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Loose 6-cycle decompositions of complete 3-uniform hypergraphs

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Abstract. The complete 3-uniform hypergraph $K_n^{(3)}$ of order n has a set V of cardinality n as its vertex set and the set of all 3-element subsets of V as its edge set. For $n \geq 2$, let $\mathbb{Z}_n$ denote the set of integers modulo n. For $m > 3$, let $LC_m^{(3)}$ denote the 3-uniform hypergraph with vertex set $\mathbb{Z}_{2m}$ and edge set $\{\{2i, 2i+1, 2i+2\} : i \in \{0, 1, 2, \ldots, m-1\}\}$. Any hypergraph isomorphic to $LC_m^{(3)}$ is a 3-*uniform loose* m-*cycle*. Given hypergraphs $\mathscr{K}$ and $\mathscr{H}$, a decomposition of $\mathscr{K}$ into $\mathscr{H}$ is a partition $\{\mathscr{E}_1, \mathscr{E}_2, \ldots, \mathscr{E}_b\}$ of the edge set of $\mathscr{K}$ such that, for each $i \in \{1, 2, \ldots, b\}$, the subhypergraph induced by $\mathscr{E}_i$ is isomorphic to $\mathscr{H}$. We show that there exists a decomposition of $K_n^{(3)}$ into $LC_6^{(3)}$ if and only if $n \geq 12$ and $n \equiv 0, 1, 2, 9, 10, 18, 20, 28$ or $29 \pmod{36}$.

1. Introduction

A hypergraph $\mathscr{F}$ consists of a finite nonempty set V of *vertices* and a set $\mathscr{E}$ of nonempty subsets of V called *hyperedges* or simply *edges*.

A *decomposition* of a hypergraph $\mathscr{K}$ is a set $\Delta = \{\mathscr{H}_1, \mathscr{H}_2, \ldots, \mathscr{H}_b\}$ of subhypergraphs of $\mathscr{K}$ such that $\mathscr{E}(\mathscr{H}_1) \cup \mathscr{E}(\mathscr{H}_2) \cup \ldots \cup \mathscr{E}(\mathscr{H}_b) = \mathscr{E}(\mathscr{K})$ and $\mathscr{E}(\mathscr{H}_i) \cap \mathscr{E}(\mathscr{H}_j) = \emptyset$ for all i and j with $1 \leq i < j \leq b$. We denote this fact by $\mathscr{K} = \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \cdots \oplus \mathscr{H}_b$. It follows from the definition that

$$|\mathscr{E}(\mathscr{H}_1)| + |\mathscr{E}(\mathscr{H}_2)| + \cdots + |\mathscr{E}(\mathscr{H}_b)| = |\mathscr{E}(\mathscr{K})|.$$

If each element $\mathscr{H}_i$ of Δ is isomorphic to a fixed hypergraph $\mathscr{H}$, then $\mathscr{H}_i$ is called an $\mathscr{H}$-*block*, and Δ is called an $\mathscr{H}$-*decomposition* of $\mathscr{K}$. In this case, we say that $\mathscr{H}$ *decomposes* $\mathscr{K}$, and we write $\mathscr{H} \mid \mathscr{K}$. Also, in this case, we have

$$b|\mathscr{E}(\mathscr{H})| = |\mathscr{E}(\mathscr{K})|.$$

Hence, a necessary condition for the existence of an $\mathscr{H}$-decomposition of $\mathscr{K}$ is that

$$|\mathscr{E}(\mathscr{H})| \text{ divides } |\mathscr{E}(\mathscr{K})|.$$

The *degree* of a vertex x in a hypergraph $\mathscr{F}$ is the number of edges of $\mathscr{F}$ containing x.

Another necessary condition for the existence of an $\mathscr{H}$-decomposition of $\mathscr{K}$ is that

the g.c.d. of the degrees of vertices in $\mathscr{H}$ divides the g.c.d. of the degrees of vertices in $\mathscr{K}$.

If each vertex x in a hypergraph $\mathscr{F}$ has the same degree, then we say that the hypergraph $\mathscr{F}$ is regular, or $\mathscr{F}$ is k-*regular* if the degree of x is k.

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If for each edge e in a hypergraph $\mathscr{F}$, we have $|e| = t$, then $\mathscr{F}$ is said to be *t-uniform*. Thus simple graphs are 2-uniform hypergraphs.

A *cycle* of length m, in a hypergraph $\mathscr{F}$ is a sequence of the form $v_1, e_1, v_2, e_2, \ldots, v_m, e_m, v_1$ where $v_1, v_2, \ldots, v_m$ are distinct vertices and $e_1, e_2, \ldots, e_m$ are distinct edges satisfying $v_i, v_{i+1} \in e_i$ for $i \in \{1, 2, \ldots, m-1\}$ and $v_m, v_1 \in e_m$. This cycle is known as a *Berge cycle* having been introduced by Berge in [1]. For $i \in \{1, 2, \ldots, m\}$, if $|e_i| = t$, then we denote this Berge cycle by $BC_m^{(t)}$.

For $n \geq 2$, let $\mathbb{Z}_n$ denote the set of integers modulo n.

For $m > t \geq 2$, let $LC_m^{(t)}$ denote the t-uniform hypergraph with vertex set $\mathbb{Z}_{(t-1)m}$ and edge set $\{\{it-i, it-i+1, it-i+2, \ldots, it-i+(t-1)\} : i \in \{0, 1, \ldots, m-1\}\}$. Any hypergraph isomorphic to $LC_m^{(t)}$ is a *t-uniform loose m-cycle*. In particular, for $t = 3$, a 3-uniform loose m-cycle $LC_m^{(3)}$ is a 3-uniform hypergraph with vertex set $\mathbb{Z}_{2m}$ and edge set $\{\{2i, 2i+1, 2i+2\} : i \in \{0, 1, \ldots, m-1\}\}$.

Let $\mathscr{F}$ be a t-uniform hypergraph. It follows from the definitions that every loose cycle of $\mathscr{F}$ is a Berge cycle of $\mathscr{F}$. Observe that, for $t = 2$, $BC_m^{(2)} \cong LC_m^{(2)}$.

Let $\mathscr{K}$ be a t-uniform hypergraph, $t \geq 3$. The necessary conditions for the existence of:

$BC_m^{(t)}$-decomposition of $\mathscr{K}$ are $|V(\mathscr{K})| \geq m$ and m divides $|\mathscr{E}(\mathscr{K})|$;

$LC_m^{(t)}$-decomposition of $\mathscr{K}$ are $|V(\mathscr{K})| \geq (t-1)m$ and m divides $|\mathscr{E}(\mathscr{K})|$.

As every loose cycle of $\mathscr{K}$ is a Berge cycle of $\mathscr{K}$, we have: every $LC_m^{(t)}$-decomposition of $\mathscr{K}$ is a $BC_m^{(t)}$-decomposition of $\mathscr{K}$.

A t-uniform hypergraph $\mathscr{F} = (V, \mathscr{E})$ is said to be *complete* if every t-element subset of V is in $\mathscr{E}$. We denote such a hypergraph by $K_V^{(t)}$ or by $K_n^{(t)}$ if $|V| = n$. $K_n^{(t)}$ is $\binom{n-1}{t-1}$-regular and it has $\binom{n}{t}$ edges. An $\mathscr{H}$-decomposition of $K_n^{(t)}$ is also known as an *$\mathscr{H}$-design* of order n. Given a t-uniform hypergraph $\mathscr{H}$, the problem of determining all values of n for which there exists an $\mathscr{H}$-design of order n is known as the *spectrum problem* for $\mathscr{H}$.

If $\mathscr{K} = K_n^{(t)}$, then the above necessary conditions for the existence of:

$BC_m^{(t)}$-decomposition of $K_n^{(t)}$ are $n \geq m$ and $m \,|\, \binom{n}{t}$;

$LC_m^{(t)}$-decomposition of $K_n^{(t)}$ are $n \geq (t-1)m$ and $m \,|\, \binom{n}{t}$.

Assume $3 \leq t < n$. A $BC_n^{(t)}$ of $K_n^{(t)}$ is called a *Hamilton cycle* of $K_n^{(t)}$ and a $BC_n^{(t)}$-decomposition of $K_n^{(t)}$ is called a *Hamilton cycle decomposition* of $K_n^{(t)}$. The necessary condition for the existence of $BC_n^{(t)} \,|\, K_n^{(t)}$ is $n \,|\, \binom{n}{t}$. In [3], Bermond et al. conjectured that this necessary condition is sufficient and proved this conjecture for n a prime. In [9], Kühn and Osthus, proved that for $t \geq 4$ and $n \geq 30$, if $n \,|\, \binom{n}{t}$, then $BC_n^{(t)} \,|\, K_n^{(t)}$. For $t = 3$, the necessary condition $n \,|\, \binom{n}{3}$ is: $n \equiv 1, 2, 4$ or $5 \pmod 6$; in [2], Bermond proved that: if $n \equiv 2, 4$ or $5 \pmod 6$, then $BC_n^{(3)} \,|\, K_n^{(3)}$, and in [16], Verrall proved that: if $n \equiv 1 \pmod 6$, then $BC_n^{(3)} \,|\, K_n^{(3)}$.

Let $\mathscr{E}_n^{(t)}$ be the set of all t-element subsets of $\mathbb{Z}_n$, where $1 < t < n$. If $E \in \mathscr{E}_n^{(t)}$ and $r \in \mathbb{Z}_n$, let $E + r$ be formed by replacing each element $x \in E$ with $x + r$; so $(r, E) \mapsto E + r$ maps $\mathbb{Z}_n \times \mathscr{E}_n^{(t)}$ into $\mathscr{E}_n^{(t)}$. It can be seen that the group $\mathbb{Z}_n$ acts on the set $\mathscr{E}_n^{(t)}$ partitioning it into $\mathbb{Z}_n$-orbits, where $E_1, E_2 \in \mathscr{E}_n^{(t)}$ are in the same orbit if and only if $E_1 + r = E_2$ for some $r \in \mathbb{Z}_n$. We define $[E]$ to be $\{E + r : r \in \mathbb{Z}_n\}$, which we refer to as the *$\mathbb{Z}_n$-orbit* of E. If $\mathscr{S} \subseteq \mathscr{E}_n^{(t)}$ and $r \in \mathbb{Z}_n$, let $\mathscr{S} + r = \{E + r : E \in \mathscr{S}\}$. By *clicking* $\mathscr{S}$, we shall mean replacing $\mathscr{S}$ with $\mathscr{S} + 1$.

Let $\mathscr{H}$ be a subhypergraph of $K_n^{(t)}$, where $V(K_n^{(t)}) = \mathbb{Z}_n$ and let Γ be a $\mathscr{H}$-decomposition of $K_n^{(t)}$. Then Γ is said to be *cyclic* if Γ is closed under clicking. Thus if $\mathscr{H}_i \in \Gamma$, then $\mathscr{H}_i + 1 \in \Gamma$. If we partition $\mathscr{E}_n^{(t)}$ into k distinct $\mathbb{Z}_n$-orbits each of size n and if $\mathscr{H}$ is a subhypergraph of $K_n^{(t)}$ consisting of one edge from each k distinct $\mathbb{Z}_n$-orbits, then $\Gamma = \{\mathscr{H} + i : i \in \mathbb{Z}_n\}$ is a *cyclic $\mathscr{H}$*-decomposition of $K_n^{(t)}$.

Petecki [12], showed that $K_n^{(t)}$ admits a cyclic Hamilton cycle decomposition if and only if $g.c.d.(n, t) = 1$ and $\lambda = \min\{d > 1 : d \,|\, n\} > \frac{n}{t}$.

Jordon et al. [8] proved that the necessary conditions are sufficient for the existence of a $BC_4^{(3)}$-decomposition of $K_n^{(3)}$. In [10, 11], Lakshmi and Poovaragavan proved that the necessary conditions are sufficient for the existence of a $BC_6^{(3)}$-decomposition of $K_n^{(3)}$ and for the existence of a $BC_p^{(3)}$-decomposition of $K_n^{(3)}$, for $p \geq 5$ is prime.

In [4], Bryant et al. proved that there exists an $LC_3^{(3)}$-decomposition of $K_n^{(3)}$ if and only if $n \equiv 0, 1$ or $2 \pmod 9$. Bunge et al. [5] shown that there exists an $LC_4^{(3)}$-decomposition of $K_n^{(3)}$ if and only if $n \equiv 0, 1, 2, 4$ or $6 \pmod 8$ and $n \notin \{4, 6\}$. In [6], Bunge et al. studied $LC_5^{(3)}$ decompositions of $K_n^{(3)}$. In [14], we have shown that $LC_7^{(3)}|K_n^{(3)}$ if and only if $n \geq 14$ and $n \equiv 0, 1$ or $2 \pmod 7$.

In this paper, we prove:

Theorem 1.1. $LC_6^{(3)}|K_n^{(3)}$ *if and only if* $n \geq 12$ *and* $n \equiv 0, 1, 2, 9, 10, 18, 20, 28,$ *or* $29 \pmod{36}$.

For convenience, in $K_n^{(3)}$, we denote the loose 6-cycle
$[(i, i+j_1, i+j_2), (i+j_2, i+j_3, i+j_4), (i+j_4, i+j_5, i+j_6), (i+j_6, i+j_7, i+j_8), (i+j_8, i+j_9, i+j_{10}), (i+j_{10}, i+j_{11}, i)]$
by
$i + [(0, j_1, j_2), (j_2, j_3, j_4), (j_4, j_5, j_6), (j_6, j_7, j_8), (j_8, j_9, j_{10}), (j_{10}, j_{11}, 0)]$.

Graphs K_n, C_n, P_n and $K_{m,n}$, respectively, denote the complete graph with n vertices, the cycle with n ($n \geq 3$) vertices, the path with n vertices and the complete bipartite graph with partite sizes m and n.

2. Loose 6-cycle decompositions of complete 3-uniform hypergraphs of small order

2.1. *Difference technique*

Following 'difference technique' method was introduced by Gionfriddo et al. [7]. Assume that the vertices of $K_n^{(3)}$ are $0, 1, \ldots, n-1$ and that they are arranged in a cyclic order. The distance between vertices i and j is defined to be

$$||i - j|| = min\{|i - j|, n - |i - j|\}.$$

Using this, define a difference triplet

$$t_{i,j,k} = (||i - j||, ||j - k||, ||k - i||)$$

to any three vertices i, j, k with $0 \leq i < j < k \leq n - 1$.

Note that the ordering condition $i < j < k$ is important in the definition. By taking $t_{j,k,i} = (||j - k||, ||k - i||, ||i - j||)$ and $t_{k,i,j} = (||k - i||, ||i - j||, ||j - k||)$, we assume that $t_{i,j,k} = t_{j,k,i} = t_{k,i,j}$ for all choices of $\{i, j, k\}$. Moreover, difference triplets are rotation-invariant, i.e. $t_{i,j,k} = t_{i+1,j+1,k+1}$ holds for all $\{i, j, k\}$.

From [7], we have: if n is not a multiple of 3, then there can occur two kinds of difference triplets:
• *symmetric triplets*: of the form (a, a, b), where $2a = b$ or $2a + b = n$, and
• *reflected triplets*: of the form (a, b, c) or (a, c, b), where $a + b = c$ or $a + b + c = n$, and $a \neq b \neq c \neq a$.
If n is a multiple of 3, then we have an additional triplet $(\frac{n}{3}, \frac{n}{3}, \frac{n}{3})$.

In what follows, the decompositions are obtained by using the method of difference triplets; in particular, when 6^2 divides $(n-1)(n-2)$, the decompositions are cyclic.

2.2. 6^2 *divides* $(n - 1)(n - 2)$

Lemma 2.1. $LC_6^{(3)}|K_{29}^{(3)}$.

