamma}\zeta(x)=I^{\beta}\eta(x,\zeta(x)). \tag{4.2}$$

Let $x=1$

$$\phi(\zeta)-\zeta'(0)+I^{\beta}_{x=1}\lambda(x)I^{\gamma}\zeta(x)=I^{\beta}_{x=1}\eta(x,\zeta(x)),\qquad(\text{since }\zeta(1)=\phi(\zeta)),$$

then

$$\zeta'(0)=\phi(\zeta)+I^{\beta}_{x=1}\lambda(x)I^{\gamma}\zeta(x)-I^{\beta}_{x=1}\eta(x,\zeta(x)).$$

From equation (4.2), we come to

$$(1+I^{\beta}\lambda(x)I^{\gamma})\,\zeta(x)=I^{\beta}\eta(x,\zeta(x))+\zeta'(0)x.$$

We will show that the inverse operator of $1+I^{\beta}\lambda(x)I^{\gamma}$ is

$$V=\sum_{k=0}^{\infty}(-1)^k\left(I^{\beta}\lambda(x)I^{\gamma}\right)^k$$

in the space $C(I,\mathbb{R})$. Indeed, for any function $\phi\in C(I,\mathbb{R})$,

$$\|V\phi\|_\infty=\left\|\sum_{k=0}^{\infty}(-1)^k\left(I^{\beta}\lambda(x)I^{\gamma}\right)^k\phi\right\|\le\|\phi\|_\infty\sum_{k=0}^{\infty}\|\lambda\|_\infty^k\left\|I^{k(\beta+\gamma)}\right\|\quad\text{(integral operator norm)}$$
$$\le\|\phi\|_\infty\sum_{k=0}^{\infty}\|\lambda\|_\infty^k\frac{1}{\Gamma(k(\beta+\gamma)+1)}=\|\phi\|_\infty\,E_{\beta+\gamma,\,1}(\|\lambda\|_\infty)<+\infty.$$

Therefore, V is a continuous mapping from $C(I,\mathbb{R})$ to itself and the series is uniformly convergent. Further, we prove that

$$V(1+I^{\beta}\lambda(x)I^{\gamma}) = (1+I^{\beta}\lambda(x)I^{\gamma})V = 1, \quad \text{(identity operator)}.$$

In fact,

$$\begin{aligned} V(1+I^{\beta}\lambda(x)I^{\gamma}) &= V + \sum_{k=0}^{\infty}(-1)^k\left(I^{\beta}\lambda(x)I^{\gamma}\right)^{k+1} = 1 + \sum_{k=1}^{\infty}(-1)^k\left(I^{\beta}\lambda(x)I^{\gamma}\right)^{k} + \sum_{k=0}^{\infty}(-1)^k\left(I^{\beta}\lambda(x)I^{\gamma}\right)^{k+1} \\ &= 1 + \sum_{k=0}^{\infty}(-1)^{k+1}\left(I^{\beta}\lambda(x)I^{\gamma}\right)^{k+1} + \sum_{k=0}^{\infty}(-1)^k\left(I^{\beta}\lambda(x)I^{\gamma}\right)^{k+1} = 1. \end{aligned}$$

Similarly,

$$(1+I^{\beta}\lambda(x)I^{\gamma})V = 1,$$

and the uniqueness follows easily. This implies that

$$\begin{aligned} &\zeta(x) = (1+I^{\beta}\lambda(x)I^{\gamma})^{-1}\left[I^{\beta}\eta(x,\zeta(x)) + x\left(\phi(\zeta) + I^{\beta}_{x=1}\lambda(x)I^{\gamma}\zeta(x) - I^{\beta}_{x=1}\eta(x,\zeta(x))\right)\right] \\ &= \sum_{k=0}^{\infty}(-1)^k\left(I^{\beta}\lambda(x)I^{\gamma}\right)^k I^{\beta}\eta(x,\zeta(x)) + \phi(\zeta)\sum_{k=0}^{\infty}(-1)^k\left(I^{\beta}\lambda(x)I^{\gamma}\right)^k x \\ &\quad + I^{\beta}_{x=1}\lambda(x)I^{\gamma}\zeta(x)\sum_{k=0}^{\infty}(-1)^k\left(I^{\beta}\lambda(x)I^{\gamma}\right)^k x - I^{\beta}_{x=1}\eta(x,\zeta(x))\sum_{k=0}^{\infty}(-1)^k\left(I^{\beta}\lambda(x)I^{\gamma}\right)^k x. \end{aligned}$$

Hence, ζ is a solution of equation (0.2) if and only if it is equivalent to the implicit integral equation (4.1), since all the above steps are reversible.

Moreover,

$$\begin{aligned} \|\zeta\|_\infty &\le \sum_{k=0}^{\infty}\frac{\|\lambda\|_\infty^k}{\Gamma(k(\beta+\gamma)+\beta+1)}\|\eta\|_\infty + |\phi(\zeta)|\sum_{k=0}^{\infty}\frac{\|\lambda\|_\infty^k}{\Gamma(k(\beta+\gamma)+1)} \\ &+ \frac{\|\lambda\|_\infty\,\|\zeta\|_\infty}{\Gamma(\beta+\gamma+1)}\sum_{k=0}^{\infty}\frac{\|\lambda\|_\infty^k}{\Gamma(k(\beta+\gamma)+1)} + \frac{1}{\Gamma(\beta+1)}\sum_{k=0}^{\infty}\frac{\|\lambda\|_\infty^k}{\Gamma(k(\beta+\gamma)+1)}\|\eta\|_\infty \\ &\le E_{\beta+\gamma,\,\beta+1}(\|\lambda\|_\infty)\|\eta\|_\infty + |\phi(\zeta)|\;E_{\beta+\gamma,\,1}(\|\lambda\|_\infty) + \frac{\|\lambda\|_\infty\,\|\zeta\|_\infty}{\Gamma(\beta+\gamma+1)}E_{\beta+\gamma,\,1}(\|\lambda\|_\infty) \\ &+ \frac{1}{\Gamma(\beta+1)}E_{\beta+\gamma,\,1}(\|\lambda\|_\infty)\|\eta\|_\infty. \end{aligned}$$

