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Sharp and general bounds for Hermite–Hadamard type inequalities via Jensen's inequality in q-calculus

Zoya Abdullah[a], Awais Yousaf[a], Hüseyin Budak[b,*]

[a]*Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur, Pakistan*
[b]*Department of Mathematics, Faculty of Sciences and Art, Kocaeli University, Kocaeli 41001, Türkiye*

Abstract. This paper establishes sharp bounds for Hermite–Hadamard inequalities within the framework of q-calculus by employing q-integrals. To achieve this, the Jensen–Mercer inequality, a generalization of Jensen's inequality, is utilized with multiple points to derive new and more precise bounds for q-Hermite–Hadamard inequalities. Previous results in classical calculus focused on convex functions and were limited to two points in Jensen's inequality. By extending the analysis to general points, this work broadens the applicability of these inequalities. The inclusion of left and right q-integrals presents challenges due to the generalized values in the Jensen–Mercer inequality, which are addressed by dividing the analysis into distinct cases. The ability to refine bounds using general points in Jensen–Mercer inequality is a significant outcome, as it unifies and extends many classical results by taking the limit as $q \to 1^-$. Numerical examples highlight the effectiveness of this approach, demonstrating that utilizing more points in Jensen–Mercer inequality produces sharper bounds for different values of q-parameter lies in $(0, 1)$.

1. Introduction

The Jensen's Inequality (JI) has a big role in pure and applied mathematics. Particularly probability theory, JI is useful in establishing bounds for variances and expectations [1]. In optimization theory, JI is useful in algorithms involving gradient descent method [2] and in information theory is helpful in establishing inequalities involving mutual information and entropy [3]. The applications of JI can be found in mathematical analysis and economics [4–7]. The JI is defines for convex function over an interval $[m, M]$ as:

$$f\left(\sum_{j=1}^{n} w_j x_j\right) \leq \sum_{j=1}^{n} w_j f\left(x_j\right), \tag{1}$$

where $w_j \in [0, 1]$, $\sum_{j=1}^{n} w_j = 1$ and $x_j \in [m, M]$, $j = 1, 2, 3, ..., n$. An extensions of inequality (1) can be found in [8].

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Communicated by Dragan S. Djordjević
* Corresponding author: Hüseyin Budak
Email addresses: zoya@cuisahiwal.edu.pk (Zoya Abdullah), awais.yousaf@iub.edu.pk (Awais Yousaf), hsyn.budak@gmail.com (Hüseyin Budak)
ORCID iDs: https://orcid.org/0009-0005-9351-0396 (Zoya Abdullah), https://orcid.org/0000-0002-3507-9367 (Awais Yousaf), https://orcid.org/0000-0001-8843-955X (Hüseyin Budak)

Mercer [9] also extended to inequality (1) and give the following inequality for convex function defined over an interval $[m, M]$:

$$f\left(m+M-\sum_{j=1}^{n} w_j x_j\right) \leq f(m)+f(M)-\sum_{j=1}^{n} w_j f\left(x_j\right), \tag{2}$$

where $w_j \in [0,1]$, $\sum_{j=1}^{n} w_j = 1$ and $x_j \in [m, M]$, $j = 1, 2, 3, ..., n$. The inequality (2) is known as Jensen–Mercer inequality and some extensions of this inequality are also given for convex and general convex functions in [10, 11].

In [12], the authors gave a new extension of the single inequality (2) in the form of double inequality for a convex function f over an interval $[m, M]$ which is defined as:

$$\begin{aligned} f(m+M-a^*) &\leq \sum_{j=1}^{n} w_j f\left(m+M-\left(t a_j+(1-t) a^*\right)\right) \\ &\leq f(m)+f(M)-\sum_{j=1}^{n} w_j f\left(x_j\right), \end{aligned} \tag{3}$$

where $a^* = \sum_{j=1}^{n} w_j x_j$, $w_j \in [0,1]$, $\sum_{j=1}^{n} w_j = 1$ and $x_j \in [m, M]$, $j = 1, 2, 3, ..., n$.

From inequality (3), one can observe that there is another term lies between left and right part of the inequality (2). These findings are very interesting and significant in the field of convex analysis. The inequality (3) may get more sharp bounds a compare to the inequalities (1) and (2).

In JI (1), if we set $n = 2$, $w_1 = w_2 = \frac{1}{2}$, $x_1 = m$ and $x_2 = M$, then we have the following inequality:

$$f\left(\frac{m+M}{2}\right) \leq \frac{1}{M-m} \int_{m}^{M} f(x)\, dx \leq \frac{f(m)+f(M)}{2}. \tag{4}$$

The inequality (4) is know as Hermite–Hadamard inequality. This inequality has many applications in different areas of mathematics particularly in the field of error analysis of numerical integration [6, 7].

The following extension of the inequality (4) was given in [13] by using the Jensen–Mercer inequality (2) with $n = 2$, $w_1 = w_2 = \frac{1}{2}$ and $x_1 < x_2$:

$$f\left(m+M-\frac{x_1+x_2}{2}\right) \leq \frac{1}{x_2-x_1} \int_{m+M-x_2}^{m+M-x_1} f(u)\, du \leq f(m)+f(M)-\frac{f(x_1)+f(x_2)}{2}. \tag{5}$$

The inequality (5) is convertible to (4) if we set $x_1 = m$ and $x_2 = M$.

