



## Hermite-Hadamard-Mercer type inequalities for twice differentiable convex involving Riemann-Liouville fractional integrals

Talib Hussain<sup>a,\*</sup>, Loredana Ciurdariu<sup>b</sup>, Merfa Sittar<sup>a</sup>, Juan E. Nápoles Valdés<sup>c</sup>

<sup>a</sup>Department of Mathematics and Statistics, University of Agriculture Faisalabad, Faisalabad 38000, Pakistan

<sup>b</sup>Department of Mathematics, Politehnica University of Timișoara, 300006 Timișoara, Romania

<sup>c</sup>Faculty of Exact and Natural Sciences and Surveying, National University of the Northeast, Corrientes 3400, Argentina

**Abstract.** In this study, authors use Riemann-Liouville fractional integrals to get several new inequalities of Hermite-Hadamard-Mercer type. We establish some trapezoid and midpoint type inequalities for functions whose twice derivatives in absolute value are convex involving Riemann-Liouville fractional integrals. The results of the paper are extensions and refinements of Hermite-Hadamard and Hermite-Hadamard-Mercer type inequalities. We discuss special cases of our main results and give new inequalities of the Hermite-Hadamard and Hermite-Hadamard-Mercer type. These results are accompanied by further remarks and observations. Next, we see the efficiency of our inequalities via some applications on special means. Lastly, a couple of examples through graphical visualizations are provided to illustrate the key findings of our research.

### 1. Introduction

The remarkable concept of mathematical inequalities has long been a subject of discussion among mathematicians; some interesting applications are found in the following areas: operator equations, numerical analysis, fractional calculus, quantum information theory, quantum calculus, network theory, and operator theory. At the moment, this is a very active research topic, enhanced by the interaction between different fields. Numerical integration and definite integral estimate have significance in applied science.

Fractional calculus is popular among researchers due to its behavior and applications not only in the field of mathematical sciences but also in different fields of applied sciences such as epidemiology [6], nanotechnology [7], physics [11], bio-engineering [14], and control systems [17]. Researchers have used fractional calculus to construct various fractional integral inequalities because of their significance in approximation theory. Based on their fundamental properties, researchers derive novel fractional operators and apply them to numerous real-world problems. There are numerous articles and monographs, where new fractional operators are employed to enhance integral inequalities including the Ostrowski inequality,

---

2020 *Mathematics Subject Classification.* Primary 26A33; Secondary 26A51, 26D15, 26D10

*Keywords.* Hermite-Hadamard inequality, Jensen Mercer inequality, Hermite-Hadamard Mercer inequality, Riemann-Liouville fractional integrals, Convex function

Received: 05 August 2024; Revised: 21 July 2025; Accepted: 28 September 2025

Communicated by Miodrag Spalević

\* Corresponding author: Talib Hussain

*Email addresses:* talibuaf2915@gmail.com (Talib Hussain), loredana.ciurdariu@upt.ro (Loredana Ciurdariu), merfasittar786@gmail.com (Merfa Sittar), jnapoles@exa.unne.edu.ar (Juan E. Nápoles Valdés)

ORCID iDs: <https://orcid.org/0000-0002-9238-3509> (Talib Hussain), <https://orcid.org/0000-0002-1234-7939> (Loredana Ciurdariu), <https://orcid.org/0000-0003-2470-1090> (Juan E. Nápoles Valdés)

Simpson inequality, Fejer type inequality, Hermite-Hadamard inequality, Jensen-Mercer type inequality, and so on. The famous Hermite-Hadamard inequality associated with convex functions is one such important integral inequality.

The concept of a convex function is extremely important in both pure and applied mathematics. Convex functions are widely used in finance, economy, science and engineering. A function  $h : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ ,  $J := [\delta_1, \delta_2]$  is said to be convex if

$$h(\xi\delta_1 + (1 - \xi)\delta_2) \leq \xi h(\delta_1) + (1 - \xi)h(\delta_2),$$

holds for all  $\delta_1, \delta_2 \in J$  and  $\xi \in [0, 1]$ . If the above inequality is reversed, then the function  $h$  will be the concave on  $[\delta_1, \delta_2]$ . Readers interested in having a broader overview of the development of different notions of convexity can consult [19], which covers the various ramifications and extensions of the classical notion of a convex function.

Let  $0 < x_1 \leq x_2 \leq \dots \leq x_n$  and let  $\theta_i = (\theta_1, \theta_2, \dots, \theta_n)$  non-negative weights such that  $\sum_{i=1}^n \theta_i = 1$ . The Jensen inequality [9] states that  $h$  is a convex function on the interval  $[\delta_1, \delta_2]$ ; then

$$h\left(\sum_{i=1}^n \theta_i x_i\right) \leq \sum_{i=1}^n \theta_i h(x_i),$$

where for all  $x_i \in [\delta_1, \delta_2]$  and  $\theta_i \in [0, 1]$ ,  $(i = \overline{1, n})$ . In particular, if  $n = 2$  and  $\theta_1 = \theta_2 = \frac{1}{2}$  we have the equation (1) of [12]:

$$h\left(\frac{x_1 + x_2}{2}\right) \leq \frac{h(x_1) + h(x_2)}{2}.$$

The Hermite-Hadamard inequality states that if a mapping  $h : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is a convex function on  $J$  with  $\delta_1, \delta_2 \in I$ ,  $\delta_1 < \delta_2$  then,

$$h\left(\frac{\delta_1 + \delta_2}{2}\right) \leq \frac{1}{\delta_2 - \delta_1} \int_{\delta_1}^{\delta_2} h(x) dx \leq \frac{h(\delta_1) + h(\delta_2)}{2}. \quad (1)$$

In other words, the integral mean value of a convex function, interpolates the Jensen Inequality for  $n = 2$  and we note that this inequality can make a bounded estimation of the integral mean on  $[\delta_1, \delta_2]$ , so it has wide applications in numerical integration, so the Hermite-Hadamard Inequality is one of the topics that attracts the most attention in the Mathematical Sciences today (see [1, 18]).

The double inequality (1) holds in the reversed direction if  $h$  is concave [10].

**Theorem 1.1.** (See [16]). If  $h$  is convex function on  $J = [\delta_1, \delta_2]$ , then

$$h\left(\delta_1 + \delta_2 - \sum_{i=1}^n \theta_i x_i\right) \leq h(\delta_1) + h(\delta_2) - \sum_{i=1}^n \theta_i h(x_i),$$

for each  $x_i \in [\delta_1, \delta_2]$  and  $\theta_i \in [0, 1]$ ,  $(i = \overline{1, n})$  with  $\sum_{i=1}^n \theta_i = 1$ .

**Remark 1.2.** For some results related with Jensen-Mercer inequality, see([2], [3], [13], [15],[16], [20]).

After these necessary inequalities about convex functions, we will now give the definitions which we will use in this paper.

**Definition 1.3.** (see [8], [23], [24], [27]). Let  $h \in L[\delta_1, \delta_2]$ . The Left sided and right sided Riemann-Liouville fractional integrals  $I_{\delta_1^+}^\alpha h$  and  $I_{\delta_2^-}^\alpha h$  of order  $\alpha > 0$  with  $\delta_1 \geq 0$  can be defined respectively by

$$I_{\delta_1^+}^\alpha h(x) = \frac{1}{\Gamma(\alpha)} \int_{\delta_1}^x (x - \xi)^{\alpha-1} h(\xi) d\xi, \quad x > \delta_1$$

$$I_{\delta_2}^{\alpha} \hbar(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\delta_2} (\xi - x)^{\alpha-1} \hbar(\xi) d\xi, \quad x < \delta_2$$

here  $\Gamma(\alpha)$  is Euler Gamma function. and  $I_{\delta_1}^0 \hbar(x) = I_{\delta_2}^0 \hbar(x) = \hbar(x)$ .