Proof. Let $V(K_{29}^{(3)}) = \mathbb{Z}_{29}$. Following $LC_6^{(3)}$'s decompose $K_{29}^{(3)}$:
$i + [(13, 12, 11), (11, 9, 8), (8, 5, 4), (4, 28, 0), (0, 1, 6), (6, 7, 13)]$
(edges having difference triplets (1, 1, 2), (1, 2, 3), (1, 3, 4), (1, 4, 5), (1, 5, 6), (1, 6, 7), respectively),
$i + [(2, 1, 9), (9, 10, 18), (18, 17, 27), (27, 8, 26), (26, 14, 15), (15, 3, 2)]$
(edges having difference triplets (1, 7, 8), (1, 8, 9), (1, 9, 10), (1, 10, 11), (1, 11, 12), (1, 12, 13), respectively),
$i + [(15, 1, 2), (2, 17, 3), (3, 4, 19), (19, 18, 6), (6, 23, 5), (5, 16, 15)]$

(edges having difference triplets (1, 13, 14), (14, 14, 1), (13, 1, 14), (12, 1, 13), (11, 1, 12), (10, 1, 11), respectively),
$i + [(12, 11, 2), (2, 23, 3), (3, 4, 25), (25, 26, 19), (19, 18, 13), (13, 8, 12)]$
(edges having difference triplets (9, 1, 10), (8, 1, 9), (7, 1, 8), (6, 1, 7), (5, 1, 6), (4, 1, 5), respectively),
$i + [(1, 2, 27), (27, 26, 24), (24, 20, 22), (22, 17, 19), (19, 21, 25), (25, 23, 1)]$
(edges having difference triplets (3, 1, 4), (2, 1, 3), (2, 2, 4), (2, 3, 5), (2, 4, 6), (2, 5, 7), respectively),
$i + [(1, 3, 9), (9, 7, 16), (16, 6, 8), (8, 19, 10), (10, 22, 12), (12, 28, 1)]$
(edges having difference triplets (2, 6, 8), (2, 7, 9), (2, 8, 10), (2, 9, 11), (2, 10, 12), (2, 11, 13), respectively),
$i + [(22, 20, 5), (5, 3, 18), (18, 2, 4), (4, 21, 6), (6, 8, 24), (24, 12, 22)]$
(edges having difference triplets (2, 12, 14), (2, 13, 14), (2, 14, 13), (12, 2, 14), (11, 2, 13), (10, 2, 12), respectively),
$i + [(13, 11, 2), (2, 4, 23), (23, 1, 3), (3, 5, 26), (26, 24, 19), (19, 17, 13)]$
(edges having difference triplets (9, 2, 11), (8, 2, 10), (7, 2, 9), (6, 2, 8), (5, 2, 7), (4, 2, 6), respectively),
$i + [(28, 25, 1), (1, 4, 7), (7, 0, 3), (3, 6, 11), (11, 5, 2), (2, 9, 28)]$
(edges having difference triplets (3, 2, 5), (3, 3, 6), (3, 4, 7), (3, 5, 8), (3, 6, 9), (3, 7, 10), respectively),
$i + [(15, 7, 4), (4, 1, 13), (13, 0, 3), (3, 6, 17), (17, 5, 2), (2, 28, 15)]$
(edges having difference triplets (3, 8, 11), (3, 9, 12), (3, 10, 13), (3, 11, 14), (3, 12, 14), (13, 13, 3), respectively),
$i + [(17, 0, 3), (3, 21, 6), (6, 9, 25), (25, 16, 28), (28, 2, 20), (20, 10, 17)]$
(edges having difference triplets (3, 14, 12), (11, 3, 14), (10, 3, 13), (9, 3, 12), (8, 3, 11), (7, 3, 10), respectively),
$i + [(0, 3, 23), (23, 26, 18), (18, 14, 21), (21, 17, 13), (13, 8, 4), (4, 10, 0)]$
(edges having difference triplets (6, 3, 9), (5, 3, 8), (4, 3, 7), (4, 4, 8), (4, 5, 9), (4, 6, 10), respectively),
$i + [(0, 4, 11), (11, 7, 19), (19, 10, 6), (6, 2, 16), (16, 27, 12), (12, 25, 0)]$
(edges having difference triplets (4, 7, 11), (4, 8, 12), (4, 9, 13), (4, 10, 14), (4, 11, 14), (4, 12, 13), respectively),
$i + [(17, 0, 4), (4, 8, 22), (22, 7, 3), (3, 16, 12), (12, 20, 24), (24, 28, 17)]$
(edges having difference triplets (4, 13, 12), (4, 14, 11), (10, 4, 14), (9, 4, 13), (8, 4, 12), (7, 4, 11), respectively),
$i + [(15, 11, 5), (5, 0, 9), (9, 4, 14), (14, 25, 19), (19, 12, 7), (7, 2, 15)]$
(edges having difference triplets (6, 4, 10), (5, 4, 9), (5, 5, 10), (5, 6, 11), (5, 7, 12), (5, 8, 13), respectively),
$i + [(26, 2, 11), (11, 6, 21), (21, 10, 5), (5, 0, 17), (17, 1, 12), (12, 7, 26)]$
(edges having difference triplets (5, 9, 14), (5, 10, 14), (5, 11, 13), (12, 12, 5), (5, 13, 11), (5, 14, 10), respectively),
$i + [(14, 28, 23), (23, 18, 10), (10, 17, 22), (22, 16, 27), (27, 15, 21), (21, 8, 14)]$
(edges having difference triplets (9, 5, 14), (8, 5, 13), (7, 5, 12), (6, 5, 11), (6, 6, 12), (6, 7, 13), respectively),
$i + [(1, 15, 7), (7, 22, 13), (13, 19, 0), (0, 17, 6), (6, 12, 24), (24, 14, 1)]$
(edges having difference triplets (6, 8, 14), (6, 9, 14), (6, 10, 13), (6, 11, 12), (6, 12, 11), (6, 13, 10), respectively),
$i + [(15, 21, 6), (6, 27, 12), (12, 5, 18), (18, 4, 11), (11, 3, 25), (25, 8, 15)]$
(edges having difference triplets (6, 14, 9), (8, 6, 14), (7, 6, 13), (7, 7, 14), (7, 8, 14), (7, 10, 12), respectively),
$i + [(21, 28, 8), (8, 15, 26), (26, 14, 7), (7, 0, 20), (20, 27, 12), (12, 4, 21)]$
(edges having difference triplets (7, 9, 13), (7, 11, 11), (7, 12, 10), (7, 13, 9), (7, 14, 8), (8, 9, 12), respectively),
$i + [(0, 8, 16), (16, 24, 5), (5, 15, 23), (23, 3, 11), (11, 2, 20), (20, 10, 0)]$
(edges having difference triplets (8, 8, 13), (8, 10, 11), (8, 11, 10), (8, 12, 9), (9, 9, 11), (10, 10, 9), respectively),
where $i \in \mathbb{Z}_{29}$. □

Lemma 2.2. $LC_6^{(3)} | K_{37}^{(3)}$.

Proof. Let $V(K_{37}^{(3)}) = \mathbb{Z}_{37}$. Following $LC_6^{(3)}$'s decompose $K_{37}^{(3)}$:
$i + [(11, 12, 13), (13, 14, 16), (16, 20, 17), (17, 22, 18), (18, 24, 19), (19, 10, 11)]$
(edges having difference triplets (1, 1, 2), (1, 2, 3), (1, 3, 4), (1, 4, 5), (1, 5, 6), (1, 8, 9), respectively),
$i + [(18, 11, 12), (12, 4, 5), (5, 15, 6), (6, 17, 7), (7, 8, 19), (19, 31, 18)]$
(edges having difference triplets (1, 6, 7), (1, 7, 8), (1, 9, 10), (1, 10, 11), (1, 11, 12), (1, 12, 13), respectively),
$i + [(11, 30, 12), (12, 32, 13), (13, 34, 14), (14, 15, 36), (36, 0, 23), (23, 24, 11)]$
(edges having difference triplets (1, 18, 18), (17, 1, 18), (16, 1, 17), (15, 1, 16), (13, 1, 14), (12, 1, 13), respectively),
$i + [(11, 12, 0), (0, 27, 1), (1, 29, 2), (2, 31, 3), (3, 33, 4), (4, 10, 11)]$
(edges having difference triplets (11, 1, 12), (10, 1, 11), (9, 1, 10), (8, 1, 9), (7, 1, 8), (6, 1, 7), respectively),
$i + [(0, 32, 1), (1, 34, 2), (2, 36, 3), (3, 5, 6), (6, 8, 4), (4, 35, 0)]$
(edges having difference triplets (5, 1, 6), (4, 1, 5), (3, 1, 4), (2, 1, 3), (2, 2, 4), (2, 4, 6), respectively),

$i + [(12, 10, 15), (15, 13, 20), (20, 28, 22), (22, 31, 24), (24, 16, 14), (14, 23, 12)]$
(edges having difference triplets $(2, 3, 5)$, $(2, 5, 7)$, $(2, 6, 8)$, $(2, 7, 9)$, $(2, 8, 10)$, $(2, 9, 11)$, respectively),
$i + [(22, 10, 12), (12, 25, 14), (14, 28, 16), (16, 31, 18), (18, 34, 20), (20, 0, 22)]$
(edges having difference triplets $(2, 10, 12)$, $(2, 11, 13)$, $(2, 12, 14)$, $(2, 13, 15)$, $(2, 14, 16)$, $(2, 15, 17)$, respectively),
$i + [(15, 13, 31), (31, 12, 14), (14, 34, 16), (16, 18, 0), (0, 22, 2), (2, 17, 15)]$
(edges having difference triplets $(2, 16, 18)$, $(2, 17, 18)$, $(2, 18, 17)$, $(16, 2, 18)$, $(15, 2, 17)$, $(13, 2, 15)$, respectively),
$i + [(26, 28, 12), (12, 0, 14), (14, 3, 16), (16, 18, 6), (6, 15, 17), (17, 24, 26)]$
(edges having difference triplets $(14, 2, 16)$, $(12, 2, 14)$, $(11, 2, 13)$, $(10, 2, 12)$, $(9, 2, 11)$, $(7, 2, 9)$, respectively),
$i + [(29, 0, 2), (2, 33, 4), (4, 6, 36), (36, 5, 3), (3, 1, 35), (35, 32, 29)]$
(edges having difference triplets $(8, 2, 10)$, $(6, 2, 8)$, $(5, 2, 7)$, $(4, 2, 6)$, $(3, 2, 5)$, $(3, 3, 6)$, respectively),
$i + [(3, 0, 7), (7, 10, 15), (15, 18, 24), (24, 27, 34), (34, 5, 31), (31, 28, 3)]$
(edges having difference triplets $(3, 4, 7)$, $(3, 5, 8)$, $(3, 6, 9)$, $(3, 7, 10)$, $(3, 8, 11)$, $(3, 9, 12)$, respectively),
$i + [(30, 27, 3), (3, 17, 6), (6, 21, 9), (9, 12, 25), (25, 8, 11), (11, 33, 30)]$
(edges having difference triplets $(3, 10, 13)$, $(3, 11, 14)$, $(3, 12, 15)$, $(3, 13, 16)$, $(3, 14, 17)$, $(3, 15, 18)$, respectively),
$i + [(2, 20, 23), (23, 3, 6), (6, 27, 9), (9, 31, 12), (12, 29, 26), (26, 5, 2)]$
(edges having difference triplets $(3, 16, 18)$, $(3, 17, 17)$, $(3, 18, 16)$, $(15, 3, 18)$, $(14, 3, 17)$, $(13, 3, 16)$, respectively),
$i + [(25, 0, 3), (3, 29, 6), (6, 9, 33), (33, 36, 24), (24, 27, 16), (16, 22, 25)]$
(edges having difference triplets $(12, 3, 15)$, $(11, 3, 14)$, $(10, 3, 13)$, $(9, 3, 12)$, $(8, 3, 11)$, $(6, 3, 9)$, respectively),
$i + [(27, 20, 30), (30, 35, 1), (1, 34, 4), (4, 8, 12), (12, 16, 21), (21, 17, 27)]$
(edges having difference triplets $(7, 3, 10)$, $(5, 3, 8)$, $(4, 3, 7)$, $(4, 4, 8)$, $(4, 5, 9)$, $(4, 6, 10)$, respectively),
$i + [(18, 7, 11), (11, 15, 23), (23, 27, 36), (36, 3, 13), (13, 35, 2), (2, 6, 18)]$
(edges having difference triplets $(4, 7, 11)$, $(4, 8, 12)$, $(4, 9, 13)$, $(4, 10, 14)$, $(4, 11, 15)$, $(4, 12, 16)$, respectively),
$i + [(17, 0, 4), (4, 8, 22), (22, 3, 7), (7, 27, 11), (11, 15, 32), (32, 36, 17)]$
(edges having difference triplets $(4, 13, 17)$, $(4, 14, 18)$, $(4, 15, 18)$, $(4, 16, 17)$, $(4, 17, 16)$, $(4, 18, 15)$, respectively),
$i + [(5, 19, 23), (23, 36, 3), (3, 7, 28), (28, 2, 6), (6, 10, 33), (33, 9, 5)]$
(edges having difference triplets $(14, 4, 18)$, $(13, 4, 17)$, $(12, 4, 16)$, $(11, 4, 15)$, $(10, 4, 14)$, $(9, 4, 13)$, respectively),
$i + [(0, 4, 29), (29, 36, 3), (3, 34, 7), (7, 2, 11), (11, 16, 6), (6, 32, 0)]$
(edges having difference triplets $(8, 4, 12)$, $(7, 4, 11)$, $(6, 4, 10)$, $(5, 4, 9)$, $(5, 5, 10)$, $(5, 6, 11)$, respectively),
$i + [(19, 7, 12), (12, 17, 25), (25, 30, 2), (2, 24, 29), (29, 3, 24), (24, 36, 19)]$
(edges having difference triplets $(5, 7, 12)$, $(5, 8, 13)$, $(5, 9, 14)$, $(5, 10, 15)$, $(5, 11, 16)$, $(5, 12, 17)$, respectively),
$i + [(26, 8, 13), (13, 18, 32), (32, 12, 17), (17, 22, 1), (1, 16, 21), (21, 7, 26)]$
(edges having difference triplets $(5, 13, 18)$, $(5, 14, 18)$, $(5, 15, 17)$, $(5, 16, 16)$, $(5, 17, 15)$, $(5, 18, 14)$, respectively),
$i + [(24, 0, 5), (5, 30, 10), (10, 15, 36), (36, 31, 21), (21, 7, 16), (16, 29, 24)]$
(edges having difference triplets $(13, 5, 18)$, $(12, 5, 17)$, $(11, 5, 16)$, $(10, 5, 15)$, $(9, 5, 14)$, $(8, 5, 13)$, respectively),
$i + [(0, 30, 5), (5, 36, 10), (10, 4, 16), (16, 3, 9), (9, 23, 15), (15, 6, 0)]$
(edges having difference triplets $(7, 5, 12)$, $(6, 5, 11)$, $(6, 6, 12)$, $(6, 7, 13)$, $(6, 8, 14)$, $(6, 9, 15)$, respectively),
$i + [(7, 13, 23), (23, 6, 12), (12, 30, 18), (18, 36, 5), (5, 28, 22), (22, 1, 7)]$
(edges having difference triplets $(6, 10, 16)$, $(6, 11, 17)$, $(6, 12, 18)$, $(6, 13, 18)$, $(6, 14, 17)$, $(6, 15, 16)$, respectively),
$i + [(21, 27, 6), (6, 12, 29), (29, 23, 10), (10, 16, 35), (35, 9, 15), (15, 5, 21)]$
(edges having difference triplets $(6, 16, 15)$, $(6, 17, 14)$, $(6, 18, 13)$, $(12, 6, 18)$, $(11, 6, 17)$, $(10, 6, 16)$, respectively),
$i + [(14, 23, 29), (29, 0, 6), (6, 19, 13), (13, 20, 27), (27, 34, 5), (5, 35, 14)]$
(edges having difference triplets $(9, 6, 15)$, $(8, 6, 14)$, $(7, 6, 13)$, $(7, 7, 14)$, $(7, 8, 15)$, $(7, 9, 16)$, respectively),
$i + [(3, 10, 20), (20, 27, 1), (1, 19, 26), (26, 33, 9), (9, 25, 32), (25, 18, 3)]$
(edges having difference triplets $(7, 10, 17)$, $(7, 11, 18)$, $(7, 12, 18)$, $(7, 13, 17)$, $(7, 14, 16)$, $(7, 15, 15)$, respectively),
$i + [(17, 10, 33), (33, 3, 20), (20, 8, 27), (27, 34, 16), (16, 36, 9), (9, 25, 17)]$
(edges having difference triplets $(7, 16, 14)$, $(7, 17, 13)$, $(7, 18, 12)$, $(11, 7, 18)$, $(10, 7, 17)$, $(8, 8, 16)$, respectively),
$i + [(7, 35, 14), (14, 29, 22), (22, 30, 2), (2, 20, 10), (10, 28, 36), (36, 19, 7)]$
(edges having difference triplets $(9, 7, 16)$, $(8, 7, 15)$, $(8, 9, 17)$, $(8, 10, 18)$, $(8, 11, 18)$, $(8, 12, 17)$, respectively),
$i + [(18, 10, 31), (31, 9, 17), (17, 25, 3), (3, 11, 27), (27, 19, 7), (7, 26, 18)]$
(edges having difference triplets $(8, 13, 16)$, $(8, 14, 15)$, $(8, 15, 14)$, $(8, 16, 13)$, $(8, 17, 12)$, $(8, 18, 11)$, respectively),
$i + [(1, 30, 20), (20, 29, 0), (0, 9, 18), (18, 36, 8), (8, 28, 17), (17, 26, 1)]$
(edges having difference triplets $(10, 8, 18)$, $(9, 8, 17)$, $(9, 9, 18)$, $(9, 10, 18)$, $(9, 11, 17)$, $(9, 12, 16)$, respectively),