In [12], the authors used the double inequality (3) and proved the following more general Hermite–Hadamard inequality:

$$f(m+M-a^*) \leq \sum_{j=1}^{n} \frac{w_j}{a^*-x_j} \int_{m+M-a^*}^{m+M-x_j} f(u)\, du \leq f(m)+f(M)-\sum_{j=1}^{n} w_j f\left(x_j\right). \tag{6}$$

The inequality (6) is convertible to (5) if we set $n = 2$, $w_1 = w_2 = \frac{1}{2}$ and $x_1 < x_2$. One can easily understand that the double inequality (6) is generalization of both inequalities (4) and (5).

The authors of [12] also proved the following inequalities by using the double inequality (3):

$$\begin{aligned} f(m+M-a^*) &\leq \sum_{j=1}^{n} w_j f\left(m+M-\frac{a^*+x_j}{2}\right) \\ &\leq \sum_{j=1}^{n} \frac{w_j}{a^*-x_j} \int_{m+M-a^*}^{m+M-x_j} f(u)\, du \end{aligned} \tag{7}$$

$$\leq \quad f(m)+f(M)-\sum_{j=1}^{n} w_j f\left(x_j\right),$$

and

$$f(a^*) \leq \sum_{j=1}^{n} w_j f\left(\frac{a^*+x_j}{2}\right) \leq \sum_{j=1}^{n} \frac{w_j}{a^*-x_j} \int_{x_j}^{a^*} f(u)\,du \leq \sum_{j=1}^{n} w_j f\left(x_j\right). \tag{8}$$

The inequalities (7) and (8) are very interesting because with the help of these inequalities we can get more sharp left bound as compare to the (4)-(6) which is the main motivation of establishment of (7) and (8).

On the other hand, q-calculus, also known as quantum calculus, is a generalization of classical calculus that eliminates the concept of limits, replacing them with q-differences and q-integrals. It is built on the idea of q-numbers, which depend on a parameter $0<q<1$ that interpolates between discrete and continuous mathematics. Widely applied in number theory, combinatorics, and mathematical physics, q-calculus provides a flexible framework for analyzing systems with non-uniform scaling or fractal-like structures.

We have the following two widely used q-integrals:

Definition 1.1. *[14, 15] Let $f:[m,M]\to\mathbb{R}$ be a continuous function, the left and right quantum integrals for $q\in(0,1)$ are*

$${}^{L}\mathcal{J}_{m}^{q} f(x)=(1-q)(x-m)\sum_{n=0}^{\infty} q^n f\left(q^n x+\left(1-q^n\right) m\right),\ x\in[m,M]$$

and

$${}^{R}\mathcal{J}_{M}^{q} f(x)=(1-q)(M-x)\sum_{n=0}^{\infty} q^n f\left(q^n x+\left(1-q^n\right) M\right),\ x\in[m,M]$$

respectively.

q-calculus finds different applications in many areas of mathematics and physics. The application of q-calculus has given in numerical integration formulas. The error bounds for midpoint and trapezoidal formulas for convex functions in q-calculus are proved in [16–19]. The error bounds for the Simpson's formula for convex functions are established in [20, 21] and the error bounds for the Newton's formula for convex and general convex functions can be found in [22, 23]. For more results regarding the error bounds for numerical integration formulas, one can consult [24–26] and references cited therein. A q-version of Hermite–Hadamard inequality can be stated as:

$$f\left(\frac{m+M}{2}\right) \leq \frac{1}{2(M-m)}\left[{}^{L}\mathcal{J}_{m}^{q} f(M)+{}^{R}\mathcal{J}_{M}^{q} f(m)\right] \leq \frac{f(m)+f(M)}{2}. \tag{9}$$

The following two q-variants of Hermite–Hadamard-Mercer inequality (5) are proved by Kara and Budak in [27]:

$$\begin{aligned} & f\left(m+M-\frac{q x_1+x_2}{1+q}\right) \qquad (10)\\ \leq\ & \frac{1}{x_2-x_1}\, {}^{L}\mathcal{J}_{(m+M-x_2)}^{q} f\left(m+M-x_1\right) \\ \leq\ & f(m)+f(M)-\frac{q f\left(x_1\right)+f\left(x_2\right)}{2}. \end{aligned}$$

$$f\left(m+M-\frac{x_1+q x_2}{1+q}\right) \tag{11}$$

$$
\begin{aligned}
&\leq \frac{1}{x_2 - x_1} {}^{R}J^{\beta}_{(m+M-x_1)-} f(m + M - x_2) \\
&\leq f(m) + f(M) - \frac{f(x_1) + qf(x_2)}{2}.
\end{aligned}
$$

Remark 1.2. *For the limit of $q \to 1^-$ in (10) and (11), we obtain the inequality (5).*

To the best of our knowledge, Hermite–Hadamard-type inequalities in the context of q-calculus, derived using q-integrals and generalized through the Jensen–Mercer inequality (7), have not been established so far. The newly proposed inequalities not only generalize existing results but also provide sharper bounds compared to their classical counterparts. These inequalities allow for improved lower and upper bounds by incorporating varying q-parameters and the number of points n in the Jensen–Mercer inequality (7). Numerical examples for different values of n and the q-parameter highlight the validity and efficiency of the proposed approach.The motivation for these results lies in the growing importance of q-calculus as a versatile tool in mathematical analysis, particularly in extending classical and fractional frameworks to more generalized settings. Traditional Hermite–Hadamard inequalities are often restricted to convex functions and specific points, limiting their applicability. By employing the Jensen–Mercer inequality (7) with generalized points, this work aims to address these limitations and provide sharper, more comprehensive bounds. The refined inequalities not only unify existing results but also offer new insights into error analysis and numerical integration, demonstrating the practical relevance and mathematical elegance of q-calculus.

