



Some combinatorial identities via Riordan arrays and k -order Fibonacci matrices

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Abstract. In this paper, we investigate interesting relationships between Riordan arrays and k -order Fibonacci matrices, which offer a unified approach to studying some lower triangular matrices, including Fuss-Catalan matrices, harmonic matrices, and others. We present factorizations of Riordan arrays via k -order Fibonacci matrices. Furthermore, based on matrix representations, we derive various combinatorial identities.

1. Introduction

The Riordan arrays, first introduced by Shapiro et al. [9], have numerous applications in combinatorial matrices and identities. Since their introduction, the theory of Riordan arrays has emerged as a major research focus in combinatorial mathematics. Sprugnoli et al. [12] explored Riordan arrays and demonstrated their utility as a practical tool to solve combinatorial sums through generating functions and the Lagrange inversion formula. Wang and Wang [13] further advanced the field by introducing the concept of generalized Riordan arrays. Subsequently, the relationships between generalized Riordan arrays and generalized Sheffer sequences, as well as matrix factorizations of generalized Riordan arrays and their applications, have been extensively studied. Numerous works have since expanded on these ideas, as seen in [1–5, 15].

Let $f(t) = \sum_{i \geq 0} f_i t^i$ be a formal power series in indeterminate t . A Riordan array $(g(t), f(t))$ is a pair of formal power series with $g(t) = \sum_{i \geq 0} g_i t^i$, $f(t) = \sum_{i \geq 1} f_i t^i$, and $g_0, f_1 \neq 0$. It defines an infinite lower triangular array $[d_{i,j}]_{i,j \in \mathbb{N}}$ according to the rule

$$d_{i,j} = [t^j]g(t)f(t)^i.$$

Denote the set of all Riordan arrays by $\mathcal{R}$. This set forms a group under matrix multiplication, where the product of two Riordan arrays $(g(t), f(t))$ and $(h(t), l(t))$ is given by

$$(g(t), f(t))(h(t), l(t)) = (g(t)h(f(t)), l(f(t))). \quad (1.1)$$

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The identity element of this group is $I = (1, t)$, and the inverse of $(g(t), f(t))$ is $\left(\frac{1}{g(f(t))}, \bar{f}(t)\right)$. The Riordan array $(g(t), f(t))$ can also be characterized by two sequences: the A -sequence $(a_i)_{i \in \mathbb{N}}$ and the Z -sequence $(z_i)_{i \in \mathbb{N}}$, as follows (see [5]):

$$\begin{aligned} d_{0,0} &= 1, \quad d_{0,j} = 0 \quad \text{for } j \geq 1, \\ d_{i,j} &= a_0 d_{i-1,j-1} + a_1 d_{i-1,j} + a_2 d_{i-1,j+1} + \cdots \quad \text{for } i, j \geq 1, \\ d_{i,0} &= z_0 d_{i-1,0} + z_1 d_{i-1,1} + z_2 d_{i-1,2} + \cdots \quad \text{for } i \geq 1. \end{aligned} \quad (1.2)$$

If $A(t)$ and $Z(t)$ are the generating functions for the A -sequence and Z -sequence, respectively, then it follows that

$$\begin{aligned} f(t) &= tA(f(t)), \\ g(t) &= \frac{1}{1 - tZ(f(t))}. \end{aligned} \quad (1.3)$$

Thus, $(g(t), f(t))$ is uniquely determined by $A(t)$ and $Z(t)$.

The Fibonacci number is one of the most well-known sequences and has numerous applications in mathematics, computer science, statistics, and other fields. In recent years, Fibonacci numbers and their various generalizations have been extensively studied in works [6–8, 14, 16].

The k -order Fibonacci numbers $\{F_n^{(k)}\}_{n \in \mathbb{N}}$ are defined by

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)} \quad \text{for } n \geq 2,$$

and $F_1^{(k)} = 1, F_n^{(k)} = 0$ for $n \leq 0$. For example, when $k = 2$, $\{F_n^{(2)}\}_{n \in \mathbb{N}}$ corresponds to the Fibonacci sequence A000045 in the OEIS [10] and when $k = 3$, $\{F_n^{(3)}\}_{n \in \mathbb{N}}$ corresponds to the Tribonacci sequence A000073 in the OEIS [10].

For any positive integer n , the $(n+1) \times (n+1)$ k -order Fibonacci matrix $\mathcal{F}_n^{(k)}$ is defined by

$$[\mathcal{F}_n^{(k)}]_{i,j} = \begin{cases} F_{i-j+1}^{(k)} & \text{if } i \geq j, \\ 0 & \text{otherwise,} \end{cases} \quad (i, j = 0, 1, \dots, n),$$

that is,

$$\mathcal{F}_n^{(k)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ F_2^{(k)} & 1 & 0 & 0 & 0 & 0 \\ F_3^{(k)} & F_2^{(k)} & 1 & 0 & 0 & 0 \\ F_4^{(k)} & F_3^{(k)} & F_2^{(k)} & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 & 0 \\ F_{n+1}^{(k)} & F_n^{(k)} & F_{n-1}^{(k)} & \cdots & F_2^{(k)} & 1 \end{bmatrix}.$$

