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Description of the (≤ 3)-hypomorphic multiposets; application to the (≤ 3)-reconstruction

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Abstract. Consider a *multiposet* $\mathfrak{R} := (V, (P_i)_{i\in I})$ made of a family $(P_i)_{i\in I}$ (the *components* of $\mathfrak{R}$) of strict orders on a **possibly infinite** set V (*the vertex set* of $\mathfrak{R}$). $\mathfrak{R}$ is *linear* if at least one of its components is a chain and its other components which are not anti-chains are equal up to duality to this chain. For $i \in I$, a subset M of V is a *module* of P_i if for every $x \in V \setminus M$, all the elements of M share the same comparability with x in P_i. A *module* of $\mathfrak{R}$ is a common module of its components. A *linear-module* of $\mathfrak{R}$ is a module M of $\mathfrak{R}$ such that the restriction $\mathfrak{R} \restriction M$ of $\mathfrak{R}$ to M is linear. $\mathfrak{R}$ is prime if $|V| \geq 3$ and its only modules are the empty set, the singletons of its vertex set, and its own vertex set. Let k be a positive integer. Two multiposets $\mathfrak{R}$ and $\mathfrak{R}'$ on a same vertex set are ($\leq k$)-hypomorphic if for every set K of at most k vertices, the two restrictions $\mathfrak{R} \restriction K$ and $\mathfrak{R}' \restriction K$ are isomorphic. A multiposet $\mathfrak{R}$ is ($\leq k$)-reconstructible if every multiposet $\mathfrak{R}'$ that is ($\leq k$)-hypomorphic to $\mathfrak{R}$ is isomorphic to $\mathfrak{R}$. In this paper, we begin by obtaining a morphological description of the difference classes, introduced by Lopez in 1972, of the pairs of (≤ 3)-hypomorphic multiposets. Then we use this result to describe the pairs of (≤ 3)-hypomorphic multiposets. As a first corollary, we obtain that *a multiposet is* (≤ 3)-*reconstructible if and only if its linear-modules are finite.* As a second corollary, we obtain that *given two* (≤ 3)-*hypomorphic multiposets* $\mathfrak{R}$ *and* $\mathfrak{R}'$ *with at least four vertices, if* $\mathfrak{R}$ *is prime, then* $\mathfrak{R}' = \mathfrak{R}$.

1. Introduction

In this paper, all binary relations are assumed to be irreflexive. Given a positive integer k, two binary relations $\mathcal{R}_1$ and $\mathcal{R}_2$ are said to be ($\leq k$)-hypomorphic if they have the same vertex set and their restrictions $\mathcal{R}_1 \restriction K$ and $\mathcal{R}_2 \restriction K$ on each set K of at most k vertices are isomorphic. A binary relation $\mathcal{R}$ is ($\leq k$)-reconstructible if every binary relation $\mathcal{R}'$ that is ($\leq k$)-hypomorphic to $\mathcal{R}$ is isomorphic to $\mathcal{R}$. In [15–17], Lopez showed that finite binary relations are (≤ 6)-reconstructible. This work was extended to the infinite case by Hagendorf in [13]. These works make an essential use of difference classes introduced by Lopez [15–17]. Based on the description by Lopez and Rauzy [18] of the difference classes of finite (≤ 4)-hypomorphic binary relations, Boudabbous [3] provided a characterization of the (≤ 5)-reconstructible finite binary relations, that generalizes to (≤ 4)-reconstructibility. On the other hand, in [5] Boudabbous and Delhommé characterized the ($\leq k$)-reconstructible binary relations (finite or not), for each $k \geq 4$. For

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the (≤ 3)-reconstruction, Boudabbous and Lopez [7] characterized the finite binary relations that are (≤ 3)-reconstructible. Hagendorf [13] proved that every finite poset is (≤ 3)-reconstructible. In [2], we studied the (≤ 3)-reconstruction of posets and bichains. Boudabbous and Delhommé suggested the question about the morphological description of the difference classes of the pairs of (≤ 3)-hypomorphic multiposets and the characterization of the (≤ 3)-reconstructible multiposets (finite or not). In this paper, we give an answer to this question. In order to present our results that we obtained, we need to recall and introduce some useful notions for which we refer essentially to [6] and [9] for the terminology.

A multiposet is a pair $\mathfrak{R} := (V, (P_i)_{i\in I})$ made of a set V and a family $(P_i)_{i\in I}$ of *strict orders* on V which we call sometimes the *components* of $\mathfrak{R}$. A strict order P (or simply an order) on V is an irreflexive, antisymmetric and transitive binary relation on V. Occasionally, (V, P) is called a *partially ordered set* or simply a *poset*. For $X \subseteq V$, the restriction of the poset (V, P) to X, denoted by $P \upharpoonright X$, is the poset $(X, P \cap (X \times X))$. The restriction of the multiposet $\mathfrak{R}$ to X, denoted by $\mathfrak{R} \upharpoonright X$, is the multiposet $(X, (P_i \upharpoonright X)_{i\in I})$. The *dual* P^* of P is the poset (V, P^*), where for $x, y \in V$, $(x, y) \in P^*$ if and only if $(y, x) \in P$. The dual multiposet $\mathfrak{R}^*$ of $\mathfrak{R}$ is made of the set V and the family $(P^*_i)_{i\in I}$ of the dual components of $\mathfrak{R}$.

Consider a multiposet $\mathfrak{R} = (V, (P_i)_{i\in I})$. The multiposet $\mathfrak{R}$ is a *linear-ordering* (or *linear*) if there exists $i \in I$ such that P_i is a chain on V and for all $j \in I \setminus \{i\}$, we have: $P_j = P_i$, or $P_j = P^*_i$ or P_j is an anti-chain on V. Consider now a linear-ordering $\mathfrak{R} = (V, (P_i)_{i\in I})$, where $|V| \geq 3$, with at least one component which is an anti-chain and all the components which are chains have the same ends vertices a and c of V. For such a linear-ordering, by adding a comparability between a and c in at least one component of $\mathfrak{R}$ which is an anti-chain, we obtain what we call a *proper pot* multiposet (or simply a *proper pot* or an SDC). A *pot* is a linear-ordering multiposet or a proper pot. In particular, a *3-consecutivity* is a proper pot with exactly three vertices. Note that, this notion of pots that we have just developed is a generalization of that introduced in [4] for binary relations.

Given a poset (V, P), consider $x \neq y \in V$. By $x <_P y$ we denote the fact that $(x, y) \in P$. If $(x, y) \notin P$ and $(y, x) \notin P$, we set $x \parallel_P y$. A subset M of V is a *module* [8, 19] (is an *interval* [11], is an *autonomous* set [12] or a *clan* [10]) of P whenever for every $y \in V \setminus M$, either for all $x \in M$, $x <_P y$, or for all $x \in M$, $y <_P x$, or for all $x \in M$, $x \parallel_P y$. Given a multiposet $\mathfrak{R} = (V, (P_i)_{i\in I})$, consider a subset M of V. The subset M is a module of $\mathfrak{R}$ if M is a module of P_i, for all $i \in I$. In other words, M is a module of $\mathfrak{R}$ if M is a common module of all of its components. The module M is a *pot-module* of $\mathfrak{R}$ if $\mathfrak{R} \upharpoonright M$ is a pot. If in addition $\mathfrak{R} \upharpoonright M$ is linear we talk about *linear-module*. The empty set, the singletons $\{x\}$ where $x \in V$, and the set V are modules of $\mathfrak{R}$ and said to be *trivial*. The multiposet $\mathfrak{R}$ is *indecomposable* if all its modules are trivial; otherwise $\mathfrak{R}$ is said to be *decomposable*. The multiposet $\mathfrak{R}$ is *prime* if it is indecomposable with at least 3 vertices.

For our first contribution, we introduce the notion of pot multiposets and obtain the following result on the decomposition of multiposets.

Proposition 1.1. *The maximal pot-modules of each multiposet $\mathfrak{R}$, form a modular partition of $\mathfrak{R}$.*

Our second main result, which is the key in the proofs of all our below results, gives a morphological description of the difference classes, introduced by Lopez in [16], of two (≤ 3)-hypomorphic multiposets finite or not.

Theorem 1.2. *Consider two (≤ 3)-hypomorphic multiposets $\mathfrak{R}$ and $\mathfrak{R}'$ on a same vertex set. For each difference class C of $\mathfrak{R}$ and $\mathfrak{R}'$, $\mathfrak{R} \upharpoonright C$ is a pot.*

Our third main result gives a description of the pairs of (≤ 3)-hypomorphic multiposets.

Theorem 1.3. *Two multiposets on a same vertex set are (≤ 3)-hypomorphic if and only if they have the same partition into maximal pot-modules, the two corresponding quotients are equal, and their restrictions to each maximal pot-module are (≤ 3)-hypomorphic.*

For prime multiposets, we obtain the following immediate consequence of Theorem 1.3.

Corollary 1.4. *Consider two* (≤ 3)*-hypomorphic multiposets* $\mathfrak{R}$ *and* $\mathfrak{R}'$ *on a same vertex set such that* $\mathfrak{R}$ *is prime. Then either* $\mathfrak{R}' = \mathfrak{R}$*, or* $\mathfrak{R}' = \mathfrak{R}^*$ *and* $\mathfrak{R}$ *is a 3-consecutivity.*

Finally, as a second consequence of Theorem 1.3, we obtain the following characterization of the (≤ 3)-reconstructible multiposets which improves the result obtained by Hagendorf in [13].

Corollary 1.5. *A multiposet is* (≤ 3)*-reconstructible if and only if its linear-modules are finite.*

2. Preliminaries

For the terminology and definitions used in this section and in Section 3, we refer essentially to [6] and [9].