The organization of the paper as follows: The Section 2 derives three new inequalities of Hermite–Hadamard type inequalities for Riemann–Liouville fractional integrals via double inequality (3). In Section 3, we establish an integral identity firts and then prove the general error bounds of numerical integration formulas for convex functions. Section 4 devoted for the numerical examples to show the validity and efficiency of newly established inequalities in Sections 2 and 3. We conclude the whole work in Section 4 and give some future directions.

2. Main Results

In this section, we derive new Hermite–Hadamard and Mercer-type inequalities for general values of n using a modified version of the Jensen–Mercer inequality with q-integrals. These results hold substantial importance in q-calculus, as they necessitate careful attention to the definitions of left and right q-integrals, which are influenced by the relationship between a^* and $a_j's$. A detailed discussion of these relationships and their implications will be presented in this section to provide a deeper insight into the proposed inequalities.

Theorem 2.1. *Let $f : [m, M] \to \mathbb{R}$ be a convex functons, then we have the following double inequalities:*

1. *For $a_i < a^* < a_{i+1}$ with $a^* \neq a_i$ for any $i = 2, 3, ..., n-1$. Next, we get the following inequalities for $a_i < a^* < a_{i+1}$*

$$
\begin{aligned}
f(m + M - a^*) &\leq \sum_{j=1}^{i} \frac{w_j}{\left(a^* - a_j\right)} \left[{}^{L}\mathcal{J}^{q}_{(m+M-a^*)} f\left(m + M - a_j\right)\right] \\
&\quad + \sum_{j=i+1}^{n} \frac{w_j}{\left(a_j - a^*\right)} \left[{}^{R}\mathcal{J}^{q}_{(m+M-a^*)} f\left(m + M - a_j\right)\right] \\
&\leq f(m) + f(M) - \sum_{j=1}^{n} w_j f\left(a_j\right).
\end{aligned} \tag{12}
$$

2. *If $w_{j-1} = w_j$ for every j and $a^* = a_j$, which is only feasible for odd n when $a_j's$ has same distance, then*

$$
\begin{aligned}
& f(m + M - a^*) - w_{\frac{n+1}{2}} f(m + M - a^*) \\
\leq & \sum_{j=1}^{\frac{n-1}{2}} \frac{w_j}{\left(a^* - a_j\right)} \left[{}^{L}\mathcal{J}^{q}_{(m+M-a^*)} f\left(m + M - a_j\right)\right]
\end{aligned} \tag{13}
$$

$$+\sum_{j=\frac{n+3}{2}}^{n}\frac{w_j}{\left(a_j-a^*\right)}\left[{}^{R}\mathcal{J}^{q}_{(m+M-a^*)}f\left(m+M-a_j\right)\right]$$
$$\leq f(m)+f(M)-w_{\frac{n+1}{2}}f(m+M-a^*)-\sum_{j=1}^{n}w_jf\left(a_j\right),$$

where $a^*=\sum_{j=1}^{n}w_ja_j$, $a^*,a_j\in[m,M]$ *with* $a_j\geq a_{j-1}$, $w_j\in[0,1]$ *and* $\sum_{j=1}^{n}w_j=1$.

Proof. With the convexity of f, we have

$$\begin{aligned} f(m+M-a^*) &= f\left(\sum_{j=1}^{n}w_j\left(m+M-\left((1-t)a^*+ta_j\right)\right)\right) \\ &\leq \sum_{j=1}^{n}w_jf\left(m+M-\left((1-t)a^*+ta_j\right)\right) \\ &= \sum_{j=1}^{n}w_jf\left((1-t)(m+M-a^*)+t\left(m+M-a_j\right)\right) \\ &\leq \sum_{j=1}^{n}w_j\left[(1-t)f(m+M-a^*)+tf\left(m+M-a_j\right)\right] \\ &\leq \sum_{j=1}^{n}w_j\left[(1-t)(f(m)+f(M)-f(a^*))+t\left(f(m)+f(M)-f\left(a_j\right)\right)\right]. \end{aligned} \tag{14}$$

By the definition of a^*, we have

$$\begin{aligned} f(m+M-a^*) &\leq \sum_{j=1}^{n}w_jf\left(m+M-\left((1-t)a^*+ta_j\right)\right) \\ &= \sum_{j=1}^{n}w_jf\left((1-t)(m+M-a^*)+t\left(m+M-a_j\right)\right) \\ &\leq f(m)+f(M)-\sum_{j=1}^{n}w_jf\left(a_j\right). \end{aligned} \tag{15}$$

Applying q-integral on both sides of (15), we have

$$\begin{aligned} f(m+M-a^*) &\leq \sum_{j=1}^{n}w_j\int_0^1 f\left((1-t)(m+M-a^*)+t\left(m+M-a_j\right)\right)_0 d_qt \\ &\leq f(m)+f(M)-\sum_{j=1}^{n}w_jf\left(a_j\right). \end{aligned}$$

Because of the definition of a^*, we have the following cases:

1. It is usually the case that a_n is more than a^* and a_1 is smaller than a^* but for any $i=2,3...n-1$, there is a chance that $a^*\neq a_i$. In this situation, for $a_i<a^*<a_{i+1}$, we have

$$f(m+M-a^*) \leq \sum_{j=1}^{i}\frac{w_j}{\left(a^*-a_j\right)}\left[{}^{L}\mathcal{J}^{q}_{(m+M-a^*)}f\left(m+M-a_j\right)\right]$$

$$
\begin{aligned}
&+\sum_{j=i+1}^{n}\frac{w_j}{\left(a_j-a^*\right)}\left[{}^{R}\mathcal{J}^{q}_{(m+M-a^*)}f\left(m+M-a_j\right)\right]\\
\leq\quad & f(m)+f(M)-\sum_{j=1}^{n}w_jf\left(a_j\right).
\end{aligned}
$$