$i + [(14, 36, 23), (23, 32, 9), (9, 18, 33), (33, 5, 21), (21, 32, 4), (4, 24, 14)]$
(edges having difference triplets $(9, 13, 15)$, $(9, 14, 14)$, $(9, 15, 13)$, $(9, 16, 12)$, $(9, 17, 11)$, $(17, 10, 10)$, respectively),
$i + [(10, 19, 0), (0, 27, 11), (11, 1, 23), (23, 13, 36), (36, 12, 22), (22, 32, 10)]$
(edges having difference triplets $(10, 9, 18)$, $(10, 11, 16)$, $(10, 12, 15)$, $(10, 13, 14)$, $(10, 14, 13)$, $(12, 10, 15)$, respectively),
$i + [(0, 10, 26), (26, 4, 15), (15, 29, 3), (3, 27, 14), (14, 2, 25), (25, 13, 0)]$
(edges having difference triplets $(10, 16, 11)$, $(15, 11, 11)$, $(11, 12, 14)$, $(11, 13, 13)$, $(12, 11, 14)$, $(13, 12, 12)$, respectively),
$i + [(0, 23, 24), (24, 25, 2), (2, 3, 18), (18, 17, 34), (34, 33, 14), (14, 15, 0)]$
(edges having difference triplets $(1, 13, 14)$, $(1, 14, 15)$, $(1, 15, 16)$, $(1, 16, 17)$, $(1, 17, 18)$, $(14, 1, 15)$, respectively), where $i \in \mathbb{Z}_{37}$. □

2.3. 6^2 does not divide $(n-1)(n-2)$

Lemma 2.3. $LC_6^{(3)} | K_{18}^{(3)}$.

Proof. Let $V(K_{18}^{(3)}) = \mathbb{Z}_{18}$. Following $LC_6^{(3)}$'s decompose $K_{18}^{(3)}$:
For each $i \in \mathbb{Z}_{18}$, consider
$i + [(0, 1, 2), (2, 3, 5), (5, 6, 9), (9, 14, 10), (10, 16, 11), (11, 12, 0)]$
(edges having difference triplets $(1, 1, 2)$, $(1, 2, 3)$, $(1, 3, 4)$, $(1, 4, 5)$, $(1, 5, 6)$, $(1, 6, 7)$, respectively),
$i + [(1, 3, 5), (5, 7, 10), (10, 6, 4), (4, 2, 9), (9, 11, 17), (17, 8, 1)]$
(edges having difference triplets $(2, 2, 4)$, $(2, 3, 5)$, $(2, 4, 6)$, $(2, 5, 7)$, $(2, 6, 8)$, $(2, 7, 9)$, respectively),
$i + [(10, 0, 2), (2, 4, 13), (13, 3, 1), (1, 6, 8), (8, 14, 12), (12, 7, 10)]$
(edges having difference triplets $(8, 8, 2)$, $(7, 2, 9)$, $(6, 2, 8)$, $(5, 2, 7)$, $(4, 2, 6)$, $(3, 2, 5)$, respectively),
$i + [(15, 12, 9), (9, 6, 13), (13, 8, 5), (5, 2, 11), (11, 1, 4), (4, 7, 15)]$,
(edges having difference triplets $(3, 3, 6)$, $(3, 4, 7)$, $(3, 5, 8)$, $(3, 6, 9)$, $(3, 7, 8)$, $(3, 8, 7)$, respectively),
$i + [(0, 12, 3), (3, 6, 16), (16, 2, 5), (5, 9, 1), (1, 14, 10), (10, 4, 0)]$
(edges having difference triplets $(3, 9, 6)$, $(5, 3, 8)$, $(4, 3, 7)$, $(4, 4, 8)$, $(4, 5, 9)$, $(4, 6, 8)$, respectively),
$i + [(1, 15, 8), (8, 4, 16), (16, 3, 7), (7, 12, 17), (17, 11, 6), (6, 13, 1)]$
(edges having difference triplets $(7, 7, 4)$, $(4, 8, 6)$, $(4, 9, 5)$, $(5, 5, 8)$, $(5, 6, 7)$, $(5, 7, 6)$, respectively).
Remaining triplets are: $(6, 6, 6)$, $(1, 7, 8)$, $(1, 8, 9)$, $(1, 9, 8)$, $(2, 1, 3)$, $(3, 1, 4)$, $(4, 1, 5)$, $(5, 1, 6)$, $(6, 1, 7)$ and $(7, 1, 8)$.
For each $j \in \mathbb{Z}_{18} \setminus \{17\}$, consider
$j + [(14, 4, 5), (5, 16, 6), (6, 7, 0), (0, 17, 12), (12, 8, 11), (11, 13, 14)]$
(edges having difference triplets $(1, 9, 8)$, $(7, 1, 8)$, $(6, 1, 7)$, $(5, 1, 6)$, $(3, 1, 4)$, $(2, 1, 3)$, respectively).
Finally, consider the following $LC_6^{(3)}$'s:
$[(0, 8, 1), (1, 9, 2), (2, 10, 3), (3, 11, 4), (4, 5, 12), (12, 6, 0)]$
(edge $(12, 6, 0)$ is of triplet $(6, 6, 6)$ and the remaining 5 edges are $(1, 7, 8)$),
$[(13, 5, 6), (6, 14, 7), (7, 15, 8), (8, 4, 9), (9, 10, 0), (0, 17, 13)]$
(edge $(9, 10, 0)$ is of triplet $(1, 8, 9)$, two edges $(8, 4, 9)$ and $(0, 17, 13)$ are $(4, 1, 5)$ and the remaining 3 edges are $(1, 7, 8)$),
$[(3, 9, 15), (15, 5, 16), (16, 6, 17), (17, 0, 7), (7, 1, 13), (13, 14, 3)]$
(edges $(3, 9, 15)$ and $(7, 1, 13)$ are of triplet $(6, 6, 6)$ and the remaining 4 edges are $(1, 7, 8)$),
$[(2, 6, 7), (7, 16, 8), (8, 17, 9), (9, 5, 10), (10, 1, 11), (11, 12, 2)]$
(edges $(2, 6, 7)$ and $(9, 5, 10)$ are of triplet $(4, 1, 5)$ and the remaining 4 edges are $(1, 8, 9)$),
$[(9, 13, 14), (14, 10, 15), (15, 11, 16), (16, 12, 17), (17, 8, 0), (0, 1, 9)]$
(edges $(17, 8, 0)$ and $(0, 1, 9)$ are of triplet $(1, 8, 9)$ and the remaining 4 edges are $(4, 1, 5)$),
$[(3, 12, 13), (13, 4, 14), (14, 5, 15), (15, 6, 16), (16, 17, 7), (7, 8, 3)]$
(edge $(7, 8, 3)$ is of triplet $(4, 1, 5)$ and the remaining 5 edges are $(1, 8, 9)$),
$[(7, 6, 15), (15, 1, 2), (2, 16, 3), (3, 4, 17), (17, 5, 11), (11, 12, 7)]$
(edge $(17, 5, 11)$ is of triplet $(6, 6, 6)$ and the remaining 5 edges are $(4, 1, 5)$),

[(13, 12, 8), (8, 16, 9), (9, 17, 10), (10, 11, 6), (6, 1, 5), (5, 4, 13)]
(edge (5, 4, 13) is of triplet (1, 8, 9), two edges (8, 16, 9) and (9, 17, 10) are (1, 7, 8) and the remaining 3 edges are (4, 1, 5)),
[(14, 0, 1), (1, 10, 2), (2, 11, 3), (3, 12, 4), (4, 15, 5), (5, 6, 14)]
(edge (14, 0, 1) is of triplet (4, 1, 5), edge (4, 15, 5) is (7, 1, 8) and the remaining 4 edges are (1, 8, 9)),
[(13, 3, 4), (4, 0, 5), (5, 6, 17), (17, 16, 11), (11, 7, 10), (10, 12, 13)]
(edges having difference triplets (1, 9, 8), (4, 1, 5), (6, 1, 7), (5, 1, 6), (3, 1, 4), (2, 1, 3), respectively),
[(10, 0, 11), (11, 1, 12), (12, 13, 2), (2, 8, 14), (14, 15, 4), (4, 16, 10)]
(edges (2, 8, 14) and (4, 16, 10) are of triplet (6, 6, 6) and the remaining 4 edges are (1, 7, 8)). □

Lemma 2.4. $LC_6^{(3)}|K_{20}^{(3)}$.

Proof. Let $V(K_{20}^{(3)}) = \mathbb{Z}_{20}$. Following $LC_6^{(3)}$'s decompose $K_{20}^{(3)}$:
For each $i \in \mathbb{Z}_{20}$, consider
$i + [(0, 1, 2), (2, 3, 5), (5, 6, 9), (9, 10, 14), (14, 8, 7), (7, 19, 0)]$
(edges having difference triplets (1, 1, 2), (1, 2, 3), (1, 3, 4), (1, 4, 5), (1, 6, 7), (1, 7, 8), respectively),
$i + [(1, 0, 9), (9, 8, 19), (19, 18, 10), (10, 11, 3), (3, 16, 2), (2, 17, 1)]$
(edges having difference triplets (1, 8, 9), (1, 10, 9), (8, 1, 9), (7, 1, 8), (6, 1, 7), (4, 1, 5), respectively),
$i + [(17, 0, 1), (1, 3, 4), (4, 6, 8), (8, 10, 13), (13, 19, 15), (15, 2, 17)]$
(edges having difference triplets (3, 1, 4), (2, 1, 3), (2, 2, 4), (2, 3, 5), (2, 4, 6), (2, 5, 7), respectively),
$i + [(2, 0, 8), (8, 10, 17), (17, 9, 7), (7, 18, 16), (16, 6, 4), (4, 15, 2)]$
(edges having difference triplets (2, 6, 8), (2, 7, 9), (2, 8, 10), (2, 9, 9), (2, 10, 8), (7, 2, 9) respectively),
$i + [(15, 9, 17), (17, 2, 4), (4, 10, 8), (8, 11, 13), (13, 16, 19), (19, 12, 15)]$
(edges having difference triplets (6, 2, 8), (5, 2, 7), (4, 2, 6), (3, 2, 5), (3, 3, 6), (3, 4, 7), respectively),
$i + [(0, 3, 8), (8, 5, 14), (14, 7, 4), (4, 16, 13), (13, 2, 10), (10, 17, 0)]$
(edges having difference triplets (3, 5, 8), (3, 6, 9), (3, 7, 10), (3, 8, 9), (3, 9, 8), (3, 10, 7), respectively),
$i + [(0, 14, 3), (3, 18, 6), (6, 2, 9), (9, 13, 5), (5, 1, 10), (10, 4, 0)]$
(edges having difference triplets (6, 3, 9), (5, 3, 8), (4, 3, 7), (4, 4, 8), (4, 5, 9), (4, 6, 10), respectively),
$i + [(11, 0, 4), (4, 8, 16), (16, 3, 7), (7, 13, 17), (17, 2, 6), (6, 1, 11)]$
(edges having difference triplets (4, 7, 9), (4, 8, 8), (4, 9, 7), (4, 10, 6), (5, 4, 9), (5, 5, 10), respectively),
$i + [(0, 11, 5), (5, 10, 17), (17, 9, 4), (4, 19, 13), (13, 1, 7), (7, 14, 0)]$
(edges having difference triplets (5, 6, 9), (5, 7, 8), (5, 8, 7), (5, 9, 6), (6, 6, 8), (6, 7, 7), respectively).
Remaining triplets are: (1, 5, 6), (1, 9, 10) and (5, 1, 6). We decompose 60 edges of these 3 triplets into 10 $LC_6^{(3)}$'s.
[(8, 2, 3), (3, 9, 4), (4, 10, 5), (5, 11, 6), (6, 12, 7), (7, 13, 8)]
(edges having difference triplet (1, 5, 6)),
[(0, 14, 15), (15, 1, 16), (16, 2, 17), (17, 3, 18), (18, 4, 19), (19, 5, 0)]
(edges having difference triplet (1, 5, 6)),
[(6, 1, 7), (7, 2, 8), (8, 3, 9), (9, 4, 10), (10, 5, 11), (11, 12, 6)]
(edges having difference triplet (5, 1, 6)),
[(14, 8, 9), (9, 19, 10), (10, 16, 11), (11, 17, 12), (12, 18, 13), (13, 3, 14)]
(edges having difference triplets (1, 5, 6), (1, 9, 10)),
[(15, 9, 10), (10, 0, 11), (11, 1, 12), (12, 2, 13), (13, 19, 14), (14, 4, 15)]
(edges having difference triplets (1, 5, 6), (1, 9, 10)),
[(0, 15, 1), (1, 11, 2), (2, 17, 3), (3, 18, 4), (4, 19, 5), (5, 6, 0)]
(edges having difference triplets (1, 9, 10), (5, 1, 6)),
[(12, 7, 13), (13, 8, 14), (14, 9, 15), (15, 5, 16), (16, 11, 17), (17, 18, 12)]
(edges having difference triplets (1, 9, 10), (5, 1, 6)),
[(16, 1, 2), (2, 12, 3), (3, 13, 4), (4, 14, 5), (5, 15, 6), (6, 7, 16)]
(edges having difference triplets (1, 9, 10), (5, 1, 6)),
[(10, 15, 16), (16, 6, 17), (17, 7, 18), (18, 8, 19), (19, 9, 0), (0, 1, 10)]
(edges having difference triplets (1, 9, 10), (5, 1, 6)),

$[(18,13,19),(19,14,0),(0,6,1),(1,2,7),(7,17,8),(8,9,18)]$
(edges having difference triplets $(1,5,6)$, $(1,9,10)$, $(5,1,6)$). □

Lemma 2.5. $LC_6^{(3)}|K_{28}^{(3)}$.