A binary structure or 2*-structure* is a pair $\mathfrak{R} := (V, (\mathcal{R}_i)_{i\in I})$ made of a set V and a family $(\mathcal{R}_i)_{i\in I}$ of binary relations on V called the components of $\mathfrak{R}$. The binary structure is a *multiposet* when, for all $i \in I$, $(V, \mathcal{R}_i)$ is a *partially ordered set* or simply a *poset*. In other words, a multiposet is a binary structure where all its components are strict orders on V.

2.1. Labelled 2-structure or ℓ2-structure

Convention 2.1. *For a set* V*, the set of ordered pairs of distinct elements of* V *is denoted by* V^2_* *(or* $E_2(V)$*). For each* $(a,b) \in V^2_*$*, we denote by* $(a,b)^*$ *the reversed ordered pair* (b,a)*.*

The 2-structures considered below correspond to the reversible labelled 2-structures of Ehrenfeucht et al. [9].

Definition 2.2. *[6] Given a set of labels* Λ *endowed with an involution* $\lambda \in \Lambda \mapsto \lambda^* \in \Lambda$ *(i.e.* $(\lambda^*)^* = \lambda$*), a* Λ*-2-structure (or a* ℓ*2-structure for short), of vertex set* V*, is a mapping* $\mathfrak{R} : V^2_* \longrightarrow \Lambda$ *satisfying:*

$$\forall (a,b) \in V^2_* : \mathfrak{R}(b,a) = (\mathfrak{R}(a,b))^*.$$

We may write $a \xrightarrow{\lambda} b$*, to mean* $\mathfrak{R}(a,b) = \lambda$*. Thus* $a \xrightarrow{\lambda} b \Leftrightarrow b \xrightarrow{\lambda^*} a$*.*

Given a Λ-2-structure $\mathfrak{R}$ of vertex set V, consider a subset X of V. The Λ-2*-structure induced* by $\mathfrak{R}$ on X is the Λ-2-structure $\mathfrak{R} \upharpoonright X$, of vertex set X, restriction of $\mathfrak{R}$ to X^2_*: $(\mathfrak{R} \upharpoonright X)(a,b) = \mathfrak{R}(a,b)$ for any $(a,b) \in X^2_*$.

2.2. Isomorphism and $(\leq k)$-hypomorphy of 2-structures

An *isomorphism* of a Λ-2-structure $\mathfrak{R}$ onto a Λ-2-structure $\mathfrak{R}'$, of respective vertex sets V and V', is any bijection $f : V \longrightarrow V'$ satisfying:

$$\forall (a,b) \in V^2_* : \mathfrak{R}'(f(a), f(b)) = \mathfrak{R}(a,b).$$

Two ℓ2-structures $\mathfrak{R}$ and $\mathfrak{R}'$ are isomorphic if there exists an isomorphism between them, which is denoted by $\mathfrak{R} \simeq \mathfrak{R}'$. Otherwise, we denote $\mathfrak{R} \not\simeq \mathfrak{R}'$.

Consider a Λ-2-structure $\mathfrak{R}$ of vertex set V. The *dual* of $\mathfrak{R}$ is the Λ-2-structure $\mathfrak{R}^*$ on the same vertex set V defined by $\mathfrak{R}^*(a,b) = \mathfrak{R}(b,a) = (\mathfrak{R}(a,b))^*$, for all $(a,b) \in V^2_*$. The Λ-2-structure $\mathfrak{R}$ is *selfdual* if it is isomorphic to its dual i.e. $\mathfrak{R} \simeq \mathfrak{R}^*$.

We recall the basic notions of the *reconstruction* problems in the theory of relations that we apply to the case of ℓ2-structures. Let $\mathfrak{R}$ and $\mathfrak{R}'$ be two ℓ2-structures on a same vertex set V and let k be a positive integer. The ℓ2-structure $\mathfrak{R}'$ is $(\leq k)$*-hypomorphic* to $\mathfrak{R}$ if for each subset K of V with at most k vertices, the induced ℓ2-structures $\mathfrak{R}' \upharpoonright K$ and $\mathfrak{R} \upharpoonright K$ are isomorphic. The ℓ2-structure $\mathfrak{R}$ is $(\leq k)$*-reconstructible*, if each ℓ2-structure $(\leq k)$-hypomorphic to $\mathfrak{R}$ is isomorphic to $\mathfrak{R}$.

2.3. Color and pair

Given a Λ-2-structure $\mathfrak{R}$, consider a label $\lambda \in \Lambda$. Say that λ is *selfdual* (or *neutral*, or *symmetric*) if $\lambda^* = \lambda$; otherwise speak of an *asymmetric* label. The color of λ is $\widehat{\lambda} := \{\lambda, \lambda^*\}$: this is a singleton if and only if λ is selfdual, and otherwise this is an unordered pair of dual labels. In the sequel, we use the notations below.

- $\widehat{\Lambda} := \{\widehat{\lambda} : \lambda \in \Lambda\}$ is the set of colors.
- $\overrightarrow{\Lambda} := \{\lambda \in \Lambda : \lambda^* \neq \lambda\}$ is the set of non selfdual (or asymmetric) labels.
- $\overset{\leftrightarrow}{\Lambda} := \{\lambda \in \Lambda : \lambda^* = \lambda\}$ is the set of selfdual (or symmetric or neutral) labels.

Clearly, Λ is the disjoint union of $\overrightarrow{\Lambda}$ and $\overset{\leftrightarrow}{\Lambda}$.

Now, consider two distinct vertices a and b of $\mathfrak{R}$, and let $\lambda := \mathfrak{R}(a, b)$. The unordered pair $\{a, b\}$ is called an *asymmetric pair*, if $\mathfrak{R}(a, b) \neq \mathfrak{R}(b, a)$, equivalently if λ is not selfdual. If $\mathfrak{R}(a, b) = \mathfrak{R}(b, a)$ (i.e. λ is selfdual), the unordered pair $\{a, b\}$ is a neutral or a symmetric pair. The *color* of (a, b) (resp. of $\{a, b\}$) is the color $\widehat{\lambda}$ of λ.

Given two distinct vertices a and b of a Λ-2-structure $\mathfrak{R}$, by *reversing* the ordered pair (a, b), or the unordered pair $\{a, b\}$, we mean considering the structure $\mathfrak{R}'$ obtained from $\mathfrak{R}$ by modifying its values just on $\{(a, b), (b, a)\}$ so that $\mathfrak{R}'(a, b) = \mathfrak{R}(b, a) = (\mathfrak{R}(a, b))^*$ (thus likewise $\mathfrak{R}'(b, a) = (\mathfrak{R}'(a, b))^* = \mathfrak{R}(a, b) = (\mathfrak{R}(b, a))^*$).

2.4. Reversed and unreversed pair

Consider two (≤ 2)-hypomorphic Λ-2-structures $\mathfrak{R}$ and $\mathfrak{R}'$ on a same vertex set V. Say that an asymmetric pair $\{a, b\}$ is *reversed* if $\mathfrak{R}'(a, b) = \mathfrak{R}(b, a)$; otherwise say that it is *unreversed*.

2.5. Modular partition, Quotient, Dilatation

Consider a Λ-2-structure $\mathfrak{R}$ of vertex set V.

Definition 2.3. *A module of $\mathfrak{R}$ is any set M of vertices such that:*

$$\forall x \in V \setminus M,\ \exists\, \lambda \in \Lambda \text{ such that } \forall\, y \in M : \mathfrak{R}(x, y) = \lambda.$$

The empty set, the singletons $\{x\}$ where $x \in V$, and the set V are modules of $\mathfrak{R}$ and said to be *trivial*. The Λ-2-structure is *indecomposable* if all its modules are trivial; otherwise, $\mathfrak{R}$ is said to be *decomposable*. The Λ-2-structure $\mathfrak{R}$ is *prime* if it is indecomposable with at least 3 vertices.

Definition 2.4. *A modular partition of $\mathfrak{R}$ is any partition of its vertex set of which the members are modules. Given a modular partition $\mathcal{P}$ of $\mathfrak{R}$, note that for all $M \neq N \in \mathcal{P}$, there exists $\lambda \in \Lambda$ such that $\mathfrak{R}(x, y) = \lambda$ for all $(x, y) \in M \times N$. The quotient Λ-2-structure $\mathfrak{R}/\mathcal{P}$ of $\mathfrak{R}$ by $\mathcal{P}$, is defined on $\mathcal{P}$ as follows:*

$$\forall M \neq N \in \mathcal{P}, (\mathfrak{R}/\mathcal{P})(M, N) = \mathfrak{R}(x, y),\ \text{where } (x, y) \in M \times N.$$

Definition 2.5. *Given two labelled-2-structures $\mathfrak{R}$ and $\mathfrak{R}'$ with disjoint vertex sets V and V' respectively, consider a vertex d of $\mathfrak{R}$. The dilated labelled-2-structure $\mathfrak{R}''$ of $\mathfrak{R}$ by $\mathfrak{R}'$ at the vertex d is defined on the vertex set $V'' = (V \setminus \{d\}) \cup V'$ as follows. For $x \neq y \in V''$:*

- $\mathfrak{R}''(x, y) = \mathfrak{R}(x, y)$ *when* $x, y \in V \setminus \{d\}$,
- $\mathfrak{R}''(x, y) = \mathfrak{R}'(x, y)$ *when* $x, y \in V'$,
- $\mathfrak{R}''(x, y) = \mathfrak{R}(x, d)$ *when* $x \in V \setminus \{d\}$ *and* $y \in V'$.

Clearly, V' is a module of the labelled-2-structure $\mathfrak{R}''$.

The following is easy to check.