2. When $a_j's$ has uniform mesh with $w_{j-1}=w_j$ for every j, there is a chance for a single value $a^*=a_j$ for odd n. That specific value is $a_{\frac{n+1}{2}}$, then

$$
\begin{aligned}
& f(m+M-a^*)-w_{\frac{n+1}{2}}f(m+M-a^*)\\
\leq\quad & \sum_{j=1}^{\frac{n-1}{2}}\frac{w_j}{\left(a^*-a_j\right)}\left[{}^{L}\mathcal{J}^{q}_{(m+M-a^*)}f\left(m+M-a_j\right)\right]\\
& +\sum_{j=\frac{n+3}{2}}^{n}\frac{w_j}{\left(a_j-a^*\right)}\left[{}^{R}\mathcal{J}^{q}_{(m+M-a^*)}f\left(m+M-a_j\right)\right]\\
\leq\quad & f(m)+f(M)-w_{\frac{n+1}{2}}f(m+M-a^*)-\sum_{j=1}^{n}w_jf\left(a_j\right)
\end{aligned}
$$

Thus, the proof is completed. □

Remark 2.2. *With $n=2$, $w_1=w_2=\frac{1}{2}$ with $a_1=x_1$ and $a_2=x_2$ in Theorem 2.1, we have*

$$
\begin{aligned}
& f\left(m+m-\frac{x_1+x_2}{2}\right) \qquad (16)\\
\leq\quad & \frac{1}{2(x_2-x_1)}\left[{}^{L}\mathcal{J}^{q}_{\left(m+M-\frac{x_1+x_2}{2}\right)}f(m+M-x_1)+{}^{R}\mathcal{J}^{q}_{\left(m+M-\frac{x_1+x_2}{2}\right)}f(m+M-x_2)\right]\\
\leq\quad & f(m)+f(M)-\frac{f(x)+f(y)}{2}.
\end{aligned}
$$

These inequalities are already proved in [27, Theorems 5 and 6].

Remark 2.3. *With $q\to 1^-$ in Theorem 2.1, we reattain the inequality (6).*

Remark 2.4. *It is important to mention here that one can establish different new inequalities for different values of n and $w_j's$ in Theorem 2.1.*

Example 2.5. *Consider the function $f:[1,5]\to\mathbb{R}$ such that $f(x)=x^3$ and let $n=3$ with $w_1=w_2=w_3=\frac{1}{3}$ and $a_1=2$, $a_2=3$ and $a_3=4$. Then we have $a^*=3$. Unther these assuption, we obtain*

$$
\begin{aligned}
& f(m+M-a^*)-w_{\frac{n+1}{2}}f(m+M-a^*)\\
=\quad & f(3)-w_2f(3)\\
=\quad & \frac{2}{3}f(3)\\
=\quad & 18
\end{aligned}
$$

and

$$
\begin{aligned}
& f(m)+f(M)-w_{\frac{n+1}{2}}f(m+M-a^*)-\sum_{j=1}^{n}w_jf\left(a_j\right)\\
=\quad & f(1)+f(5)-w_2f(3)-\sum_{j=1}^{n}w_jf\left(a_j\right)
\end{aligned}
$$

$$\begin{aligned} &= 126 - \frac{27}{2} - \frac{8+27+64}{3} \\ &= \frac{159}{2}. \end{aligned}$$

We also have

$$\begin{aligned} &\sum_{j=1}^{\frac{n-1}{2}} \frac{w_j}{\left(a^* - a_j\right)} \left[{}^{L}\mathcal{J}^{q}_{(m+M-a^*)} f\left(m+M-a_j\right)\right] \\ &+ \sum_{j=\frac{n+3}{2}}^{n} \frac{w_j}{\left(a_j - a^*\right)} \left[{}^{R}\mathcal{J}^{q}_{(m+M-a^*)} f\left(m+M-a_j\right)\right] \\ = \;& \frac{w_1}{(3-a_1)} \, {}^{L}\mathcal{J}^{q}_{3} f(6-a_1) + \frac{w_3}{(a_3-3)} \, {}^{R}\mathcal{J}^{q}_{3} f(6-a_3) \\ = \;& \frac{1}{2}\left[{}^{L}\mathcal{J}^{q}_{3} f(4) + {}^{R}\mathcal{J}^{q}_{3} f(2)\right] \\ = \;& \frac{1}{2}\left[27 + \frac{27}{[2]_q} + \frac{9}{[3]_q} + \frac{1}{[4]_q} + 27 - \frac{27}{[2]_q} + \frac{9}{[3]_q} - \frac{1}{[4]_q}\right] \\ = \;& 27 + \frac{9}{[3]_q}. \end{aligned}$$

Then the inequality (13) reduce to

$$18 \le 27 + \frac{9}{[3]_q} \le \frac{159}{2}.$$

That is,

$$-9 \le \frac{9}{[3]_q} \le \frac{141}{2}. \tag{17}$$

It is obvious that the inequality (17) is satisfied.