Proof. Let $V(K_{28}^{(3)}) = \mathbb{Z}_{28}$. Following $LC_6^{(3)}$'s decompose $K_{28}^{(3)}$:
For each $i \in \mathbb{Z}_{28}$, consider
$i+[(1,0,2),(2,3,5),(5,6,9),(9,10,14),(14,15,21),(21,22,1)]$
(edges having difference triplets $(1,1,2)$, $(1,2,3)$, $(1,3,4)$, $(1,4,5)$, $(1,6,7)$, $(1,7,8)$, respectively),
$i+[(1,0,9),(9,18,8),(8,26,25),(25,12,13),(13,27,14),(14,15,1)]$
(edges having difference triplets $(1,8,9)$, $(1,9,10)$, $(1,10,11)$, $(1,12,13)$, $(1,13,14)$, $(1,14,13)$, respectively),
$i+[(6,19,18),(18,1,2),(2,20,3),(3,4,22),(22,21,13),(13,14,6)]$
(edges having difference triplets $(12,1,13)$, $(11,1,12)$, $(10,1,11)$, $(9,1,10)$, $(8,1,9)$, $(7,1,8)$, respectively),
$i+[(7,0,6),(6,5,1),(1,2,26),(26,25,23),(23,21,19),(19,9,7)]$
(edges having difference triplets $(6,1,7)$, $(4,1,5)$, $(3,1,4)$, $(2,1,3)$, $(2,2,4)$, $(2,10,12)$, respectively),
$i+[(4,2,7),(7,5,11),(11,13,18),(18,24,16),(16,23,14),(14,6,4)]$
(edges having difference triplets $(2,3,5)$, $(2,4,6)$, $(2,5,7)$, $(2,6,8)$, $(2,7,9)$, $(2,8,10)$, respectively),
$i+[(23,21,4),(4,2,15),(15,3,1),(1,16,14),(14,0,12),(12,25,23)]$
(edges having difference triplets $(2,9,11)$, $(2,11,13)$, $(2,12,14)$, $(13,13,2)$, $(2,14,12)$, $(11,2,13)$, respectively),
$i+[(11,13,1),(1,20,3),(3,23,5),(5,7,26),(26,6,4),(4,9,11)]$
(edges having difference triplets $(10,2,12)$, $(9,2,11)$, $(8,2,10)$, $(7,2,9)$, $(6,2,8)$, $(5,2,7)$, respectively),
$i+[(0,24,2),(2,27,4),(4,7,10),(10,13,17),(17,20,25),(25,6,0)]$
(edges having difference triplets $(4,2,6)$, $(3,2,5)$, $(3,3,6)$, $(3,4,7)$, $(3,5,8)$, $(3,6,9)$, respectively),
$i+[(10,0,3),(3,14,6),(6,18,9),(9,22,12),(12,15,26),(26,23,10)]$
(edges having difference triplets $(3,7,10)$, $(3,8,11)$, $(3,9,12)$, $(3,10,13)$, $(3,11,14)$, $(3,12,13)$, respectively),
$i+[(16,0,3),(3,20,6),(6,24,9),(9,21,18),(18,1,26),(26,23,16)]$
(edges having difference triplets $(3,13,12)$, $(3,14,11)$, $(10,3,13)$, $(9,3,12)$, $(8,3,11)$, $(7,3,10)$, respectively),
$i+[(7,4,26),(26,3,6),(6,10,13),(13,17,21),(21,2,25),(25,1,7)]$
(edges having difference triplets $(6,3,9)$, $(5,3,8)$, $(4,3,7)$, $(4,4,8)$, $(4,5,9)$, $(4,6,10)$, respectively),
$i+[(5,9,16),(16,4,8),(8,21,12),(12,26,2),(2,6,17),(17,1,5)]$
(edges having difference triplets $(4,7,11)$, $(4,8,12)$, $(4,9,13)$, $(4,10,14)$, $(4,11,13)$, $(12,12,4)$, respectively),
$i+[(7,3,20),(20,2,6),(6,10,25),(25,9,5),(5,22,1),(1,11,7)]$
(edges having difference triplets $(4,13,11)$, $(4,14,10)$, $(9,4,13)$, $(8,4,12)$, $(7,4,11)$, $(6,4,10)$, respectively),
$i+[(11,15,6),(6,1,24),(24,18,13),(13,8,20),(20,5,25),(25,16,11)]$
(edges having difference triplets $(5,4,9)$, $(5,5,10)$, $(5,6,11)$, $(5,7,12)$, $(5,8,13)$, $(5,9,14)$, respectively),
$i+[(0,23,10),(10,5,21),(21,4,9),(9,14,27),(27,22,13),(13,8,0)]$
(edges having difference triplets $(5,10,13)$, $(5,11,12)$, $(5,12,11)$, $(5,13,10)$, $(5,14,9)$, $(8,5,13)$, respectively),
$i+[(20,15,8),(8,3,25),(25,13,19),(19,6,12),(12,18,26),(26,7,20)]$
(edges having difference triplets $(7,5,12)$, $(6,5,11)$, $(6,6,12)$, $(6,7,13)$, $(6,8,14)$, $(6,9,13)$, respectively),
$i+[(0,16,6),(6,23,12),(12,18,2),(2,8,21),(21,27,13),(13,7,0)]$
(edges having difference triplets $(6,10,12)$, $(6,11,11)$, $(6,12,10)$, $(6,13,9)$, $(6,14,8)$, $(7,6,13)$, respectively),
$i+[(14,0,7),(7,27,20),(20,4,11),(11,1,22),(22,5,15),(5,21,14)]$
(edges having difference triplets $(7,7,14)$, $(7,8,13)$, $(7,9,12)$, $(7,10,11)$, $(7,11,10)$, $(7,12,9)$, respectively),
$i+[(15,23,2),(2,10,18),(18,1,9),(9,27,17),(17,26,6),(6,24,15)]$
(edges having difference triplets $(7,13,8)$, $(8,8,12)$, $(8,9,11)$, $(10,10,8)$, $(8,11,9)$, $(9,9,10)$, respectively).
Remaining triplets are: $(1,5,6)$, $(1,11,12)$ and $(5,1,6)$. We decompose 84 edges of these 3 triplets into 14 $LC_6^{(3)}$'s.
Let $j \in \{0,6,12,18\}$.
$j+[(2,14,3),(3,9,4),(4,26,27),(27,5,0),(0,6,1),(1,7,2)]$
(first edge is of triplet $(1,11,12)$ and the remaining are $(1,5,6)$),
$j+[(0,23,1),(1,24,2),(2,25,3),(3,26,4),(4,27,5),(5,6,0)]$
(edges having difference triplet $(5,1,6)$),

$j + [(8, 2, 3), (3, 15, 4), (4, 16, 5), (5, 17, 6), (6, 18, 7), (7, 19, 8)]$
(first edge is of triplet $(1, 5, 6)$ and the remaining are $(1, 11, 12)$).

Remaining two $LC_6^{(3)}$'s are:

$[(24, 19, 25), (25, 20, 26), (26, 21, 27), (27, 11, 0), (0, 22, 23), (23, 1, 24)]$
(first three edges are of triplet $(5, 1, 6)$, $(27, 11, 0)$ is of $(1, 11, 12)$ and the latter two are $(1, 5, 6)$)

$[(2, 24, 25), (25, 3, 26), (26, 10, 27), (27, 22, 0), (0, 12, 1), (1, 13, 2)]$
(first two edges are of triplet $(1, 5, 6)$, $(27, 22, 0)$ is of $(5, 1, 6)$ and the remaining three are $(1, 11, 12)$). □

Lemma 2.6. $LC_6^{(3)} | K_{45}^{(3)}$.

Proof. Let $V(K_{45}^{(3)}) = \mathbb{Z}_{45}$.

Following $LC_6^{(3)}$'s decompose $K_{45}^{(3)}$:

For each $i \in \mathbb{Z}_{45}$, consider

$i + [(20, 37, 21), (21, 22, 23), (23, 24, 26), (26, 27, 30), (30, 34, 29), (29, 19, 20)]$
(edges having difference triplets $(1, 16, 17)$, $(1, 1, 2)$, $(1, 2, 3)$, $(1, 3, 4)$, $(1, 4, 5)$, $(1, 9, 10)$, respectively),

$i + [(6, 13, 7), (7, 8, 15), (15, 16, 24), (24, 25, 36), (36, 4, 37), (37, 38, 6)]$
(edges having difference triplets $(1, 6, 7)$, $(1, 7, 8)$, $(1, 8, 9)$, $(1, 11, 12)$, $(1, 12, 13)$, $(1, 13, 14)$, respectively),

$i + [(41, 36, 40), (40, 19, 20), (20, 5, 6), (6, 31, 32), (32, 13, 14), (14, 42, 41)]$
(edges having difference triplets $(4, 1, 5)$, $(1, 20, 21)$, $(1, 14, 15)$, $(1, 19, 20)$, $(1, 18, 19)$, $(1, 17, 18)$, respectively),

$i + [(18, 17, 44), (44, 22, 23), (23, 1, 24), (24, 25, 3), (3, 4, 28), (28, 29, 18)]$
(edges having difference triplets $(18, 1, 19)$, $(1, 21, 22)$, $(22, 22, 1)$, $(21, 1, 22)$, $(20, 1, 21)$, $(10, 1, 11)$, respectively),

$i + [(20, 19, 2), (2, 3, 31), (31, 30, 15), (15, 14, 0), (0, 1, 32), (32, 33, 20)]$
(edges having difference triplets $(17, 1, 18)$, $(16, 1, 17)$, $(15, 1, 16)$, $(14, 1, 15)$, $(13, 1, 14)$, $(12, 1, 13)$, respectively),

$i + [(0, 2, 7), (7, 9, 15), (15, 17, 24), (24, 26, 34), (34, 32, 43), (43, 10, 0)]$
(edges having difference triplets $(2, 5, 7)$, $(2, 6, 8)$, $(2, 7, 9)$, $(2, 8, 10)$, $(2, 9, 11)$, $(2, 10, 12)$, respectively),

$i + [(3, 9, 8), (8, 7, 4), (4, 5, 2), (2, 43, 0), (0, 40, 42), (42, 44, 3)]$
(edges having difference triplets $(5, 1, 6)$, $(3, 1, 4)$, $(2, 1, 3)$, $(2, 2, 4)$, $(2, 3, 5)$, $(2, 4, 6)$, respectively),

$i + [(2, 4, 15), (15, 13, 27), (27, 29, 42), (42, 26, 28), (28, 30, 0), (0, 18, 2)]$
(edges having difference triplets $(2, 11, 13)$, $(2, 12, 14)$, $(2, 13, 15)$, $(2, 14, 16)$, $(2, 15, 17)$, $(2, 16, 18)$, respectively),

$i + [(9, 35, 37), (37, 19, 17), (17, 15, 36), (36, 11, 34), (34, 10, 32), (32, 30, 9)]$
(edges having difference triplets $(2, 17, 19)$, $(2, 18, 20)$, $(2, 19, 21)$, $(2, 20, 22)$, $(2, 21, 22)$, $(2, 22, 21)$, respectively),

$i + [(34, 9, 11), (11, 13, 37), (37, 39, 19), (19, 21, 2), (2, 31, 4), (4, 6, 34)]$
(edges having difference triplets $(20, 2, 22)$, $(19, 2, 21)$, $(18, 2, 20)$, $(17, 2, 19)$, $(16, 2, 18)$, $(15, 2, 17)$, respectively),

$i + [(40, 9, 11), (11, 13, 43), (43, 41, 29), (29, 27, 16), (16, 18, 6), (6, 4, 40)]$
(edges having difference triplets $(14, 2, 16)$, $(13, 2, 15)$, $(12, 2, 14)$, $(11, 2, 13)$, $(10, 2, 12)$, $(9, 2, 11)$, respectively),

$i + [(43, 6, 8), (8, 10, 1), (1, 3, 40), (40, 42, 35), (35, 39, 41), (41, 38, 43)]$
(edges having difference triplets $(8, 2, 10)$, $(7, 2, 9)$, $(6, 2, 8)$, $(5, 2, 7)$, $(4, 2, 6)$, $(3, 2, 5)$, respectively),

$i + [(3, 0, 6), (6, 9, 13), (13, 16, 21), (21, 24, 30), (30, 33, 40), (40, 37, 3)]$
(edges having difference triplets $(3, 3, 6)$, $(3, 4, 7)$, $(3, 5, 8)$, $(3, 6, 9)$, $(3, 7, 10)$, $(3, 8, 11)$, respectively),

$i + [(3, 0, 12), (12, 15, 25), (25, 28, 39), (39, 42, 9), (9, 22, 6), (6, 20, 3)]$
(edges having difference triplets $(3, 9, 12)$, $(3, 10, 13)$, $(3, 11, 14)$, $(3, 12, 15)$, $(3, 13, 16)$, $(3, 14, 17)$, respectively),

$i + [(21, 36, 18), (18, 15, 34), (34, 31, 6), (6, 9, 27), (27, 24, 1), (1, 43, 21)]$
(edges having difference triplets $(3, 15, 18)$, $(3, 16, 19)$, $(3, 17, 20)$, $(3, 18, 21)$, $(3, 19, 22)$, $(3, 20, 22)$, respectively),

$i + [(18, 42, 21), (21, 24, 1), (1, 4, 27), (27, 30, 9), (9, 12, 37), (37, 34, 18)]$
(edges having difference triplets $(21, 21, 3)$ $(3, 22, 20)$, $(19, 3, 22)$, $(18, 3, 21)$, $(17, 3, 20)$, $(16, 3, 19)$, respectively),

$i + [(12, 27, 30), (30, 33, 16), (16, 29, 32), (32, 35, 20), (20, 23, 9), (9, 44, 12)]$
(edges having difference triplets $(15, 3, 18)$, $(14, 3, 17)$, $(13, 3, 16)$, $(12, 3, 15)$, $(11, 3, 14)$, $(10, 3, 13)$, respectively),

$i + [(12, 9, 0), (0, 3, 37), (37, 40, 30), (30, 33, 24), (24, 27, 19), (19, 16, 12)]$
(edges having difference triplets $(9, 3, 12)$, $(8, 3, 11)$, $(7, 3, 10)$, $(6, 3, 9)$, $(5, 3, 8)$, $(4, 3, 7)$, respectively),

$i + [(22, 18, 14), (14, 10, 19), (19, 23, 29), (29, 33, 40), (40, 3, 36), (36, 26, 22)]$
(edges having difference triplets $(4, 4, 8)$, $(4, 5, 9)$, $(4, 6, 10)$, $(4, 7, 11)$, $(4, 8, 12)$, $(4, 10, 14)$, respectively),

$i + [(4, 0, 13), (13, 17, 28), (28, 32, 44), (44, 3, 16), (16, 12, 30), (30, 34, 4)]$