For example, the 3-order Fibonacci matrix and 5-order Fibonacci matrix are as follows:

$$\mathcal{F}_6^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 4 & 2 & 1 & 1 & 0 & 0 & 0 \\ 7 & 4 & 2 & 1 & 1 & 0 & 0 \\ 13 & 7 & 4 & 2 & 1 & 1 & 0 \\ 24 & 13 & 7 & 4 & 2 & 1 & 1 \end{bmatrix}, \quad \mathcal{F}_6^{(5)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 4 & 2 & 1 & 1 & 0 & 0 & 0 \\ 8 & 4 & 2 & 1 & 1 & 0 & 0 \\ 16 & 8 & 4 & 2 & 1 & 1 & 0 \\ 31 & 16 & 8 & 4 & 2 & 1 & 1 \end{bmatrix}.$$

For any positive integer n , the $(n+1) \times (n+1)$ inverse matrix $\mathcal{L}_n^{(k)} = [l_{i,j}]_{i,j \in \mathbb{N}}$ of the k -order Fibonacci matrix is

$$[\mathcal{L}_n^{(k)}]_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ -1 & \text{if } i - k \leq j \leq i - 1, \quad (i, j = 0, 1, \dots, n), \\ 0, & \text{otherwise,} \end{cases}$$

Lee et al. [7] studied factorizations of the generalized Pascal matrix, the Fibonacci matrix, and the Pell matrix. Zhang and Wang [16] presented a factorization of the Pascal matrix and derived various combinatorial properties using the Fibonacci matrix. Köme [6] investigated relationships between the Pascal matrix, the Stirling matrix, and the k -order Fibonacci matrices. It is well known that many combinatorial matrices, such as the Pascal matrix, Stirling matrices of both kinds, Fuss-Catalan matrices, and harmonic matrices, are special cases of Riordan arrays. Therefore, by studying Riordan arrays through k -order Fibonacci matrices, we obtain a unified approach to analyzing various combinatorial matrices.

The article is organized as follows. In Section 2, we explore the relationships between Riordan arrays and k -order Fibonacci matrices. In Section 3, we derive several interesting identities related to k -order Fibonacci numbers.

2. Factorization of the Riordan array

In this section, we investigate the factorization of Riordan arrays using k -order Fibonacci matrices. Let $(g, f)_n$ denote the submatrix consisting of the first $n+1$ rows and columns of the Riordan array (g, f) .

Definition 2.1. Let $R_n = (g(t), f(t))_n = [d_{i,j}]_{i,j=0,1,\dots,n}$ be a Riordan matrix. For $n \in \mathbb{N}$, the $(n+1) \times (n+1)$ matrices $M_n^{(k)} = [m_{i,j}]_{i,j=0,1,\dots,n}$ and $N_n^{(k)} = [n_{i,j}]_{i,j=0,1,\dots,n}$ are defined by

$$[M_n^{(k)}]_{i,j} = m_{i,j} = d_{i,j} - \sum_{s=1}^k d_{i-s,j}, \quad (2.1)$$

$$[N_n^{(k)}]_{i,j} = n_{i,j} = d_{i,j} - \sum_{s=1}^k d_{i,j+s}. \quad (2.2)$$

Lemma 2.2. We have

$$M_n^{(k)} = \left(\sum_{i=0}^n \left(d_{i,0} - \sum_{s=1}^k d_{i-s,0} \right) t^i, f(t) \right)_n, \quad (2.3)$$

$$N_n^{(k)} = \left(\sum_{i=0}^n \left(d_{i,0} - \sum_{s=1}^k d_{i,s} \right) t^i, f(t) \right)_n. \quad (2.4)$$

Proof. The generating function of the first column of $M_n^{(k)}$ is

$$\sum_{i \geq 0} \left(d_{i,0} - \sum_{s=1}^k d_{i-s,0} \right) t^i.$$

For $i, j \geq 1$, the recurrence relation holds:

$$m_{i,j} = a_1 m_{i-1,j-1} + a_2 m_{i-1,j} + \dots + a_{i+j+1} m_{i-1,i-1},$$

where $(a_0, a_1, \dots)$ is the A -sequence of the Riordan array $(g(t), f(t))$. Using (1.2), $M_n^{(k)}$ is a Riordan matrix and satisfies (2.3).

Similarly, the generating function of the first column of $N_n^{(k)}$ is

$$\sum_{i \geq 0} \left(d_{i,0} - \sum_{s=1}^k d_{i,s} \right) t^i.$$

For $i, j \geq 2$, the recurrence relation holds:

$$n_{i,j} = a_1 n_{i-1,j-1} + a_2 n_{i-1,j} + \cdots + a_{i+j+1} n_{i-1,i-1}.$$

Applying (1.2) again, we obtain (2.4). $\square$

Now, we can give the factorization of the Riordan arrays via the k -order Fibonacci matrices by the following theorem.

Theorem 2.3. Let $M_n^{(k)}$ and $N_n^{(k)}$ be the matrices defined in (2.1) and (2.2), respectively. For the Riordan array R_n and the k -order Fibonacci matrix $\mathcal{F}_n^{(k)}$, we have

$$R_n = \mathcal{F}_n^{(k)} M_n^{(k)},$$

$$R_n = N_n^{(k)} \mathcal{F}_n^{(k)}.$$

Proof. Since the matrix $\mathcal{L}_n^{(k)}$ is the inverse of $\mathcal{F}_n^{(k)}$, let us prove the equation $\mathcal{L}_n^{(k)} R_n = M_n^{(k)}$. Since $l_{0,j} = d_{0,j} = 0$ for $j \geq 1$, then $\sum_{s=0}^n l_{0,s} d_{s,0} = l_{0,0} d_{0,0} = d_{0,0} = m_{0,0}$, $\sum_{s=0}^n l_{0,s} d_{s,j} = l_{0,0} d_{0,j} = 0 = m_{0,j}$ for $j \geq 1$. Since $l_{1,j} = 0$ for $j \geq 2$, $l_{1,0} = -1$ and $l_{1,1} = 1$, we have $\sum_{s=0}^n l_{1,s} d_{s,0} = d_{1,0} - d_{0,0} = m_{1,0}$. By (2.1), we have $\sum_{s=0}^n l_{i,s} d_{s,0} = m_{i,0}$ for $i = 2, 3, \dots, n$.