Theorem 2.6. *Let $f : [m, M] \to \mathbb{R}$ be a convex functons, then we have the following double inequalities:*

1. *For $a_i < a^* < a_{i+1}$ with $a^* \neq a_i$ for any $i = 2, 3, ..., n-1$. Next, we get the following inequalities for $a_i < a^* < a_{i+1}$*

$$\begin{aligned} & f(m+M-a^*) \qquad (18) \\ \le \;& \sum_{j=1}^{n} w_j f\left(m+M-\frac{a^*+a_j}{2}\right) \\ \le \;& \sum_{j=1}^{i} \frac{w_j}{2\left(a^*-a_j\right)} \left[{}^{L}\mathcal{J}^{q}_{(m+M-a^*)} f\left(m+M-a_j\right) + {}^{R}\mathcal{J}^{q}_{\left(m+M-a_j\right)} f(m+M-a^*)\right] \\ &+ \sum_{j=i+1}^{n} \frac{w_j}{2\left(a_j-a^*\right)} \left[{}^{R}\mathcal{J}^{q}_{(m+M-a^*)} f\left(m+M-a_j\right) + {}^{L}\mathcal{J}^{q}_{\left(m+M-a_j\right)} f(m+M-a^*)\right] \\ \le \;& \frac{f(m+M-a^*) + \sum_{j=1}^{n} w_j f\left(m+M-a_j\right)}{2} \\ \le \;& f(m) + f(M) - \sum_{j=1}^{n} w_j f\left(a_j\right). \end{aligned}$$

2. *If $a^* = a_j$ which is only possible for odd n when a_j's has equal distance and $w_{j-1} = w_j$ for every j, then*

$$\begin{aligned}
& f(m+M-a^*) - w_{\frac{n+1}{2}}\left[f(m+M-a^*) + f\left(m+M-a_{\frac{n+1}{2}}\right)\right] && (19)\\
\le\ & \sum_{j=1}^{n} w_j f\left(m+M-\frac{a^*+a_j}{2}\right) - w_{\frac{n+1}{2}}\left[f(m+M-a^*) + f\left(m+M-a_{\frac{n+1}{2}}\right)\right]\\
\le\ & \sum_{j=1}^{\frac{n-1}{2}} \frac{w_j}{2\left(a^*-a_j\right)}\left[{}^{L}\mathcal{J}^{q}_{(m+M-a^*)} f\left(m+M-a_j\right) + {}^{R}\mathcal{J}^{q}_{(m+M-a_j)} f(m+M-a^*)\right]\\
& + \sum_{j=\frac{n+3}{2}}^{n} \frac{w_j}{2\left(a_j-a^*\right)}\left[{}^{R}\mathcal{J}^{q}_{(m+M-a^*)} f\left(m+M-a_j\right) + {}^{L}\mathcal{J}^{q}_{(m+M-a_j)} f(m+M-a^*)\right]\\
\le\ & \frac{f(m+M-a^*) + \sum_{j=1}^{n} w_j f\left(m+M-a_j\right)}{2} - w_{\frac{n+1}{2}}\left[f(m+M-a^*) + f\left(m+M-a_{\frac{n+1}{2}}\right)\right]\\
\le\ & f(m) + f(M) - \sum_{j=1}^{n} w_j f\left(a_j\right) - w_{\frac{n+1}{2}}\left[f(m+M-a^*) + f\left(m+M-a_{\frac{n+1}{2}}\right)\right],
\end{aligned}$$

where $a^* = \sum_{j=1}^{n} w_j a_j$, $a^*, a_j \in [m, M]$ with $a_j \ge a_{j-1}$, $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$.

Proof. From convexity of f and $a^*, a_j \in [m, M]$, we have

$$\begin{aligned}
& f\left(m+M-\frac{a^*+a_j}{2}\right) && (20)\\
\le\ & \frac{f\left(m+M-\left(ta^*+(1-t)a_j\right)\right) + f\left(m+M-\left((1-t)a^*+ta_j\right)\right)}{2}\\
\le\ & \frac{f(m+M-a^*) + f\left(m+M-a_j\right)}{2}.
\end{aligned}$$

q-Integrating (20) over $[0,1]$, we have

$$\begin{aligned}
f\left(m+M-\frac{a^*+a_j}{2}\right) \le\ & \frac{1}{2}\left[\int_0^1 f\left(m+M-\left(ta^*+(1-t)a_j\right)\right)\ {}_0d_qt\right.\\
& \left. + \int_0^1 f\left(m+M-\left((1-t)a^*+ta_j\right)\right)\ {}_0d_qt\right]\\
\le\ & \frac{f(m+M-a^*) + f\left(m+M-a_j\right)}{2}.
\end{aligned}$$

Because of the definition of a^*, we have the following cases:

1. The a_n is always greater than a^* and a_1 is always less than a^* but there is possibility of $a^* \ne a_i$ for any $i = 2, 3 \ldots n-1$, then for $a_i < a^* < a_{i+1}$, we have

$$\begin{aligned}
& \sum_{j=1}^{n} w_j f\left(m+M-\frac{a^*+a_j}{2}\right) && (21)\\
\le\ & \sum_{j=1}^{i} \frac{w_j}{2\left(a^*-a_j\right)}\left[{}^{L}\mathcal{J}^{q}_{(m+M-a^*)} f\left(m+M-a_j\right) + {}^{R}\mathcal{J}^{q}_{(m+M-a_j)} f(m+M-a^*)\right]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=i+1}^{n}\frac{w_j}{2\left(a_j-a^*\right)}\left[{}^{R}\mathcal{J}^{q}_{(m+M-a^*)}f\left(m+M-a_j\right)+{}^{L}\mathcal{J}^{q}_{\left(m+M-a_j\right)}f\left(m+M-a^*\right)\right]\\
\leq\;&\frac{f\left(m+M-a^*\right)+\sum\limits_{j=1}^{n}w_jf\left(m+M-a_j\right)}{2}.
\end{aligned}
$$