Recently, Öğülüş and Sarıkaya [21] introduced several integral inequalities of Hermite–Hadamard–Mercer type for functions whose first derivatives in absolute value are convex functions via Riemann–Liouville fractional integrals. We present two of their Lemmas here for the purpose of our new developments.

**Lemma 1.4.** ([21]). Let  $\hbar : J = [\delta_1, \delta_2] \rightarrow \mathbb{R}$  be a twice differentiable mapping on  $(\delta_1, \delta_2)$  with  $\delta_1 < \delta_2$ . If  $\hbar' \in L[\delta_1, \delta_2]$ , then the following identity for fractional integrals holds:

$$\begin{aligned} & \frac{\hbar(\delta_1 + \delta_2 - s_1) + \hbar(\delta_1 + \delta_2 - s_2)}{2} - \frac{\Gamma(\alpha + 1)}{2(s_2 - s_1)^{\alpha}} \left[ I_{(\delta_1 + \delta_2 - s_2)^+}^{\alpha} \hbar(\delta_1 + \delta_2 - s_1) + I_{(\delta_1 + \delta_2 - s_1)^-}^{\alpha} \hbar(\delta_1 + \delta_2 - s_2) \right] \\ &= \frac{s_2 - s_1}{2} \int_0^1 (\xi^{\alpha} - (1 - \xi)^{\alpha}) \hbar'(\delta_1 + \delta_2 - (\xi s_1 + (1 - \xi)s_2)) d\xi, \end{aligned} \quad (2)$$

for all  $s_1, s_2 \in [\delta_1, \delta_2]$ ,  $\alpha > 0$  and  $\xi \in [0, 1]$ .

**Lemma 1.5.** ([21]). Assume that  $\hbar : J = [\delta_1, \delta_2] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(\delta_1, \delta_2)$  such that  $\hbar' \in L[\delta_1, \delta_2]$  where  $\delta_1, \delta_2 \in J$  with  $\delta_1 < \delta_2$ . Then the following identity for fractional integrals holds:

$$\begin{aligned} & \frac{2^{\alpha-1} \Gamma(\alpha + 1)}{2(s_2 - s_1)^{\alpha}} \left[ I_{(\delta_1 + \delta_2 - \frac{s_1 + s_2}{2})^+}^{\alpha} \hbar(\delta_1 + \delta_2 - s_1) + I_{(\delta_1 + \delta_2 - \frac{s_1 + s_2}{2})^-}^{\alpha} \hbar(\delta_1 + \delta_2 - s_2) \right] - \hbar\left(\delta_1 + \delta_2 - \frac{s_1 + s_2}{2}\right) \\ &= \frac{s_2 - s_1}{4} \int_0^1 \xi^{\alpha} \left[ \hbar'\left(\delta_1 + \delta_2 - \left(\frac{2 - \xi}{2}s_1 + \frac{\xi}{2}s_2\right)\right) - \hbar'\left(\delta_1 + \delta_2 - \left(\frac{\xi}{2}s_1 + \frac{2 - \xi}{2}s_2\right)\right) \right] d\xi, \end{aligned} \quad (3)$$

for all  $s_1, s_2 \in [\delta_1, \delta_2]$ ,  $\alpha > 0$  and  $\xi \in [0, 1]$ .

The structure of the current article is as follows: After reviewing some basic ideas and concepts about fractional calculus and Jensen–Mercer type inequalities, in Section 2 we establish trapezoid type for twice differential convex functions in the setting of Riemann–Liouville fractional integrals. After that, in the context of Riemann–Liouville fractional integrals, we develop Midpoint type for twice differential convex functions in Section 3. Next, applications to special means of real numbers of the outcomes in Section 4 are examined. Furthermore, the main findings of our study are demonstrated with certain examples through graphical visualizations. in section 5. Finally, a brief conclusion of the findings is provided in Section 6.

## 2. Riemann–Liouville fractional integrals inequalities of trapezoid type for twice differentiable convex functions

Here we give trapezoid type for twice differential convex functions via Riemann–Liouville fractional integrals. We need the following lemma for obtaining those results:

**Lemma 2.1.** Assume that  $\hbar : J = [\delta_1, \delta_2] \rightarrow \mathbb{R}$  be two times differentiable mapping on  $(\delta_1, \delta_2)$  such that  $\hbar'' \in L[\delta_1, \delta_2]$  where  $\delta_1, \delta_2 \in J$  with  $\delta_1 < \delta_2$ . Then the following identity for fractional integrals holds:

$$\begin{aligned} & \frac{\hbar(\delta_1 + \delta_2 - s_1) + \hbar(\delta_1 + \delta_2 - s_2)}{2} - \frac{\Gamma(\alpha + 1)}{2(s_2 - s_1)^{\alpha}} \left[ I_{(\delta_1 + \delta_2 - s_2)^+}^{\alpha} \hbar(\delta_1 + \delta_2 - s_1) + I_{(\delta_1 + \delta_2 - s_1)^-}^{\alpha} \hbar(\delta_1 + \delta_2 - s_2) \right] \\ &= \frac{(s_2 - s_1)^2}{2(\alpha + 1)} \int_0^1 [1 - (1 - \xi)^{\alpha+1} - \xi^{\alpha+1}] \hbar''(\delta_1 + \delta_2 - (\xi s_1 + (1 - \xi)s_2)) d\xi, \end{aligned} \quad (4)$$

for all  $s_1, s_2 \in [\delta_1, \delta_2]$ ,  $\alpha > 0$  and  $\xi \in [0, 1]$ .

*Proof.* By using integration by parts, we have

$$\begin{aligned}
 & \int_0^1 [1 - (1 - \xi)^{\alpha+1} - \xi^{\alpha+1}] \hbar''(\delta_1 + \delta_2 - (\xi s_1 + (1 - \xi)s_2)) d\xi \\
 &= \left. \frac{[1 - (1 - \xi)^{\alpha+1} - \xi^{\alpha+1}] \hbar'(\delta_1 + \delta_2 - (\xi s_1 + (1 - \xi)s_2))}{s_2 - s_1} \right|_0^1 \\
 &+ \frac{\alpha + 1}{(s_2 - s_1)} \int_0^1 (\xi^\alpha - (1 - \xi)^\alpha) \hbar'(\delta_1 + \delta_2 - (\xi s_1 + (1 - \xi)s_2)) d\xi \\
 &= \frac{\alpha + 1}{(s_2 - s_1)} \int_0^1 (\xi^\alpha - (1 - \xi)^\alpha) \hbar'(\delta_1 + \delta_2 - (\xi s_1 + (1 - \xi)s_2)) d\xi.
 \end{aligned} \tag{5}$$

The intended identity in (4) follows from (5) by using (2) and rearranging the terms.  $\square$

**Remark 2.2.** If we take  $s_1 = \delta_1$  and  $s_2 = \delta_2$ , in Lemma 2.1, then Lemma 2.1 gives [26, Lemma 2.1].

**Remark 2.3.** If we take  $s_1 = \delta_1$ ,  $s_2 = \delta_2$  and  $\alpha = 1$ , in Lemma 2.1, then Lemma 2.1 gives [5, Lemma 1].