Next, we consider $i \geq 2$ and $j \geq 1$. From (2.1) and the recurrence relation of $m_{i,j}$, we have $\sum_{s=0}^n l_{i,s} d_{s,j} = m_{i,j}$. Therefore, we have $\mathcal{L}_n^{(k)} R_n = M_n^{(k)}$.

Similarly to the preceding process, we can also prove $R_n = N_n^{(k)} \mathcal{F}_n^{(k)}$. $\square$

Using Theorem 2.3, we have the following corollary.

Corollary 2.4. For $1 \leq j \leq i$, we have

$$d_{i,j} = \sum_{l=j}^i F_{i-l+1}^{(k)} \left(d_{l,j} - \sum_{s=1}^k d_{l-s,j} \right), \quad (2.5)$$

$$d_{i,j} = \sum_{l=j}^i \left(d_{i,l} - \sum_{s=1}^k d_{i,l+s} \right) F_{l-j+1}^{(k)}. \quad (2.6)$$

Let I_n be the identity matrix of order $n+1$, and let L_k be the $(k+1) \times (k+1)$ lower triangular matrix as follows:

$$L_k = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Set $S_l = L_{k+1} \oplus I_l$, $l = 0, 1, 2, \dots$. Let $G_0 = I_n$, $G_1 = I_{n-1} \oplus L_1$, $G_2 = I_{n-2} \oplus L_2$, $G_3 = I_{n-3} \oplus L_3, \dots, G_{k+1} = I_{n-k-1} \oplus L_{k+1}$, and, for $k+2 \leq l \leq n$, $G_l = I_{n-l} \oplus S_{l-k-1}$. In particular $S_0 = L_{k+1}$ and $G_n = S_{n-k-1}$. In [8], the authors gave the factorization of the k -order Fibonacci matrix as: $\mathcal{F}_n^{(k)} = G_0 G_1 G_2 G_3 \cdots G_n$.

For $k \geq 2$, we define $(n+1) \times (n+1)$ matrices $H_n^{(k)}, J_n^{(k)}$ and $\overline{H}_i, \overline{J}_i$ by

$$H_n^{(k)} = \begin{bmatrix} d_{0,0} & 0 & 0 & 0 & \cdots & 0 & 0 \\ d_{1,0} - d_{0,0} & f_1 & 0 & 0 & \cdots & 0 & 0 \\ d_{2,0} - d_{1,0} - d_{0,0} & f_2 & f_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & 0 & 0 \\ d_{k,0} - \sum_{l=1}^k d_{k-l,0} & f_k & f_{k-1} & f_{k-2} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & f_1 & 0 \\ d_{k,0} - \sum_{l=1}^k d_{k-l,0} & f_n & f_{n-1} & f_{n-2} & \cdots & f_2 & f_1 \end{bmatrix},$$

$$J_n^{(k)} = \begin{bmatrix} d_{0,0} & 0 & 0 & 0 & \cdots & 0 & 0 \\ d_{1,0} - d_{1,1} & f_1 & 0 & 0 & \cdots & 0 & 0 \\ d_{2,0} - d_{2,1} - d_{2,2} & f_2 & f_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & 0 & 0 \\ d_{k,0} - \sum_{l=1}^k d_{k,l} & f_k & f_{k-1} & f_{k-2} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & f_1 & 0 \\ d_{k,0} - \sum_{l=1}^k d_{k,l} & f_n & f_{n-1} & f_{n-2} & \cdots & f_2 & f_1 \end{bmatrix},$$

$$\overline{H}_i = I_{n-i} \oplus H_i^{(k)} \text{ and } \overline{J}_i = I_{n-i} \oplus J_i^{(k)}.$$

Lemma 2.5. We have

$$M_n^{(k)} = \overline{H}_n \overline{H}_{n-1} \cdots \overline{H}_2 \overline{H}_1 \overline{H}_0, \quad (2.7)$$

$$N_n^{(k)} = \overline{J}_n \overline{J}_{n-1} \cdots \overline{J}_2 \overline{J}_1 \overline{J}_0. \quad (2.8)$$

Proof. We obtain (2.7) and (2.8) by using Lemma 2.2 and the definition of the $M_n^{(k)}$ and $N_n^{(k)}$. $\square$

Theorem 2.6. For the $(n+1) \times (n+1)$ Riordan matrix R_n , we have

$$R_n = \prod_{i=0}^n G_i \prod_{i=0}^n \overline{H}_{n-i} = \prod_{i=0}^n \overline{J}_{n-i} \prod_{i=0}^n G_i.$$