2. For the case $a_i = a^*$, we follow the same method used in Theorem 2.1 and we have

$$
\begin{aligned}
&\sum_{j=1}^{n}w_jf\left(m+M-\frac{a^*+a_j}{2}\right)\\
&-w_{\frac{n+1}{2}}\left[f\left(m+M-a^*\right)+f\left(m+M-a_{\frac{n+1}{2}}\right)\right]\\
\leq\;&\sum_{j=1}^{\frac{n-1}{2}}\frac{w_j}{2\left(a^*-a_j\right)}\left[{}^{L}\mathcal{J}^{q}_{(m+M-a^*)}f\left(m+M-a_j\right)+{}^{R}\mathcal{J}^{q}_{\left(m+M-a_j\right)}f\left(m+M-a^*\right)\right]\\
&+\sum_{j=\frac{n+3}{2}}^{n}\frac{w_j}{2\left(a_j-a^*\right)}\left[{}^{R}\mathcal{J}^{q}_{(m+M-a^*)}f\left(m+M-a_j\right)+{}^{L}\mathcal{J}^{q}_{\left(m+M-a_j\right)}f\left(m+M-a^*\right)\right]\\
\leq\;&\frac{f\left(m+M-a^*\right)+\sum\limits_{j=1}^{n}w_jf\left(m+M-a_j\right)}{2}-w_{\frac{n+1}{2}}\left[f\left(m+M-a^*\right)+f\left(m+M-a_{\frac{n+1}{2}}\right)\right]
\end{aligned}
\tag{22}
$$

Now, it is easy to understand for a convex function f

$$
\begin{aligned}
f\left(m+M-a^*\right)&=f\left(\sum_{j=1}^{n}w_j\left(m+M-\frac{a^*+a_j}{2}\right)\right)\\
&\leq\sum_{j=1}^{n}w_jf\left(m+M-\frac{a^*+a_j}{2}\right)
\end{aligned}
\tag{23}
$$

and by definition of a^*, we have

$$
\begin{aligned}
&\frac{f\left(m+M-a^*\right)+\sum\limits_{j=1}^{n}w_jf\left(m+M-a_j\right)}{2}\\
\leq\;&\frac{f(m)+f(M)-\sum\limits_{j=1}^{n}w_jf\left(a_j\right)+f(m)+f(M)-\sum\limits_{j=1}^{n}w_jf\left(a_j\right)}{2}\\
=\;&f(m)+f(M)-\sum_{j=1}^{n}w_jf\left(a_j\right).
\end{aligned}
\tag{24}
$$

Thus, we obtain the required results by combining the above inequalities. □

Remark 2.7. *If we set the limit as $q \to 1^-$ in Theorem 2.6, then we recapture the inequality (7).*

Corollary 2.8. *If we set $n = 2$, $w_1 = w_2 = \frac{1}{2}$ with $a_1 = x_1$ and $a_2 = x_2$ in Theorem 2.6, then we have the following new inequality:*

$$
\begin{aligned}
&f\left(m+M-\frac{x_1+x_2}{2}\right)\\
\leq\;&\frac{1}{2}\left[f\left(m+M-\frac{3x_1+x_2}{2}\right)+f\left(m+M-\frac{x_1+3x_2}{2}\right)\right]
\end{aligned}
$$

$$\begin{aligned}
&\le \frac{1}{2(x_2-x_1)}\Big[{}^{L}\mathcal{J}^{q}_{\left(m+M-\frac{x_1+x_2}{2}\right)}f(m+M-x_1) \\
&\quad + {}^{R}\mathcal{J}^{q}_{\left(m+M-\frac{x_1+x_2}{2}\right)}f(m+M-x_2) + {}^{L}\mathcal{J}^{q}_{(m+M-x_2)}f\left(m+M-\frac{x_1+x_2}{2}\right) \\
&\quad + {}^{R}\mathcal{J}^{q}_{(m+M-x_1)-}f\left(m+M-\frac{x_1+x_2}{2}\right)\Big] \\
&\le \frac{1}{2}\left[f\left(m+M-\frac{x_1+x_2}{2}\right)+\frac{f(m+M-x_1)+f(m+M-x_2)}{2}\right] \\
&\le f(m)+f(M)-\frac{f(x_1)+f(x_2)}{2}.
\end{aligned}$$

These inequalities are also new in the literature and extend the results of [15, 27].

Example 2.9. *Consider the function $f:[1,5]\to\mathbb{R}$ such that $f(x)=x^2$ and let $n=3$ with $w_1=\frac{2}{3}$, $w_2=w_3=\frac{1}{6}$ and $a_1=2$, $a_2=3$ and $a_3=4$. Then we have $a^*=\frac{5}{2}$. Unther these assuption, we obtain*

$$f(m+M-a^*)=f\left(\frac{7}{2}\right)=\frac{49}{4},$$

$$\begin{aligned}
&\sum_{j=1}^{n} w_j f\left(m+M-\frac{a^*+a_j}{2}\right) \\
&= \frac{2}{3}f\left(\frac{15}{4}\right)+\frac{1}{6}f\left(\frac{13}{4}\right)+\frac{1}{6}f\left(\frac{11}{4}\right) \\
&= \frac{595}{48},
\end{aligned}$$