**Theorem 2.4.** Assume that  $\hbar : J = [\delta_1, \delta_2] \rightarrow \mathbb{R}$  be two times differentiable mapping on  $(\delta_1, \delta_2)$  with  $\delta_1 < \delta_2$ . If  $|\hbar''|^q$  is convex on  $[\delta_1, \delta_2]$ ,  $q \geq 1$ , then for all  $s_1, s_2 \in [\delta_1, \delta_2]$  and  $\alpha > 0$ , the following inequality for fractional integrals holds:

$$\begin{aligned}
 & \left| \frac{\hbar(\delta_1 + \delta_2 - s_1) + \hbar(\delta_1 + \delta_2 - s_2)}{2} - \frac{\Gamma(\alpha + 1)}{2(s_2 - s_1)^\alpha} \left[ I_{(\delta_1 + \delta_2 - s_2)^+}^\alpha \hbar(\delta_1 + \delta_2 - s_1) + I_{(\delta_1 + \delta_2 - s_1)^-}^\alpha \hbar(\delta_1 + \delta_2 - s_2) \right] \right| \\
 & \leq \frac{\alpha(s_2 - s_1)^2}{2(\alpha + 1)(\alpha + 2)} \left( |\hbar''(\delta_1)|^q + |\hbar''(\delta_2)|^q - \frac{|\hbar''(s_1)|^q + |\hbar''(s_2)|^q}{2} \right)^{\frac{1}{q}}.
 \end{aligned} \tag{6}$$

*Proof.* From Lemma 2.1, using the Power mean inequality, we obtain

$$\begin{aligned}
 & \left| \frac{\hbar(\delta_1 + \delta_2 - s_1) + \hbar(\delta_1 + \delta_2 - s_2)}{2} - \frac{\Gamma(\alpha + 1)}{2(s_2 - s_1)^\alpha} \left[ I_{(\delta_1 + \delta_2 - s_2)^+}^\alpha \hbar(\delta_1 + \delta_2 - s_1) + I_{(\delta_1 + \delta_2 - s_1)^-}^\alpha \hbar(\delta_1 + \delta_2 - s_2) \right] \right| \\
 & \leq \frac{(s_2 - s_1)^2}{2(\alpha + 1)} \left( \int_0^1 [1 - (1 - \xi)^{\alpha+1} - \xi^{\alpha+1}] d\xi \right)^{1 - \frac{1}{q}} \\
 & \times \left( \int_0^1 [1 - (1 - \xi)^{\alpha+1} - \xi^{\alpha+1}] |\hbar''(\delta_1 + \delta_2 - (\xi s_1 + (1 - \xi)s_2))|^q d\xi \right)^{\frac{1}{q}}.
 \end{aligned}$$

Using Jensen-Mercer inequality because of the convexity of  $|\hbar''|^q$ , we obtain

$$\begin{aligned}
 & \left| \frac{\hbar(\delta_1 + \delta_2 - s_1) + \hbar(\delta_1 + \delta_2 - s_2)}{2} - \frac{\Gamma(\alpha + 1)}{2(s_2 - s_1)^\alpha} \left[ I_{(\delta_1 + \delta_2 - s_2)^+}^\alpha \hbar(\delta_1 + \delta_2 - s_1) + I_{(\delta_1 + \delta_2 - s_1)^-}^\alpha \hbar(\delta_1 + \delta_2 - s_2) \right] \right| \\
 & \leq \frac{(s_2 - s_1)^2}{2(\alpha + 1)} \left( \int_0^1 [1 - (1 - \xi)^{\alpha+1} - \xi^{\alpha+1}] d\xi \right)^{1 - \frac{1}{q}} \\
 & \times \left( \int_0^1 [1 - (1 - \xi)^{\alpha+1} - \xi^{\alpha+1}] (|\hbar''(\delta_1)|^q + |\hbar''(\delta_2)|^q - (\xi |\hbar''(s_1)|^q + (1 - \xi) |\hbar''(s_2)|^q)) d\xi \right)^{\frac{1}{q}} \\
 & = \frac{(s_2 - s_1)^2}{2(\alpha + 1)} \left( \int_0^1 [1 - (1 - \xi)^{\alpha+1} - \xi^{\alpha+1}] d\xi \right)^{1 - \frac{1}{q}} \\
 & \times \left( (|\hbar''(\delta_1)|^q + |\hbar''(\delta_2)|^q) \int_0^1 [1 - (1 - \xi)^{\alpha+1} - \xi^{\alpha+1}] d\xi \right. \\
 & \left. - \int_0^1 [1 - (1 - \xi)^{\alpha+1} - \xi^{\alpha+1}] (\xi |\hbar''(s_1)|^q + (1 - \xi) |\hbar''(s_2)|^q) d\xi \right)^{\frac{1}{q}}.
 \end{aligned}$$

The desired result follows from the above inequality and using the following computations:

$$\int_0^1 [1 - (1 - \xi)^{\alpha+1} - \xi^{\alpha+1}] d\xi = \frac{\alpha}{\alpha + 2},$$

$$\int_0^1 [1 - (1 - \xi)^{\alpha+1} - \xi^{\alpha+1}] \xi d\xi = \int_0^1 [1 - (1 - \xi)^{\alpha+1} - \xi^{\alpha+1}] (1 - \xi) d\xi = \frac{\alpha}{2(\alpha + 2)},$$

this completes the proof.  $\square$

**Corollary 2.5.** *If we take  $s_1 = \delta_1$  and  $s_2 = \delta_2$ , in Theorem 2.4, then we have*

$$\left| \frac{\hbar(\delta_1) + \hbar(\delta_2)}{2} - \frac{\Gamma(\alpha + 1)}{2(\delta_2 - \delta_1)^\alpha} [I_{(\delta_1)^+}^\alpha \hbar(\delta_2) + I_{(\delta_2)^-}^\alpha \hbar(\delta_1)] \right| \leq \frac{\alpha(\delta_2 - \delta_1)^2}{2(\alpha + 1)(\alpha + 2)} \left( \frac{|\hbar''(\delta_1)|^q + |\hbar''(\delta_2)|^q}{2} \right)^{\frac{1}{q}}.$$

**Remark 2.6.** *If we take  $s_1 = \delta_1$ ,  $s_2 = \delta_2$  and  $\alpha = 1$ , in Theorem 2.4, then Theorem 2.4 gives [25, Remark 3.11].*

**Theorem 2.7.** *Assume that  $\hbar : J = [\delta_1, \delta_2] \rightarrow \mathbb{R}$  be two times differentiable mapping on  $(\delta_1, \delta_2)$  with  $\delta_1 < \delta_2$ . If  $|\hbar''|^q$  is convex on  $[\delta_1, \delta_2]$ ,  $q > 1$ , then for all  $s_1, s_2 \in [\delta_1, \delta_2]$  and  $\alpha > 0$ , the following inequality for fractional integrals holds:*

$$\begin{aligned} & \left| \frac{\hbar(\delta_1 + \delta_2 - s_1) + \hbar(\delta_1 + \delta_2 - s_2)}{2} - \frac{\Gamma(\alpha + 1)}{2(s_2 - s_1)^\alpha} [I_{(\delta_1 + \delta_2 - s_2)^+}^\alpha \hbar(\delta_1 + \delta_2 - s_1) + I_{(\delta_1 + \delta_2 - s_1)^-}^\alpha \hbar(\delta_1 + \delta_2 - s_2)] \right| \\ & \leq \frac{(s_2 - s_1)^2}{2(\alpha + 1)} \left( \int_0^1 [1 - (1 - \xi)^{\alpha+1} - \xi^{\alpha+1}]^r d\xi \right)^{\frac{1}{r}} \left( |\hbar''(\delta_1)|^q + |\hbar''(\delta_2)|^q - \frac{|\hbar''(s_1)|^q + |\hbar''(s_2)|^q}{2} \right)^{\frac{1}{q}} \\ & \leq \frac{(s_2 - s_1)^2}{2(\alpha + 1)} \left( \frac{r(\alpha + 1) - 1}{r(\alpha + 1) + 1} \right)^{\frac{1}{r}} \left( |\hbar''(\delta_1)|^q + |\hbar''(\delta_2)|^q - \frac{|\hbar''(s_1)|^q + |\hbar''(s_2)|^q}{2} \right)^{\frac{1}{q}}, \end{aligned} \quad (7)$$

where  $\frac{1}{r} + \frac{1}{q} = 1$ .