Lemma 2.6. *[4] The collection $\mathcal{M}$ of modules of a Λ-2-structure $\mathfrak{R}$ with vertex set V satisfies the following properties.*

1. *It contains the empty set, the singletons and the vertex set (trivial modules).*
2. *It is closed under arbitrary intersection, i.e. $\forall \mathcal{N} \subseteq \mathcal{M}$: $\cap\mathcal{N} \in \mathcal{M}$(with the convention that $\cap\varnothing = V$).*
3. *It contains the union of any subcollection with a non-empty intersection, i.e. $\forall \mathcal{N} \subseteq \mathcal{M}(\cap\mathcal{N} \neq \varnothing \Rightarrow \cup\mathcal{N} \in \mathcal{M})$.*
4. *It is closed under balanced difference, i.e. $\forall M, N \in \mathcal{M}(M \setminus N \neq \varnothing \Rightarrow N \setminus M \in \mathcal{M})$.*
5. *Given a module M of $\mathfrak{R}$ and a vertex subset W, $M \cap W$ is a module of the restriction $\mathfrak{R} \upharpoonright W$.*

Lemma 2.7. *[6] Consider two Λ-2-structures $\mathfrak{R}$ and $\mathfrak{R}'$ with the same vertex set and a common modular partition $\mathcal{M}$ such that $\mathfrak{R}/\mathcal{M} = \mathfrak{R}'/\mathcal{M}$. Then the following assertions hold.*

1. *If the restrictions $\mathfrak{R} \upharpoonright M$ and $\mathfrak{R}' \upharpoonright M$ are isomorphic for each member M of $\mathcal{M}$, then $\mathfrak{R}$ and $\mathfrak{R}'$ are isomorphic.*
2. *For $k \geq 1$, if $\mathfrak{R} \upharpoonright M$ and $\mathfrak{R}' \upharpoonright M$ are $(\leq k)$-hypomorphic for each $M \in \mathcal{M}$, then $\mathfrak{R}$ and $\mathfrak{R}'$ are $(\leq k)$-hypomorphic.*

2.6. *Difference graph of two (≤ 2)-hypomorphic 2-structures*

Definition 2.8. *Consider two Λ-2-structures $\mathfrak{R}$ and $\mathfrak{R}'$ on a same vertex set V. Let $DiffGr(\mathfrak{R}, \mathfrak{R}')$ denote their difference graph on V: two distinct vertices a and b are adjacent in the difference graph if and only if $\mathfrak{R}'(a, b) = \mathfrak{R}(b, a) \neq \mathfrak{R}(a, b)$. The difference classes of $\mathfrak{R}$ and $\mathfrak{R}'$ are the connected components of their difference graph. The set of difference classes is denoted by $DiffCl(\mathfrak{R}, \mathfrak{R}')$.*

Given two (≤ 2)-hypomorphic Λ-2-structures $\mathfrak{R}$ and $\mathfrak{R}'$ on a same vertex set V, a *difference path* is a path in their difference graph. Thus, a difference path of length n is a finite sequence $x_0, \ldots, x_n$ of vertices such that $\mathfrak{R}(x_{i-1}, x_i) \neq \mathfrak{R}'(x_{i-1}, x_i)$, for $1 \leq i \leq n$. A *minimal* difference path between two vertices is a difference path of minimum length between them.

Remark 2.9. *Given two (≤ 2)-hypomorphic Λ-2-structures $\mathfrak{R}$ and $\mathfrak{R}'$ on a same vertex set V, consider $a \neq b \in V$:*

1. *The vertices a and b are not adjacent in $DiffGr(\mathfrak{R}, \mathfrak{R}')$ if and only if $\mathfrak{R}(a, b)$ is a symmetric label or $\mathfrak{R}(a, b) = \mathfrak{R}'(a, b)$ with $\mathfrak{R}(a, b)$ is an asymmetric label.*
2. *If $a, b \in C \in DiffCl(\mathfrak{R}, \mathfrak{R}')$ such that a and b are not adjacent in $DiffGr(\mathfrak{R}, \mathfrak{R}')$, then there exists a difference path between them.*

3. Multiposets

3.1. *Labelled 2-structures and multiposets*

Given a multiposet $\mathfrak{R} = (V, (P_i)_{i \in I})$, consider $x \neq y \in V$. For $i \in I$, write: $P_i(x, y) = +$ (resp. $P_i(x, y) = -$) if $x <_{P_i} y$ (resp. $y <_{P_i} x$) and $P_i(x, y) = 0$ if $x \parallel_{P_i} y$ (i.e. $x \not<_{P_i} y$ and $y \not<_{P_i} x$). In another words, the comparability of x and y in P_i can be labelled by $-$, 0 and $+$. Using these notations, we can consider the multiposet $\mathfrak{R}$ as a Λ-labelled 2-structure where $\Lambda := \{(\alpha_i)_{i \in I} \mid \alpha_i \in \{-, 0, +\} \textit{ for all } i \in I\}$ and $\alpha^* = ((\alpha_i)_{i \in I})^* := (\alpha_i^*)_{i \in I}$, for $\alpha \in \Lambda$, with $+^* := -, -^* := +$ and $0^* := 0$. For example, consider the multiposet $\mathfrak{R} = (V = \{a, b, c, d\}, (P_i)_{i \in \{1,2,3\}})$ with $P_1 := a <_{P_1} b <_{P_1} c <_{P_1} d$, $P_2 := a <_{P_2} d, a \parallel_{P_2} b, a \parallel_{P_2} c, b \parallel_{P_2} c, b \parallel_{P_2} d, c \parallel_{P_2} d$ and $P_3 := d <_{P_3} c <_{P_3} b <_{P_3} a$. The multiposet $\mathfrak{R}$ is a Λ-labelled 2-structure with range $\{(+, +, -), (-, -, +), (+, 0, -), (-, 0, +)\}$.

Remark 3.1. *Consider a multiposet $\mathfrak{R} = (V, (P_i)_{i \in I})$. If there exists $i \in I$ and $a \neq b \in V$ such that $P_i(a, b) \neq 0$, then $\mathfrak{R}(a, b) \neq \mathfrak{R}(b, a)$, and thus $\mathfrak{R}(a, b)$ is an asymmetric label. More precisely, the unique possible symmetric label of $\mathfrak{R}$ (if it exists) is $\mathbf{0} = (0, 0, \ldots)$. In other words, $\mathfrak{R}$ has a symmetric label if there exist $x \neq y \in V$ such that $P_i(x, y) = 0$, for all $i \in I$.*

Remark 3.2. *In the sequel, each multiposet $\mathfrak{R} = (V, (P_i)_{i \in I})$ is considered as a Λ-labelled 2-structure, with $\Lambda := \{(\alpha_i)_{i \in I} \mid \alpha_i \in \{-, 0, +\} \textit{ for all } i \in I\}$.*

3.2. *Particular Multiposets*

In this subsection, we describe all 3-vertex multiposets that are usefull for the sequel.
Consider a multiposet $\mathfrak{R} = (V, (P_i)_{i\in I})$ with the set of labels Λ.

Definition 3.3. *The multiposet $\mathfrak{R}$ is said to be empty if all its components are anti-chains i.e. for all $i \in I, P_i$ is an anti-chain on V.*

Using the labelling concept already mentioned in the above subsection, we can obviously remark the following.

Remark 3.4.

1. *The multiposet $\mathfrak{R}$ is said to be empty if it is a monochromatic symmetric multiposet i.e. $\mathfrak{R}(a,b) = \mathbf{0}$, for all $(a,b) \in V_*^2$. We denote by E_n the class of n-vertex empty multiposets.*
2. *The multiposet $\mathfrak{R}$ is a linear-ordering (or linear) if it is a monochromatic asymmetric multiposet i.e. there exists $\lambda \in \overrightarrow{\Lambda}$, such that $\widehat{\mathfrak{R}(a,b)} = \widehat{\lambda}$, for all $(a,b) \in V_*^2$. For this, we say that the linear-ordering $\mathfrak{R}$ is a λ-chain (or simply λ-linear). We denote by λ-L_n the class of n-vertex λ-chains. In particular, for $n = 3$, the class λ-L_3 (for $\lambda \in \overrightarrow{\Lambda}$), of linear 3-vertex structures is of the form presented in Figure 1.*

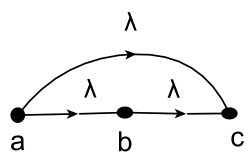

Fig. 1 a λ-L_3, for $\lambda \in \overrightarrow{\Lambda}$.

3. *The multiposet $\mathfrak{R}$ is a 3-consecutivity if it is a 3-vertex multiposet of the form $\overrightarrow{a \xrightarrow{\alpha} b \xrightarrow{\alpha} c}^{\,\mu}$ for two labels $\alpha \in \overrightarrow{\Lambda}$ and $\mu \in \Lambda \setminus \widehat{\alpha}$. Such a 3-consecutivity is called an (α,μ)-3-consecutivity (see Figure 2).*

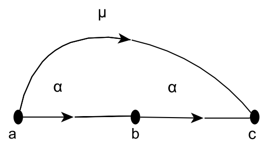

Fig. 2 an (α,μ)-3-consecutivity, for $\alpha \in \overrightarrow{\Lambda}$ and $\mu \in \Lambda \setminus \widehat{\alpha}$.

We talk also about a 3-consecutivity at b. The 3-consecutivities are prime and selfdual.

Finally, it remains two 3-vertex multiposets to be defined: the one-module 3-vertex multiposets and the flags.

Definition 3.5.