(edges having difference triplets $(4, 9, 13)$, $(4, 11, 15)$, $(4, 12, 16)$, $(4, 13, 17)$, $(4, 14, 18)$, $(4, 15, 19)$, respectively),
$i + [(24, 40, 20), (20, 16, 37), (37, 10, 33), (33, 29, 7), (7, 32, 28), (28, 4, 24)]$
(edges having difference triplets $(4, 16, 20)$, $(4, 17, 21)$, $(4, 18, 22)$, $(4, 19, 22)$, $(4, 20, 21)$, $(4, 21, 20)$, respectively),
$i + [(4, 0, 26), (26, 30, 8), (8, 36, 12), (12, 16, 41), (41, 37, 22), (22, 18, 4)]$
(edges having difference triplets $(4, 22, 19)$, $(18, 4, 22)$, $(17, 4, 21)$, $(16, 4, 20)$, $(15, 4, 19)$, $(14, 4, 18)$, respectively),
$i + [(41, 24, 37), (37, 4, 8), (8, 42, 12), (12, 16, 2), (2, 43, 34), (34, 0, 41)]$
(edges having difference triplets $(13, 4, 17)$, $(12, 4, 16)$, $(11, 4, 15)$, $(10, 4, 14)$, $(9, 4, 13)$, $(7, 4, 11)$, respectively),
$i + [(4, 0, 37), (37, 41, 31), (31, 35, 26), (26, 21, 16), (16, 22, 11), (11, 44, 4)]$
(edges having difference triplets $(8, 4, 12)$, $(6, 4, 10)$, $(5, 4, 9)$, $(5, 5, 10)$, $(5, 6, 11)$, $(5, 7, 12)$, respectively),
$i + [(8, 21, 13), (13, 18, 27), (27, 32, 42), (42, 26, 31), (31, 36, 3), (3, 22, 8)]$
(edges having difference triplets $(5, 8, 13)$, $(5, 9, 14)$, $(5, 10, 15)$, $(5, 11, 16)$, $(5, 12, 17)$, $(5, 14, 19)$, respectively),
$i + [(5, 0, 18), (18, 23, 38), (38, 17, 22), (22, 27, 44), (44, 21, 26), (26, 31, 5)]$
(edges having difference triplets $(5, 13, 18)$, $(5, 15, 20)$, $(5, 16, 21)$, $(5, 17, 22)$, $(5, 18, 22)$, $(5, 19, 21)$, respectively),
$i + [(20, 0, 25), (25, 30, 6), (6, 1, 28), (28, 33, 11), (11, 16, 40), (40, 35, 20)]$
(edges having difference triplets $(20, 20, 5)$, $(5, 21, 19)$, $(5, 22, 18)$, $(17, 5, 22)$, $(16, 5, 21)$, $(15, 5, 20)$, respectively),
$i + [(12, 26, 31), (31, 36, 18), (18, 13, 1), (1, 6, 35), (35, 40, 25), (25, 20, 12)]$
(edges having difference triplets $(14, 5, 19)$, $(13, 5, 18)$, $(12, 5, 17)$, $(11, 5, 16)$, $(10, 5, 15)$, $(8, 5, 13)$, respectively),
$i + [(21, 26, 12), (12, 5, 17), (17, 23, 28), (28, 16, 22), (22, 9, 15), (15, 29, 21)]$
(edges having difference triplets $(9, 5, 14)$, $(7, 5, 12)$, $(6, 5, 11)$, $(6, 6, 12)$, $(6, 7, 13)$, $(6, 8, 14)$, respectively),
$i + [(24, 9, 15), (15, 21, 31), (31, 37, 3), (3, 30, 36), (36, 42, 10), (10, 4, 24)]$
(edges having difference triplets $(6, 9, 15)$, $(6, 10, 16)$, $(6, 11, 17)$, $(6, 12, 18)$, $(6, 13, 19)$, $(6, 14, 20)$, respectively),
$i + [(6, 0, 21), (21, 15, 37), (37, 14, 20), (20, 26, 44), (44, 19, 25), (25, 31, 6)]$
(edges having difference triplets $(6, 15, 21)$, $(6, 16, 22)$, $(6, 17, 22)$, $(6, 18, 21)$, $(6, 19, 20)$, $(6, 20, 19)$, respectively),
$i + [(22, 16, 43), (43, 15, 21), (21, 5, 27), (27, 42, 3), (3, 9, 34), (34, 40, 22)]$
(edges having difference triplets $(6, 21, 18)$, $(6, 22, 17)$, $(16, 6, 22)$, $(15, 6, 21)$, $(14, 6, 20)$, $(12, 6, 18)$, respectively),
$i + [(20, 7, 26), (26, 32, 15), (15, 9, 44), (44, 5, 35), (35, 41, 27), (27, 33, 20)]$
(edges having difference triplets $(13, 6, 19)$, $(11, 6, 17)$, $(10, 6, 16)$, $(9, 6, 15)$, $(8, 6, 14)$, $(7, 6, 13)$, respectively),
$i + [(35, 28, 21), (21, 14, 29), (29, 22, 38), (38, 3, 31), (31, 24, 42), (42, 9, 35)]$
(edges having difference triplets $(7, 7, 14)$, $(7, 8, 15)$, $(7, 9, 16)$, $(7, 10, 17)$, $(7, 11, 18)$, $(7, 12, 19)$, respectively),
$i + [(21, 28, 41), (41, 20, 27), (27, 34, 4), (4, 26, 33), (33, 12, 40), (40, 2, 21)]$
(edges having difference triplets $(7, 13, 20)$, $(7, 14, 21)$, $(7, 15, 22)$, $(7, 16, 22)$, $(7, 17, 21)$, $(19, 19, 7)$, respectively),
$i + [(25, 0, 7), (7, 34, 14), (14, 42, 21), (21, 28, 5), (5, 35, 12), (12, 32, 25)]$
(edges having difference triplets $(7, 18, 20)$, $(7, 20, 18)$, $(7, 21, 17)$, $(7, 22, 16)$, $(15, 7, 22)$, $(13, 7, 20)$, respectively),
$i + [(31, 0, 7), (7, 14, 40), (40, 6, 13), (13, 20, 3), (3, 10, 39), (39, 1, 31)]$
(edges having difference triplets $(14, 7, 21)$, $(12, 7, 19)$, $(11, 7, 18)$, $(10, 7, 17)$, $(9, 7, 16)$, $(8, 7, 15)$, respectively),
$i + [(40, 32, 24), (24, 16, 33), (33, 25, 43), (43, 35, 9), (9, 29, 17), (17, 3, 40)]$
(edges having difference triplets $(8, 8, 16)$, $(8, 9, 17)$, $(8, 10, 18)$, $(8, 11, 19)$, $(8, 12, 20)$, $(8, 14, 22)$, respectively),
$i + [(8, 0, 21), (21, 36, 13), (13, 5, 29), (29, 37, 9), (9, 17, 35), (35, 16, 8)]$
(edges having difference triplets $(8, 13, 21)$, $(8, 15, 22)$, $(8, 16, 21)$, $(8, 17, 20)$, $(8, 18, 19)$, $(8, 19, 18)$, respectively),
$i + [(28, 0, 8), (8, 37, 16), (16, 24, 1), (1, 9, 32), (32, 19, 40), (40, 3, 28)]$
(edges having difference triplets $(8, 20, 17)$, $(8, 21, 16)$, $(8, 22, 15)$, $(14, 8, 22)$, $(13, 8, 21)$, $(12, 8, 20)$, respectively),
$i + [(17, 9, 43), (43, 8, 16), (16, 7, 24), (24, 15, 6), (6, 32, 41), (41, 5, 17)]$
(edges having difference triplets $(11, 8, 19)$, $(10, 8, 18)$, $(9, 8, 17)$, $(9, 9, 18)$, $(9, 10, 19)$, $(9, 12, 21)$, respectively),
$i + [(38, 27, 18), (18, 9, 31), (31, 22, 0), (0, 21, 30), (30, 39, 10), (10, 29, 38)]$
(edges having difference triplets $(9, 11, 20)$, $(9, 13, 22)$, $(9, 14, 22)$, $(9, 15, 21)$, $(9, 16, 20)$, $(9, 17, 19)$, respectively),
$i + [(19, 1, 37), (37, 9, 18), (18, 34, 43), (43, 7, 28), (28, 6, 42), (42, 10, 19)]$
(edges having difference triplets $(18, 18, 9)$, $(9, 19, 17)$, $(9, 20, 16)$, $(9, 21, 15)$, $(9, 22, 14)$, $(13, 9, 22)$, respectively),
$i + [(6, 30, 42), (42, 33, 22), (22, 31, 12), (12, 2, 37), (37, 26, 16), (16, 28, 6)]$
(edges having difference triplets $(12, 9, 21)$, $(11, 9, 20)$, $(10, 9, 19)$, $(10, 10, 20)$, $(10, 11, 21)$, $(10, 12, 22)$, respectively),
$i + [(26, 3, 13), (13, 23, 37), (37, 27, 7), (7, 17, 33), (33, 6, 16), (16, 44, 26)]$
(edges having difference triplets $(10, 13, 22)$, $(10, 14, 21)$, $(10, 15, 20)$, $(10, 16, 19)$, $(10, 17, 18)$, $(10, 18, 17)$,

respectively),

$i + [(29, 0, 10), (10, 40, 20), (20, 30, 6), (6, 28, 41), (41, 31, 19), (19, 8, 29)]$
(edges having difference triplets $(10, 19, 16)$, $(10, 20, 15)$, $(10, 21, 14)$, $(10, 22, 13)$, $(12, 10, 22)$, $(11, 10, 21)$, respectively),

$i + [(20, 9, 31), (31, 8, 19), (19, 30, 43), (43, 32, 12), (12, 23, 38), (38, 4, 20)]$
(edges having difference triplets $(11, 11, 22)$, $(11, 12, 22)$, $(11, 13, 21)$, $(11, 14, 20)$, $(11, 15, 19)$, $(11, 16, 18)$, respectively),

$i + [(28, 0, 11), (11, 22, 40), (40, 21, 10), (10, 44, 30), (30, 41, 17), (17, 5, 28)]$
(edges having difference triplets $(17, 17, 11)$, $(11, 18, 16)$, $(11, 19, 15)$, $(11, 20, 14)$, $(11, 21, 13)$, $(11, 22, 12)$, respectively),

$i + [(12, 0, 24), (24, 36, 4), (4, 16, 30), (30, 3, 15), (15, 43, 27), (27, 39, 12)]$
(edges having difference triplets $(12, 12, 21)$, $(12, 13, 20)$, $(12, 14, 19)$, $(12, 15, 18)$, $(12, 16, 17)$, $(12, 18, 15)$, respectively),

$i + [(7, 40, 24), (24, 12, 43), (43, 10, 30), (30, 4, 17), (17, 3, 35), (35, 21, 7)]$
(edges having difference triplets $(12, 17, 16)$, $(12, 19, 14)$, $(12, 20, 13)$, $(13, 13, 19)$, $(13, 14, 18)$, $(14, 14, 17)$, respectively),

$i + [(43, 11, 26), (26, 39, 10), (10, 23, 40), (40, 27, 13), (13, 44, 28), (28, 12, 43)]$
(edges having difference triplets $(13, 15, 17)$, $(16, 16, 13)$, $(13, 17, 15)$, $(13, 18, 14)$, $(14, 15, 16)$, $(16, 15, 14)$, respectively).

Remaining triplets are: $(1, 5, 6)$, $(1, 10, 11)$, $(1, 15, 16)$, $(6, 1, 7)$, $(7, 1, 8)$, $(8, 1, 9)$, $(9, 1, 10)$, $(11, 1, 12)$, $(19, 1, 20)$ and $(15, 15, 15)$.

For each $j \in \mathbb{Z}_{45} \setminus \{44\}$, consider

$j + [(39, 33, 40), (40, 2, 3), (3, 37, 4), (4, 13, 14), (14, 22, 23), (23, 24, 39)]$
(edges having difference triplets $(6, 1, 7)$, $(7, 1, 8)$, $(11, 1, 12)$, $(9, 1, 10)$, $(8, 1, 9)$, $(1, 15, 16)$, respectively).

Remaining edges are: $\{38, 32, 39\}$, $\{39, 1, 2\}$, $\{2, 36, 3\}$, $\{3, 12, 13\}$, $\{13, 21, 22\}$ and $\{22, 23, 38\}$ and edges of the triplets: $(1, 5, 6)$, $(1, 10, 11)$, $(19, 1, 20)$ and $(15, 15, 15)$.

Finally, consider the following 26 $LC_6^{(3)}$'s containing these 156 edges.

$[(0, 11, 1), (1, 27, 2), (2, 13, 3), (3, 14, 4), (4, 5, 15), (15, 30, 0)]$,
$[(5, 16, 6), (6, 17, 7), (7, 18, 8), (8, 19, 9), (9, 10, 20), (20, 35, 5)]$,
$[(21, 10, 11), (11, 17, 12), (12, 23, 13), (13, 24, 14), (14, 25, 15), (15, 16, 21)]$,
$[(16, 27, 17), (17, 28, 18), (18, 29, 19), (19, 0, 20), (20, 21, 31), (31, 1, 16)]$,
$[(21, 32, 22), (22, 33, 23), (23, 34, 24), (24, 35, 25), (25, 26, 36), (36, 6, 21)]$,
$[(26, 37, 27), (27, 38, 28), (28, 39, 29), (29, 35, 30), (30, 31, 41), (41, 11, 26)]$,
$[(32, 43, 33), (33, 44, 34), (34, 15, 35), (35, 1, 36), (36, 37, 2), (2, 17, 32)]$,
$[(43, 4, 44), (44, 0, 5), (5, 39, 40), (40, 6, 41), (41, 7, 42), (42, 3, 43)]$,
$[(3, 29, 4), (4, 30, 5), (5, 31, 6), (6, 32, 7), (7, 8, 33), (33, 18, 3)]$,
$[(8, 34, 9), (9, 35, 10), (10, 36, 11), (11, 37, 12), (12, 13, 38), (38, 23, 8)]$,
$[(13, 39, 14), (14, 40, 15), (15, 26, 16), (16, 42, 17), (17, 18, 43), (43, 28, 13)]$,
$[(30, 19, 20), (20, 1, 21), (21, 2, 22), (22, 3, 23), (23, 4, 24), (24, 25, 30)]$,
$[(24, 5, 25), (25, 6, 26), (26, 7, 27), (27, 8, 28), (28, 29, 9), (9, 39, 24)]$,
$[(14, 44, 29), (29, 10, 30), (30, 11, 31), (31, 37, 32), (32, 13, 33), (33, 34, 14)]$,
$[(0, 6, 1), (1, 7, 2), (2, 28, 3), (3, 9, 4), (4, 5, 10), (10, 44, 0)]$,
$[(6, 12, 7), (7, 13, 8), (8, 14, 9), (9, 15, 10), (10, 16, 11), (11, 5, 6)]$,
$[(18, 12, 13), (13, 19, 14), (14, 20, 15), (15, 41, 16), (16, 22, 17), (17, 23, 18)]$,
$[(24, 18, 19), (19, 25, 20), (20, 26, 21), (21, 27, 22), (22, 28, 23), (23, 29, 24)]$,
$[(40, 10, 25), (25, 31, 26), (26, 32, 27), (27, 33, 28), (28, 34, 29), (29, 30, 40)]$,
$[(36, 30, 31), (31, 12, 32), (32, 38, 33), (33, 39, 34), (34, 40, 35), (35, 41, 36)]$,
$[(42, 36, 37), (37, 3, 38), (38, 4, 39), (39, 0, 40), (40, 1, 41), (41, 2, 42)]$,
$[(39, 20, 40), (40, 21, 41), (41, 22, 42), (42, 23, 43), (43, 24, 44), (44, 38, 39)]$,
$[(34, 0, 35), (35, 16, 36), (36, 17, 37), (37, 38, 18), (18, 44, 19), (19, 4, 34)]$,
$[(1, 12, 2), (2, 3, 8), (8, 42, 43), (43, 9, 44), (44, 25, 0), (0, 26, 1)]$,

$[(32,39,38),(38,43,37),(37,7,22),(22,11,12),(12,27,42),(42,31,32)]$,
$[(38,19,39),(39,1,2),(2,36,3),(3,12,13),(13,21,22),(22,23,38)]$. □

3. Loose 6-cycle decompositions of complete bipartite 3-uniform hypergraphs

For disjoint sets X and Y, the hypergraph with vertex set $X \cup Y$ and edge set consisting of all 3-sets having at most 2 vertices in each of X and Y is denoted either by $K^{(3)}_{X,Y}$ or by $K^{(3)}_{|X|,|Y|}$. We partition the edge set of $K^{(3)}_{X,Y}$ into two sets one consisting of all 3-sets having exactly 2 vertices in X and the other consisting of all 3-sets having exactly 2 vertices in Y. We denote the subhypergraph induced by the former edge set by $K^{(3)}_{X,\overline{Y}}$ or by $K^{(3)}_{|X|,\overline{|Y|}}$ and the latter by $K^{(3)}_{\overline{X},Y}$ or $K^{(3)}_{\overline{|X|},|Y|}$. Clearly, $K^{(3)}_{X,Y} = K^{(3)}_{X,\overline{Y}} \oplus K^{(3)}_{\overline{X},Y}$.

Lemma 3.1. $LC^{(3)}_6 \mid K^{(3)}_{6,6}$.

Proof. Consider the complete graph K_6 with vertex set $\mathbb{Z}_6$. Decompose K_6 into two copies of C_6 : $(0,2,1,5,3,4,0)$, $(0,3,1,4,2,5,0)$ and a 1-factor: $\{01, 23, 45\}$.

Now, consider $K^{(3)}_{6,6} = K^{(3)}_{X,Y}$ with $X = \{x_0, x_1, \ldots, x_5\}$ and $Y = \{y_0, y_1, \ldots, y_5\}$. By above, decompose each of the complete graphs K_X and K_Y into two copies of C_6 and a 1-factor:

$$K_X = (x_0,x_2,x_1,x_5,x_3,x_4,x_0) \oplus (x_0,x_3,x_1,x_4,x_2,x_5,x_0) \oplus \{x_0x_1, x_2x_3, x_4x_5\},$$
$$K_Y = (y_0,y_2,y_1,y_5,y_3,y_4,y_0) \oplus (y_0,y_3,y_1,y_4,y_2,y_5,y_0) \oplus \{y_0y_1, y_2y_3, y_4y_5\}.$$

From each of the above C_6's, we produce six $LC^{(3)}_6$'s in $K^{(3)}_{X,Y}$ as follows:
$[(x_0,y_i,x_2),(x_2,y_{i+1},x_1),(x_1,y_{i+2},x_5),(x_5,y_{i+3},x_3),(x_3,y_{i+4},x_4),(x_4,y_{i+5},x_0)]$,
$[(x_0,,y_i,x_3),(x_3,y_{i+1},x_1),(x_1,y_{i+2},x_4),(x_4,y_{i+3},x_2),(x_2,y_{i+4},x_5),(x_5,y_{i+5},x_0)]$,
$[(y_0,x_i,y_2),(y_2,x_{i+1},y_1),(y_1,x_{i+2},y_5),(y_5,x_{i+3},y_3),(y_3,x_{i+4},y_4),(y_4,x_{i+5},y_0)]$,
$[(y_0,,x_i,y_3),(y_3,x_{i+1},y_1),(y_1,x_{i+2},y_4),(y_4,x_{i+3},y_2),(y_2,x_{i+4},y_5),(y_5,x_{i+5},y_0)]$,
where $i \in \mathbb{Z}_6$.
From the two 1-factors, we produce six $LC^{(3)}_6$'s in $K^{(3)}_{X,Y}$ as follows:
$[(x_0,x_1,y_j),(y_j,y_{j+1},x_2),(x_2,x_3,y_{j+2}),(y_{j+2},y_{j+3},x_4),(x_4,x_5,y_{j+4}),(y_{j+4},y_{j+5},x_0)]$,
where $j \in \{0,2,4\}$;
$[(x_1,x_0,y_k),(y_k,y_{k+5},x_3),(x_3,x_2,y_{k+2}),(y_{k+2},y_{k+1},x_5),(x_5,x_4,y_{k+4}),(y_{k+4},y_{k+3},x_1)]$,
where $k \in \{1,3,5\}$.