*Proof.* From Lemma 2.1, using the Holder's inequality, we obtain

$$\begin{aligned} & \left| \frac{\hbar(\delta_1 + \delta_2 - s_1) + \hbar(\delta_1 + \delta_2 - s_2)}{2} - \frac{\Gamma(\alpha + 1)}{2(s_2 - s_1)^\alpha} [I_{(\delta_1 + \delta_2 - s_2)^+}^\alpha \hbar(\delta_1 + \delta_2 - s_1) + I_{(\delta_1 + \delta_2 - s_1)^-}^\alpha \hbar(\delta_1 + \delta_2 - s_2)] \right| \\ & \leq \frac{(s_2 - s_1)^2}{2(\alpha + 1)} \left( \int_0^1 [1 - (1 - \xi)^{\alpha+1} - \xi^{\alpha+1}]^r d\xi \right)^{\frac{1}{r}} \left( \int_0^1 |\hbar''(\delta_1 + \delta_2 - (\xi s_1 + (1 - \xi)s_2))|^q d\xi \right)^{\frac{1}{q}}. \end{aligned}$$

Using Jensen-Mercer inequality because of the convexity of  $|\hbar''|^q$ , we obtain

$$\begin{aligned} & \left| \frac{\hbar(\delta_1 + \delta_2 - s_1) + \hbar(\delta_1 + \delta_2 - s_2)}{2} - \frac{\Gamma(\alpha + 1)}{2(s_2 - s_1)^\alpha} [I_{(\delta_1 + \delta_2 - s_2)^+}^\alpha \hbar(\delta_1 + \delta_2 - s_1) + I_{(\delta_1 + \delta_2 - s_1)^-}^\alpha \hbar(\delta_1 + \delta_2 - s_2)] \right| \\ & \leq \frac{(s_2 - s_1)^2}{2(\alpha + 1)} \left( \int_0^1 [1 - (1 - \xi)^{\alpha+1} - \xi^{\alpha+1}]^r d\xi \right)^{\frac{1}{r}} \\ & \quad \times \left( \int_0^1 (|\hbar''(\delta_1)|^q + |\hbar''(\delta_2)|^q - (\xi |\hbar''(s_1)|^q + (1 - \xi) |\hbar''(s_2)|^q)) d\xi \right)^{\frac{1}{q}} \\ & = \frac{(s_2 - s_1)^2}{2(\alpha + 1)} \left( \int_0^1 [1 - (1 - \xi)^{\alpha+1} - \xi^{\alpha+1}]^r d\xi \right)^{\frac{1}{r}} \left( |\hbar''(\delta_1)|^q + |\hbar''(\delta_2)|^q - \frac{|\hbar''(s_1)|^q + |\hbar''(s_2)|^q}{2} \right)^{\frac{1}{q}}. \end{aligned}$$

This proves the first inequality. To prove the second inequality, we observe that for any  $A > B \geq 0$  and  $r \geq 1$ , we have  $(A - B)^r \leq A^r - B^r$ . Thus, it follows that

$$[1 - (1 - \xi)^{\alpha+1} - \xi^{\alpha+1}]^r \leq 1 - (1 - \xi)^{r(\alpha+1)} - \xi^{r(\alpha+1)},$$

for all  $\xi \in [0, 1]$ . Hence we have that

$$\begin{aligned} \int_0^1 [1 - (1 - \xi)^{\alpha+1} - \xi^{\alpha+1}]^r d\xi &\leq \int_0^1 [1 - (1 - \xi)^{r(\alpha+1)} - \xi^{r(\alpha+1)}] d\xi \\ &= \frac{r(\alpha+1) - 1}{r(\alpha+1) + 1}. \end{aligned}$$

This completes the proof of the Theorem.  $\square$

**Corollary 2.8.** *If we take  $s_1 = \delta_1$  and  $s_2 = \delta_2$ , in Theorem 2.7, then we have*

$$\left| \frac{\hbar(\delta_1) + \hbar(\delta_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\delta_2 - \delta_1)^\alpha} [I_{(\delta_1)^+}^\alpha \hbar(\delta_2) + I_{(\delta_2)^-}^\alpha \hbar(\delta_1)] \right| \leq \frac{(\delta_2 - \delta_1)^2}{2(\alpha+1)} \left( \frac{r(\alpha+1) - 1}{r(\alpha+1) + 1} \right)^{\frac{1}{r}} \left( \frac{|\hbar''(\delta_1)|^q + |\hbar''(\delta_2)|^q}{2} \right)^{\frac{1}{q}}.$$

**Corollary 2.9.** *If we take  $s_1 = \delta_1$ ,  $s_2 = \delta_2$  and  $\alpha = 1$ , in Theorem 2.7, then we have*

$$\left| \frac{\hbar(\delta_1) + \hbar(\delta_2)}{2} - \frac{1}{\delta_2 - \delta_1} \int_{\delta_1}^{\delta_2} \hbar(x) dx \right| \leq \frac{(\delta_2 - \delta_1)^2}{4} \left( \frac{2r-1}{2r+1} \right)^{\frac{1}{r}} \left( \frac{|\hbar''(\delta_1)|^q + |\hbar''(\delta_2)|^q}{2} \right)^{\frac{1}{q}}.$$

### 3. Riemann-Liouville fractional integrals inequalities of midpoint type for twice differentiable convex functions

Here we give Midpoint type for twice differential convex functions via Riemann-Liouville fractional integrals. We need the following lemma for obtaining those results:

**Lemma 3.1.** *Assume that  $\hbar : J = [\delta_1, \delta_2] \rightarrow \mathbb{R}$  be two times differentiable mapping on  $(\delta_1, \delta_2)$  such that  $\hbar'' \in L[\delta_1, \delta_2]$  where  $\delta_1, \delta_2 \in J$  with  $\delta_1 < \delta_2$ . Then the following identity for fractional integrals holds:*

$$\begin{aligned} &\frac{2^{\alpha-1}\Gamma(\alpha+2)}{(s_2 - s_1)^\alpha} \left[ I_{(\delta_1 + \delta_2 - \frac{s_1 + s_2}{2})^+}^\alpha \hbar(\delta_1 + \delta_2 - s_1) + I_{(\delta_1 + \delta_2 - \frac{s_1 + s_2}{2})^-}^\alpha \hbar(\delta_1 + \delta_2 - s_2) \right] - (\alpha+1)\hbar\left(\delta_1 + \delta_2 - \frac{s_1 + s_2}{2}\right) \\ &= \frac{(s_2 - s_1)^2}{8} \int_0^1 \xi^{\alpha+1} \left[ \hbar''\left(\delta_1 + \delta_2 - \left(\frac{2-\xi}{2}s_1 + \frac{\xi}{2}s_2\right)\right) + \hbar''\left(\delta_1 + \delta_2 - \left(\frac{\xi}{2}s_1 + \frac{2-\xi}{2}s_2\right)\right) \right] d\xi, \end{aligned} \quad (8)$$

for all  $s_1, s_2 \in [\delta_1, \delta_2]$ ,  $\alpha > 0$  and  $\xi \in [0, 1]$ .