Proof. By virtue of Lemma 2.5. $\square$

Example 2.7. The Catalan numbers, $C_i = \frac{1}{2i+1} \binom{2i+1}{i}$, a sequence of integers that occur in many counting situations. The generating function of the Catalan numbers can be written as $C(t) = \sum_{i \geq 0} C_n t^i = \frac{1 - \sqrt{1-4t}}{2t}$. The Fuss-Catalan numbers [4] are defined by

$$F_m(i, r) := \frac{r}{mi+r} \binom{mi+r}{i}.$$

For $m = 2, r = 1$, the Fuss-Catalan numbers are Catalan numbers. The generating function $F_m(t)$ for the Fuss-Catalan numbers $\{F_m(i, 1)\}$ satisfies

$$F_m(t)^r = \sum_{i \geq 0} \frac{r}{mi+r} \binom{mi+r}{i} t^i.$$

The Riordan arrays $(F_m(t)^q, tF_m(t)^r)$ are called Fuss-Catalan matrices. Particularly, $(F_2(t), tF_2(t)) = (C(t), tC(t))$, $(F_2(t)^2, tF_2(t)^2) = (C(t)^2, tC(t)^2)$ and $(F_2(t), tF_2(t)^2) = (C(t), tC(t)^2)$. Setting $k = 3$, from Theorem 2.3, we have

$$\begin{aligned}
(C(t)^2, tC(t)^2)_5 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 5 & 4 & 1 & 0 & 0 & 0 \\ 14 & 14 & 6 & 1 & 0 & 0 \\ 42 & 48 & 27 & 8 & 1 & 0 \\ 132 & 165 & 110 & 44 & 10 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 4 & 2 & 1 & 1 & 0 & 0 \\ 7 & 4 & 2 & 1 & 1 & 0 \\ 13 & 7 & 4 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 & 0 \\ 6 & 9 & 5 & 1 & 0 & 0 \\ 21 & 29 & 20 & 7 & 1 & 0 \\ 71 & 99 & 76 & 35 & 9 & 1 \end{bmatrix} \\
&= \mathcal{F}_5^{(3)}(1+t+2t^2+6t^3+21t^4+71t^5, tC(t)^2)_5 \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 \\ -7 & 7 & 5 & 1 & 0 & 0 \\ -41 & 12 & 18 & 7 & 1 & 0 \\ -187 & 1 & 55 & 33 & 9 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 4 & 2 & 1 & 1 & 0 & 0 \\ 7 & 4 & 2 & 1 & 1 & 0 \\ 13 & 7 & 4 & 2 & 1 & 1 \end{bmatrix} \\
&= (1+t-7t^3-41t^4-187t^5, tC(t)^2)_5 \mathcal{F}_5^{(3)}.
\end{aligned}$$

Setting $k = 5$, we have

$$\begin{aligned}
(C(t), tC(t)^2)_5 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 & 0 \\ 5 & 9 & 5 & 1 & 0 & 0 \\ 14 & 28 & 20 & 7 & 1 & 0 \\ 42 & 90 & 75 & 35 & 9 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 4 & 2 & 1 & 1 & 0 & 0 \\ 8 & 4 & 2 & 1 & 1 & 0 \\ 16 & 8 & 4 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 1 & 5 & 4 & 1 & 0 & 0 \\ 5 & 15 & 14 & 6 & 1 & 0 \\ 19 & 49 & 49 & 27 & 8 & 1 \end{bmatrix} \\
&= \mathcal{F}_5^{(5)}(1+t^3+5t^4+19t^5, tC(t)^2)_5 \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 2 & 1 & 0 & 0 & 0 \\ -10 & 3 & 4 & 1 & 0 & 0 \\ -42 & 0 & 12 & 6 & 1 & 0 \\ -168 & -30 & 30 & 25 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 4 & 2 & 1 & 1 & 0 & 0 \\ 8 & 4 & 2 & 1 & 1 & 0 \\ 16 & 8 & 4 & 2 & 1 & 1 \end{bmatrix} \\
&= (1-2t^2-10t^3-42t^4-168t^5, tC(t)^2)_5 \mathcal{F}_5^{(5)}.
\end{aligned}$$

The general element of the $(F_m(t)^q, tF_m(t)^r)$ is $F_m(i-j, p+jr)$. By Corollary 2.4, we have

$$F_m(i-j, p+jr) = \sum_{l=j}^i F_{i-l+1}^{(k)} \left(F_m(l-j, p+jr) - \sum_{s=1}^k F_m(l-s-j, p+jr) \right)$$

$$= \sum_{l=j}^i \left(F_m(i-l, p+lr) - \sum_{s=1}^k F_m(i-l-s, p+lr+sr) \right) F_{l-j+1}^{(k)}.$$

In particular, if $m = 2, p = 1$ and $r = 1$, we obtain

$$\begin{aligned} \frac{j+1}{2i-j+1} \binom{2i-j+1}{i-j} &= \sum_{l=j}^i F_{i-l+1}^{(k)} \left(\frac{j+1}{2l-j+1} \binom{2l-j+1}{l-j} - \sum_{s=1}^k \frac{j+1}{2(l-s)-j+1} \binom{2(l-s)-j+1}{l-s-j} \right) \\ &= \sum_{l=j}^i \left(\frac{l+1}{2i-l+1} \binom{2i-l+1}{i+1} - \sum_{s=1}^k \frac{l+s+1}{2i-l-s+1} \binom{2i-l-s+1}{i+1} \right) F_{l-j+1}^{(k)}. \end{aligned}$$