$$\begin{aligned}
&\frac{f(m+M-a^*)+\sum\limits_{j=1}^{n} w_j f\left(m+M-a_j\right)}{2} \\
&= \frac{49}{8}+\frac{1}{2}\left(\frac{2}{3}f(4)+\frac{1}{6}f(3)+\frac{1}{6}f(2)\right) \\
&= \frac{49}{8}+\frac{1}{2}\left(\frac{2}{3}(4)^2+\frac{1}{6}(3)^2+\frac{1}{6}(2)^2\right) \\
&= \frac{301}{24},
\end{aligned}$$

and

$$\begin{aligned}
&f(m)+f(M)-\sum_{j=1}^{n} w_j f\left(a_j\right) \\
&= 1+25-\left(\frac{2}{3}f(2)+\frac{1}{6}f(3)+\frac{1}{6}f(4)\right) \\
&= \frac{115}{6}.
\end{aligned}$$

On the other hand, since $a_1<a^*<a_2$, we have

$$\sum_{j=1}^{i}\frac{w_j}{2\left(a^*-a_j\right)}\left[{}^{L}\mathcal{J}^{q}_{(m+M-a^*)}f\left(m+M-a_j\right)+{}^{R}\mathcal{J}^{q}_{\left(m+M-a_j\right)}f(m+M-a^*)\right]$$

$$
\begin{aligned}
&+\sum_{j=i+1}^{n} \frac{w_j}{2\left(a_j-a^*\right)}\left[{}^{R}\mathcal{J}^{q}_{(m+M-a^*)}f\left(m+M-a_j\right)+{}^{L}\mathcal{J}^{q}_{\left(m+M-a_j\right)}f\left(m+M-a^*\right)\right]\\
=\;& \frac{w_1}{2\left(a^*-a_1\right)}\left[{}^{L}\mathcal{J}^{q}_{(m+M-a^*)}f\left(m+M-a_1\right)+{}^{R}\mathcal{J}^{q}_{(m+M-a_1)}f\left(m+M-a^*\right)\right]\\
&+\frac{w_2}{2\left(a_2-a^*\right)}\left[{}^{R}\mathcal{J}^{q}_{(m+M-a^*)}f\left(m+M-a_2\right)+{}^{L}\mathcal{J}^{q}_{(m+M-a_2)}f\left(m+M-a^*\right)\right]\\
&+\frac{w_3}{2\left(a_3-a^*\right)}\left[{}^{R}\mathcal{J}^{q}_{(m+M-a^*)}f\left(m+M-a_3\right)+{}^{L}\mathcal{J}^{q}_{(m+M-a_3)}f\left(m+M-a^*\right)\right]\\
=\;& \frac{2}{3}\left[{}^{L}\mathcal{J}^{q}_{\left(\frac{7}{2}\right)}f(4)+{}^{R}\mathcal{J}^{q}_{4}f\left(\frac{7}{2}\right)\right]\\
&+\frac{1}{6}\left[{}^{R}\mathcal{J}^{q}_{\left(\frac{7}{2}\right)}f(3)+{}^{L}\mathcal{J}^{q}_{3}f\left(\frac{7}{2}\right)\right]\\
&+\frac{1}{18}\left[{}^{R}\mathcal{J}^{q}_{\left(\frac{7}{2}\right)}f(2)+{}^{L}\mathcal{J}^{q}_{2}f\left(\frac{7}{2}\right)\right]\\
=\;& \frac{2}{3}\left[\frac{1}{8}\left(49+\frac{14}{[2]_q}+\frac{1}{[3]_q}\right)+\frac{1}{8}\left(64-\frac{16}{[2]_q}+\frac{1}{[3]_q}\right)\right]\\
&+\frac{1}{6}\left[\frac{1}{8}\left(49-\frac{14}{[2]_q}+\frac{1}{[3]_q}\right)+\frac{1}{8}\left(36+\frac{12}{[2]_q}+\frac{1}{[3]_q}\right)\right]\\
&+\frac{1}{18}\left[\frac{3}{8}\left(49-\frac{42}{[2]_q}+\frac{9}{[3]_q}\right)+\frac{3}{8}\left(16+\frac{24}{[2]_q}+\frac{9}{[3]_q}\right)\right]\\
=\;& \frac{1}{12}\left[113-\frac{2}{[2]_q}+\frac{2}{[3]_q}\right]+\frac{1}{48}\left[85-\frac{2}{[2]_q}+\frac{2}{[3]_q}\right]+\frac{1}{48}\left[65-\frac{18}{[2]_q}+\frac{18}{[3]_q}\right]\\
=\;& \frac{1}{24}\left[301-\frac{14}{[2]_q}+\frac{14}{[3]_q}\right].
\end{aligned}
$$

Then, from the inequality (18), we have the inequality

$$\frac{49}{4} \leq \frac{595}{48} \leq \frac{1}{24}\left[301-\frac{14}{[2]_q}+\frac{14}{[3]_q}\right] \leq \frac{301}{24} \leq \frac{115}{6}. \tag{25}$$

It is obvious that the first and the last inequalities in (25) are satisfied. One can see from Figure 1 that the second ans the third inequalities in (25) are also satisfied.