Since

$$\mathcal{W} = 1 - \frac{\|\lambda\|_\infty}{\Gamma(\beta+\gamma+1)}E_{\beta+\gamma,\,1}(\|\lambda\|_\infty) > 0,$$

we get

$$\|\zeta\|_\infty \le \frac{1}{\mathcal{W}}\left(E_{\beta+\gamma,\,\beta+1}(\|\lambda\|_\infty)\|\eta\|_\infty + |\phi(\zeta)|\;E_{\beta+\gamma,\,1}(\|\lambda\|_\infty) + \frac{1}{\Gamma(\beta+1)}E_{\beta+\gamma,\,1}(\|\lambda\|_\infty)\|\eta\|_\infty\right) < +\infty.$$

This completes the proof. □

Theorem 4.2. *Let η be a continuous and bounded function on $[0,1]\times\mathbb{R}$, satisfying the following Lipschitz condition for a non-negative constant $\mathcal{L}_1$:*

$$|\eta(x,y_1)-\eta(x,y_2)|\leq\mathcal{L}_1\,|y_1-y_2|,\quad y_1,y_2\in\mathbb{R},\ \textit{and}\ \ x\in[0,1]$$

and ϕ: $C(I,\mathbb{R})\to\mathbb{R}$ be a functional satisfying the condition for a non-negative constant $\mathcal{L}_2$:

$$|\phi(\zeta_1)-\phi(\zeta_2)|\leq\mathcal{L}_2\,\|\zeta_1-\zeta_2\|_\infty,$$

for ζ_1, $\zeta_2\in C(I,\mathbb{R})$. Furthermore, if

$$\Omega=\mathcal{L}_1E_{\beta+\gamma,\,\beta+1}(\|\lambda\|_\infty)+\left(\mathcal{L}_2+\frac{\|\lambda\|_\infty}{\Gamma(\beta+\gamma+1)}+\frac{\mathcal{L}_1}{\Gamma(\beta+1)}\right)E_{\beta+\gamma,\,1}(\|\lambda\|_\infty)<1,$$

then equation (0.2) has a unique solution in $C(I,\mathbb{R})$.

Proof. We first define a nonlinear mapping $\mathcal{M}$ over $C(I,\mathbb{R})$ as

$$(\mathcal{M}\zeta)(x)=\sum_{k=0}^{\infty}(-1)^k(I^\beta\lambda(x)I^\gamma)^kI^\beta\eta(x,\zeta(x))+\phi(\zeta)\sum_{k=0}^{\infty}(-1)^k(I^\beta\lambda(x)I^\gamma)^kx$$
$$+I^\beta_{x=1}\lambda(x)\,I^\gamma\zeta(x)\sum_{k=0}^{\infty}(-1)^k(I^\beta\lambda(x)I^\gamma)^kx-I^\beta_{x=1}\eta(x,\zeta(x))\sum_{k=0}^{\infty}(-1)^k(I^\beta\lambda(x)I^\gamma)^kx.$$

It follows from the proof of Theorem 4.1 that

$$\|\mathcal{M}\zeta\|_\infty<+\infty,$$

which claims that $\mathcal{M}\zeta\in C(I,\mathbb{R})$.

Next, we show that $\mathcal{M}$ is contractive.

$$\mathcal{M}\zeta_1-\mathcal{M}\zeta_2=\sum_{k=0}^{\infty}(-1)^k(I^\beta\lambda(x)I^\gamma)^kI^\beta\left(\eta(x,\zeta_1(x))-\eta(x,\zeta_2(x))\right)$$
$$+\left(\phi(\zeta_1)-\phi(\zeta_2)\right)\sum_{k=0}^{\infty}(-1)^k(I^\beta\lambda(x)I^\gamma)^kx+I^\beta_{x=1}\lambda(x)I^\gamma\left(\zeta_1(x)-\zeta_2(x)\right)\sum_{k=0}^{\infty}(-1)^k(I^\beta\lambda(x)I^\gamma)^kx$$
$$-I^\beta_{x=1}\left(\eta(x,\zeta_1(x))-\eta(x,\zeta_2(x))\right)\sum_{k=0}^{\infty}(-1)^k(I^\beta\lambda(x)I^\gamma)^kx.$$

Hence,

$$\begin{aligned}\|\mathcal{M}\zeta_1-\mathcal{M}\zeta_2\|_\infty&\leq\mathcal{L}_1E_{\beta+\gamma,\,\beta+1}(\|\lambda\|_\infty)\,\|\zeta_1-\zeta_2\|_\infty+\mathcal{L}_2E_{\beta+\gamma,\,1}(\|\lambda\|_\infty)\,\|\zeta_1-\zeta_2\|_\infty\\&\quad+\frac{\|\lambda\|_\infty\,\|\zeta_1-\zeta_2\|_\infty}{\Gamma(\beta+\gamma+1)}\,E_{\beta+\gamma,\,1}(\|\lambda\|_\infty)+\frac{\mathcal{L}_1}{\Gamma(\beta+1)}E_{\beta+\gamma,1}(\|\lambda\|_\infty)\,\|\zeta_1-\zeta_2\|_\infty\\&\leq\Omega\,\|\zeta_1-\zeta_2\|_\infty.\end{aligned}$$

Since $\Omega<1$, equation (0.2) has a unique solution in $C(I,\mathbb{R})$ by using the Banach contraction principle. □

Example 4.3. The following nonlinear fractional integro-differential equation with variable coefficient and a functional boundary condition:

$$\begin{cases}{}_CD^{1.5}\zeta(x)+\dfrac{1}{22(x^2+1)}\,I^{1.1}\zeta(x)=\dfrac{1}{20}\cos\left((x^3+2)\zeta(x)\right),\quad x\in[0,1],\\ \zeta(0)=0,\ \ \zeta(1)=\dfrac{1}{18}\displaystyle\int_0^1x^2|\zeta(x)|dx,\end{cases}\tag{4.3}$$

has a unique solution in $C(I,\mathbb{R})$.