If $m = 2, p = 1$ and $r = 2$, we have

$$\begin{aligned} \frac{2j+1}{2i+1} \binom{2i+1}{i-j} &= \sum_{l=j}^i F_{i-l+1}^{(k)} \left(\frac{2j+1}{2l+1} \binom{2l+1}{l-j} - \sum_{s=1}^k \frac{2j+1}{2(l-s)+1} \binom{2(l-s)+1}{l-s-j} \right) \\ &= \sum_{l=j}^i \left(\frac{2l+1}{2i+1} \binom{2i+1}{i-l} - \sum_{s=1}^k \frac{2l+2s+1}{2i+1} \binom{2i+1}{i-l-s} \right) F_{l-j+1}^{(k)}. \end{aligned}$$

If $m = 3, p = 1$ and $r = 1$, we obtain

$$\begin{aligned} \frac{j+1}{3i-2j+1} \binom{3i-2j+1}{i-j} &= \sum_{l=j}^i F_{i-l+1}^{(k)} \left(\frac{j+1}{3l-2j+1} \binom{3l-2j+1}{l-j} - \sum_{s=1}^k \frac{j+1}{3(l-s)-2j+1} \binom{3(l-s)-2j+1}{l-s-j} \right) \\ &= \sum_{l=j}^i \left(\frac{l+1}{3i-2l+1} \binom{3i-2l+1}{i-l} - \sum_{s=1}^k \frac{l+s+1}{3i-2l-2s+1} \binom{3i-2l-2s+1}{i-l-s} \right) F_{l-j+1}^{(k)}. \end{aligned}$$

Example 2.8. The harmonic numbers are defined by

$$H_0 = 0 \text{ and } H_n = \sum_{l=1}^n \frac{1}{l}, \text{ for } n = 1, 2, \dots,$$

and the generating function of the harmonic number is $\frac{-\ln(1-x)}{1-x}$. The following generalization of the harmonic number can be found in [3, 11, 15].

$$H_0^{(k)} = 0 \text{ and } H_n^{(k)} = \sum_{l=1}^n \frac{1}{l^k}, \text{ for } n, k \geq 1,$$

$$H_n^0 = \frac{1}{n} \text{ and } H_n^k = \sum_{l=1}^n H_l^{k-1}, \text{ for } n, k \geq 1,$$

$$H(n, k) = \sum_{1 \leq n_0 + n_1 + \dots + n_k \leq n} \frac{1}{n_0 n_1 \dots n_k}, \text{ for } n \geq 1, k \geq 0.$$

Letting $A = \left(\frac{-\ln(1-t)}{t(1-t)}, -\ln(1-t) \right)$ and $B = \left(\frac{-\ln(1-t)}{t(1-t)}, \frac{t}{1-t} \right)$ be two Riordan arrays. Then, the general element of the A and B are $H(i+1, j)$ and H_{i-j+1}^{j+1} , respectively. Setting $k = 3$, from Theorem 2.3, we get

$$A_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 3/2 & 1 & 0 & 0 & 0 & 0 \\ 11/6 & 2 & 1 & 0 & 0 & 0 \\ 25/12 & 35/12 & 5/2 & 1 & 0 & 0 \\ 137/60 & 15/4 & 17/4 & 3 & 1 & 0 \\ 49/20 & 203/45 & 49/8 & 35/6 & 7/2 & 1 \end{bmatrix}$$

$$\begin{aligned}
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 4 & 2 & 1 & 1 & 0 & 0 \\ 7 & 4 & 2 & 1 & 1 & 0 \\ 13 & 7 & 4 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 & 0 & 0 \\ -2/3 & 1 & 1 & 0 & 0 & 0 \\ -27/12 & -1/12 & 3/2 & 1 & 0 & 0 \\ -47/15 & -13/6 & 3/4 & 2 & 1 & 0 \\ -45/12 & -187/45 & -13/8 & 11/6 & 5/2 & 1 \end{bmatrix} \\
&= \mathcal{F}_5^{(3)} \left(1 + \frac{1}{2}t - \frac{2}{3}t^2 - \frac{27}{12}t^3 - \frac{47}{15}t^4 - \frac{45}{12}t^5, -\ln(1-t) \right)_5 \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 & 0 & 0 \\ -7/6 & 1 & 1 & 0 & 0 & 0 \\ -13/3 & -7/12 & 3/2 & 1 & 0 & 0 \\ -523/60 & -9/2 & 1/4 & 2 & 1 & 0 \\ -5047/360 & -3941/360 & -101/24 & 4/3 & 5/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 4 & 2 & 1 & 1 & 0 & 0 \\ 7 & 4 & 2 & 1 & 1 & 0 \\ 13 & 7 & 4 & 2 & 1 & 1 \end{bmatrix} \\
&= \left(1 + \frac{1}{2}t - \frac{7}{6}t^2 - \frac{13}{3}t^3 - \frac{523}{60}t^4 - \frac{5047}{360}t^5, -\ln(1-t) \right)_5 \mathcal{F}_5^{(3)}.
\end{aligned}$$