1. *A 3-vertex multiposet $\mathfrak{R}$ is said to be a one-module 3-vertex multiposet if $\mathfrak{R}$ has exactelly one non trivial module. The one-module 3-vertex multiposets are of the form $b \xrightarrow{\alpha} (a \xrightarrow{\mu} c)$ for two labels α and μ of different colors: $\widehat{\mu} \neq \widehat{\alpha}$ (see Figure 3). Among them are distinguished the peaks at b, corresponding to $\alpha \in \overrightarrow{\Lambda}$. Such a peak is said to be asymmetric if μ is not selfdual.*
2. *A 3-vertex multiposet is a flag if its three unordered pairs have different colors. The flags are prime and not selfdual.*

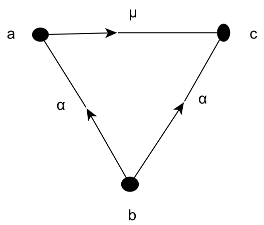

Fig. 3 One-module 3-vertex multiposet, for $\mu, \alpha \in \Lambda$ and $\widehat{\mu} \neq \widehat{\alpha}$.

By the above description presented in this subsection, we can partitioned the classes of 3-vertex multiposets as follows by the following easily checked lemma.

Lemma 3.6. *A 3-vertex mutliposet is one of the following non isomorphic multiposets: an element of E_3, a flag, a one-module 3-vertex multiposet, a 3-consecutivity or an element of L_3.*

Notation 3.7. *Consider a multiposet $\mathfrak{R} = (V, (P_i)_{i\in I})$. For $\lambda = (\lambda_i)_{i\in I} \in \overrightarrow{\Lambda}$, write:*

$$\underset{\neq 0}{\lambda} := \{i \in I : \ \lambda_i \neq 0\}.$$

Lemma 3.8. *Consider a 3-consecutivity $\mathfrak{R} = (\{a, b, c\}, (P_i)_{i\in I})$ at b, where $\mathfrak{R}(a, b) = \mathfrak{R}(b, c) = \alpha = (\alpha_i)_{i\in I}$ and $\mathfrak{R}(a, c) = \mu = (\mu_i)_{i\in I}$. Then $\underset{\neq 0}{\alpha} \neq I$, and μ is an asymmetric label such that $\underset{\neq 0}{\alpha}$ is a proper subset of $\underset{\neq 0}{\mu}$ and $\mu_i = \alpha_i$, for each element i of $\underset{\neq 0}{\alpha}$.*

Proof. Since $\alpha = (\alpha_i)_{i\in I} \in \overrightarrow{\Lambda}$, by Remark 3.1 there exists $i \in I$ such that $P_i(a, b) \neq 0$ and $P_i(b, c) \neq 0$. It ensues that, by transitivity of P_i, $P_i(a, c) = \mu_i \neq 0$. Thus, by Remark 3.1, $\mathfrak{R}(a, c) = \mu$ is an asymmetric label. For $i \in I$, clearly if $\alpha_i = P_i(a, b) = P_i(b, c) \neq 0$, then, by transitivity of P_i, $\mu_i = P_i(a, c) = P_i(a, b) = \alpha_i \neq 0$. It follows that, $\underset{\neq 0}{\alpha} \subseteq \underset{\neq 0}{\mu}$. Besides, since $\mu \in \Lambda \setminus \widehat{\alpha}$, there exists $j \in I \setminus \{i\}$ such that $\alpha_j = P_j(a, b) = P_j(b, c) = 0$ and $\mu_j = P_j(a, c) \neq 0$. Thus, $\underset{\neq 0}{\alpha} \neq I$ and $\underset{\neq 0}{\alpha} \subsetneq \underset{\neq 0}{\mu}$. □

At the end of this section, we will make the link between the 3-consecutivity as a labelled 2-structure and the SDC using the dilation concept.

Observation and notation 3.9. *Consider a 3-consecutivity $\mathfrak{R}$ at b on a 3-element set $\{a, b, c\}$ with $\alpha = \mathfrak{R}(a, b) = \mathfrak{R}(b, c)$ and $\mathfrak{R}(a, c) = \mu$. An SDC can be defined as the dilated multiposet $\mathfrak{R}'$ of $\mathfrak{R}$ at b by an α-linear multiposet with a vertex set V (see Figure 4). Note that the vertex set of $\mathfrak{R}'$ is the set $V' = V \cup \{a, c\}$, and for all $x \in V$, $\mathfrak{R}' \upharpoonright \{a, x, c\}$ is a 3-consecutivity at x with $\alpha = \mathfrak{R}(a, x) = \mathfrak{R}(x, c)$ and $\mathfrak{R}(a, c) = \mu$. The vertices a and c are called the end vertices of the SDC, and we say that $\mathfrak{R}'$ is an SDC (or simply an (λ, μ)-SDC) with ends a and c. For this, an SDC is considered as a special dilated consecutivity.*

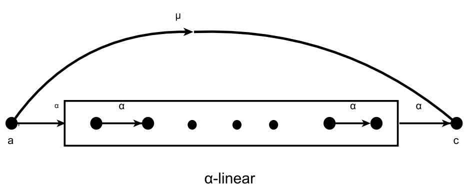

Fig. 4 **An SDC**

3.3. Pot-modules

In this subsection, we study some useful properties of pot-modules.

Observation 3.10. *Let $\mathfrak{R}$ be an SDC on a vertex set V. Then:*

1. *For all $X \subseteq V$, $\mathfrak{R} \upharpoonright X$ is a pot.*
2. *There is no modular partition of $\mathfrak{R}$ into two non-empty modules.*

Lemma 3.11. *Given a multiposet $\mathfrak{R}$ on a vertex set V, consider two pot-modules M and N of $\mathfrak{R}$ such that M is a proper pot and $M \cap N \neq \emptyset$. Then the following assertions hold:*

1. *If N is a proper pot, then $N = M$.*
2. *If N is a linear-module, then $N \subsetneq M$.*

Proof. Assume that M is an (λ, μ)-SDC with ends a and c such that $\mathfrak{R}(a, c) = \mu$, $\mathfrak{R}(a, x) = \mathfrak{R}(x, c) = \lambda$, for all $x \in M \setminus \{a, c\}$. If M and N overlap, then Lemma 2.6 implies that $\{M \cap N, M \setminus N\}$ is a modular partition of $\mathfrak{R} \upharpoonright M$ into two modules; which is in contradiction with Assertion two of Observation 3.10. Hence, $N \subseteq M$ or $M \subseteq N$. First, assume that N is linear. Since M has two different colors and N has a unique color or N is a singleton, necessarily $N \subsetneq M$. Second, assume that N is an SDC and let us prove that $N = M$. Without loss of generalities, we may assume that $N \subseteq M$. Thus by Lemma 2.6, N is a module of $\mathfrak{R} \upharpoonright M$. Clearly, a and c must be the ends of N. Moreover, if $M \setminus N \neq \emptyset$, then for each element x of $M \setminus N$ we have $\mathfrak{R}(a, x) = \mathfrak{R}(x, c) = \lambda$. Thus, $\mathfrak{R}(a, x) \neq \mathfrak{R}(c, x)$, which contradicts the fact that N is a module of $\mathfrak{R} \upharpoonright M$. Hence $N = M$. $\square$

The following result will be useful for the description of a modular partition of a given multiposet into maximal pot-modules.

Lemma 3.12. *Given a multiposet $\mathfrak{R}$, the union of any collection of linear-modules containing a given vertex is a linear-module. Every non-empty linear-module of $\mathfrak{R}$ is included in a unique maximal one. In particular, the maximal linear-modules of $\mathfrak{R}$ form a modular partition of $\mathfrak{R}$.*

Proof. Consider a multiposet $\mathfrak{R}$. We start by the following two claims.

Claim 3.13. *If M and N are two linear-modules such that $M \cap N \neq \emptyset$, then there exists $\lambda \in \overrightarrow{\Lambda}$ such that M and N are λ-linear.*

Indeed: The result is immediate when $M \subseteq N$ or $N \subseteq M$. Now, assume that M and N overlap. In this case, $|M| \geq 2$ and $|N| \geq 2$. By Lemma 2.6, $N \setminus M$ and $M \setminus N$ are modules of $\mathfrak{R}$. To the contrary, suppose that M is λ-linear and N is α-linear, for $\widehat{\lambda} \neq \widehat{\alpha}$. It follows that, $|M \cap N| = 1$. Let denote by x the unique element of $M \cap N$. Observe that, x must be the smallest element or the largest one of the linear multiposet $\mathfrak{R} \upharpoonright N$ because $N \setminus M$ is a module of $\mathfrak{R}$. By interchanging $\mathfrak{R}$ and $\mathfrak{R}^*$, we may assume that x is the smallest element of the linear multiposet $\mathfrak{R} \upharpoonright N$, i.e. $x \xrightarrow{\alpha} N \setminus M$. Therefore, the fact that $x \xrightarrow{\alpha} N \setminus M$ and M is a module implies that $M \setminus N \xrightarrow{\alpha} N \setminus M$. Since N is a module, $M \setminus N \xrightarrow{\alpha} x$. Consequently, $M \setminus N \xrightarrow{\beta} x$ and $M \setminus N \xrightarrow{\alpha} x$, for $\beta \in \widehat{\lambda}$, which contradicts the fact that M is λ-linear. $\square$

Claim 3.14. *If M and N are two λ-linear-modules, for $\lambda \in \overrightarrow{\Lambda}$, such that $M \cap N \neq \emptyset$, then $M \cup N$ is a λ-linear-module.*

Indeed: We may assume that M and N overlap. Lemma 2.6 implies that $M \cup N$, $M \cap N$, $M \setminus N$ and $N \setminus M$ are modules of $\mathfrak{R}$. Thus, $N \setminus M \xrightarrow{\mu} M \cap N$, where $\mu \in \widehat{\lambda}$. Since M is a module of $\mathfrak{R}$, $N \setminus M \xrightarrow{\mu} M \setminus N$. Hence, $\mathfrak{R} \upharpoonright (M \cup N)$ has a unique color $\widehat{\lambda}$. Consequently, $M \cup N$ is a linear-module of $\mathfrak{R}$. $\square$

Now, let us prove the first assertion. Let denote by $\mathcal{U}$ the union of some collection C_x of linear-modules containing a given vertex x. We will prove that $\mathcal{U}$ is also a linear-module. Clearly, by Lemma 2.6, $\mathcal{U}$ is a module of $\mathfrak{R}$. By Claim 3.13, all the members of C_x are linear with the same color. Assume that all the members of C_x are λ-linear, for $\lambda \in \overrightarrow{\Lambda}$. We will prove that $\mathcal{U}$ is λ-linear. Let $a \neq b \in \mathcal{U}$. There exist two members A and B of C_x such that $a \in A$ and $b \in B$. By Claim 3.14, $A \cup B$ is λ-linear and hence $\mathfrak{R}(a, b) \in \widehat{\lambda}$. It follows that, $\mathfrak{R} \restriction \mathcal{U}$ is λ-linear.