Proof. Clearly,

$$\beta = 1.5, \ \ \|\lambda\|_\infty = \frac{1}{22}, \ \ \gamma = 1.1,$$

and

$$\eta(x,\zeta) = \frac{1}{20}\cos((x^3+2)\zeta)$$

is a continuous and bounded function on $[0,1]\times\mathbb{R}$, satisfying

$$\begin{aligned}|\eta(x,\zeta_1)-\eta(x,\zeta_2)| &\le \frac{1}{20}\left|\cos((x^3+2)\zeta_1)-\cos((x^3+2)\zeta_2)\right| \quad (\text{by noting that } |\cos x-\cos y|\le|x-y|)\\ &\le \frac{3}{20}|\zeta_1-\zeta_2| \quad \left(\text{since } \max_{x\in[0,1]}\left|x^3+2\right| = 3\right),\end{aligned}$$

which infers that $\mathcal{L}_1 = \frac{3}{20}$.

On the other hand,

$$\phi(\zeta) = \frac{1}{18}\int_0^1 x^2|\zeta(x)|dx$$

is a functional satisfying

$$\begin{aligned}|\phi(\zeta_1)-\phi(\zeta_2)| &\le \frac{1}{18}\left|\int_0^1 x^2(|\zeta_1(x)|-|\zeta_2(x)|)dx\right|\\ &\le \frac{1}{18}\int_0^1\left|x^2(|\zeta_1(x)|-|\zeta_2(x)|)\right|dx \quad \left(\text{since } \left||x|-|y|\right|\le|x-y|\right)\\ &\le \frac{1}{18}\int_0^1\left|\zeta_1(x)-\zeta_2(x)\right|dx\\ &\le \frac{1}{18}\|\zeta_1-\zeta_2\|_\infty,\end{aligned}$$

which implies that $\mathcal{L}_2 = \frac{1}{18}$. Thus,

$$\begin{aligned}\Omega &= \mathcal{L}_1 E_{\beta+\gamma,\,\beta+1}(\|\lambda\|_\infty) + \left(\mathcal{L}_2 + \frac{\|\lambda\|_\infty}{\Gamma(\beta+\gamma+1)} + \frac{\mathcal{L}_1}{\Gamma(\beta+1)}\right)E_{\beta+\gamma,\,1}(\|\lambda\|_\infty)\\ &= \frac{3}{20}E_{2.6,\,2.5}\left(\frac{1}{22}\right) + \left(\frac{1}{18} + \frac{1}{22\Gamma(3.6)} + \frac{3}{20\Gamma(2.5)}\right)E_{2.6,\,1}\left(\frac{1}{22}\right)\\ &\approx \frac{3}{20}(0.753881) + (0.180622)(1.01224)\\ &\approx 0.29591496328 < 1.\end{aligned}$$

By Theorem 4.2, equation (4.3) has a unique solution in $C(I,\mathbb{R})$. □

Theorem 4.4. *Let η be a continuous and bounded function on $[0,1]\times\mathbb{R}$ and $\phi : C(I,\mathbb{R}) \to \mathbb{R}$ be a functional satisfying the condition for a non-negative constant $\mathcal{L}_2$:*

$$|\phi(\zeta_1)-\phi(\zeta_2)| \le \mathcal{L}_2\,\|\zeta_1-\zeta_2\|_\infty,$$

for $\zeta_1, \zeta_2 \in C(I, \mathbb{R})$. In addition, if

$$Q = 1 - \left(\mathcal{L}_2 + \frac{\|\lambda\|_\infty}{\Gamma(\beta+\gamma+1)}\right) E_{\beta+\gamma,\,1}(\|\lambda\|_\infty) > 0,$$

then there exists at least one solution to equation (0.2) in the space $C(I, \mathbb{R})$.

Proof. Clearly,

$$|\phi(\zeta)| \leq |\phi(\zeta) - \phi(0) + \phi(0)| \leq |\phi(\zeta) - \phi(0)| + |\phi(0)| \leq \mathcal{L}_2 \, \|\zeta\|_\infty + |\phi(0)| < +\infty,$$

if $\zeta \in C(I, \mathbb{R})$. We define the nonlinear mapping $\mathcal{M}$ over $C(I, \mathbb{R})$ again as

$$\mathcal{M}\zeta(x) = \sum_{k=0}^{\infty} (-1)^k (I^\beta \lambda(x) I^\gamma)^k I^\beta \eta(x, \zeta(x)) + \phi(\zeta) \sum_{k=0}^{\infty} (-1)^k (I^\beta \lambda(x) I^\gamma)^k x$$
$$+ I^\beta_{x=1} \lambda(x) \, I^\gamma \zeta(x) \sum_{k=0}^{\infty} (-1)^k (I^\beta \lambda(x) I^\gamma)^k x - I^\beta_{x=1} \eta(x, \zeta(x)) \sum_{k=0}^{\infty} (-1)^k (I^\beta \lambda(x) I^\gamma)^k x.$$

It follows the proof of Theorem 4.1 that the mapping $\mathcal{M}$ is from $C(I, \mathbb{R})$ to itself.