*Proof.* It suffices to note that

$$\begin{aligned} I &= \int_0^1 \xi^{\alpha+1} \left[ \hbar''\left(\delta_1 + \delta_2 - \left(\frac{2-\xi}{2}s_1 + \frac{\xi}{2}s_2\right)\right) + \hbar''\left(\delta_1 + \delta_2 - \left(\frac{\xi}{2}s_1 + \frac{2-\xi}{2}s_2\right)\right) \right] d\xi \\ &= \int_0^1 \xi^{\alpha+1} \hbar''\left(\delta_1 + \delta_2 - \left(\frac{2-\xi}{2}s_1 + \frac{\xi}{2}s_2\right)\right) d\xi + \int_0^1 \xi^{\alpha+1} \hbar''\left(\delta_1 + \delta_2 - \left(\frac{\xi}{2}s_1 + \frac{2-\xi}{2}s_2\right)\right) d\xi \\ &= I_1 + I_2. \end{aligned}$$

Integrating by parts, we have

$$\begin{aligned} I_1 &= \int_0^1 \xi^{\alpha+1} h'' \left( \delta_1 + \delta_2 - \left( \frac{2-\xi}{2} s_1 + \frac{\xi}{2} s_2 \right) \right) d\xi \\ &= - \frac{2\xi^{\alpha+1} h' \left( \delta_1 + \delta_2 - \left( \frac{2-\xi}{2} s_1 + \frac{\xi}{2} s_2 \right) \right)}{s_2 - s_1} \Big|_0^1 + \frac{2(\alpha+1)}{s_2 - s_1} \int_0^1 \xi^\alpha h' \left( \delta_1 + \delta_2 - \left( \frac{2-\xi}{2} s_1 + \frac{\xi}{2} s_2 \right) \right) d\xi \\ &= - \frac{2h \left( \delta_1 + \delta_2 - \frac{s_1+s_2}{2} \right)}{s_2 - s_1} + \frac{2(\alpha+1)}{s_2 - s_1} \int_0^1 \xi^\alpha h' \left( \delta_1 + \delta_2 - \left( \frac{2-\xi}{2} s_1 + \frac{\xi}{2} s_2 \right) \right) d\xi. \end{aligned}$$

By a similar argument, one gets:

$$\begin{aligned} I_2 &= \int_0^1 \xi^{\alpha+1} h'' \left( \delta_1 + \delta_2 - \left( \frac{\xi}{2} s_1 + \frac{2-\xi}{2} s_2 \right) \right) d\xi \\ &= \frac{2\xi^{\alpha+1} h' \left( \delta_1 + \delta_2 - \left( \frac{\xi}{2} s_1 + \frac{2-\xi}{2} s_2 \right) \right)}{s_2 - s_1} \Big|_0^1 - \frac{2(\alpha+1)}{s_2 - s_1} \int_0^1 \xi^\alpha h' \left( \delta_1 + \delta_2 - \left( \frac{\xi}{2} s_1 + \frac{2-\xi}{2} s_2 \right) \right) d\xi \\ &= \frac{2h \left( \delta_1 + \delta_2 - \frac{s_1+s_2}{2} \right)}{s_2 - s_1} - \frac{2(\alpha+1)}{s_2 - s_1} \int_0^1 \xi^\alpha h' \left( \delta_1 + \delta_2 - \left( \frac{\xi}{2} s_1 + \frac{2-\xi}{2} s_2 \right) \right) d\xi, \end{aligned}$$

we can write

$$I = \frac{2(\alpha+1)}{s_2 - s_1} \int_0^1 \xi^\alpha \left[ h' \left( \delta_1 + \delta_2 - \left( \frac{2-\xi}{2} s_1 + \frac{\xi}{2} s_2 \right) \right) - h' \left( \delta_1 + \delta_2 - \left( \frac{\xi}{2} s_1 + \frac{2-\xi}{2} s_2 \right) \right) \right] d\xi. \quad (9)$$

The desired identity in (8) follows from (9) by using (3) and rearranging the terms.  $\square$

**Remark 3.2.** If we take  $\alpha = 1$ , in Lemma 3.1, then Lemma 3.1 gives [4, Lemma 1].

**Remark 3.3.** If we take  $s_1 = \delta_1$ ,  $s_2 = \delta_2$  and  $\alpha = 1$ , in Lemma 3.1, then Lemma 3.1 gives [22, Lemma 2].

**Theorem 3.4.** Assume that  $h : J = [\delta_1, \delta_2] \rightarrow \mathbb{R}$  be two times differentiable mapping on  $(\delta_1, \delta_2)$  with  $\delta_1 < \delta_2$ . If  $|h''|$  is convex on  $[\delta_1, \delta_2]$ . Then the following inequality for fractional integrals holds:

$$\begin{aligned} & \left| \frac{2^{\alpha-1} \Gamma(\alpha+2)}{(s_2 - s_1)^\alpha} \left[ I_{(\delta_1 + \delta_2 - \frac{s_1+s_2}{2})^+}^\alpha h(\delta_1 + \delta_2 - s_1) + I_{(\delta_1 + \delta_2 - \frac{s_1+s_2}{2})^-}^\alpha h(\delta_1 + \delta_2 - s_2) \right] - (\alpha+1) h \left( \delta_1 + \delta_2 - \frac{s_1+s_2}{2} \right) \right| \\ & \leq \frac{(s_2 - s_1)^2}{8(\alpha+2)} \left[ 2 \left[ |h''(\delta_1)| + |h''(\delta_2)| \right] - \left[ |h''(s_1)| + |h''(s_2)| \right] \right], \end{aligned}$$

for all  $s_1, s_2 \in [\delta_1, \delta_2]$  and  $\alpha > 0$ .

*Proof.* By means of Lemma 3.1 and Jensen-mercer inequality, we find that

$$\begin{aligned} & \left| \frac{2^{\alpha-1} \Gamma(\alpha+2)}{(s_2 - s_1)^\alpha} \left[ I_{(\delta_1 + \delta_2 - \frac{s_1+s_2}{2})^+}^\alpha h(\delta_1 + \delta_2 - s_1) + I_{(\delta_1 + \delta_2 - \frac{s_1+s_2}{2})^-}^\alpha h(\delta_1 + \delta_2 - s_2) \right] - (\alpha+1) h \left( \delta_1 + \delta_2 - \frac{s_1+s_2}{2} \right) \right| \\ & \leq \frac{(s_2 - s_1)^2}{8} \left\{ \int_0^1 \xi^{\alpha+1} \left| h'' \left( \delta_1 + \delta_2 - \left( \frac{2-\xi}{2} s_1 + \frac{\xi}{2} s_2 \right) \right) \right| d\xi + \int_0^1 \xi^{\alpha+1} \left| h'' \left( \delta_1 + \delta_2 - \left( \frac{\xi}{2} s_1 + \frac{2-\xi}{2} s_2 \right) \right) \right| d\xi \right\} \\ & \leq \frac{(s_2 - s_1)^2}{8} \left\{ \int_0^1 \xi^{\alpha+1} \left[ |h''(\delta_1)| + |h''(\delta_2)| - \left( \frac{2-\xi}{2} |h''(s_1)| + \frac{\xi}{2} |h''(s_2)| \right) \right] d\xi \right. \\ & \quad \left. + \int_0^1 \xi^{\alpha+1} \left[ |h''(\delta_1)| + |h''(\delta_2)| - \left( \frac{\xi}{2} |h''(s_1)| + \frac{2-\xi}{2} |h''(s_2)| \right) \right] d\xi \right\} \\ & = \frac{(s_2 - s_1)^2}{4(\alpha+2)} \left[ |h''(\delta_1)| + |h''(\delta_2)| - \frac{|h''(s_1)| + |h''(s_2)|}{2} \right]. \end{aligned}$$

Which completed the proof.  $\square$

**Remark 3.5.** If we take  $\alpha = 1$ , in Theorem 3.4, then Theorem 3.4 gives [4, Theorem 3].

**Theorem 3.6.** Assume that  $h : J = [\delta_1, \delta_2] \rightarrow \mathbb{R}$  be two times differentiable mapping on  $(\delta_1, \delta_2)$  with  $\delta_1 < \delta_2$ . If  $|h''|^q$  is convex on  $[\delta_1, \delta_2]$ ,  $q > 1$ , then for all  $s_1, s_2 \in [\delta_1, \delta_2]$  and  $\alpha > 0$ , the following inequality for fractional integrals holds:

$$\begin{aligned} & \left| \frac{2^{\alpha-1}\Gamma(\alpha+2)}{(s_2-s_1)^\alpha} \left[ I_{(\delta_1+\delta_2-\frac{s_1+s_2}{2})^+}^\alpha h(\delta_1+\delta_2-s_1) + I_{(\delta_1+\delta_2-\frac{s_1+s_2}{2})^-}^\alpha h(\delta_1+\delta_2-s_2) \right] - (\alpha+1)h\left(\delta_1+\delta_2-\frac{s_1+s_2}{2}\right) \right| \\ & \leq \frac{(s_2-s_1)^2}{8} \left( \frac{1}{(\alpha+1)r+1} \right)^{\frac{1}{r}} \left[ \left( |h''(\delta_1)|^q + |h''(\delta_2)|^q - \frac{3|h''(s_1)|^q + |h''(s_2)|^q}{4} \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left( |h''(\delta_1)|^q + |h''(\delta_2)|^q - \frac{|h''(s_1)|^q + 3|h''(s_2)|^q}{4} \right)^{\frac{1}{q}} \right], \end{aligned} \quad (10)$$

where  $\frac{1}{r} + \frac{1}{q} = 1$ .