If $k = 5$, we obtain

$$\begin{aligned}
B_5 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 3/2 & 1 & 0 & 0 & 0 & 0 \\ 11/6 & 5/2 & 1 & 0 & 0 & 0 \\ 25/12 & 13/3 & 7/2 & 1 & 0 & 0 \\ 137/60 & 77/12 & 47/6 & 9/2 & 1 & 0 \\ 49/20 & 87/10 & 57/4 & 37/3 & 11/2 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 4 & 2 & 1 & 1 & 0 & 0 \\ 8 & 4 & 2 & 1 & 1 & 0 \\ 16 & 8 & 4 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 & 0 & 0 \\ -2/3 & 3/2 & 1 & 0 & 0 & 0 \\ -27/12 & 5/6 & 5/2 & 1 & 0 & 0 \\ -62/15 & -17/12 & 10/3 & 7/2 & 1 & 0 \\ -75/12 & -111/20 & -23/12 & 41/6 & 9/2 & 1 \end{bmatrix} \\
&= \mathcal{F}_5^{(5)} \left(1 + \frac{1}{2}t - \frac{2}{3}t^2 - \frac{27}{12}t^3 - \frac{62}{15}t^4 - \frac{75}{12}t^5, \frac{t}{1-t} \right)_5 \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 & 0 & 0 \\ -5/3 & 3/2 & 1 & 0 & 0 & 0 \\ -27/4 & -1/6 & 5/2 & 1 & 0 & 0 \\ -262/15 & -83/12 & 7/3 & 7/2 & 1 & 0 \\ -118/3 & -1463/60 & -55/12 & 35/6 & 9/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 4 & 2 & 1 & 1 & 0 & 0 \\ 8 & 4 & 2 & 1 & 1 & 0 \\ 16 & 8 & 4 & 2 & 1 & 1 \end{bmatrix} \\
&= \left(1 + \frac{1}{2}t - \frac{5}{3}t^2 - \frac{27}{4}t^3 - \frac{262}{15}t^4 - \frac{118}{3}t^5, \frac{t}{1-t} \right)_5 \mathcal{F}_5^{(5)}.
\end{aligned}$$

By using Corollary 2.4, we have

$$\begin{aligned}
H(i+1, j) &= \sum_{l=j}^i F_{i-l+1}^{(k)} \left(H(l+1, j) - \sum_{s=1}^k H(l-s+1, j) \right) \\
&= \sum_{l=j}^i \left(H(i+1, l) - \sum_{s=1}^k H(i+1, l+s) \right) F_{l-j+1}^{(k)}, \\
H_{i-j+1}^{j+1} &= \sum_{l=j}^i F_{i-l+1}^{(k)} \left(H_{l-j+1}^{j+1} - \sum_{s=1}^k H_{l-s-j+1}^{j+1} \right)
\end{aligned}$$

$$= \sum_{l=j}^i \left(H_{i-l+1}^{l+1} - \sum_{s=1}^k H_{i-l-s+1}^{l+s+1} \right) F_{l-j+1}^{(k)}.$$

In particular, if $j = 0$ get

$$\begin{aligned} H_{i+1} &= \sum_{l=j}^i F_{i-l+1}^{(k)} \left(H_{l+1} - \sum_{s=1}^k H_{l-s+1} \right) = \sum_{l=j}^i \left(H(i+1, l) - \sum_{s=1}^k H(i+1, l+s) \right) F_{l+1}^{(k)}, \\ H_{i+1} &= \sum_{l=j}^i \left(H_{i-l+1}^{l+1} - \sum_{s=1}^k H_{i-l-s+1}^{l+s+1} \right) F_{l+1}^{(k)}. \end{aligned}$$

The generalized Riordan array with respect to the sequence $\{c_i\}_{i \in \mathbb{N}}$ is a pair $(g(t), f(t))$, where $g(t) = \sum_{i \geq 0} g_i t^i / c_i$ and $f(t) = \sum_{i \geq 1} f_i t^i / c_i$ with $g_0, f_1 \neq 0$. It defines an infinite lower triangular array $(d_{i,j})_{i,j \in \mathbb{N}}$ according to the rule

$$d_{i,j} = \left[\frac{t^i}{c_i} \right] g(t) \frac{f(t)^j}{c_j},$$

where $g(t)f(t)^j/c_j$ is the generating function of the j -th column. By definition, the classical Riordan arrays correspond to the case $c_i = 1$, and the exponential Riordan arrays correspond to the case $c_i = i!$.

It is straightforward to verify that Theorem 2.3 and Corollary 2.4 also hold when the Riordan array is replaced by a generalized Riordan array with respect to c_n . Consequently, a broader class of combinatorial matrices can be factorized using k -order Fibonacci matrices. As a special case, the Riordan arrays $(1, \log(1+t))$ and $(1, e^t - 1)$ with respect to $c_i = i!$ correspond to the Stirling matrices of the first kind and second kind, respectively. Matrix factorizations of these have been studied in [6]. Additionally, similarly to Example 2.7 and Example 2.8, we can derive numerous combinatorial identities from their matrix representations. However, we have chosen not to present them here.

3. Identities related to the k -order Fibonacci matrix

In this section, we derive some identities involving the k -order Fibonacci numbers. By setting the Riordan matrix $(g(t), t) = [d_{i,j}]$ and $j = 0$ in (2.5), we obtain the following theorem.