We first show that (i) $\mathcal{M}$ is continuous. In fact,

$$\begin{aligned}\|\mathcal{M}\zeta_1 - \mathcal{M}\zeta_2\|_\infty \leq{}& E_{\beta+\gamma,\,\beta+1}(\|\lambda\|_\infty) \sup_{x \in [0,1]} |\eta(x, \zeta_1) - \eta(x, \zeta_2)| \\ &+ \mathcal{L}_2 \, \|\zeta_1 - \zeta_2\|_\infty \, E_{\beta+\gamma,\,1}(\|\lambda\|_\infty) + \frac{\|\lambda\|_\infty \, \|\zeta_1 - \zeta_2\|_\infty}{\Gamma(\beta+\gamma+1)} \, E_{\beta+\gamma,\,1}(\|\lambda\|_\infty) \\ &+ \frac{1}{\Gamma(\beta+1)} E_{\beta+\gamma,1}(\|\lambda\|_\infty) \sup_{x \in [0,1]} |\eta(x, \zeta_1) - \eta(x, \zeta_2)|.\end{aligned}$$

This implies $\mathcal{M}$ is continuous since η is continuous.

(ii) $\mathcal{M}$ is a mapping from bounded sets to bounded sets. Let $\mathcal{S}$ be a bounded set in $C(I, R)$. Then for $\zeta \in \mathcal{S}$,

$$|\phi(\zeta)| \leq \mathcal{L}_2 \, \|\zeta\|_\infty + |\phi(0)| < C,$$

where C is a positive constant. From the inequality,

$$\begin{aligned}\|\mathcal{M}\zeta\|_\infty \leq{}& E_{\beta+\gamma,\,\beta+1}(\|\lambda\|_\infty)\|\eta\|_\infty + \; C E_{\beta+\gamma,\,1}(\|\lambda\|_\infty) + \frac{\|\lambda\|_\infty \, \|\zeta\|_\infty}{\Gamma(\beta+\gamma+1)} E_{\beta+\gamma,\,1}(\|\lambda\|_\infty) \\ &+ \frac{1}{\Gamma(\beta+1)} E_{\beta+\gamma,\,1}(\|\lambda\|_\infty)\|\eta\|_\infty,\end{aligned}$$

which implies that $\mathcal{M}\zeta$ is uniformly bounded if $\zeta \in S$, as η is bounded.

In order to show thar $\mathcal{M}$ is equicontinuous on every bounded set $\mathcal{S}$ of $C(I, \mathbb{R})$, we first define the following:

$$\phi_k(\tau) = \lambda(\tau) I^\gamma_{x=\tau} \left(I^\beta \lambda(\tau) I^\gamma\right)^{k-1} I^\beta \eta(x, \zeta(x))$$

and

$$\psi_k(\tau) = \lambda(\tau) I^\gamma_{x=\tau} \left(I^\beta \lambda(\tau) I^\gamma\right)^{k-1} x,$$

for $k \geq 1$. Then,

$$\|\phi_k\|_\infty \leq \|\lambda\|^k \|\eta\|_\infty \|I^{k(\beta+\gamma)}\| \leq \frac{\|\lambda\|_\infty^k}{\Gamma(k(\beta+\gamma)+1)} \|\eta\|_\infty,$$

$$\|\psi_k\|_\infty \leq \|\lambda\|_\infty^k \|I^{(k-1)(\beta+\gamma)+\gamma}\| \leq \frac{\|\lambda\|_\infty^k}{\Gamma((k-1)(\beta+\gamma)+\gamma+1)}.$$

(iii) $\mathcal{M}$ is completely continuous from $C(I, \mathbb{R})$ to itself. By the Arzela–Ascoli theorem, we need to show that $\mathcal{M}$ is equicontinuous on every bounded set $\mathcal{S}$ of $C(I, \mathbb{R})$. For $0 \leq t_1 < t_2 \leq 1$ and $\zeta \in \mathcal{S}$, we have

$$\begin{aligned}
&(\mathcal{M}\zeta)(t_2) - (\mathcal{M}\zeta)(t_1) = \\
&I^{\beta}_{x=t_2}\eta(x,\zeta(x)) - I^{\beta}_{x=t_1}\eta(x,\zeta(x)) + \sum_{k=1}^{\infty}(-1)^k\left(I^{\beta}_{x=t_2}\phi_k(\tau) - I^{\beta}_{x=t_1}\phi_k(\tau)\right) \quad (= I_1) \\
&+ \phi(\zeta)(t_2-t_1) + \phi(\zeta)\sum_{k=1}^{\infty}(-1)^k\left(I^{\beta}_{x=t_2}\psi_k(\tau) - I^{\beta}_{x=t_1}\psi_k(\tau)\right) \quad (= I_2) \\
&+ I^{\beta}_{x=1}\lambda(x)I^{\gamma}\zeta(x)\left[(t_2-t_1) + \sum_{k=1}^{\infty}(-1)^k\left(I^{\beta}_{x=t_2}\psi_k(\tau) - I^{\beta}_{x=t_1}\psi_k(\tau)\right)\right] \quad (= I_3) \\
&- I^{\beta}_{x=1}\eta(x,\zeta(x))\left[(t_2-t_1) + \sum_{k=1}^{\infty}(-1)^k\left(I^{\beta}_{x=t_2}\psi_k(\tau) - I^{\beta}_{x=t_1}\psi_k(\tau)\right)\right]. \quad (= I_4)
\end{aligned}$$