The collection of these loose 6-cycles yield the required decomposition of $K^{(3)}_{6,6}$. □

Lemma 3.2. $LC^{(3)}_6 \mid K^{(3)}_{10,18}$.

Proof. Since $K^{(3)}_{10,18} = K^{(3)}_{10,\overline{18}} \oplus K^{(3)}_{\overline{10},18}$, it is enough to show that $LC^{(3)}_6 \mid K^{(3)}_{10,\overline{18}}$ and $LC^{(3)}_6 \mid K^{(3)}_{\overline{10},18}$. Let $X = \{x_i | i \in \mathbb{Z}_{10}\}$.

First, consider $K^{(3)}_{10,\overline{18}} = K^{(3)}_{X,\overline{Y'}}$ with $Y' = \{y_j | j \in \mathbb{Z}_{18}\}$. We use the (Hamilton path) P_{10}-decomposition

$$\{x_ix_{1+i}x_{9+i}x_{2+i}x_{8+i}x_{3+i}x_{7+i}x_{4+i}x_{6+i}x_{5+i} \;:\; i = 0,1,2,3,4\}$$

of the complete graph $K_{10} = K_X$. Following $LC^{(3)}_6$'s decompose $K^{(3)}_{X,\overline{Y'}}$:
$[(y_1,x_i,x_{1+i}),(x_{1+i},y_2,x_{9+i}),(x_{9+i},y_3,x_{2+i}),(x_{2+i},y_4,x_{8+i}),(x_{8+i},y_5,x_{3+i}),(x_{3+i},x_{7+i},y_1)]$,
$[(y_2,x_i,x_{1+i}),(x_{1+i},y_3,x_{9+i}),(x_{9+i},y_{12},x_{2+i}),(x_{2+i},y_0,x_{8+i}),(x_{8+i},y_6,x_{3+i}),(x_{3+i},x_{7+i},y_2)]$,
$[(y_9,x_i,x_{1+i}),(x_{1+i},y_4,x_{9+i}),(x_{9+i},y_5,x_{2+i}),(x_{2+i},y_{13},x_{8+i}),(x_{8+i},y_3,x_{3+i}),(x_{3+i},x_{7+i},y_9)]$,
$[(y_4,x_i,x_{1+i}),(x_{1+i},y_5,x_{9+i}),(x_{9+i},y_6,x_{2+i}),(x_{2+i},y_7,x_{8+i}),(x_{8+i},y_8,x_{3+i}),(x_{3+i},x_{7+i},y_4)]$,
$[(x_{7+i},y_6,x_{4+i}),(x_{4+i},y_0,x_{6+i}),(x_{6+i},x_{5+i},y_8),(y_8,x_{2+i},x_{8+i}),(x_{8+i},y_9,x_{3+i}),(x_{3+i},y_{10},x_{7+i})]$,
$[(x_{7+i},y_7,x_{4+i}),(x_{4+i},y_{13},x_{6+i}),(x_{6+i},x_{5+i},y_9),(y_9,x_{2+i},x_{8+i}),(x_{8+i},y_{10},x_{3+i}),(x_{3+i},y_{11},x_{7+i})]$,
$[(x_{7+i},y_8,x_{4+i}),(x_{4+i},y_{14},x_{6+i}),(x_{6+i},x_{5+i},y_{10}),(y_{10},x_{2+i},x_{8+i}),(x_{8+i},y_{11},x_{3+i}),(x_{3+i},y_{12},x_{7+i})]$,
$[(x_{7+i},y_9,x_{4+i}),(x_{4+i},y_{15},x_{6+i}),(x_{6+i},x_{5+i},y_{11}),(y_{11},x_{2+i},x_{8+i}),(x_{8+i},y_{12},x_{3+i}),(x_{3+i},y_{13},x_{7+i})]$,
$[(x_{7+i},y_{16},x_{4+i}),(x_{4+i},y_7,x_{6+i}),(x_{6+i},x_{5+i},y_{12}),(y_{12},x_{2+i},x_{8+i}),(x_{8+i},y_{13},x_{3+i}),(x_{3+i},y_{14},x_{7+i})]$,
$[(y_8,x_i,x_{1+i}),(x_{1+i},y_6,x_{9+i}),(x_{9+i},y_7,x_{2+i}),(x_{2+i},y_3,x_{8+i}),(x_{8+i},y_{14},x_{3+i}),(x_{3+i},x_{7+i},y_8)]$,

$[(y_{16}, x_i, x_{1+i}), (x_{1+i}, y_7, x_{9+i}), (x_{9+i}, y_8, x_{2+i}), (x_{2+i}, y_{14}, x_{8+i}), (x_{8+i}, y_{15}, x_{3+i}), (x_{3+i}, x_{7+i}, y_{16})]$,
$[(y_{17}, x_i, x_{1+i}), (x_{1+i}, y_8, x_{9+i}), (x_{9+i}, y_9, x_{2+i}), (x_{2+i}, y_{15}, x_{8+i}), (x_{8+i}, y_{16}, x_{3+i}), (x_{3+i}, x_{7+i}, y_{17})]$,
$[(y_5, x_i, x_{1+i}), (x_{1+i}, y_9, x_{9+i}), (x_{9+i}, y_{10}, x_{2+i}), (x_{2+i}, y_{16}, x_{8+i}), (x_{8+i}, y_{17}, x_{3+i}), (x_{3+i}, x_{7+i}, y_5)]$,
$[(y_6, x_i, x_{1+i}), (x_{1+i}, y_{10}, x_{9+i}), (x_{9+i}, y_{11}, x_{2+i}), (x_{2+i}, y_{17}, x_{8+i}), (x_{8+i}, y_0, x_{3+i}), (x_{3+i}, x_{7+i}, y_6)]$,
$[(y_{11}, x_{7+i}, x_{4+i}), (x_{4+i}, y_{16}, x_{6+i}), (x_{6+i}, x_{5+i}, y_{13}), (y_{13}, x_{2+i}, x_{9+i}), (x_{9+i}, y_{12}, x_{1+i}), (x_{1+i}, x_i, y_{11})]$,
$[(y_{12}, x_{7+i}, x_{4+i}), (x_{4+i}, y_8, x_{6+i}), (x_{6+i}, x_{5+i}, y_{14}), (y_{14}, x_{2+i}, x_{9+i}), (x_{9+i}, y_{13}, x_{1+i}), (x_{1+i}, x_i, y_{12})]$,
$[(y_{13}, x_{7+i}, x_{4+i}), (x_{4+i}, y_9, x_{6+i}), (x_{6+i}, x_{5+i}, y_{15}), (y_{15}, x_{2+i}, x_{9+i}), (x_{9+i}, y_{14}, x_{1+i}), (x_{1+i}, x_i, y_{13})]$,
$[(y_{14}, x_{7+i}, x_{4+i}), (x_{4+i}, y_{10}, x_{6+i}), (x_{6+i}, x_{5+i}, y_{16}), (y_{16}, x_{2+i}, x_{9+i}), (x_{9+i}, y_{15}, x_{1+i}), (x_{1+i}, x_i, y_{14})]$,
$[(y_{15}, x_{7+i}, x_{4+i}), (x_{4+i}, y_{11}, x_{6+i}), (x_{6+i}, x_{5+i}, y_{17}), (y_{17}, x_{2+i}, x_{9+i}), (x_{9+i}, y_{16}, x_{1+i}), (x_{1+i}, x_i, y_{15})]$,
$[(y_5, x_{2+i}, x_{8+i}), (x_{8+i}, y_1, x_{3+i}), (x_{3+i}, y_7, x_{7+i}), (x_{7+i}, y_3, x_{4+i}), (x_{4+i}, y_4, x_{6+i}), (x_{6+i}, x_{5+i}, y_5)]$,
$[(y_1, x_{2+i}, x_{8+i}), (x_{8+i}, y_2, x_{3+i}), (x_{3+i}, y_{15}, x_{7+i}), (x_{7+i}, y_4, x_{4+i}), (x_{4+i}, y_5, x_{6+i}), (x_{6+i}, x_{5+i}, y_1)]$,
$[(y_2, x_{2+i}, x_{8+i}), (x_{8+i}, y_7, x_{3+i}), (x_{3+i}, y_3, x_{7+i}), (x_{7+i}, y_5, x_{4+i}), (x_{4+i}, y_6, x_{6+i}), (x_{6+i}, x_{5+i}, y_2)]$,
$[(y_0, x_{7+i}, x_{4+i}), (x_{4+i}, y_1, x_{6+i}), (x_{6+i}, x_{5+i}, y_7), (y_7, x_i, x_{1+i}), (x_{1+i}, y_{17}, x_{9+i}), (x_{9+i}, x_{2+i}, y_0)]$,
$[(y_1, x_{7+i}, x_{4+i}), (x_{4+i}, y_2, x_{6+i}), (x_{6+i}, x_{5+i}, y_3), (y_3, x_i, x_{1+i}), (x_{1+i}, y_0, x_{9+i}), (x_{9+i}, x_{2+i}, y_1)]$,
$[(y_{10}, x_i, x_{1+i}), (x_{1+i}, y_{11}, x_{9+i}), (x_{9+i}, x_{2+i}, y_4), (y_4, x_{5+i}, x_{6+i}), (x_{6+i}, y_3, x_{4+i}), (x_{4+i}, x_{7+i}, y_{10})]$,
$[(y_2, x_{7+i}, x_{4+i}), (x_{4+i}, y_{17}, x_{6+i}), (x_{6+i}, x_{5+i}, y_0), (y_0, x_i, x_{1+i}), (x_{1+i}, y_1, x_{9+i}), (x_{9+i}, x_{2+i}, y_2)]$,
$[(y_6, x_{2+i}, x_{8+i}), (x_{8+i}, y_4, x_{3+i}), (x_{3+i}, y_0, x_{7+i}), (x_{7+i}, y_{17}, x_{4+i}), (x_{4+i}, y_{12}, x_{6+i}), (x_{6+i}, x_{5+i}, y_6)]$,
where $i = 0, 1, 2, 3, 4$.

Next, consider $K^{(3)}_{10,18} = K^{(3)}_{\overline{X},Y''}$ with $Y'' = \{y_\infty\} \cup \{y_k | k \in \mathbb{Z}_{17}\}$. We use the P_{10}-decomposition

$$\{y_\infty y_k y_{1+k} y_{16+k} y_{2+k} y_{15+k} y_{3+k} y_{14+k} y_{4+k} y_{13+k} \ : \ k \in \mathbb{Z}_{17}\}$$

of $K_{18} = K_{Y''}$ (decomposition arise out of a ρ-valuation of P_9). Following $LC^{(3)}_6$'s decompose $K^{(3)}_{\overline{X},Y''}$:
$[(x_0, y_\infty, y_k), (y_k, x_1, y_{1+k}), (y_{1+k}, x_2, y_{16+k}), (y_{16+k}, x_9, y_{2+k}), (y_{2+k}, x_4, y_{15+k}), (y_{15+k}, y_{3+k}, x_0)]$,
$[(x_1, y_\infty, y_k), (y_k, x_2, y_{1+k}), (y_{1+k}, x_7, y_{16+k}), (y_{16+k}, x_0, y_{2+k}), (y_{2+k}, x_5, y_{15+k}), (y_{15+k}, y_{3+k}, x_1)]$,
$[(x_2, y_\infty, y_k), (y_k, x_3, y_{1+k}), (y_{1+k}, x_8, y_{16+k}), (y_{16+k}, x_5, y_{2+k}), (y_{2+k}, x_6, y_{15+k}), (y_{15+k}, y_{3+k}, x_2)]$,
$[(x_3, y_\infty, y_k), (y_k, x_4, y_{1+k}), (y_{1+k}, x_9, y_{16+k}), (y_{16+k}, x_6, y_{2+k}), (y_{2+k}, x_7, y_{15+k}), (y_{15+k}, y_{3+k}, x_3)]$,
$[(x_4, y_\infty, y_k), (y_k, x_5, y_{1+k}), (y_{1+k}, x_6, y_{16+k}), (y_{16+k}, x_7, y_{2+k}), (y_{2+k}, x_8, y_{15+k}), (y_{15+k}, y_{3+k}, x_4)]$,
$[(x_8, y_{16+k}, y_{2+k}), (y_{2+k}, x_9, y_{15+k}), (y_{15+k}, x_5, y_{3+k}), (y_{3+k}, x_6, y_{14+k}), (y_{14+k}, x_7, y_{4+k}), (y_{4+k}, y_{13+k}, x_8)]$,
$[(x_3, y_{16+k}, y_{2+k}), (y_{2+k}, x_0, y_{15+k}), (y_{15+k}, x_6, y_{3+k}), (y_{3+k}, x_7, y_{14+k}), (y_{14+k}, x_8, y_{4+k}), (y_{4+k}, y_{13+k}, x_3)]$,
$[(x_4, y_{16+k}, y_{2+k}), (y_{2+k}, x_1, y_{15+k}), (y_{15+k}, x_7, y_{3+k}), (y_{3+k}, x_2, y_{14+k}), (y_{14+k}, x_9, y_{4+k}), (y_{4+k}, y_{13+k}, x_4)]$,
$[(x_1, y_{16+k}, y_{2+k}), (y_{2+k}, x_2, y_{15+k}), (y_{15+k}, x_8, y_{3+k}), (y_{3+k}, x_9, y_{14+k}), (y_{14+k}, x_0, y_{4+k}), (y_{4+k}, y_{13+k}, x_1)]$,
$[(x_2, y_{16+k}, y_{2+k}), (y_{2+k}, x_3, y_{15+k}), (y_{15+k}, x_9, y_{3+k}), (y_{3+k}, x_0, y_{14+k}), (y_{14+k}, x_1, y_{4+k}), (y_{4+k}, y_{13+k}, x_2)]$,
$[(x_3, y_{3+k}, y_{14+k}), (y_{14+k}, x_4, y_{4+k}), (y_{4+k}, y_{13+k}, x_5), (x_5, y_\infty, y_k), (y_k, x_6, y_{1+k}), (y_{1+k}, y_{16+k}, x_3)]$,
$[(x_4, y_{3+k}, y_{14+k}), (y_{14+k}, x_5, y_{4+k}), (y_{4+k}, y_{13+k}, x_6), (x_6, y_\infty, y_k), (y_k, x_7, y_{1+k}), (y_{1+k}, y_{16+k}, x_4)]$,
$[(x_9, y_\infty, y_k), (y_k, x_0, y_{1+k}), (y_{1+k}, y_{16+k}, x_1), (x_1, y_{3+k}, y_{14+k}), (y_{14+k}, x_2, y_{4+k}), (y_{4+k}, y_{13+k}, x_9)]$,
$[(x_0, y_{13+k}, y_{4+k}), (y_{4+k}, x_3, y_{14+k}), (y_{14+k}, y_{3+k}, x_8), (x_8, y_\infty, y_k), (y_k, x_9, y_{1+k}), (y_{1+k}, y_{16+k}, x_0)]$,
$[(x_7, y_\infty, y_k), (y_k, x_8, y_{1+k}), (y_{1+k}, y_{16+k}, x_5), (x_5, y_{3+k}, y_{14+k}), (y_{14+k}, x_6, y_{4+k}), (y_{4+k}, y_{13+k}, x_7)]$,
where $k \in \mathbb{Z}_{17}$. □

Lemma 3.3. *For an integer $\ell \geq 5$, we have $LC^{(3)}_6 \,|\, K^{(3)}_{\overline{\ell,36}}$.*

Proof. Consider $K^{(3)}_{\overline{\ell,36}} = K^{(3)}_{\overline{X},Y}$ with $X = \{x_i \,|\, i \in \mathbb{Z}_\ell\}$ and $Y = \{y_\infty\} \cup \{y_j \,|\, j \in \mathbb{Z}_{35}\}$. We use the P_{19}-decomposition

$$\{y_\infty\, y_j\, y_{j+1}\, y_{j+34}\, y_{j+2}\, y_{j+33}\, y_{j+3}\, y_{j+32}\, y_{j+4}\, y_{j+31}\, y_{j+5}\, y_{j+30}\, y_{j+6}\, y_{j+29}\, y_{j+7}\, y_{j+28}\, y_{j+8}\, y_{j+27}\, y_{j+9} \ : j \in \mathbb{Z}_{35}\}$$

of $K_{36} = K_Y$. For each $j \in \mathbb{Z}_{35}$, we produce 3ℓ loose 6-cycles of $K^{(3)}_{\overline{\ell,36}}$ as follows:
$[(x_i, y_\infty, y_j), (y_j, x_{i+1}, y_{j+1}), (y_{j+1}, x_{i+2}, y_{j+34}), (y_{j+34}, x_{i+3}, y_{j+2}), (y_{j+2}, x_{i+4}, y_{j+33}), (y_{j+33}, y_{j+3}, x_i)]$,
$[(x_i, y_{j+3}, y_{j+32}), (y_{j+32}, x_{i+1}, y_{j+4}), (y_{j+4}, x_{i+2}, y_{j+31}), (y_{j+31}, x_{i+3}, y_{j+5}), (y_{j+5}, x_{i+4}, y_{j+30}), (y_{j+30}, y_{j+6}, x_i)]$,
$[(x_i, y_{j+6}, y_{j+29}), (y_{j+29}, x_{i+1}, y_{j+7}), (y_{j+7}, x_{i+2}, y_{j+28}), (y_{j+28}, x_{i+3}, y_{j+8}), (y_{j+8}, x_{i+4}, y_{j+27}), (y_{j+27}, y_{j+9}, x_i)]$,
where $i \in \mathbb{Z}_\ell$.