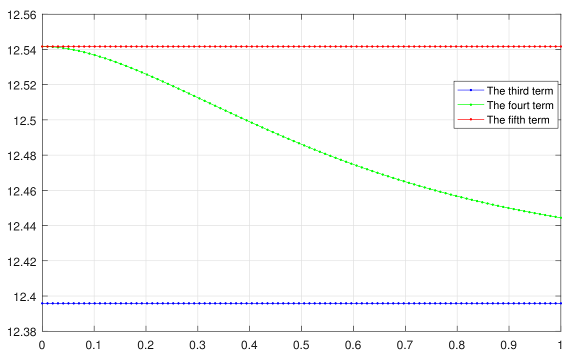

Figure 1: Comparison of third, fourth and fifth terms in (25)

Theorem 2.10. *Let f be a convex function over a real interval $[m, M]$, then the following double inequalities hold:*

1. *For $a_i < a^* < a_{i+1}$ with $a^* \neq a_i$ for any $i = 2, 3, ..., n-1$. Next, we get the following inequalities for $a_i < a^* < a_{i+1}$*

$$
\begin{aligned}
& f(a^*) \leq \sum_{j=1}^{n} w_j f\left(\frac{a^* + a_j}{2}\right) \\
\leq\ & \sum_{j=1}^{i} \frac{w_j}{2\left(a^* - a_j\right)} \left[{}^{R}\mathcal{J}_{a^*}^{q} f\left(a_j\right) + {}^{L}\mathcal{J}_{a_j}^{q} f\left(a^*\right)\right] \\
& + \sum_{j=i+1}^{n} \frac{w_j}{2\left(a_j - a^*\right)} \left[{}^{L}\mathcal{J}_{a^*}^{q} f\left(a_j\right) + {}^{R}\mathcal{J}_{a_j}^{q} f\left(a^*\right)\right] \\
\leq\ & \frac{f(a^*) + \sum_{j=1}^{n} w_j f\left(a_j\right)}{2} \leq \sum_{j=1}^{n} w_j f\left(a_j\right).
\end{aligned}
$$

2. *If $a^* = a_j$ which is only possible for odd n when a_j's has equal distance and $w_{j-1} = w_j$ for every j, then*

$$
\begin{aligned}
& f(a^*) - w_{\frac{n+1}{2}} \left[f\left(a_{\frac{n+1}{2}}\right) + f(a^*)\right] \\
\leq\ & \sum_{j=1}^{n} w_j f\left(\frac{a^* + a_j}{2}\right) - w_{\frac{n+1}{2}} \left[f\left(a_{\frac{n+1}{2}}\right) + f(a^*)\right] \\
\leq\ & \sum_{j=1}^{\frac{n-1}{2}} \frac{w_j}{2\left(a^* - a_j\right)} \left[{}^{R}\mathcal{J}_{a^*}^{q} f\left(a_j\right) + {}^{L}\mathcal{J}_{a_j}^{q} f\left(a^*\right)\right] \\
& + \sum_{j=\frac{n+3}{2}}^{n} \frac{w_j}{2\left(a_j - a^*\right)} \left[{}^{L}\mathcal{J}_{a^*}^{q} f\left(a_j\right) + {}^{R}\mathcal{J}_{a_j}^{q} f\left(a^*\right)\right] \\
\leq\ & \frac{f(a^*) + \sum_{j=1}^{n} w_j f\left(a_j\right)}{2} - w_{\frac{n+1}{2}} \left[f\left(a_{\frac{n+1}{2}}\right) + f(a^*)\right] \\
\leq\ & \sum_{j=1}^{n} w_j f\left(a_j\right) - w_{\frac{n+1}{2}} \left[f\left(a_{\frac{n+1}{2}}\right) + f(a^*)\right],
\end{aligned}
$$

where $a^* = \sum_{j=1}^{n} w_j a_j$, $a^*, a_j \in [m, M]$ with $a_j \geq a_{j-1}$, $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$.

Proof. The proof of this theorem is same as the proof of Theorem 2.6. □

Remark 2.11. *If we set the limit $q \to 1^-$ in Theorem 2.10, then we recapture the inequality (8).*

Corollary 2.12. *If we set $n = 2$, $w_1 = w_2 = \frac{1}{2}$ with $a_1 = x_1$ and $a_2 = x_2$ in Theorem 2.10, then we have the following new inequality:*

$$
\begin{aligned}
& f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{1}{2} \left[f\left(\frac{3x_1 + x_2}{4}\right) + f\left(\frac{x_1 + 3x_2}{4}\right)\right] \\
\leq\ & \frac{1}{2(x_2 - x_1)} \left[{}^{R}\mathcal{J}_{\left(\frac{x_1+x_2}{2}\right)}^{\beta} f(x_1) + {}^{L}\mathcal{J}_{\left(\frac{x_1+x_2}{2}\right)}^{q} f(x_2)\right. \\
& \left. + {}^{R}\mathcal{J}_{x_2}^{q} f\left(\frac{x_1 + x_2}{2}\right) + {}^{L}\mathcal{J}_{x_1}^{q} f\left(\frac{x_1 + x_2}{2}\right)\right] \\
\leq\ & \frac{f\left(\frac{x_1+x_2}{2}\right) + \frac{1}{2}\left[f(x_1) + f(x_2)\right]}{2} \leq \frac{f(x_1) + f(x_2)}{2}.
\end{aligned}
$$

3. Conclusion

This study successfully extends the Hermite–Hadamard inequalities to the framework of q-calculus, providing sharper and more generalized bounds through the application of the Jensen–Mercer inequality with multiple points. By addressing the challenges associated with q-integrals and leveraging a case-based approach, the findings unify and enhance existing results in classical calculus. The numerical examples validate the significance of this approach, highlighting the potential of q-calculus in refining mathematical inequalities and advancing error analysis in numerical integration. This work opens new pathways for further exploration of q-calculus in various mathematical and applied contexts.

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