As for I_1,

$$\begin{aligned}
I_1 &= I^{\beta}_{x=t_2}\eta(x,\zeta(x)) - I^{\beta}_{x=t_1}\eta(x,\zeta(x)) + \sum_{k=1}^{\infty}(-1)^k\left(I^{\beta}_{x=t_2}\phi_k(\tau) - I^{\beta}_{x=t_1}\phi_k(\tau)\right) \\
&= I_{12} + I_{13}.
\end{aligned}$$

Clearly,

$$\begin{aligned}
I_{12} &= I^{\beta}_{x=t_2}\eta(x,\zeta(x)) - I^{\beta}_{x=t_1}\eta(x,\zeta(x)) \\
&= \frac{1}{\Gamma(\beta)}\left(\int_0^{t_2}(t_2-\tau)^{\beta-1}\eta(\tau,\zeta(\tau))d\tau - \int_0^{t_1}(t_1-\tau)^{\beta-1}\eta(\tau,\zeta(\tau))d\tau\right),
\end{aligned}$$

and

$$\int_0^{t_2}(t_2-\tau)^{\beta-1}\eta(\tau,\zeta(\tau))d\tau = \int_0^{t_1}(t_2-\tau)^{\beta-1}\eta(\tau,\zeta(\tau))d\tau + \int_{t_1}^{t_2}(t_2-\tau)^{\beta-1}\eta(\tau,\zeta(\tau))d\tau.$$

Thus,

$$\begin{aligned}
&\int_0^{t_2}(t_2-\tau)^{\beta-1}\eta(\tau,\zeta(\tau))d\tau - \int_0^{t_1}(t_1-\tau)^{\beta-1}\eta(\tau,\zeta(\tau))d\tau \\
&= \int_0^{t_1}\left((t_2-\tau)^{\beta-1} - (t_1-\tau)^{\beta-1}\right)\eta(\tau,\zeta(\tau))d\tau + \int_{t_1}^{t_2}(t_2-\tau)^{\beta-1}\eta(\tau,\zeta(\tau))d\tau \\
&= I_{121} + I_{122},
\end{aligned}$$

and

$$\begin{aligned}
|I_{121}| &\leq \int_0^{t_1}\left((t_2-\tau)^{\beta-1} - (t_1-\tau)^{\beta-1}\right)d\tau\,\|\eta\|_\infty \\
&= \left(-\frac{(t_2-t_1)^\beta}{\beta} + \frac{t_2^\beta}{\beta} - \frac{t_1^\beta}{\beta}\right)\|\eta\|_\infty \leq \left(\frac{t_2^\beta}{\beta} - \frac{t_1^\beta}{\beta}\right)\|\eta\|_\infty \leq (t_2-t_1)\|\eta\|_\infty,
\end{aligned}$$

by the mean value theorem. On the other hand,

$$|I_{122}| \leq \int_{t_1}^{t_2} (t_2-\tau)^{\beta-1} d\tau \left\|\eta\right\|_\infty \leq (t_2-t_1) \ \left\|\eta\right\|_\infty .$$

Regarding I_{13}, we have

$$I^{\beta}_{x=t_2}\phi_k(\tau) - I^{\beta}_{x=t_1}\phi_k(\tau) = \frac{1}{\Gamma(\beta)}\left(\int_0^{t_2}(t_2-\tau)^{\beta-1}\phi_k(\tau)d\tau - \int_0^{t_1}(t_1-\tau)^{\beta-1}\phi_k(\tau)d\tau\right),$$

and

$$\int_0^{t_2}(t_2-\tau)^{\beta-1}\phi_k(\tau)d\tau = \int_0^{t_1}(t_2-\tau)^{\beta-1}\phi_k(\tau)d\tau + \int_{t_1}^{t_2}(t_2-\tau)^{\beta-1}\phi_k(\tau)d\tau.$$

Hence,

$$\begin{aligned}
&\int_0^{t_2}(t_2-\tau)^{\beta-1}\phi_k(\tau)d\tau - \int_0^{t_1}(t_1-\tau)^{\beta-1}\phi_k(\tau)d\tau \\
&= \int_0^{t_1}\left((t_2-\tau)^{\beta-1} - (t_1-\tau)^{\beta-1}\right)\phi_k(\tau)d\tau + \int_{t_1}^{t_2}(t_2-\tau)^{\beta-1}\phi_k(\tau)d\tau \\
&= I_{131} + I_{132},
\end{aligned}$$

and

$$\begin{aligned}
|I_{131}| &\leq \int_0^{t_1}\left((t_2-\tau)^{\beta-1} - (t_1-\tau)^{\beta-1}\right)d\tau \cdot \|\phi_k\|_\infty \\
&= \left(-\frac{(t_2-t_1)^\beta}{\beta} + \frac{t_2^\beta}{\beta} - \frac{t_1^\beta}{\beta}\right)\|\phi_k\|_\infty \leq \left(\frac{t_2^\beta}{\beta} - \frac{t_1^\beta}{\beta}\right)\|\phi_k\|_\infty \leq (t_2-t_1)\ \|\phi_k\|_\infty,
\end{aligned}$$

as well as

$$|I_{132}| \leq \int_{t_1}^{t_2}(t_2-\tau)^{\beta-1}d\tau\|\phi_k\|_\infty \leq (t_2-t_1)\ \|\phi_k\|_\infty.$$

In summary,

$$|I_{13}| \leq \sum_{k=1}^{\infty}\left|\left(I^{\beta}_{x=t_2}\phi_k(\tau) - I^{\beta}_{x=t_2}\phi_k(\tau)\right)\right| \leq \frac{2}{\Gamma(\beta)}(t_2-t_1)\sum_{k=1}^{\infty}\frac{\|\lambda\|_\infty^k}{\Gamma(k(\beta+\gamma)+1)}\left\|\eta\right\|_\infty .$$