*Proof.* From Lemma 3.1, using the Holder's inequality, we obtain

$$\begin{aligned} & \left| \frac{2^{\alpha-1}\Gamma(\alpha+2)}{(s_2-s_1)^\alpha} \left[ I_{(\delta_1+\delta_2-\frac{s_1+s_2}{2})^+}^\alpha h(\delta_1+\delta_2-s_1) + I_{(\delta_1+\delta_2-\frac{s_1+s_2}{2})^-}^\alpha h(\delta_1+\delta_2-s_2) \right] - (\alpha+1)h\left(\delta_1+\delta_2-\frac{s_1+s_2}{2}\right) \right| \\ & \leq \frac{(s_2-s_1)^2}{8} \left( \int_0^1 \xi^{(\alpha+1)r} d\xi \right)^{\frac{1}{r}} \left\{ \left( \int_0^1 \left| h''\left(\delta_1+\delta_2-\left(\frac{2-\xi}{2}s_1+\frac{\xi}{2}s_2\right)\right) \right|^q d\xi \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left( \int_0^1 \left| h''\left(\delta_1+\delta_2-\left(\frac{\xi}{2}s_1+\frac{2-\xi}{2}s_2\right)\right) \right|^q d\xi \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

Using Jensen-Mercer inequality because of the convexity of  $|h''|^q$ , we obtain

$$\begin{aligned} & \left| \frac{2^{\alpha-1}\Gamma(\alpha+2)}{(s_2-s_1)^\alpha} \left[ I_{(\delta_1+\delta_2-\frac{s_1+s_2}{2})^+}^\alpha h(\delta_1+\delta_2-s_1) + I_{(\delta_1+\delta_2-\frac{s_1+s_2}{2})^-}^\alpha h(\delta_1+\delta_2-s_2) \right] - (\alpha+1)h\left(\delta_1+\delta_2-\frac{s_1+s_2}{2}\right) \right| \\ & \leq \frac{(s_2-s_1)^2}{8} \left( \frac{1}{(\alpha+1)r+1} \right)^{\frac{1}{r}} \left\{ \left( \int_0^1 \left( |h''(\delta_1)|^q + |h''(\delta_2)|^q - \left(\frac{2-\xi}{2}|h''(s_1)|^q + \frac{\xi}{2}|h''(s_2)|^q\right) \right) d\xi \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left( \int_0^1 \left( |h''(\delta_1)|^q + |h''(\delta_2)|^q - \left(\frac{\xi}{2}|h''(s_1)|^q + \frac{2-\xi}{2}|h''(s_2)|^q\right) \right) d\xi \right)^{\frac{1}{q}} \right\} \\ & = \frac{(s_2-s_1)^2}{8} \left( \frac{1}{(\alpha+1)r+1} \right)^{\frac{1}{r}} \left[ \left( |h''(\delta_1)|^q + |h''(\delta_2)|^q - \frac{3|h''(s_1)|^q + |h''(s_2)|^q}{4} \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left( |h''(\delta_1)|^q + |h''(\delta_2)|^q - \frac{|h''(s_1)|^q + 3|h''(s_2)|^q}{4} \right)^{\frac{1}{q}} \right]. \end{aligned}$$

And so the proof is completed.  $\square$

**Remark 3.7.** If we take  $\alpha = 1$ , in Theorem 3.6, then Theorem 3.6 gives [4, Theorem 5].

**Theorem 3.8.** Assume that  $h : J = [\delta_1, \delta_2] \rightarrow \mathbb{R}$  be two times differentiable mapping on  $(\delta_1, \delta_2)$  with  $\delta_1 < \delta_2$ . If  $|h''|^q$  is convex on  $[\delta_1, \delta_2]$ ,  $q \geq 1$ , then for all  $s_1, s_2 \in [\delta_1, \delta_2]$  and  $\alpha > 0$ , the following inequality for fractional integrals holds:

$$\begin{aligned} & \left| \frac{2^{\alpha-1}\Gamma(\alpha+2)}{(s_2-s_1)^\alpha} \left[ I_{(\delta_1+\delta_2-\frac{s_1+s_2}{2})^+}^\alpha h(\delta_1+\delta_2-s_1) + I_{(\delta_1+\delta_2-\frac{s_1+s_2}{2})^-}^\alpha h(\delta_1+\delta_2-s_2) \right] - (\alpha+1)h\left(\delta_1+\delta_2-\frac{s_1+s_2}{2}\right) \right| \\ & \leq \frac{(s_2-s_1)^2}{8(\alpha+2)^{1-\frac{1}{q}}} \left[ \left( \frac{|h''(\delta_1)|^q + |h''(\delta_2)|^q}{\alpha+2} - |h''(s_1)|^q \left( \frac{1}{\alpha+2} - \frac{1}{2(\alpha+3)} \right) - \frac{1}{2(\alpha+3)} |h''(s_2)|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left( \frac{|h''(\delta_1)|^q + |h''(\delta_2)|^q}{\alpha+2} - \frac{1}{2(\alpha+3)} |h''(s_1)|^q - \left( \frac{1}{\alpha+2} - \frac{1}{2(\alpha+3)} \right) |h''(s_2)|^q \right)^{\frac{1}{q}} \right], \end{aligned}$$

for all  $s_1, s_2 \in [\delta_1, \delta_2]$  and  $\alpha > 0$ .

*Proof.* From Lemma 3.1 and using power mean inequality for  $q \geq 1$ , we have

$$\begin{aligned} & \left| \frac{2^{\alpha-1}\Gamma(\alpha+2)}{(s_2-s_1)^\alpha} \left[ I_{(\delta_1+\delta_2-\frac{s_1+s_2}{2})^+}^\alpha \hbar(\delta_1+\delta_2-s_1) + I_{(\delta_1+\delta_2-\frac{s_1+s_2}{2})^-}^\alpha \hbar(\delta_1+\delta_2-s_2) \right] - (\alpha+1)\hbar\left(\delta_1+\delta_2-\frac{s_1+s_2}{2}\right) \right| \\ & \leq \frac{(s_2-s_1)^2}{8} \left( \int_0^1 \xi^{\alpha+1} d\xi \right)^{1-\frac{1}{q}} \left\{ \left( \int_0^1 \xi^{\alpha+1} \left| \hbar''\left(\delta_1+\delta_2-\left(\frac{2-\xi}{2}s_1+\frac{\xi}{2}s_2\right)\right) \right|^q d\xi \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left( \int_0^1 \xi^{\alpha+1} \left| \hbar''\left(\delta_1+\delta_2-\left(\frac{\xi}{2}s_1+\frac{2-\xi}{2}s_2\right)\right) \right|^q d\xi \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