Theorem 3.1. For a sequence $\{g_n\} = \{g_0, g_1, \dots, g_n, \dots\}$ with $g_n = 0$ for $n < 0$, the following identity holds:

$$g_n = \sum_{l=0}^n F_{n-l+1}^{(k)} \left(g_l - \sum_{s=1}^k g_{l-s} \right). \quad (3.1)$$

By choosing different sequences in (3.1), we can derive interesting identities related to k -order Fibonacci matrices. For example, if $g_n = 1$, we obtain

$$1 = F_{n+1}^{(k)} - \sum_{l=0}^{n-2} (l+1) F_{n-l-1}^{(k)}, \quad (n \leq k), \quad (3.2)$$

$$1 = F_{n+1}^{(k)} - \sum_{l=0}^{k-2} (l+1) F_{n-l-1}^{(k)} - (k-1) \sum_{l=k+1}^n F_{n-l+1}^{(k)}, \quad (n > k). \quad (3.3)$$

Corollary 3.2. We have the sum of the first n terms of k -order Fibonacci numbers are

$$\sum_{l=1}^n F_l^{(k)} = \frac{F_{n+1}^{(k)} + \sum_{i=1}^{k-1} i F_{n-k+i+1}^{(k)} - 1}{k-1}, \quad (n \geq k). \quad (3.4)$$

In particular, setting $k = 2, 3, 4, 5$, we have

$$\begin{aligned}\sum_{l=1}^n F_l &= F_{n+2} - 1, \quad (n \geq 2), \\ \sum_{l=1}^n F_l^{(3)} &= \frac{F_{n+2}^{(3)} + F_n^{(3)} - 1}{2}, \quad (n \geq 3), \\ \sum_{l=1}^n F_l^{(4)} &= \frac{F_{n+2}^{(4)} + 2F_n^{(4)} + F_{n-1}^{(4)} - 1}{3}, \quad (n \geq 4), \\ \sum_{l=1}^n F_l^{(5)} &= \frac{F_{n+2}^{(5)} + 3F_n^{(5)} + 2F_{n-1}^{(5)} + F_{n-2}^{(5)} - 1}{4}, \quad (n \geq 5).\end{aligned}$$

Setting $g_n := n^r$ in (3.1), where r is a nonnegative integer, we derive the following theorem.

Theorem 3.3. For each integer r , we have

$$q^r = \sum_{l=1}^q \left(l^r - \sum_{i=1}^{l-1} i^r \right) F_{q-l+1}^{(k)}, \quad (q \leq k), \quad (3.5)$$

$$q^r = \sum_{l=1}^k \left(l^r - \sum_{i=1}^{l-1} i^r \right) F_{q-l+1}^{(k)} + \sum_{l=k+1}^q \left(l^r - \sum_{s=1}^k (l-s)^r \right) F_{q-l+1}^{(k)}, \quad (q > k), \quad (3.6)$$

$$\sum_{q=1}^n q^r = \sum_{l=1}^n \left(l^r - \sum_{i=1}^{l-1} i^r \right) \frac{F_{n-l+2}^{(k)} + \sum_{i=1}^{k-1} i F_{n-k+i-l+2}^{(k)} - 1}{k-1}, \quad (n \leq k), \quad (3.7)$$

$$\begin{aligned}\sum_{q=1}^n q^r &= \sum_{l=1}^k \left(l^r - \sum_{i=1}^{l-1} i^r \right) \frac{F_{n-l+2}^{(k)} + \sum_{i=1}^{k-1} i F_{n-k+i-l+2}^{(k)} - 1}{k-1} \\ &\quad + \sum_{l=k+1}^n \left(l^r - \sum_{i=1}^k (l-i)^r \right) \frac{F_{n-l+2}^{(k)} + \sum_{i=1}^{k-1} i F_{n-k+i-l+2}^{(k)} - 1}{k-1}, \quad (n > k).\end{aligned} \quad (3.8)$$

Proof. (3.5) and (3.6) are direct consequences of (3.1). We now prove (3.7) and (3.8) by using (3.1) and (3.4). If $n \leq k$, we have

$$\begin{aligned}\sum_{q=1}^n q^r &= \sum_{q=1}^n \sum_{l=1}^q \left(l^r - \sum_{i=1}^{l-1} i^r \right) F_{q-l+1}^{(k)} \\ &= \sum_{l=1}^n F_l^{(k)} + (2^r - 1) \sum_{l=1}^{n-1} F_l^{(k)} + (3^r - 2^r - 1) \sum_{l=1}^{n-2} F_l^{(k)} + \cdots + \left(n^r - \sum_{i=1}^{n-1} i^r \right) F_1 \\ &= \sum_{l=1}^n \left(l^r - \sum_{i=1}^{l-1} i^r \right) \frac{F_{n-l+2}^{(k)} + \sum_{i=1}^{k-1} i F_{n-k+i-l+2}^{(k)} - 1}{k-1}.\end{aligned}$$

If $n > k$, we have

$$\begin{aligned}\sum_{q=1}^n q^r &= \sum_{q=1}^k \sum_{l=1}^q \left(l^r - \sum_{i=1}^{l-1} i^r \right) F_{q-l+1}^{(k)} + \sum_{q=k+1}^n \left(\sum_{l=1}^k \left(l^r - \sum_{i=1}^{l-1} i^r \right) F_{q-l+1}^{(k)} + \sum_{l=k+1}^q \left(l^r - \sum_{i=1}^k (l-i)^r \right) F_{q-l+1}^{(k)} \right) \\ &= \sum_{l=1}^{n+k} F_l^{(k)} + (2^r - 1) \sum_{l=1}^{n+k-1} F_l^{(k)} + (3^r - 2^r - 1) \sum_{l=1}^{n+k-2} F_l^{(k)} + \cdots\end{aligned}$$

$$\begin{aligned}
& + \left(k^r - \sum_{i=1}^{k-1} i^r \right) \sum_{l=1}^{n+1} F_l^{(k)} + \left((k+1)^r - \sum_{i=1}^k i^r \right) \sum_{l=1}^n F_l^{(k)} + \cdots + \left(n^r - \sum_{i=n}^{n-1} i^r \right) F_1^{(k)} \\
& = \sum_{l=1}^k \left(l^r - \sum_{i=1}^{l-1} i^r \right) \frac{F_{n-l+2}^{(k)} + \sum_{i=1}^{k-1} i F_{n-k+i-l+2}^{(k)} - 1}{k-1} + \sum_{l=k+1}^n \left(l^r - \sum_{i=1}^k (l-i)^r \right) \frac{F_{n-l+2}^{(k)} + \sum_{i=1}^{k-1} i F_{n-k+i-l+2}^{(k)} - 1}{k-1}.
\end{aligned}$$

This completes the proof. $\square$

Setting $k = 3, 4$ in Theorem 3.3, we have the following corollary.