Therefore,

$$\begin{aligned}
|I_1| &\leq |I_{12}| + |I_{13}| \leq \frac{2}{\Gamma(\beta)}(t_2-t_1)\left\|\eta\right\|_\infty + \frac{2}{\Gamma(\beta)}(t_2-t_1)\sum_{k=1}^{\infty}\frac{\|\lambda\|_\infty^k}{\Gamma(k(\beta+\gamma)+1)}\left\|\eta\right\|_\infty \\
&= \frac{2}{\Gamma(\beta)}(t_2-t_1)\left\|\eta\right\|_\infty\left(1 + \sum_{k=1}^{\infty}\frac{\|\lambda\|^k}{\Gamma(k(\beta+\gamma)+1)}\right),
\end{aligned}$$

which is equicontinuous on every bounded set $\mathcal{S}$ of $C(I,\mathbb{R})$. It follows similarly that I_2, I_3 and I_4 are equicontinuous on $\mathcal{S}$.

(iv) Finally, we prove that the set for $0 < \delta \leq 1$

$$\Upsilon = \{\zeta \in C(I,\mathbb{R}) : \zeta = \delta\mathcal{M}\zeta\}$$

is bounded. Noting that

$$\begin{aligned}\|\zeta\|_\infty \le \|\mathcal{M}\zeta\|_\infty &\le E_{\beta+\gamma,\,\beta+1}(\|\lambda\|_\infty)\|\eta\|_\infty + (\mathcal{L}_2\,\|\zeta\|_\infty + |\phi(0)|)\,E_{\beta+\gamma,\,1}(\|\lambda\|_\infty)\\ &+\frac{\|\lambda\|_\infty\,\|\zeta\|_\infty}{\Gamma(\beta+\gamma+1)}E_{\beta+\gamma,\,1}(\|\lambda\|_\infty) + \frac{1}{\Gamma(\beta+1)}E_{\beta+\gamma,\,1}(\|\lambda\|_\infty)\|\eta\|_\infty,\end{aligned}$$

and

$$Q = 1-\left(\mathcal{L}_2 + \frac{\|\lambda\|_\infty}{\Gamma(\beta+\gamma+1)}\right)E_{\beta+\gamma,\,1}(\|\lambda\|_\infty) > 0.$$

Then,

$$\|\zeta\|_\infty \le \frac{1}{Q}\left(E_{\beta+\gamma,\,\beta+1}(\|\lambda\|_\infty)\|\eta\|_\infty + |\phi(0)|\,E_{\beta+\gamma,\,1}(\|\lambda\|_\infty) + \frac{1}{\Gamma(\beta+1)}E_{\beta+\gamma,1}(\|\lambda\|_\infty)\|\eta\|_\infty\right) < +\infty,$$

which indicates that Y is bounded. Using the Arzela–Ascoli theorem, we claim that $\mathcal{M}$ is compact. By Leray–Schauder's fixed point theorem, equation (0.2) has at least one solution in $C(I,\mathbb{R})$. This completes the proof. □

Example 4.5. The following nonlinear fractional integro-differential equation with variable coefficient and functional boundary condition:

$$\begin{cases} {}_CD^{1.7}\zeta(x) + (x^2+1)\,I^{1.4}\zeta(x) = x^2\cos\zeta^2(x), & x\in[0,1],\\ \zeta(0)=0,\ \ \zeta(1) = \dfrac{1}{16}\cos\zeta(3/4), \end{cases} \tag{4.4}$$

has at least one solution in $C(I,\mathbb{R})$.

Proof. Comparing equation (4.4) with equation (0.2), we get for $\lambda(x) = x^2+1$ that

$$\beta = 1.7,\ \ \|\lambda\|_\infty = \max_{x\in[0,1]}|x^2+1| = 2,\ \ \gamma = 1.4,$$

and

$$\eta(x,y) = x^2\cos y^2$$

is a continuous and bounded function on $[0,1]\times\mathbb{R}$. However, it is not a Lipschitz function with respect to y.

On the other hand,

$$\phi(\zeta) = \frac{1}{16}\cos\zeta(3/4)$$

is a functional satisfying

$$\begin{aligned}|\phi(\zeta_1)-\phi(\zeta_2)| &\le \frac{1}{16}\,|\cos\zeta_1(3/4) - \cos\zeta_2(3/4)|\\ &\le \frac{1}{16}\,|\zeta_1(3/4)-\zeta_2(3/4)| \quad (|\cos x - \cos y| \le |x-y|)\\ &\le \frac{1}{16}\,\|\zeta_1-\zeta_2\|_\infty\,,\end{aligned}$$

where $\zeta_1, \zeta_2 \in C(I,\mathbb{R})$, which implies that $\mathcal{L}_2 = \frac{1}{16}$. Thus,

$$\begin{aligned}Q &= 1-\left(\mathcal{L}_2 + \frac{\|\lambda\|_\infty}{\Gamma(\beta+\gamma+1)}\right)E_{\beta+\gamma,\,1}(\|\lambda\|_\infty) = 1-\left(\frac{1}{16}+\frac{2}{\Gamma(4.1)}\right)E_{3.1,\,1}(2)\\ &\approx 1-(0.356073)(1.29739) = 0.53803445053 > 0.\end{aligned}$$

By Theorem 4.4, equation (4.4) has at least one solution in $C(I,\mathbb{R})$. □

5. The generalized fractional heat equation

In this section, we will find an analytic solution to the generalized fractional heat equation (0.3) given in the abstract based on an inverse operator and the multivariate Mittag-Leffler function to illustrate an application of the inverse operator method in PDEs.