The collection of these $105\,\ell$ loose 6-cycles yield the required decomposition of $K^{(3)}_{\overline{\ell,36}}$. □

Lemma 3.4. *For an integer $\ell \geq 5$, we have $LC_6^{(3)} \,|\, K_{\overline{\ell,37}}^{(3)}$.*

Proof. Consider $K_{\overline{\ell,37}}^{(3)} = K_{\overline{X,Y}}^{(3)}$ with $X = \{x_i \mid i \in \mathbb{Z}_\ell\}$ and $Y = \{y_j \mid j \in \mathbb{Z}_{37}\}$. We use the P_{19}-decomposition

$$\{y_j\, y_{j+1}\, y_{j+36}\, y_{j+2}\, y_{j+35}\, y_{j+3}\, y_{j+34}\, y_{j+4}\, y_{j+33}\, y_{j+5}\, y_{j+32}\, y_{j+6}\, y_{j+31}\, y_{j+7}\, y_{j+30}\, y_{j+8}\, y_{j+29} y_{j+9}\, y_{j+28} \;:\; j \in \mathbb{Z}_{37}\}$$

of $K_{37} = K_Y$. For each $j \in \mathbb{Z}_{37}$, we produce 3ℓ loose 6-cycles of $K_{\overline{\ell,37}}^{(3)}$ as follows:

$[(x_i, y_j, y_{j+1}), (y_{j+1}, x_{i+1}, y_{j+36}), (y_{j+36}, x_{i+2}, y_{j+2}), (y_{j+2}, x_{i+3}, y_{j+35}), (y_{j+35}, x_{i+4}, y_{j+3}), (y_{j+3}, y_{j+34}, x_i)]$,
$[(x_i, y_{j+34}, y_{j+4}), (y_{j+4}, x_{i+1}, y_{j+33}), (y_{j+33}, x_{i+2}, y_{j+5}), (y_{j+5}, x_{i+3}, y_{j+32}), (y_{j+32}, x_{i+4}, y_{j+6}), (y_{j+6}, y_{j+31}, x_i)]$,
$[(x_i, y_{j+31}, y_{j+7}), (y_{j+7}, x_{i+1}, y_{j+30}), (y_{j+30}, x_{i+2}, y_{j+8}), (y_{j+8}, x_{i+3}, y_{j+29}), (y_{j+29}, x_{i+4}, y_{j+9}), (y_{j+9}, y_{j+28}, x_i)]$,
where $i \in \mathbb{Z}_\ell$.

The collection of these $111\,\ell$ loose 6-cycles yield the required decomposition of $K_{\overline{\ell,37}}^{(3)}$. $\square$

Lemma 3.5. *If $C_{11} \,|\, K_n$, then $LC_6^{(3)} \,|\, K_{\overline{n,18}}^{(3)}$.*

Proof. Consider $K_{\overline{n,18}}^{(3)} = K_{\overline{X,Y}}^{(3)}$ with $X = \{x_i \mid i = 1, 2, \ldots, n\}$ and $Y = \{y_j \mid j = 1, 2, \ldots, 18\}$. Let $\mathscr{D}$ be a C_{11}-decomposition of K_n. For each $C_{11} = x_{i_1}x_{i_2}x_{i_3}x_{i_4}x_{i_5}x_{i_6}x_{i_7}x_{i_8}\; x_{i_9}x_{i_{10}}x_{i_{11}}x_{i_1}$ in $\mathscr{D}$, we consider the following $LC_6^{(3)}$'s in $K_{\overline{n,18}}^{(3)}$.

$[(y_1, x_{i_1}, x_{i_2}), (x_{i_2}, y_4, x_{i_3}), (x_{i_3}, y_7, x_{i_4}), (x_{i_4}, y_{10}, x_{i_5}), (x_{i_5}, y_{13}, x_{i_6}), (x_{i_6}, x_{i_7}, y_{16})]$,
$[(y_2, x_{i_1}, x_{i_2}), (x_{i_2}, y_5, x_{i_3}), (x_{i_3}, y_8, x_{i_4}), (x_{i_4}, y_{11}, x_{i_5}), (x_{i_5}, y_{14}, x_{i_6}), (x_{i_6}, x_{i_7}, y_{17})]$,
$[(y_3, x_{i_1}, x_{i_2}), (x_{i_2}, y_6, x_{i_3}), (x_{i_3}, y_9, x_{i_4}), (x_{i_4}, y_{12}, x_{i_5}), (x_{i_5}, y_{15}, x_{i_6}), (x_{i_6}, x_{i_7}, y_{18})]$,
$[(y_1, x_{i_2}, x_{i_3}), (x_{i_3}, y_4, x_{i_4}), (x_{i_4}, y_7, x_{i_5}), (x_{i_5}, y_{10}, x_{i_6}), (x_{i_6}, y_{13}, x_{i_7}), (x_{i_7}, x_{i_8}, y_{16})]$,
$[(y_2, x_{i_2}, x_{i_3}), (x_{i_3}, y_5, x_{i_4}), (x_{i_4}, y_8, x_{i_5}), (x_{i_5}, y_{11}, x_{i_6}), (x_{i_6}, y_{14}, x_{i_7}), (x_{i_7}, x_{i_8}, y_{17})]$,
$[(y_3, x_{i_2}, x_{i_3}), (x_{i_3}, y_6, x_{i_4}), (x_{i_4}, y_9, x_{i_5}), (x_{i_5}, y_{12}, x_{i_6}), (x_{i_6}, y_{15}, x_{i_7}), (x_{i_7}, x_{i_8}, y_{18})]$,
$[(y_1, x_{i_3}, x_{i_4}), (x_{i_4}, y_4, x_{i_5}), (x_{i_5}, y_7, x_{i_6}), (x_{i_6}, y_{10}, x_{i_7}), (x_{i_7}, y_{13}, x_{i_8}), (x_{i_8}, x_{i_9}, y_{16})]$,
$[(y_2, x_{i_3}, x_{i_4}), (x_{i_4}, y_5, x_{i_5}), (x_{i_5}, y_8, x_{i_6}), (x_{i_6}, y_{11}, x_{i_7}), (x_{i_7}, y_{14}, x_{i_8}), (x_{i_8}, x_{i_9}, y_{17})]$,
$[(y_3, x_{i_3}, x_{i_4}), (x_{i_4}, y_6, x_{i_5}), (x_{i_5}, y_9, x_{i_6}), (x_{i_6}, y_{12}, x_{i_7}), (x_{i_7}, y_{15}, x_{i_8}), (x_{i_8}, x_{i_9}, y_{18})]$,
$[(y_1, x_{i_4}, x_{i_5}), (x_{i_5}, y_4, x_{i_6}), (x_{i_6}, y_7, x_{i_7}), (x_{i_7}, y_{10}, x_{i_8}), (x_{i_8}, y_{13}, x_{i_9}), (x_{i_9}, x_{i_{10}}, y_{16})]$,
$[(y_2, x_{i_4}, x_{i_5}), (x_{i_5}, y_5, x_{i_6}), (x_{i_6}, y_8, x_{i_7}), (x_{i_7}, y_{11}, x_{i_8}), (x_{i_8}, y_{14}, x_{i_9}), (x_{i_9}, x_{i_{10}}, y_{17})]$,
$[(y_3, x_{i_4}, x_{i_5}), (x_{i_5}, y_6, x_{i_6}), (x_{i_6}, y_9, x_{i_7}), (x_{i_7}, y_{12}, x_{i_8}), (x_{i_8}, y_{15}, x_{i_9}), (x_{i_9}, x_{i_{10}}, y_{18})]$,
$[(y_1 x_{i_5}, x_{i_6}), (x_{i_6}, y_4, x_{i_7}), (x_{i_7}, y_7, x_{i_8}), (x_{i_8}, y_{10}, x_{i_9}), (x_{i_9}, y_{13}, x_{i_{10}}), (x_{i_{10}}, x_{i_{11}}, y_{16})]$,
$[(y_2, x_{i_5}, x_{i_6}), (x_{i_6}, y_5, x_{i_7}), (x_{i_7}, y_8, x_{i_8}), (x_{i_8}, y_{11}, x_{i_9}), (x_{i_9}, y_{14}, x_{i_{10}}), (x_{i_{10}}, x_{i_{11}}, y_{17})]$,
$[(y_3, x_{i_5}, x_{i_6}), (x_{i_6}, y_6, x_{i_7}), (x_{i_7}, y_9, x_{i_8}), (x_{i_8}, y_{12}, x_{i_9}), (x_{i_9}, y_{15}, x_{i_{10}}), (x_{i_{10}}, x_{i_{11}}, y_{18})]$,
$[(y_1, x_{i_6}, x_{i_7}), (x_{i_7}, y_4, x_{i_8}), (x_{i_8}, y_7, x_{i_9}), (x_{i_9}, y_{10}, x_{i_{10}}), (x_{i_{10}}, y_{13}, x_{i_{11}}), (x_{i_{11}}, x_{i_1}, y_{16})]$,
$[(y_2, x_{i_6}, x_{i_7}), (x_{i_7}, y_5, x_{i_8}), (x_{i_8}, y_8, x_{i_9}), (x_{i_9}, y_{11}, x_{i_{10}}), (x_{i_{10}}, y_{14}, x_{i_{11}}), (x_{i_{11}}, x_{i_1}, y_{17})]$,
$[(y_3, x_{i_6}, x_{i_7}), (x_{i_7}, y_6, x_{i_8}), (x_{i_8}, y_9, x_{i_9}), (x_{i_9}, y_{12}, x_{i_{10}}), (x_{i_{10}}, y_{15}, x_{i_{11}}), (x_{i_{11}}, x_{i_1}, y_{18})]$,
$[(y_1, x_{i_7}, x_{i_8}), (x_{i_8}, y_4, x_{i_9}), (x_{i_9}, y_7, x_{i_{10}}), (x_{i_{10}}, y_{10}, x_{i_{11}}), (x_{i_{11}}, y_{13}, x_{i_1}), (x_{i_1}, x_{i_2}, y_{16})]$,
$[(y_2, x_{i_7}, x_{i_8}), (x_{i_8}, y_5, x_{i_9}), (x_{i_9}, y_8, x_{i_{10}}), (x_{i_{10}}, y_{11}, x_{i_{11}}), (x_{i_{11}}, y_{14}, x_{i_1}), (x_{i_1}, x_{i_2}, y_{17})]$,
$[(y_3, x_{i_7}, x_{i_8}), (x_{i_8}, y_6, x_{i_9}), (x_{i_9}, y_9, x_{i_{10}}), (x_{i_{10}}, y_{12}, x_{i_{11}}), (x_{i_{11}}, y_{15}, x_{i_1}), (x_{i_1}, x_{i_2}, y_{18})]$,
$[(y_1, x_{i_8}, x_{i_9}), (x_{i_9}, y_4, x_{i_{10}}), (x_{i_{10}}, y_7, x_{i_{11}}), (x_{i_{11}}, y_{10}, x_{i_1}), (x_{i_1}, y_{13}, x_{i_2}), (x_{i_2}, x_{i_3}, y_{16})]$,
$[(y_2, x_{i_8}, x_{i_9}), (x_{i_9}, y_5, x_{i_{10}}), (x_{i_{10}}, y_8, x_{i_{11}}), (x_{i_{11}}, y_{11}, x_{i_1}), (x_{i_1}, y_{14}, x_{i_2}), (x_{i_2}, x_{i_3}, y_{17})]$,
$[(y_3, x_{i_8}, x_{i_9}), (x_{i_9}, y_6, x_{i_{10}}), (x_{i_{10}}, y_9, x_{i_{11}}), (x_{i_{11}}, y_{12}, x_{i_1}), (x_{i_1}, y_{15}, x_{i_2}), (x_{i_2}, x_{i_3}, y_{18})]$,
$[(y_1, x_{i_9}, x_{i_{10}}), (x_{i_{10}}, y_4, x_{i_{11}}), (x_{i_{11}}, y_7, x_{i_1}), (x_{i_1}, y_{10}, x_{i_2}), (x_{i_2}, y_{13}, x_{i_3}), (x_{i_3}, x_{i_4}, y_{16})]$,
$[(y_2, x_{i_9}, x_{i_{10}}), (x_{i_{10}}, y_5, x_{i_{11}}), (x_{i_{11}}, y_8, x_{i_1}), (x_{i_1}, y_{11}, x_{i_2}), (x_{i_2}, y_{14}, x_{i_3}), (x_{i_3}, x_{i_4}, y_{17})]$,
$[(y_3, x_{i_9}, x_{i_{10}}), (x_{i_{10}}, y_6, x_{i_{11}}), (x_{i_{11}}, y_9, x_{i_1}), (x_{i_1}, y_{12}, x_{i_2}), (x_{i_2}, y_{15}, x_{i_3}), (x_{i_3}, x_{i_4}, y_{18})]$,
$[(y_1, x_{i_{10}}, x_{i_{11}}), (x_{i_{11}}, y_4, x_{i_1}), (x_{i_1}, y_7, x_{i_2}), (x_{i_2}, y_{10}, x_{i_3}), (x_{i_3}, y_{13}, x_{i_4}), (x_{i_4}, x_{i_5}, y_{16})]$,
$[(y_2, x_{i_{10}}, x_{i_{11}}), (x_{i_{11}}, y_5, x_{i_1}), (x_{i_1}, y_8, x_{i_2}), (x_{i_2}, y_{11}, x_{i_3}), (x_{i_3}, y_{14}, x_{i_4}), (x_{i_4}, x_{i_5}, y_{17})]$,
$[(y_3, x_{i_{10}}, x_{i_{11}}), (x_{i_{11}}, y_6, x_{i_1}), (x_{i_1}, y_9, x_{i_2}), (x_{i_2}, y_{12}, x_{i_3}), (x_{i_3}, y_{15}, x_{i_4}), (x_{i_4}, x_{i_5}, y_{18})]$,
$[(y_1, x_{i_{11}}, x_{i_1}), (x_{i_1}, y_4, x_{i_2}), (x_{i_2}, y_7, x_{i_3}), (x_{i_3}, y_{10}, x_{i_4}), (x_{i_4}, y_{13}, x_{i_5}), (x_{i_5}, x_{i_6}, y_{16})]$,
$[(y_2, x_{i_{11}}, x_{i_1}), (x_{i_1}, y_5, x_{i_2}), (x_{i_2}, y_8, x_{i_3}), (x_{i_3}, y_{11}, x_{i_4}), (x_{i_4}, y_{14}, x_{i_5}), (x_{i_5}, x_{i_6}, y_{17})]$,
$[(y_3, x_{i_{11}}, x_{i_1}), (x_{i_1}, y_6, x_{i_2}), (x_{i_2}, y_9, x_{i_3}), (x_{i_3}, y_{12}, x_{i_4}), (x_{i_4}, y_{15}, x_{i_5}), (x_{i_5}, x_{i_6}, y_{18})]$.