Using Jensen-Mercer inequality because of the convexity of  $|\hbar''|^q$ , we obtain

$$\begin{aligned} & \left| \frac{2^{\alpha-1}\Gamma(\alpha+2)}{(s_2-s_1)^\alpha} \left[ I_{(\delta_1+\delta_2-\frac{s_1+s_2}{2})^+}^\alpha \hbar(\delta_1+\delta_2-s_1) + I_{(\delta_1+\delta_2-\frac{s_1+s_2}{2})^-}^\alpha \hbar(\delta_1+\delta_2-s_2) \right] - (\alpha+1)\hbar\left(\delta_1+\delta_2-\frac{s_1+s_2}{2}\right) \right| \\ & \leq \frac{(s_2-s_1)^2}{8(\alpha+2)^{1-\frac{1}{q}}} \left\{ \left( \int_0^1 \xi^{\alpha+1} (|\hbar''(\delta_1)|^q + |\hbar''(\delta_2)|^q - \left(\frac{2-\xi}{2}|\hbar''(s_1)|^q + \frac{\xi}{2}|\hbar''(s_2)|^q\right)) d\xi \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left( \int_0^1 \xi^{\alpha+1} (|\hbar''(\delta_1)|^q + |\hbar''(\delta_2)|^q - \left(\frac{\xi}{2}|\hbar''(s_1)|^q + \frac{2-\xi}{2}|\hbar''(s_2)|^q\right)) d\xi \right)^{\frac{1}{q}} \right\} \\ & = \frac{(s_2-s_1)^2}{8(\alpha+2)^{1-\frac{1}{q}}} \left[ \left( \frac{|\hbar''(\delta_1)|^q + |\hbar''(\delta_2)|^q}{\alpha+2} - |\hbar''(s_1)|^q \left( \frac{1}{\alpha+2} - \frac{1}{2(\alpha+3)} \right) - \frac{1}{2(\alpha+3)} |\hbar''(s_2)|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left( \frac{|\hbar''(\delta_1)|^q + |\hbar''(\delta_2)|^q}{\alpha+2} - \frac{1}{2(\alpha+3)} |\hbar''(s_1)|^q - \left( \frac{1}{\alpha+2} - \frac{1}{2(\alpha+3)} \right) |\hbar''(s_2)|^q \right)^{\frac{1}{q}} \right]. \end{aligned}$$

This complete the proof.  $\square$

**Remark 3.9.** If we take  $\alpha = 1$ , in Theorem 3.8, then Theorem 3.8 gives [4, Theorem 4].

**Remark 3.10.** If we take  $q = 1$ , in Theorem 3.8, then Theorem 3.8 gives Theorem 3.4.

#### 4. Application: Special Means

In this section, we provide applications of newly discovered inequalities in the context of special means of real numbers. For arbitrary positive real numbers  $\delta_1$  and  $\delta_2$  ( $\delta_1 \neq \delta_2$ ), we consider the means as follows:

(1) The arithmetic mean:

$$A(\delta_1, \delta_2) = \frac{\delta_1 + \delta_2}{2}, \quad \delta_1, \delta_2 \geq 0.$$

(2) The harmonic mean:

$$H(\delta_1, \delta_2) = \frac{2\delta_1\delta_2}{\delta_1 + \delta_2}, \quad \delta_1, \delta_2 > 0.$$

(3) The logarithmic mean:

$$L(\delta_1, \delta_2) = \begin{cases} \frac{\delta_2 - \delta_1}{\ln \delta_2 - \ln \delta_1}; & \delta_1 \neq \delta_2, \\ \delta_1; & \delta_1 = \delta_2, \end{cases} \quad \delta_1, \delta_2 > 0.$$

(4) The generalized logarithmic mean:

$$L_n(\delta_1, \delta_2) = \begin{cases} \left[ \frac{\delta_2^{n+1} - \delta_1^{n+1}}{(n+1)(\delta_2 - \delta_1)} \right]^{\frac{1}{n}}; & \delta_1 \neq \delta_2, \\ \delta_1; & \delta_1 = \delta_2, \end{cases} \quad \delta_1, \delta_2 > 0; n \in \mathbb{Z} - \{-1, 0\}.$$

**Proposition 4.1.** Let  $0 < \delta_1 < \delta_2$ , and  $n \in \mathbb{N}$ ,  $n \geq 3$ . Then, for  $q \geq 1$  and for all  $s_1, s_2 \in [\delta_1, \delta_2]$ , we have:

$$\begin{aligned} & \left| L_n^n(\delta_1 + \delta_2 - s_2, \delta_1 + \delta_2 - s_1) - A((\delta_1 + \delta_2 - s_1)^n, (\delta_1 + \delta_2 - s_2)^n) \right| \\ & \leq \frac{n(n-1)(s_2 - s_1)^2}{12} \left[ 2A(\delta_1^{q(n-2)}, \delta_2^{q(n-2)}) - A(s_1^{q(n-2)}, s_2^{q(n-2)}) \right]^{\frac{1}{q}}. \end{aligned}$$

*Proof.* If we set in Theorem 2.4,  $\alpha = 1$  and  $h(\xi) = \xi^n$  one can obtain the result directly.  $\square$

**Proposition 4.2.** Let  $0 < \delta_1 < \delta_2$ , and  $n \in \mathbb{N}$ ,  $n \geq 3$ . Then, for  $q \geq 1$  and for all  $s_1, s_2 \in [\delta_1, \delta_2]$ , we have:

$$\begin{aligned} & \left| L^{-1}(\delta_1 + \delta_2 - s_2, \delta_1 + \delta_2 - s_1) - H^{-1}(\delta_1 + \delta_2 - s_1, \delta_1 + \delta_2 - s_2) \right| \\ & \leq \frac{(s_2 - s_1)^2}{6} \left[ 2H^{-1}(\delta_1^{3q}, \delta_2^{3q}) - H^{-1}(s_1^{3q}, s_2^{3q}) \right]^{\frac{1}{q}}. \end{aligned}$$

*Proof.* If we set in Theorem 2.4,  $\alpha = 1$  and  $h(\xi) = \frac{1}{\xi}$  one can obtain the result directly.  $\square$

**Proposition 4.3.** Let  $0 < \delta_1 < \delta_2$ , and  $n \in \mathbb{N}$ ,  $n \geq 3$ . Then, we have:

$$\left| L_n^n(\delta_1, \delta_2) - A(\delta_1^n, \delta_2^n) \right| \leq \frac{n(n-1)(\delta_2 - \delta_1)^2 \sqrt[q]{A(\delta_1^{q(n-2)}, \delta_2^{q(n-2)})}}{12},$$

and

$$\left| L^{-1}(\delta_1, \delta_2) - H^{-1}(\delta_1, \delta_2) \right| \leq \frac{(\delta_2 - \delta_1)^2 \sqrt[q]{H^{-1}(\delta_1^{3q}, \delta_2^{3q})}}{6}.$$

*Proof.* If we set  $s_1 = \delta_1$  and  $s_1 = \delta_1$  in results of Proposition 4.1 and Proposition 4.2, one can obtain the Proposition 4.3.  $\square$

**Proposition 4.4.** Let  $0 < \delta_1 < \delta_2$ , and  $n \in \mathbb{N}$ ,  $n \geq 3$ . Then, for  $q > 1$ ,  $\frac{1}{q} + \frac{1}{r} = 1$  and for all  $s_1, s_2 \in [\delta_1, \delta_2]$ , we have:

$$\begin{aligned} & \left| L_n^n(\delta_1 + \delta_2 - s_2, \delta_1 + \delta_2 - s_1) - A((\delta_1 + \delta_2 - s_1)^n, (\delta_1 + \delta_2 - s_2)^n) \right| \\ & \leq \frac{n(n-1)(s_2 - s_1)^2}{4} \left( \frac{2r-1}{2r+1} \right)^{\frac{1}{r}} \left[ 2A(\delta_1^{q(n-2)}, \delta_2^{q(n-2)}) - A(s_1^{q(n-2)}, s_2^{q(n-2)}) \right]^{\frac{1}{q}}. \end{aligned}$$

*Proof.* If we set in Theorem 2.7,  $\alpha = 1$  and  $h(\xi) = \xi^n$  one can obtain the result directly.  $\square$