Since each singleton of the vertex set of $\mathfrak{R}$ is a non-empty linear-module of $\mathfrak{R}$, the last assertion is an immediate consequence of the second one. Finally, let us prove the second assertion. Consider a non-empty linear-module M of $\mathfrak{R}$ and let $x \in M$. By the first assertion, the union of the collection of all linear-modules containing the vertex x is a linear-module including M, and hence it is the unique maximal one including it. □

The following is deduced immediately from Lemma 3.11.

Lemma 3.15. *Given a multiposet $\mathfrak{R}$, consider a maximal linear-module M of $\mathfrak{R}$. Then, there is at most one SDC module including M.*

Proof of Proposition 1.1

Clearly, each singleton of the vertex set of $\mathfrak{R}$ is a pot-module of $\mathfrak{R}$. By Lemma 3.12, the maximal linear-modules of $\mathfrak{R}$ form a modular partition of $\mathfrak{R}$ and hence each singleton is included in a maximal linear-module of $\mathfrak{R}$. Moreover, by Lemma 3.15, each maximal linear-module is either a maximal pot-module or is included in a proper pot which is unique and then it is maximal. Consider two maximal pot-modules M and N of $\mathfrak{R}$ such that $M \cap N \neq \emptyset$. Using Lemma 3.11 and Lemma 3.12, we can obviously deduce that $M = N$. Hence the maximal pot-modules form a modular partition of $\mathfrak{R}$. □

3.4. (≤ 3)-*hypomorphic multiposets*

In this subsection, we make an essential use of the difference classes to describe the pairs of (≤ 3)-hypomorphic multiposets.

Lemma 3.16. *Given two (≤ 2)-hypomorphic multiposets $\mathfrak{R}$ and $\mathfrak{R}'$ on a same vertex set, consider Λ and Λ' their sets of labels respectively. If $\mathfrak{R}$ is λ-linear for $\lambda \in \overrightarrow{\Lambda}$, then $\mathfrak{R}'$ is λ-linear.*

Proof. Since $\mathfrak{R}$ and $\mathfrak{R}'$ are (≤ 2)-hypomorphic, $\widehat{\Lambda'} = \widehat{\Lambda} = \{\widehat{\lambda}\}$. Moreover, since $\mathfrak{R}'$ is a multiposet with one asymmetric color $\widehat{\lambda}$, $\mathfrak{R}'$ is a λ-linear multiposet. □

Lemma 3.17. *Consider two (≤ 3)-hypomorphic multiposets $\mathfrak{R}$ and $\mathfrak{R}'$ on a same 3-element set $V = \{a, b, c\}$.*

1. *If $\mathfrak{R}$ is a 3-consecutivity, then $\mathfrak{R}' = \mathfrak{R}$ or $\mathfrak{R}' = \mathfrak{R}^*$.*
2. *If $\mathfrak{R}$ is a peak at b, then $\mathfrak{R}' = \mathfrak{R}$ or $\mathfrak{R}$ is an asymmetric peak at b and $\mathfrak{R}'$ is the peak at b obtained from $\mathfrak{R}$ by reversing only the ordered pair (a, c).*
3. *If $\mathfrak{R}$ is a flag, then $\mathfrak{R}' = \mathfrak{R}$.*

Proof.

1. Assume for instance that $\mathfrak{R}$ is a 3-consecutivity at b with $\alpha = \mathfrak{R}(a, b) = \mathfrak{R}(b, c)$ and $\mu = \mathfrak{R}(a, c)$. Since $\mathfrak{R}$ and $\mathfrak{R}'$ are (≤ 2)-hypomorphic, $\mathfrak{R}'$ is a multiposet with $\widehat{\mathfrak{R}(a, b)} = \widehat{\mathfrak{R}'(a, b)} = \widehat{\alpha}$, $\widehat{\mathfrak{R}(b, c)} = \widehat{\mathfrak{R}'(b, c)} = \widehat{\alpha}$ and $\widehat{\mathfrak{R}(a, c)} = \widehat{\mathfrak{R}'(a, c)} = \widehat{\mu}$. Besides, since $\mathfrak{R}$ and $\mathfrak{R}'$ are (≤ 3)-hypomorphic, there is an isomorphism f from $\mathfrak{R}$ onto $\mathfrak{R}'$. Hence, by Lemma 3.6, $\mathfrak{R}'$ must be a 3-consecutivity. Clearly, $\widehat{\mathfrak{R}(a, c)} = \widehat{\mathfrak{R}'(f(a), f(c))} = \widehat{\mu}$. Moreover, the fact that $\{a, c\}$ is the unique pair having $\widehat{\mu}$ as color in the multiposet $\mathfrak{R}'$ implies that $(f(a), f(c)) = (a, c)$ or $(f(a), f(c)) = (c, a)$ and hence $f(b) = b$. In the first case, $\mathfrak{R}' = \mathfrak{R}$ and in the second one $\mathfrak{R}' = \mathfrak{R}^*$.

2. Assume for instance that $\mathfrak{R}$ is a peak at b with two labels $\alpha = \mathfrak{R}(b,a) = \mathfrak{R}(b,c)$ and $\mu = \mathfrak{R}(a,c)$, where α is an asymmetric label and $\widehat{\mu} \neq \widehat{\alpha}$. Since $\mathfrak{R}$ and $\mathfrak{R}'$ are (≤ 2)-hypomorphic, $\mathfrak{R}'$ is a multiposet with $\widehat{\mathfrak{R}(a,b)} = \widehat{\mathfrak{R}'(a,b)} = \widehat{\alpha}$, $\widehat{\mathfrak{R}(b,c)} = \widehat{\mathfrak{R}'(b,c)} = \widehat{\alpha}$ and $\widehat{\mathfrak{R}(a,c)} = \widehat{\mathfrak{R}'(a,c)} = \widehat{\mu}$. Since $\mathfrak{R}$ and $\mathfrak{R}'$ are (≤ 3)-hypomorphic, there is an isomorphism f from $\mathfrak{R}$ onto $\mathfrak{R}'$. Thus, by Lemma 3.6, $\mathfrak{R}'$ must be a peak. Clearly, $\widehat{\mathfrak{R}(a,c)} = \widehat{\mathfrak{R}'(f(a),f(c))} = \widehat{\mu}$. Moreover, the fact that $\{a,c\}$ is the unique pair having $\widehat{\mu}$ as color in the multiposet $\mathfrak{R}'$ implies that $(f(a),f(c)) = (a,c)$ or $(f(a),f(c)) = (c,a)$ and hence $f(b) = b$. It ensues that, $\mathfrak{R}'$ is a peak at b. It follows that if $\mathfrak{R} \neq \mathfrak{R}'$, then $(f(a),f(c)) = (c,a)$ and the peak $\mathfrak{R}$ is asymmetric; which implies that $\mathfrak{R}'$ is the peak at b obtained from $\mathfrak{R}$ by reversing only the ordered pair (a,c).
3. Assume for instance that $\mathfrak{R}$ is a flag with $\widehat{\mathfrak{R}(a,b)} = \widehat{\alpha}$, $\widehat{\mathfrak{R}(a,c)} = \widehat{\lambda}$ and $\widehat{\mathfrak{R}(b,c)} = \widehat{\mu}$. Since $\mathfrak{R}$ and $\mathfrak{R}'$ are (≤ 2)-hypomorphic, $\mathfrak{R}'$ is a multiposet with three different colors: $\widehat{\mathfrak{R}'(a,b)} = \widehat{\alpha}$, $\widehat{\mathfrak{R}'(a,c)} = \widehat{\lambda}$ and $\widehat{\mathfrak{R}'(b,c)} = \widehat{\mu}$. By Lemma 3.6, $\mathfrak{R}'$ must be a flag. Since $\mathfrak{R}$ and $\mathfrak{R}'$ are (≤ 3)-hypomorphic, there is an isomorphism f from $\mathfrak{R}$ onto $\mathfrak{R}'$. Clearly, $\widehat{\mathfrak{R}(a,b)} = \widehat{\mathfrak{R}'(f(a),f(b))} = \widehat{\alpha} = \widehat{\mathfrak{R}'(a,b)}$, $\widehat{\mathfrak{R}(a,c)} = \widehat{\mathfrak{R}'(f(a),f(c))} = \widehat{\lambda} = \widehat{\mathfrak{R}'(a,c)}$ and $\widehat{\mathfrak{R}(b,c)} = \widehat{\mathfrak{R}'(f(b),f(c))} = \widehat{\mu} = \widehat{\mathfrak{R}'(b,c)}$. It follows that, $\{f(a),f(b)\} = \{a,b\}$, $\{f(a),f(c)\} = \{a,c\}$ and $\{f(b),f(c)\} = \{b,c\}$. The fact that $\{f(a),f(b)\} = \{a,b\}$ and $\{f(a),f(c)\} = \{a,c\}$, implies that $f(a) = a$ and $f(b) = b$ and hence $f(c) = c$. Consequently, $\mathfrak{R}' = \mathfrak{R}$.