The collection of these loose 6-cycles yield the required decomposition of $K_{\overline{n,18}}^{(3)}$. $\square$

Since $K^{(3)}_{n,\overline{36}} = K^{(3)}_{n,\overline{18}} \oplus K^{(3)}_{n,\overline{18}}$, we have, by above lemma,

Lemma 3.6. *If $C_{11} \mid K_n$, then $LC^{(3)}_6 \mid K^{(3)}_{n,\overline{36}}$.*

Lemma 3.7. $LC^{(3)}_6 | K^{(3)}_{11,36}$.

Proof. Write $K^{(3)}_{11,36}$ as $K^{(3)}_{11,\overline{36}} \oplus K^{(3)}_{\overline{11},36}$. By Lemma 3.3, $LC^{(3)}_6 \mid K^{(3)}_{\overline{11},36}$. As $C_{11} \mid K_{11}$, by Lemma 3.6, $LC^{(3)}_6 \mid K^{(3)}_{11,\overline{36}}$. Hence, $LC^{(3)}_6 | K^{(3)}_{11,36}$. □

Lemma 3.8. $LC^{(3)}_6 | K^{(3)}_{36,37}$.

Proof. Write $K^{(3)}_{36,37}$ as $K^{(3)}_{36,\overline{37}} \oplus K^{(3)}_{\overline{36},37}$. By Lemmas 3.3 and 3.4, we have, respectively, $LC^{(3)}_6 \mid K^{(3)}_{36,\overline{37}}$ and $LC^{(3)}_6 \mid K^{(3)}_{\overline{36},37}$. Hence, $LC^{(3)}_6 | K^{(3)}_{36,37}$. □

Lemma 3.9. $LC^{(3)}_6 | K^{(3)}_{36,45}$.

Proof. Write $K^{(3)}_{36,45}$ as $K^{(3)}_{36,\overline{45}} \oplus K^{(3)}_{\overline{36},45}$. By Lemma 3.3, $LC^{(3)}_6 \mid K^{(3)}_{36,\overline{45}}$. By Lemma 3.6, $LC^{(3)}_6 \mid K^{(3)}_{\overline{36},45}$, if $C_{11} | K_{45}$. The existence of the decomposition $C_{11} | K_{45}$ follows from the result of Sajna [13]: "Let n be an odd integer and m be an even integer with $3 \leq m \leq n$. The complete graph K_n can be decomposed into cycles of length m whenever m divides the number of edges in K_n." Hence, $LC^{(3)}_6 | K^{(3)}_{36,45}$. □

4. Loose 6-cycle decompositions of complete tripartite 3-uniform hypergraphs

For pairwise disjoint sets X, Y and Z, the hypergraph with vertex set $X \cup Y \cup Z$ and edge set consisting of all 3-sets having exactly one vertex in each of X, Y and Z is denoted by $K^{(3)}_{X,Y,Z}$ or $K^{(3)}_{|X|,|Y|,|Z|}$.

Lemma 4.1. *If $m, n \geq 6$ are even integers, $3 \mid mn$, and $\ell \geq 5$ is an integer, then $LC^{(3)}_6 \mid K^{(3)}_{m,n,\ell}$.*

To prove Lemma 4.1, we use the following:

Theorem 4.2. (Truszczyński [15]). *If k, m, n are positive integers with m, n even and $m \geq n$, then $K_{m,n}$ has a P_{k+1}-decomposition if and only if $m \geq \lceil \frac{k+1}{2} \rceil$, $n \geq \lceil \frac{k}{2} \rceil$ and $mn \equiv 0 \pmod{k}$.*

Proof of Lemma 4.1.

Let $K^{(3)}_{m,n,\ell} = K^{(3)}_{X,Y,Z}$, where $X = \{x_1, \ldots, x_m\}$, $Y = \{y_1, \ldots, y_n\}$ and $Z = \{z_i \mid i \in \mathbb{Z}_\ell\}$. Consider the complete bipartite graph $K_{m,n}$ with bipartition (X, Y). By Theorem 4.2, $P_7 \mid K_{m,n}$. Let $\mathscr{D}$ be one such decomposition. For each $P_7 := v_1v_2v_3v_4v_5v_6v_7$ in $\mathscr{D}$, construct ℓ edge-disjoint loose 6-cycles $[(z_i, v_1, v_2), (v_2, z_{i+1}, v_3), (v_3, z_{i+2}, v_4), (v_4, z_{i+3}, v_5), (v_5, z_{i+4}, v_6), (v_6, v_7, z_i)]$ of $K^{(3)}_{m,n,\ell}$, where $i \in \mathbb{Z}_\ell$. Collection of these loose 6-cycles yield a decomposition of $K^{(3)}_{m,n,\ell}$. □

5. More loose 6-cycle decompositions

Lemma 5.1. *If $n \equiv 0 \pmod{6}$, then $LC^{(3)}_6 \mid K^{(3)}_{n,n}$.*

Proof. Then, $n = 6s$ for some integer $s \geq 1$, and therefore $K^{(3)}_{n,n} = K^{(3)}_{X,Y}$, where $X = \bigcup_{i=1}^{s} X_i$ and $Y = \bigcup_{j=1}^{s} Y_j$ be disjoint union of sets $X_1, \ldots, X_s$ and $Y_1, \ldots, Y_s$, respectively, with $|X_i| = |Y_j| = 6$, where $i, j \in \{1, \ldots, s\}$. Write $K^{(3)}_{X,Y}$ as an edge-disjoint union of $K^{(3)}_{X_i,Y_j} \cong K^{(3)}_{6,6}$, $i, j \in \{1, \ldots, s\}$; $K^{(3)}_{X_{i_1},X_{i_2},Y_j} \cong K^{(3)}_{6,6,6}$, i_1, i_2, $j \in \{1, \ldots, s\}$ and $i_1 \neq i_2$; and $K^{(3)}_{X_i,Y_{j_1},Y_{j_2}} \cong K^{(3)}_{6,6,6}$, i, j_1, $j_2 \in \{1, \ldots, s\}$ and $j_1 \neq j_2$. By Lemma 3.1, $LC^{(3)}_6 | K^{(3)}_{6,6}$; and by Lemma 4.1, $LC^{(3)}_6 | K^{(3)}_{6,6,6}$. Hence, $LC^{(3)}_6 | K^{(3)}_{n,n}$. □

Lemma 5.2. $LC_6^{(3)} \mid K_{36}^{(3)}$.

Proof. Write $K_{36}^{(3)} = 2K_{18}^{(3)} \oplus K_{18,18}^{(3)}$. By Lemmas 2.3 and 5.1, we have, respectively, $LC_6^{(3)} \mid K_{18}^{(3)}$ and $LC_6^{(3)} \mid K_{18,18}^{(3)}$. Hence, $LC_6^{(3)} \mid K_{36}^{(3)}$. □

Lemma 5.3. $LC_6^{(3)} | K_{18,20}^{(3)}$.

Proof. Write $K_{18,20}^{(3)} \cong K_{20,18}^{(3)}$ as $K_{10,18}^{(3)} \oplus K_{10,18}^{(3)} \oplus K_{10,10,18}^{(3)}$. By Lemmas 3.2 and 4.1, we have, respectively, $LC_6^{(3)} \mid K_{10,18}^{(3)}$ and $LC_6^{(3)} \mid K_{18,10,10}^{(3)}$. Hence, $LC_6^{(3)} | K_{18,20}^{(3)}$. □

Lemma 5.4. $LC_6^{(3)} | K_{18,28}^{(3)}$.

Proof. Write $K_{18,28}^{(3)} \cong K_{28,18}^{(3)}$ as $K_{10,18}^{(3)} \oplus K_{18,18}^{(3)} \oplus K_{10,18,18}^{(3)}$. By Lemmas 3.2, 5.1 and 4.1, we have, respectively, $LC_6^{(3)} \mid K_{10,18}^{(3)}$, $LC_6^{(3)} \mid K_{18,18}^{(3)}$ and $LC_6^{(3)} \mid K_{10,18,18}^{(3)}$. Hence, $LC_6^{(3)} | K_{18,28}^{(3)}$. □

Lemma 5.5. $LC_6^{(3)} | K_{18,36}^{(3)}$.

Proof. Write $K_{18,36}^{(3)}$ as $K_{18,18}^{(3)} \oplus K_{18,18}^{(3)} \oplus K_{18,18,18}^{(3)}$. By Lemmas 5.1 and 4.1, we have, respectively, $LC_6^{(3)} \mid K_{18,18}^{(3)}$ and $LC_6^{(3)} \mid K_{18,18,18}^{(3)}$. Hence, $LC_6^{(3)} | K_{18,36}^{(3)}$. □

Lemma 5.6. $LC_6^{(3)} | K_{29,36}^{(3)}$.

Proof. Write $K_{29,36}^{(3)}$ as $K_{11,36}^{(3)} \oplus K_{18,36}^{(3)} \oplus K_{11,18,36}^{(3)}$. By Lemmas 3.7, 5.5 and 4.1, we have, respectively, $LC_6^{(3)} \mid K_{11,36}^{(3)}$, $LC_6^{(3)} \mid K_{18,36}^{(3)}$ and $LC_6^{(3)} \mid K_{18,36,11}^{(3)}$. Hence, $LC_6^{(3)} | K_{29,36}^{(3)}$. □

6. Proof of Theorem 1.1

The proof of the necessity is obvious. The congruence in the necessary condition follows from the divisibility condition $6 \mid \binom{n}{3}$. Now, we prove the sufficiency. We consider three cases.

Case 1. $n = 18k + \ell$, where $\ell \in \{0, 2, 10\}$.

Then, $K_n^{(3)} = K_{18k+\ell}^{(3)} = K_X^{(3)}$, where $X = X_0 \cup X_1 \cup X_2 \cup \cdots \cup X_{k-1}$ be pairwise disjoint union of sets $X_0, X_1, X_2, \ldots, X_{k-1}$ with $|X_0| = 18 + \ell$ and $|X_1| = |X_2| = \ldots = |X_{k-1}| = 18$. Write $K_X^{(3)}$ as an edge-disjoint union of $K_{X_0}^{(3)} \cong K_{18+\ell}^{(3)}$, $K_{X_i}^{(3)} \cong K_{18}^{(3)}$, $K_{X_i,X_0}^{(3)} \cong K_{18,18+\ell}^{(3)}$, $K_{X_{i_1},X_{i_2}}^{(3)} \cong K_{18,18}^{(3)}$, $K_{X_{i_1},X_{i_2},X_0}^{(3)} \cong K_{18,18,18+\ell}^{(3)}$, $K_{X_{i_1},X_{i_2},X_{i_3}}^{(3)} \cong K_{18,18,18}^{(3)}$, where $i, i_1, i_2, i_3 \in \{1, 2, \ldots, k-1\}$, $i_1 \neq i_2$, $i_1 \neq i_3$ and $i_2 \neq i_3$. By Lemmas 2.3, 2.4 and 2.5, we have, respectively, $LC_6^{(3)} \mid K_{18}^{(3)}$, $LC_6^{(3)} \mid K_{20}^{(3)}$ and $LC_6^{(3)} \mid K_{28}^{(3)}$. By Lemmas 5.1, 5.3 and 5.4, we have, respectively, $LC_6^{(3)} \mid K_{18,18}^{(3)}$, $LC_6^{(3)} \mid K_{18,20}^{(3)}$ and $LC_6^{(3)} \mid K_{18,28}^{(3)}$. By Lemma 4.1, $LC_6^{(3)} \mid K_{18,18,18+\ell}^{(3)}$, $\ell \in \{0, 2, 10\}$.

Case 2. $n = 36k + \ell$, where $\ell \in \{1, 9\}$.

Then, $K_n^{(3)} = K_{36k+\ell}^{(3)} = K_X^{(3)}$, where $X = X_0 \cup X_1 \cup X_2 \cup \cdots \cup X_{k-1}$ be pairwise disjoint union of sets $X_0, X_1, X_2, \ldots, X_{k-1}$ with $|X_0| = 36 + \ell$ and $|X_1| = |X_2| = \ldots = |X_{k-1}| = 36$. Write $K_X^{(3)}$ as an edge-disjoint union of $K_{X_0}^{(3)} \cong K_{36+\ell}^{(3)}$, $K_{X_i}^{(3)} \cong K_{36}^{(3)}$, $K_{X_i,X_0}^{(3)} \cong K_{36,36+\ell}^{(3)}$, $K_{X_{i_1},X_{i_2}}^{(3)} \cong K_{36,36}^{(3)}$, $K_{X_{i_1},X_{i_2},X_0}^{(3)} \cong K_{36,36,36+\ell}^{(3)}$, $K_{X_{i_1},X_{i_2},X_{i_3}}^{(3)} \cong K_{36,36,36}^{(3)}$, where $i, i_1, i_2, i_3 \in \{1, 2, \ldots, k-1\}$, $i_1 \neq i_2$, $i_1 \neq i_3$ and $i_2 \neq i_3$. By Lemmas 5.2, 2.2 and 2.6, we have, respectively, $LC_6^{(3)} \mid K_{36}^{(3)}$, $LC_6^{(3)} \mid K_{37}^{(3)}$ and $LC_6^{(3)} \mid K_{45}^{(3)}$. By Lemmas 5.1, 3.8 and 3.9, we have, respectively, $LC_6^{(3)} \mid K_{36,36}^{(3)}$, $LC_6^{(3)} \mid K_{36,37}^{(3)}$ and $LC_6^{(3)} \mid K_{36,45}^{(3)}$. By Lemma 4.1, $LC_6^{(3)} \mid K_{36,36,36+\ell}^{(3)}$, $\ell \in \{0, 1, 9\}$.

Case 3. $n = 36k + 29$.

Then, $K_n^{(3)} = K_{36k+29}^{(3)} = K_X^{(3)}$, where $X = X_0 \cup X_1 \cup X_2 \cup \cdots \cup X_k$ be pairwise disjoint union of sets $X_0, X_1, X_2, \ldots, X_k$ with $|X_0| = 29$ and $|X_1| = |X_2| = \ldots = |X_k| = 36$. Write $K_X^{(3)}$ as an edge-disjoint union of $K^{(3)}_{X_0} \cong K^{(3)}_{29}$, $K^{(3)}_{X_i} \cong K^{(3)}_{36}$, $K^{(3)}_{X_i,X_0} \cong K^{(3)}_{36,29}$, $K^{(3)}_{X_{i_1},X_{i_2}} \cong K^{(3)}_{36,36}$, $K^{(3)}_{X_{i_1},X_{i_2},X_0} \cong K^{(3)}_{36,36,29}$, $K^{(3)}_{X_{i_1},X_{i_2},X_{i_3}} \cong K^{(3)}_{36,36,36}$, where i, i_1, i_2, $i_3 \in \{1, 2, \dots, k\}$, $i_1 \neq i_2$, $i_1 \neq i_3$ and $i_2 \neq i_3$. By Lemmas 2.1 and 5.2, we have, respectively, $LC^{(3)}_6 \mid K^{(3)}_{29}$ and $LC^{(3)}_6 \mid K^{(3)}_{36}$. By Lemmas 5.6 and 5.1, we have, respectively, $LC^{(3)}_6 \mid K^{(3)}_{36,29}$ and $LC^{(3)}_6 \mid K^{(3)}_{36,36}$. By Lemma 4.1, $LC^{(3)}_6 \mid K^{(3)}_{36,36,36-\ell}$, $\ell \in \{0, 7\}$.

In any case, $LC^{(3)}_m \mid K^{(3)}_n$. This completes the proof. □

Declaration of competing interest

The authors declare that they have no conflict of interest.

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