Theorem 5.1. *Let $a_j(x_j)$ be a continuous function over $\mathbb{R}$, $f(t,x)$ and ψ be in W_0 defined by*

$$W_0 = \Big\{ f \in C(\mathbb{R}^+ \times \mathbb{R}^n) : \exists \text{ a constant } M_{f,a_1,\cdots,a_n} > 0 \text{ such that}$$
$$\sup_{(t,x)\in\mathbb{R}^+\times\mathbb{R}^n} \left| \left(a_1(x_1)\frac{\partial^2}{\partial x_1^2}\right)^{k_1} \cdots \left(a_n(x_n)\frac{\partial^2}{\partial x_n^2}\right)^{k_n} f(t,x) \right| \leq M_{f,a_1,\cdots,a_n}^{k_1+\cdots+k_n} \Big\},$$

where $(k_1,\cdots,k_n) \in (\mathbb{N}\cup\{0\})^n$. Then equation (0.3) has a unique solution

$$u(t,x) = \sum_{k=0}^{\infty} I_t^{\alpha k+\alpha} \triangle^k_{a_1(x_1),\cdots,a_n(x_n)} f(t,x) + \sum_{k=0}^{\infty} \frac{t^{\alpha k}}{\Gamma(\alpha k+1)} \triangle^k_{a_1(x_1),\cdots,a_n(x_n)} \psi(x).$$

Proof. Let the partial integral operator I_t^α ($\alpha > 0$) be defined as

$$I_t^\alpha f(t,x) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau,x)d\tau, \quad t \in \mathbb{R}^+.$$

Applying the operator I_t^α to both sides of equation (0.3), we have

$$u(t,x) - \psi(x) - I_t^\alpha \triangle_{a_1(x_1),\cdots,a_n(x_n)} u(t,x) = I_t^\alpha f(t,x),$$

which claims that

$$\left(1 - I_t^\alpha \triangle_{a_1(x_1),\cdots,a_n(x_n)}\right) u(t,x) = I_t^\alpha f(t,x) + \psi(x). \tag{5.1}$$

We are going to show that the inverse operator of $1 - I_t^\alpha \triangle_{a_1(x_1),\cdots,a_n(x_n)}$ is

$$\begin{aligned} V &= \sum_{k=0}^{\infty} \left(\sum_{i=1}^{n} I_t^\alpha a_i(x_i) \frac{\partial^2}{\partial x_i^2} \right)^k \\ &= \sum_{k=0}^{\infty} I_t^{\alpha k} \sum_{k_1+\cdots+k_n=k} \binom{k}{k_1,\cdots,k_n} \left(a_1(x_1)\frac{\partial^2}{\partial x_1^2}\right)^{k_1} \cdots \left(a_n(x_n)\frac{\partial^2}{\partial x_n^2}\right)^{k_n} \end{aligned}$$

in W_0. Indeed, for any $f \in W_0$ we get

$$\begin{aligned} &\|Vf\|_\infty \\ &\leq \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_n=k} \binom{k}{k_1,\cdots,k_n} \|I_t^{\alpha k}\| \sup_{(t,x)\in\mathbb{R}^+\times\mathbb{R}^n} \left| \left(a_1(x_1)\frac{\partial^2}{\partial x_1^2}\right)^{k_1} \cdots \left(a_n(x_n)\frac{\partial^2}{\partial x_n^2}\right)^{k_n} f(t,x) \right| \\ &\leq \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_n=k} \binom{k}{k_1,\cdots,k_n} \frac{\left(t^\alpha M_{f,a_1,\cdots,a_n}\right)^{k_1} \cdots \left(t^\alpha M_{f,a_1,\cdots,a_n}\right)^{k_n}}{\Gamma(\alpha k_1+\cdots+\alpha k_n+1)} \\ &= E_{(\alpha,\cdots,\alpha),1}\left(t^\alpha M_{f,a_1,\cdots,a_n},\cdots,t^\alpha M_{f,a_1,\cdots,a_n}\right) < +\infty, \quad t \in \text{any closed interval of } \mathbb{R}^+, \end{aligned}$$

which indicates that V is a well-defined operator over W_0. Furthermore,

$$V\left(1 - I_t^\alpha \triangle_{a_1(x_1),\cdots,a_n(x_n)}\right) = \left(1 - I_t^\alpha \triangle_{a_1(x_1),\cdots,a_n(x_n)}\right) V = 1.$$

In fact,

$$V\left(1 - I_t^{\alpha}\triangle_{a_1(x_1),\cdots,a_n(x_n)}\right) = 1 + \sum_{k=1}^{\infty}\left(\sum_{i=1}^{n} I_t^{\alpha} a_i(x_i)\frac{\partial^2}{\partial x_i^2}\right)^k - \sum_{k=0}^{\infty}\left(\sum_{i=1}^{n} I_t^{\alpha} a_i(x_i)\frac{\partial^2}{\partial x_i^2}\right)^{k+1} = 1.$$