□

The following remark is an immediate consequence of the second assertion of Lemma 3.17.

Remark 3.18. *Consider two (≤ 3)-hypomorphic multiposets $\mathfrak{R}$ and $\mathfrak{R}'$ on a same vertex set. If $\mathfrak{R}$ is a symmetric peak, then $\mathfrak{R}' = \mathfrak{R}$.*

By the following lemma, we will prove that the unique symmetric label does not appear in the difference classes of two (≤ 3)-hypomorphic multiposets.

Lemma 3.19. *Consider two (≤ 3)-hypomorphic multiposets $\mathfrak{R}$ and $\mathfrak{R}'$ on a same vertex set. Then, for each class $C \in$ DiffCl$(\mathfrak{R},\mathfrak{R}')$, the multiposet $\mathfrak{R} \upharpoonright C$ does not contain the unique symmetric label.*

Proof. Consider $C \in$ DiffCl$(\mathfrak{R},\mathfrak{R}')$ such that $|C| \geq 2$. To the contrary, suppose that C contains a symmetric pair. Consider a symmetric pair $\{a,b\}$ of C such that there is a difference path $\mathcal{P} := x_0 = a \xrightarrow{\alpha_1} x_1 \ldots x_{n-1} \xrightarrow{\alpha_n} x_n = b$ between a and b which is with minimum length among all the lengths of all the difference paths connecting the two vertices of any symmetric pair of C. If $\{x_1,b\}$ is a symmetric pair, then $\mathcal{P}' := x_1 \ldots x_{n-1} \xrightarrow{\alpha_n} x_n = b$ is a difference path between the vertices x_1 and b with length $(n-1)$; which is in contradiction with the minimality hypothesis assumed for $\mathcal{P}$. Now, consider the restrictions of $\mathfrak{R}$ and $\mathfrak{R}'$ on the 3-element set $\{a,x_1,b\}$. By Lemmas 3.6 and 3.8, since $\mathfrak{R} \upharpoonright \{a,x_1,b\}$ has a symmetric pair $\{a,b\}$ and two asymmetric pairs $\{x_1,b\}$ and $\{a,x_1\}$, $\mathfrak{R} \upharpoonright \{a,x_1,b\}$ must be a flag or a peak at x_1. By Lemma 3.17 and Remark 3.18, $\mathfrak{R} \upharpoonright \{a,x_1,b\} = \mathfrak{R}' \upharpoonright \{a,x_1,b\}$. Therefore, $\{a,x_1\}$ is an unreversed pair which contradicts the fact that the vertices a and x_1 are adjacent in DiffGr$(\mathfrak{R},\mathfrak{R}')$. Consequently, the multiposet $\mathfrak{R} \upharpoonright C$ does not contain the unique symmetric label. □

In [6], the authors proved that the difference classes form a common modular partition of two (≤ 3)-hypomorphic Λ-2-structures and they studied the quotient Λ-2-structures by this partition. This generalizes the study carried out in [14, 16] for binary relations. Thus the following lemmas are immediate consequences of the study done in [6].

Lemma 3.20. *Consider two (≤ 3)-hypomorphic multiposets $\mathfrak{R}$ and $\mathfrak{R}'$ on a same vertex set.*

1. *For each $C \in$ DiffCl$(\mathfrak{R},\mathfrak{R}')$, C is a module of both $\mathfrak{R}$ and $\mathfrak{R}'$.*
2. *$\mathfrak{R}/$DiffCl$(\mathfrak{R},\mathfrak{R}') = \mathfrak{R}'/$DiffCl$(\mathfrak{R},\mathfrak{R}')$.*

Lemma 3.21. *Consider two (≤ 3)-hypomorphic multiposets $\mathfrak{R}$ and $\mathfrak{R}'$ on a same vertex set. Inside a difference class:*

1. *There is no peak.*
2. *Every 3-consecutivity is reversed.*

For our contribution in this study of difference classes, we are managed to show that inside a difference class of two (≤ 3)-hypomorphic multiposets there is no flags. What we obtained is based on Lemma 3.17, Lemma 3.19, Lemma 3.21 and the following lemma.

Lemma 3.22. *[6] Consider two* (≤ 3)*-hypomorphic multiposets* $\mathfrak{R}$ *and* $\mathfrak{R}'$ *on a same vertex set, and two adjacent asymmetric pairs* $\{a, b\}$ *and* $\{b, b'\}$ *of different colors. If* $\{a, b\}$ *is unreversed and* $\{b, b'\}$ *is reversed, then* $\mathfrak{R}(a, b') = \mathfrak{R}(a, b)$*. In particular* $\mathfrak{R} \restriction \{a, b, b'\}$ *is a peak at* a*.*

Corollary 3.23. *Consider two* (≤ 3)*-hypomorphic multiposets* $\mathfrak{R}$ *and* $\mathfrak{R}'$ *on a same vertex set. Inside a difference class, there is no flag.*

Proof. Let C be an element of $\mathrm{DiffCl}(\mathfrak{R}, \mathfrak{R}')$ such that $|C| \geq 3$. To the contrary, suppose that there exists $X := \{a, b, c\} \subseteq C$ such that $\mathfrak{R} \restriction X$ is a flag and, for instance, $\mathfrak{R}(a, b) = \lambda_1$, $\mathfrak{R}(b, c) = \lambda_2$ and $\mathfrak{R}(a, c) = \lambda_3$. By Lemma 3.19, λ_1, λ_2 and λ_3 are asymmetric labels. By Lemma 3.17, $\mathfrak{R} \restriction X = \mathfrak{R}' \restriction X$ and hence $\{a, b\}$, $\{b, c\}$ and $\{a, c\}$ are unreversed pairs. Since C is a connected component of $\mathrm{DiffGr}(\mathfrak{R}, \mathfrak{R}')$, there exists a difference path $\mathcal{P} := x_0 = a \xrightarrow{\alpha_1} x_1 \ldots x_{n-1} \xrightarrow{\alpha_n} x_n = b$ between a and b with $n \geq 2$. By definition of the difference path $\{a, x_1\}$ is a reversed pair, and hence $x_1 \neq c$. Moreover, since $\mathfrak{R} \restriction X$ is a flag, $\widehat{\lambda_1} \neq \widehat{\lambda_3}$. Therefore, there exists a label $\lambda \in \{\lambda_1, \lambda_3\}$ such that $\widehat{\lambda} \neq \widehat{\mathfrak{R}(a, x_1)}$. It ensues that, the reversed pair $\{a, x_1\}$ is adjacent to an unreversed pair with a label λ such that $\widehat{\lambda} \neq \widehat{\mathfrak{R}(a, x_1)}$, where $\lambda = \mathfrak{R}(a, c) = \lambda_3$ or $\lambda = \mathfrak{R}(a, b) = \lambda_1$. Consequently, by Lemma 3.22, $\mathfrak{R} \restriction \{a, x_1, c\}$ is a peak at c or $\mathfrak{R} \restriction \{a, x_1, b\}$ is a peak at b; which contradicts Assertion one of Lemma 3.21. $\square$

The following result is an immediate consequence of Lemma 3.6, Lemma 3.19, Lemma 3.21 and Corollary 3.23.

Corollary 3.24. *Given two* (≤ 3)*-hypomorphic multiposets* $\mathfrak{R}$ *and* $\mathfrak{R}'$ *on a same vertex set, consider an element* C *of* $\mathit{DiffCl}(\mathfrak{R}, \mathfrak{R}')$*. Then, for every 3-element set* $X \subseteq C$*, either* $\mathfrak{R} \restriction X$ *is an element of* L_3 *or a 3-consecutivity.*

4. Proof of Theorem 1.2

Theorem 1.2 is an immediate consequence of the following two lemmas.