Corollary 3.4. *We have*

$$q^r = 2^r F_{q-1}^{(3)} + (3^r - 2^r) F_{q-2}^{(3)} + F_{q-3}^{(3)} + \sum_{l=4}^q (l^r - (l-1)^r - (l-2)^r - (l-3)^r) F_{q-l+1}^{(3)}, \quad (q \geq 3), \quad (3.9)$$

$$\sum_{q=1}^n q^r = \sum_{l=1}^3 \left(l^r - \sum_{i=1}^{l-1} i^r \right) \frac{F_{n-l+3}^{(3)} + F_{n-l+1}^{(3)} - 1}{2} + \sum_{l=4}^n \left(l^r - \sum_{i=1}^3 (l-i)^r \right) \frac{F_{n-l+3}^{(3)} + F_{n-l+1}^{(3)} - 1}{2}, \quad (n \geq 3), \quad (3.10)$$

$$q^r = 2^r F_{q-1}^{(4)} + (3^r - 2^r) F_{q-2}^{(4)} + (4^r - 3^r - 2^2) F_{q-3}^{(4)} + F_{q-4}^{(4)} + \sum_{l=5}^q \left(l^r - \sum_{i=1}^4 (l-i)^r \right) F_{q-l+1}^{(4)}, \quad (q \geq 4) \quad (3.11)$$

$$\begin{aligned}
\sum_{q=1}^n q^r &= \sum_{l=1}^4 \left(l^r - \sum_{i=1}^{l-1} i^r \right) \frac{F_{n-l+3}^{(4)} + 2F_{n-l+1}^{(4)} + F_{n-l}^{(4)} - 1}{3} \\
&+ \sum_{l=5}^n \left(l^r - \sum_{i=1}^4 (l-i)^r \right) \frac{F_{n-l+3}^{(4)} + 2F_{n-l+1}^{(4)} + F_{n-l}^{(4)} - 1}{3}, \quad (n \geq 4). \quad (3.12)
\end{aligned}$$

Corollary 3.4 gives the following special case.

Corollary 3.5. *For $n \geq 3$, we have*

$$q = F_q^{(3)} + F_{q-1}^{(3)} + \sum_{l=4}^q (6-2l) F_{q-l+1}^{(3)}, \quad (3.13)$$

$$\sum_{l=1}^{n-3} (l+3) (F_{n-l-2}^{(3)} + F_{n-l}^{(3)}) = \frac{F_{n+2}^{(3)} + 4F_{n+1}^{(3)} + F_n^{(3)} + 7F_{n-1}^{(3)} + 3F_{n-3}^{(3)} - 6n - 14}{2}. \quad (3.14)$$

Proof. (3.13) follows from (3.9) by letting $r = 1$.

$$\begin{aligned}
\sum_{q=1}^n q &= \frac{n(n+1)}{2} = \frac{F_{n+2}^{(3)} + F_n^{(3)} - 1}{2} + \frac{F_{n+1}^{(3)} + F_{n-1}^{(3)} - 1}{2} + \sum_{l=4}^n (6-2l) \frac{F_{n-l+3}^{(3)} + F_{n-l+1}^{(3)} - 1}{2} \\
&= \frac{F_{n+2}^{(3)} + F_n^{(3)} - 1}{2} + \frac{F_{n+1}^{(3)} + F_{n-1}^{(3)} - 1}{2} + 3 \sum_{l=4}^n (F_{n-l+3}^{(3)} + F_{n-l+1}^{(3)}) \\
&\quad - \sum_{l=4}^n l(F_{n-l+3}^{(3)} + F_{n-l+1}^{(3)}) - \sum_{l=4}^n (3-l) \\
&= \frac{F_{n+2}^{(3)} + 4F_{n+1}^{(3)} + F_n^{(3)} + 7F_{n-1}^{(3)} + 3F_{n-3}^{(3)} - 6n - 14}{2} - \sum_{l=1}^{n-3} (l+3) (F_{n-l-2}^{(3)} + F_{n-l}^{(3)}) + \frac{n(n-1)}{2}.
\end{aligned}$$

Hence we have (3.14). $\square$

Conclusion

In recent years, matrix factorizations of combinatorial matrices have become an attractive topic in mathematics. Several authors have studied relationships between combinatorial matrices, such as Pascal matrices, Stirling matrices of both kinds, and Fibonacci matrices. In this context, our study provides a new generalization of some results in the literature. For example, when the Riordan array $(g(t), f(t)) = \left(\frac{1}{1-t}, \frac{t}{1-t}\right)$, we recover the results in [6]. If we set $k = 2$, we obtain the results in [14]. Therefore, this study contributes to the literature by offering a unified approach to combinatorial matrices and identities.

Declaration

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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