Similarly,

$$\left(1 - I_t^{\alpha}\triangle_{a_1(x_1),\cdots,a_n(x_n)}\right) V = 1,$$

and the uniqueness follows easily. From equation (5.1), we derive that

$$\begin{aligned}
u(t,x) =& \sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_n=k}\binom{k}{k_1,\cdots,k_n} I_t^{\alpha k+\alpha}\left(a_1(x_1)\frac{\partial^2}{\partial x_1^2}\right)^{k_1}\cdots\left(a_n(x_n)\frac{\partial^2}{\partial x_n^2}\right)^{k_n} f(t,x)\\
&+\sum_{k=0}^{\infty}\frac{t^{\alpha k}}{\Gamma(\alpha k+1)}\sum_{k_1+\cdots+k_n=k}\binom{k}{k_1,\cdots,k_n}\left(a_1(x_1)\frac{\partial^2}{\partial x_1^2}\right)^{k_1}\cdots\left(a_n(x_n)\frac{\partial^2}{\partial x_n^2}\right)^{k_n}\psi(x)\\
=& \sum_{k=0}^{\infty} I_t^{\alpha k+\alpha}\triangle_{a_1(x_1),\cdots,a_n(x_n)}^k f(t,x) + \sum_{k=0}^{\infty}\frac{t^{\alpha k}}{\Gamma(\alpha k+1)}\triangle_{a_1(x_1),\cdots,a_n(x_n)}^k\psi(x),
\end{aligned}$$

by noting that

$$\triangle_{a_1(x_1),\cdots,a_n(x_n)}^k = \sum_{k_1+\cdots+k_n=k}\binom{k}{k_1,\cdots,k_n}\left(a_1(x_1)\frac{\partial^2}{\partial x_1^2}\right)^{k_1}\cdots\left(a_n(x_n)\frac{\partial^2}{\partial x_n^2}\right)^{k_n}.$$

The uniqueness of solutions follows from the observation that the differential equation

$$\begin{cases} {}_C D_t^{\alpha} u(t,x) = \triangle_{a_1(x_1),\cdots,a_n(x_n)} u(t,x), & (t,x)\in\mathbb{R}^+\times\mathbb{R}^n,\quad 0<\alpha\le 1,\\ u(0,x) = 0, \end{cases}$$

admits only the trivial solution $u(t,x) = 0$. This establishes the uniqueness of the solution and thus completes the proof. □

In particular, equation

$$\begin{cases} {}_C D_t^{\alpha} u(t,x) = \triangle_{1,\cdots,1} u(t,x) + f(t,x), & (t,x)\in\mathbb{R}^+\times\mathbb{R}^n,\quad 0<\alpha\le 1,\\ u(0,x) = \psi(x), \end{cases} \tag{5.2}$$

where

$$\triangle_{1,\cdots,1} = \triangle,$$

has a unique solution

$$u(t,x) = \sum_{k=0}^{\infty} I_t^{\alpha k+\alpha}\triangle^k f(t,x) + \sum_{k=0}^{\infty}\frac{t^{\alpha k}}{\Gamma(\alpha k+1)}\triangle^k\psi(x).$$

In particular for $\alpha = 1$,

$$u(t,x) = \sum_{k=0}^{\infty} I_t^{k+1}\triangle^k f(t,x) + \sum_{k=0}^{\infty}\frac{t^k}{k!}\triangle^k\psi(x).$$

Example 5.2. The following differential equation

$$\begin{cases} {}_C D_t^\alpha u(t,x) = \triangle u(t,x) + t^2 x_1 x_2 \cdots x_n, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \quad 0 < \alpha \leq 1, \\ u(0,x) = \sin x_1, \end{cases} \tag{5.3}$$

has the solution

$$\begin{aligned} u(t,x) &= \sum_{k=0}^{\infty} I_t^{\alpha k+\alpha} t^2 \triangle^k (x_1 x_2 \cdots x_n) + \sum_{k=0}^{\infty} \frac{t^{\alpha k}}{\Gamma(\alpha k+1)} \triangle^k \sin x_1 \\ &= I_t^\alpha t^2 x_1 x_2 \cdots x_n + \sum_{k=0}^{\infty} \frac{t^{\alpha k}}{\Gamma(\alpha k+1)} \sin\left(x_1 + 2k\frac{\pi}{2}\right) \\ &= \frac{t^{\alpha+2}}{\Gamma(\alpha+2)} x_1 x_2 \cdots x_n + \sin x_1 \sum_{k=0}^{\infty} \frac{t^{\alpha k}(-1)^k}{\Gamma(\alpha k+1)}. \end{aligned}$$

For $n = 1$, $a_1(x_1) = \cdots = a_n(x_n) = \mu > 0$ and $f(t,x) = 0$, then equation (0.3) turns out to be equation (1.1) with the solution given in (1.2), which coincides with our results. Indeed,

$$\begin{aligned} u(t,x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{\alpha,1}(-\mu k^2 t^\alpha) \hat{\psi}(k) e^{-ikx} dk \\ &= \sum_{j=0}^{\infty} \frac{t^{\alpha j}\mu^j}{\Gamma(\alpha j+1)} \frac{1}{2\pi} \int_{-\infty}^{\infty} (-1)^j k^{2j} \hat{\psi}(k) e^{-ikx} dk = \sum_{j=0}^{\infty} \frac{t^{\alpha j}\mu^j}{\Gamma(\alpha j+1)} \frac{d^{2j}}{dx^{2j}} \psi(x), \end{aligned}$$

using the inverse Fourier transform

$$\frac{d^{2j}}{dx^{2j}} \psi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (-1)^j k^{2j} \hat{\psi}(k) e^{-ikx} dk.$$

6. Conclusion

Applying several well-known fixed-point theorems and the inverse operator method, we studied the uniqueness and existence of solutions to equations (0.1) and (0.2) by the implicit integral equation and the Mittag-Leffler functions. In addition, we derived an analytic solution to the generalized fractional heat equation with an initial condition. This new approach works for a wide range for differential equations including PDEs, with variable coefficients and various initial or boundary conditions.

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Competing interests

The authors declare they have no competing interests.

Data Availability

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Author contributions

All authors conceived of the study, participated in its design and coordination, participated in the sequence alignment, read and approved the final version.

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