Lemma 4.1. *Given two* (≤ 3)*-hypomorphic multiposets* $\mathfrak{R}$ *and* $\mathfrak{R}'$ *on a same vertex set, consider* $C \in \mathit{DiffCl}(\mathfrak{R}, \mathfrak{R}')$ *such that* $|C| \geq 4$ *and* $\mathfrak{R} \restriction C$ *contains a 3-consecutivity. Then* $\mathfrak{R} \restriction C$ *is an SDC.*

Proof. Consider $X := \{a, b, c\} \subset C$ such that $\mathfrak{R} \restriction X$ is an (λ, μ)-3-consecutivity at b with $\mathfrak{R}(a, b) = \mathfrak{R}(b, c) = \lambda$ and $\mathfrak{R}(a, c) = \mu$. Let d be an element of $C \setminus X$. By Corollary 3.24, $\mathfrak{R} \restriction \{a, c, d\}$ must be a μ-L_3 or a 3-consecutivity. In a first step, we are managed to prove that $\mathfrak{R} \restriction \{a, c, d\}$ is a 3-consecutivity. To the contrary, suppose that $\mathfrak{R} \restriction \{a, c, d\}$ is a μ-L_3. Observe that, $\mathfrak{R} \restriction \{b, c, d\}$ has two different colors $\widehat{\mu}$ and $\widehat{\lambda}$. By Corollary 3.24, $\mathfrak{R} \restriction \{b, c, d\}$ must be a 3-consecutivity. By Lemma 3.8, $\underset{\neq 0}{\lambda} \subsetneq \underset{\neq 0}{\mu}$ and hence $\mathfrak{R} \restriction \{b, c, d\}$ must be a 3-consecutivity at b and $\mathfrak{R}(d, b) = \mathfrak{R}(b, c) = \lambda$. Thus, $\mathfrak{R} \restriction \{a, b, d\}$ is a peak at b; which contradicts the fact that inside C there is no peak by Lemma 3.21. Consequently, $\mathfrak{R} \restriction \{a, c, d\}$ must be a 3-consecutivity. In a second step, we will show that $\mathfrak{R}(a, d) = \lambda$. To the contrary, suppose that $\mathfrak{R}(a, d) \neq \lambda$. First, assume that $\widehat{\mathfrak{R}(a, d)} = \widehat{\alpha} \neq \widehat{\lambda}$ and without loss of generality we may assume that $\mathfrak{R}(a, d) = \alpha$. Since $\mathfrak{R} \restriction \{a, b, d\}$ has two different colors $\widehat{\alpha}$ and $\widehat{\lambda}$, by Corollary 3.24 $\mathfrak{R} \restriction \{a, b, d\}$ must be a 3-consecutivity and hence $\underset{\neq 0}{\lambda} \subsetneq \underset{\neq 0}{\alpha}$ or $\underset{\neq 0}{\alpha} \subsetneq \underset{\neq 0}{\lambda}$ by Lemma 3.8. If $\underset{\neq 0}{\lambda} \subsetneq \underset{\neq 0}{\alpha}$, then $\mathfrak{R} \restriction \{a, b, d\}$ is a 3-consecutivity at b, by Lemma 3.8, and $\mathfrak{R}(b, d) = \lambda$. Moreover, since $\mathfrak{R} \restriction \{a, c, d\}$ is a 3-consecutivity with $\widehat{\mathfrak{R}(a, d)} = \widehat{\alpha} \neq \widehat{\lambda}$ and $\widehat{\mathfrak{R}(a, c)} = \widehat{\mu} \neq \widehat{\lambda}$, $\widehat{\mathfrak{R}(c, d)} \neq \widehat{\lambda}$. Thus, $\mathfrak{R} \restriction \{b, c, d\}$ must be a peak at b; which contradicts the fact that inside C there is no peak by Lemma 3.21. If $\underset{\neq 0}{\alpha} \subsetneq \underset{\neq 0}{\lambda}$, then $\mathfrak{R} \restriction \{a, b, d\}$ is a 3-consecutivity at d and $\mathfrak{R}(d, b) = \alpha$. Since $\underset{\neq 0}{\alpha} \subsetneq \underset{\neq 0}{\lambda}$ and $\underset{\neq 0}{\lambda} \subsetneq \underset{\neq 0}{\mu}$, $\underset{\neq 0}{\alpha} \subsetneq \underset{\neq 0}{\mu}$ and hence $\mathfrak{R} \restriction \{a, c, d\}$ must be a 3-consecutivity at d with $\mathfrak{R}(d, c) = \alpha$. Thus $\mathfrak{R} \restriction \{b, c, d\}$ must be a peak at d; which contradicts the fact that inside C there is no peak by Lemma 3.21. Second, assume that $\mathfrak{R}(a,d) = \lambda^*$. Since $\underset{\neq 0}{\lambda} \subsetneq \underset{\neq 0}{\mu}$, $\underset{\neq 0}{\lambda^*} \subsetneq \underset{\neq 0}{\mu}$ and hence $\mathfrak{R} \upharpoonright \{a,c,d\}$ must be a 3-consecutivity at d with $\mathfrak{R}(a,d) = \mathfrak{R}(d,c) = \lambda^*$ and $\mathfrak{R}(a,c) = \mu$; which contradicts the fact that μ is an asymmetric label by Lemma 3.8. Consequently, $\mathfrak{R}(a,d) = \lambda$. Since $\underset{\neq 0}{\lambda} \subsetneq \underset{\neq 0}{\mu}$ and $\mathfrak{R}(a,d) = \lambda$, $\mathfrak{R} \upharpoonright \{a,c,d\}$ is a 3-consecutivity at d with $\mathfrak{R}(a,d) = \mathfrak{R}(d,c) = \lambda$ and $\mathfrak{R}(a,c) = \mu$. Finally, observe that $\mathfrak{R}(b,d) \in \widehat{\lambda}$ because otherwise $\mathfrak{R} \upharpoonright \{a,b,d\}$ will be a peak at a. Now, for a vertex $d' \in C \setminus \{a,b,c,d\}$, it is easily checked that $\mathfrak{R}(d,d') \in \widehat{\lambda}$. Indeed, it suffices to apply the above reasoning by considering the extension of the 3-consecutivity $\mathfrak{R} \upharpoonright \{a,c,d\}$ (witch is isomorphic to $\mathfrak{R} \upharpoonright \{a,b,c\}$) by d'. Consequently, $\mathfrak{R} \upharpoonright (C \setminus \{a,c\})$ is a λ-linear multiposet and hence $\mathfrak{R} \upharpoonright C$ is an (λ,μ)-SDC with ends a and c. □

Lemma 4.2. *Given two* (≤ 3)*-hypomorphic multiposets* $\mathfrak{R}$ *and* $\mathfrak{R}'$ *on a same vertex set, consider an element* C *of DiffCl*$(\mathfrak{R},\mathfrak{R}')$*. If* $\mathfrak{R} \upharpoonright C$ *doest not contain a* 3*-consecutivity, then* $\mathfrak{R} \upharpoonright C$ *is a linear-ordering.*

Proof. Let C be an element of DiffCl$(\mathfrak{R},\mathfrak{R}')$ such that $\mathfrak{R} \upharpoonright C$ doest not contain a 3-consecutivity. By Corollary 3.24, for each 3-element set $X \subseteq C$, $\mathfrak{R} \upharpoonright X$ is an element of L_3. Clearly, we can assume that $|C| \geq 4$. Consider a 3-element set $X = \{a,b,c\} \subsetneq C$. By the hypothesis, there exits $\lambda \in \overrightarrow{\Lambda}$ such that $\mathfrak{R} \upharpoonright X$ is λ-linear. Let $x \neq y \in C$, we have to prove that $\widehat{\mathfrak{R}(x,y)} = \widehat{\lambda}$. Clearly, we may assume that $|X \cap \{x,y\}| = 0$ or 1. In the first case, since $\widehat{\mathfrak{R}(a,b)} = \widehat{\lambda}$, $\mathfrak{R} \upharpoonright \{a,b,x\}$ and $\mathfrak{R} \upharpoonright \{a,b,y\}$ are λ-linear and hence $\widehat{\mathfrak{R}(x,y)} = \widehat{\lambda}$. In the second case, without loss of generalities assume that $x = a$ and $y \notin X$. Since $\widehat{\mathfrak{R}(b,x)} = \widehat{\lambda}$, $\mathfrak{R} \upharpoonright \{b,x,y\}$ is λ-linear. Hence, $\widehat{\mathfrak{R}(x,y)} = \widehat{\lambda}$. □

5. Proof of Theorem 1.3

The proof of Theorem 1.3 is based essentially on Lemma 2.7 and the following lemmas.

The following lemma is deduced immediately from Lemma 3.16 and Lemma 3.17.

Lemma 5.1. *Consider two* (≤ 3)*-hypomorphic multiposets* $\mathfrak{R}$ *and* $\mathfrak{R}'$ *on a same vertex set. If* $\mathfrak{R}$ *is an SDC, then* $\mathfrak{R}'$ *is an SDC with the same ends as* $\mathfrak{R}$ *and with the same ordered pair of labels likewise* $\mathfrak{R}$*.*

Lemma 5.2. *Consider two* (≤ 3)*-hypomorphic multiposets* $\mathfrak{R}$ *and* $\mathfrak{R}'$ *on a same vertex set* V*. If* M *is a maximal pot-module of* $\mathfrak{R}$*, then* M *is a union of some difference classes of DiffCl*$(\mathfrak{R},\mathfrak{R}')$*, and* M *is a maximal pot-module of* $\mathfrak{R}'$*. More precisely, the collection of maximal pot-modules is a common modular partition of both* $\mathfrak{R}$ *and* $\mathfrak{R}'$*, and the corresponding quotient structures are equal.*

Proof. Let $M \subsetneq V$ be a maximal pot-module of $\mathfrak{R}$. By Lemma 3.20 and Theorem 1.2, DiffCl$(\mathfrak{R},\mathfrak{R}')$ is a modular partition of V into pot-modules. Thus, to prove that M is a union of some difference classes it suffices to show that: for all $C \in$ DiffCl$(\mathfrak{R},\mathfrak{R}')$ such that $C \cap M \neq \emptyset$, $C \subseteq M$. Consider $C \in$ DiffCl$(\mathfrak{R},\mathfrak{R}')$ such that $C \cap M \neq \emptyset$. We distinguish the following two cases.
Case 1: M is a linear-ordering. If C is a proper pot, then by Lemma 3.11 $M \subsetneq C$; which contradicts the fact that M is a maximal pot-module. Hence, C must be a linear-module. Moreover, since $M \cap C \neq \emptyset$, $M \cup C$ must be a linear-module of $\mathfrak{R}$ by Lemma 3.12. By the maximality of M, $C \subseteq M$.
Case 2: M is a proper pot. By Lemma 3.11, since $M \cap C \neq \emptyset$, $C \subseteq M$.
In the sequel, let us prove that M is a maximal pot-module of $\mathfrak{R}'$. Clearly, by Lemma 3.16 and Lemma 5.1, M is a pot of $\mathfrak{R}'$. It remains to prove that M is a maximal module of $\mathfrak{R}'$. First, let $x \in V \setminus M$. Since M is a module of $\mathfrak{R}$, there exists $\lambda \in \Lambda$ such that $\mathfrak{R}(x,y) = \lambda$ for all $y \in M$. Since M is a union of some difference classes of DiffCl$(\mathfrak{R},\mathfrak{R}')$ and $\mathfrak{R}/$ DiffCl$(\mathfrak{R},\mathfrak{R}') = \mathfrak{R}'/$ DiffCl$(\mathfrak{R},\mathfrak{R}')$, $\mathfrak{R}'(x,y) = \lambda$ for all $y \in M$. Thus M is a pot-module of $\mathfrak{R}'$. Second, let us prove the maximality of the module M in $\mathfrak{R}'$. If M is an SDC, then by Lemma 3.11 M is a maximal module in $\mathfrak{R}'$. If M is linear, then by Proposition 1.1 there is a maximal pot-module M' including M. By interchanging $\mathfrak{R}$ and $\mathfrak{R}'$, it follows that M' is pot-module including M. Thus $M' = M$ and hence the collection of maximal pot-modules is a common modular partition of both $\mathfrak{R}$ and $\mathfrak{R}'$. Finally, since $\mathfrak{R}/$ DiffCl$(\mathfrak{R},\mathfrak{R}') = \mathfrak{R}'/$ DiffCl$(\mathfrak{R},\mathfrak{R}')$, the corresponding quotient structures are equal. □

6. (≤ 3)-reconstructibility of multiposets

In this section we give a characterization of the (≤ 3)-reconstructible multiposets based on a description of the maximal pot-modules. We use essentially the following lemma which is an immediate consequence of Lemma 2 of [1] deduced from the study of Boudabbous and Delhommé [5].

Lemma 6.1. *[1] If X is an infinite set, then there are at least two nonhemimorphic linear-orders on X: i.e. there exist two linear-orders $\mathbf{C}_0$ and $\mathbf{C}_1$ on X such that $\mathbf{C}_0 \not\simeq \mathbf{C}_1$ and $\mathbf{C}_0 \not\simeq \mathbf{C}_1^*$.*

6.1. Proof of Corollary 1.5

First, assume that a multiposet $\mathfrak{R} := (V, (P_i)_{i\in I})$ has an infinite linear-module. By Proposition 1.1, the family $\mathcal{M}$ of maximal pot-modules of $\mathfrak{R}$ forms a modular partition of $\mathfrak{R}$ and then $\mathfrak{R}$ has at least one infinite maximal pot-module. For $\lambda \in \overrightarrow{\Lambda}$, consider the family $\mathcal{F}_\lambda$ of such maximal pot-modules M such that M is λ-linear with infinite cardinality $\mathcal{K}$, or M is an SDC including a maximal λ-linear-module with infinite cardinality $\mathcal{K}$. By Lemma 6.1, there are two nonhemimorphic linear-orders $\mathbf{C}_0$ and $\mathbf{C}_1$ with cardinality $\mathcal{K}$. Under this assumption and using the deformation technic presented in [5] we will construct two (≤ 3)-hypomorphic multiposets which are nonisomorphic. Consider the following multiposets $\mathfrak{R}_0$ and $\mathfrak{R}_1$ obtained by deforming each element M of $\mathcal{F}_\lambda$ in $\mathfrak{R}$ as follows. If $\mathfrak{R} \upharpoonright M$ is a λ-linear multiposet, where $\lambda = (\lambda_i)_{i\in I} \in \overrightarrow{\Lambda}$, then the multiposet $\mathfrak{R}_0 \upharpoonright M$ (respectively, $\mathfrak{R}_1 \upharpoonright M$) is obtained by replacing $\mathfrak{R} \upharpoonright M$ by the multiposet $(M, (P'_i)_{i\in I})$ (respectively, $(M, (P''_i)_{i\in I})$) such that, for $i \in I$,
$P'_i = \begin{cases} \mathbf{C}_{0,M} & \text{if } \lambda_i = +; \\ \mathbf{C}^*_{0,M} & \text{if } \lambda_i = -; \\ \text{anti-chain} & \text{if } \lambda_i = 0. \end{cases}$ $\left(\text{respectively, } P''_i = \begin{cases} \mathbf{C}_{1,M} & \text{if } \lambda_i = +; \\ \mathbf{C}^*_{1,M} & \text{if } \lambda_i = -; \\ \text{anti-chain} & \text{if } \lambda_i = 0. \end{cases}\right)$, where $\mathbf{C}_{0,M}$ (respectively, $\mathbf{C}_{1,M}$) is a linear-order on M isomorphic to $\mathbf{C}_0$ (respectively, to $\mathbf{C}_1$). If M is an (λ, μ)-SDC with ends a and c, then the multiposet $\mathfrak{R}_0 \upharpoonright M$ (resp. $\mathfrak{R}_1 \upharpoonright M$) is obtained by replacing $\mathfrak{R} \upharpoonright (M \setminus \{a, c\})$ by the multiposet $(M \setminus \{a, c\}, (P'_i)_{i\in I})$ (respectively, $(M \setminus \{a, c\}, (P''_i)_{i\in I})$) such that, for $i \in I$,
$P'_i = \begin{cases} \mathbf{C}_{0,M} & \text{if } \lambda_i = +; \\ \mathbf{C}^*_{0,M} & \text{if } \lambda_i = -; \\ \text{anti-chain} & \text{if } \lambda_i = 0. \end{cases}$ $\left(\text{resp. } P''_i = \begin{cases} \mathbf{C}_{1,M} & \text{if } \lambda_i = +; \\ \mathbf{C}^*_{1,M} & \text{if } \lambda_i = -; \\ \text{anti-chain} & \text{if } \lambda_i = 0. \end{cases}\right)$ and $\mathfrak{R}_0 \upharpoonright M$ (respectively, $\mathfrak{R}_1 \upharpoonright M$) is an (λ, μ)-SDC with ends a and c. Clearly, for each $M \in \mathcal{F}_\lambda$, either $\mathfrak{R} \upharpoonright M$, $\mathfrak{R}_0 \upharpoonright M$ and $\mathfrak{R}_1 \upharpoonright M$ are λ-linear or each of them is an (λ, μ)-SDC with ends a and c. Now, let us prove that $\mathfrak{R}_0$ and $\mathfrak{R}_1$ are (≤ 3)-hypomorphic to $\mathfrak{R}$. Clearly, $\mathcal{M}$ is a common modular partition of $\mathfrak{R}$, $\mathfrak{R}_0$ and $\mathfrak{R}_1$ with $\mathfrak{R}/\mathcal{M} = \mathfrak{R}_0/\mathcal{M} = \mathfrak{R}_1/\mathcal{M}$. Moreover, for each $M \in \mathcal{M} \setminus \mathcal{F}_\lambda$, $\mathfrak{R} \upharpoonright M = \mathfrak{R}_0 \upharpoonright M = \mathfrak{R}_1 \upharpoonright M$ and hence they are (≤ 3)-hypomorphic. Now, let $M \in \mathcal{F}_\lambda$. By our construction, each of the multiposets $\mathfrak{R} \upharpoonright M$, $\mathfrak{R}_0 \upharpoonright M$ and $\mathfrak{R}_1 \upharpoonright M$ is λ-linear or is an (λ, μ)-SDC with ends a and c. It follows that, $\mathfrak{R} \upharpoonright M$, $\mathfrak{R}_0 \upharpoonright M$ and $\mathfrak{R}_1 \upharpoonright M$ are (≤ 3)-hypomorphic. It ensues that the multiposets $\mathfrak{R} \upharpoonright M$, $\mathfrak{R}_0 \upharpoonright M$ and $\mathfrak{R}_1 \upharpoonright M$ are (≤ 3)-hypomorphic for each $M \in \mathcal{M}$. Thus, Lemma 2.7 implies that the multiposets $\mathfrak{R}$, $\mathfrak{R}_0$ and $\mathfrak{R}_1$ are (≤ 3)-hypomorphic. Consequently, by Theorem 1.3, $\mathcal{M}$ is a common partition into maximal pot-modules of the multiposets $\mathfrak{R}$, $\mathfrak{R}_0$ and $\mathfrak{R}_1$. Since the linear-orderings $\mathbf{C}_0$ and $\mathbf{C}_1$ are not hemimomorphic, $\mathfrak{R}_0$ and $\mathfrak{R}_1$ are not isomorphic. Therefore, at least one of the multiposets $\mathfrak{R}_0$ and $\mathfrak{R}_1$ is not isomorphic to $\mathfrak{R}$, and hence $\mathfrak{R}$ is not (≤ 3)-reconstructible.

Second, let $\mathfrak{R}$ be a multiposet with a vertex set V, and assume that all its linear-modules are finite. It ensues that all its maximal pot-modules are finite. Consider a multiposet $\mathfrak{R}'$ which is (≤ 3)-hypomorphic to $\mathfrak{R}$. We shall prove that $\mathfrak{R}$ and $\mathfrak{R}'$ are isomorphic. By Theorem 1.3, $\mathfrak{R}$ and $\mathfrak{R}'$ share the same maximal pot-modules, and they have the same corresponding quotient multiposet and their restrictions to each maximal pot-module are (≤ 3)-hypomorphic. Since the maximal pot-modules of $\mathfrak{R}$ are finite, $\mathfrak{R} \upharpoonright M \simeq \mathfrak{R}' \upharpoonright M$ for each maximal pot-module M of $\mathfrak{R}$. Finally, by Lemma 2.7, $\mathfrak{R}$ and $\mathfrak{R}'$ are isomorphic. □

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