**Proposition 4.5.** Let  $0 < \delta_1 < \delta_2$ , and  $n \in \mathbb{N}$ ,  $n \geq 3$ . Then, for  $q > 1$ ,  $\frac{1}{q} + \frac{1}{r} = 1$  and for all  $s_1, s_2 \in [\delta_1, \delta_2]$ , we have:

$$\begin{aligned} & \left| L^{-1}(\delta_1 + \delta_2 - s_2, \delta_1 + \delta_2 - s_1) - H^{-1}(\delta_1 + \delta_2 - s_1, \delta_1 + \delta_2 - s_2) \right| \\ & \leq \frac{(s_2 - s_1)^2}{2} \left( \frac{2r-1}{2r+1} \right)^{\frac{1}{r}} \left[ 2H^{-1}(\delta_1^{3q}, \delta_2^{3q}) - H^{-1}(s_1^{3q}, s_2^{3q}) \right]^{\frac{1}{q}}. \end{aligned}$$

*Proof.* If we set in Theorem 2.7,  $\alpha = 1$  and  $h(\xi) = \frac{1}{\xi}$  one can obtain the result directly.  $\square$

**Proposition 4.6.** Let  $0 < \delta_1 < \delta_2$ , and  $n \in \mathbb{N}$ ,  $n \geq 3$ . Then, we have:

$$\left| L_n^n(\delta_1, \delta_2) - A(\delta_1^n, \delta_2^n) \right| \leq \frac{n(n-1)(\delta_2 - \delta_1)^2 \sqrt[2r]{2r-1} \sqrt[q]{A(\delta_1^{q(n-2)}, \delta_2^{q(n-2)})}}{4 \sqrt[2r]{2r+1}},$$

and

$$\left| L^{-1}(\delta_1, \delta_2) - H^{-1}(\delta_1, \delta_2) \right| \leq \frac{(\delta_2 - \delta_1)^2 \sqrt[2r]{2r-1} \sqrt[q]{H^{-1}(\delta_1^{3q}, \delta_2^{3q})}}{2 \sqrt[2r]{2r+1}}.$$

*Proof.* If we set  $s_1 = \delta_1$  and  $s_1 = \delta_1$  in results of Proposition 4.4 and Proposition 4.5, one can obtain the Proposition 4.6.  $\square$

**Proposition 4.7.** Let  $0 < \delta_1 < \delta_2$ , and  $n \in \mathbb{N}$ ,  $n \geq 3$ . Then, for all  $s_1, s_2 \in [\delta_1, \delta_2]$ , we have:

$$\begin{aligned} & \left| L_n^n(\delta_1 + \delta_2 - s_2, \delta_1 + \delta_2 - s_1) - (2A(\delta_1, \delta_2) - A(s_1, s_2))^n \right| \\ & \leq \frac{n(n-1)(s_2 - s_1)^2}{8} \left[ \frac{2}{3} A(\delta_1^{n-2}, \delta_2^{n-2}) - \frac{1}{3} A(s_1^{n-2}, s_2^{n-2}) \right]. \end{aligned}$$

*Proof.* If we set in Theorem 3.4,  $\alpha = 1$  and  $\hbar(\xi) = \xi^n$  one can obtain the result directly.  $\square$

**Proposition 4.8.** Let  $0 < \delta_1 < \delta_2$ , and  $n \in \mathbb{N}$ ,  $n \geq 3$ . Then, for all  $s_1, s_2 \in [\delta_1, \delta_2]$ , we have:

$$\begin{aligned} & \left| L^{-1}(\delta_1 + \delta_2 - s_2, \delta_1 + \delta_2 - s_1) - (2A(\delta_1, \delta_2) - A(s_1, s_2))^{-1} \right| \\ & \leq \frac{(s_2 - s_1)^2}{12} \left[ 2H^{-1}(\delta_1^3, \delta_2^3) - H^{-1}(s_1^3, s_2^3) \right]. \end{aligned}$$

*Proof.* If we set in Theorem 3.4,  $\alpha = 1$  and  $\hbar(\xi) = \frac{1}{\xi}$  one can obtain the result directly.  $\square$

**Proposition 4.9.** Let  $0 < \delta_1 < \delta_2$ , and  $n \in \mathbb{N}$ ,  $n \geq 3$ . Then, we have:

$$\left| L_n^n(\delta_1, \delta_2) - (A(\delta_1, \delta_2))^n \right| \leq \frac{n(n-1)(\delta_2 - \delta_1)^2}{24} \left[ A(\delta_1^{n-2}, \delta_2^{n-2}) \right],$$

and

$$\left| L^{-1}(\delta_1, \delta_2) - (A(\delta_1, \delta_2))^{-1} \right| \leq \frac{(\delta_2 - \delta_1)^2}{12} \left[ H^{-1}(\delta_1^3, \delta_2^3) \right].$$

*Proof.* If we set  $s_1 = \delta_1$  and  $s_1 = \delta_1$  in results of Proposition 4.7 and Proposition 4.8, one can obtain the Proposition 4.9.  $\square$

**Proposition 4.10.** Let  $0 < \delta_1 < \delta_2$ , and  $n \in \mathbb{N}$ ,  $n \geq 3$ . Then, for  $q > 1$ ,  $\frac{1}{q} + \frac{1}{r} = 1$  and for all  $s_1, s_2 \in [\delta_1, \delta_2]$ , we have:

$$\begin{aligned} & \left| L_n^n(\delta_1 + \delta_2 - s_2, \delta_1 + \delta_2 - s_1) - (2A(\delta_1, \delta_2) - A(s_1, s_2))^n \right| \\ & \leq \frac{n(n-1)(s_2 - s_1)^2}{16\sqrt{2r+1}} \left[ \left( 2A(\delta_1^{q(n-2)}, \delta_2^{q(n-2)}) - \frac{1}{2} A(3s_1^{q(n-2)}, s_2^{q(n-2)}) \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left( 2A(\delta_1^{q(n-2)}, \delta_2^{q(n-2)}) - \frac{1}{2} A(s_1^{q(n-2)}, 3s_2^{q(n-2)}) \right)^{\frac{1}{q}} \right]. \end{aligned}$$

*Proof.* If we set in Theorem 3.6,  $\alpha = 1$  and  $\hbar(\xi) = \xi^n$  one can obtain the result directly.  $\square$

**Proposition 4.11.** Let  $0 < \delta_1 < \delta_2$ , and  $n \in \mathbb{N}$ ,  $n \geq 3$ . Then, for  $q > 1$ ,  $\frac{1}{q} + \frac{1}{r} = 1$  and for all  $s_1, s_2 \in [\delta_1, \delta_2]$ , we have:

$$\begin{aligned} & \left| L^{-1}(\delta_1 + \delta_2 - s_2, \delta_1 + \delta_2 - s_1) - (2A(\delta_1, \delta_2) - A(s_1, s_2))^{-1} \right| \\ & \leq \frac{(s_2 - s_1)^2}{8\sqrt{2r+1}} \left[ \left( 2H^{-1}(\delta_1^{3q}, \delta_2^{3q}) - \frac{3}{2} H^{-1}(3s_1^{3q}, s_2^{3q}) \right)^{\frac{1}{q}} + \left( 2H^{-1}(\delta_1^{3q}, \delta_2^{3q}) - \frac{3}{2} H^{-1}(s_1^{3q}, 3s_2^{3q}) \right)^{\frac{1}{q}} \right]. \end{aligned}$$

*Proof.* If we set in Theorem 3.6,  $\alpha = 1$  and  $\hbar(\xi) = \frac{1}{\xi}$  one can obtain the result directly.  $\square$

## 5. Illustration Examples

In this section, a couple of examples through graphical visualizations are provided to illustrate the key findings of our research.

**Example 5.1.** Let  $h : [0, 1] \rightarrow \mathbf{R}$  be a mapping,  $\delta_1 = 0$ ,  $\delta_2 = 1$ ,  $s_1 = \frac{1}{4}$  and  $s_2 = \frac{3}{4}$ . Then  $|h''(x)|^q = x^{2q}$  is convex on  $[0, 1]$ . We compute the left-member and the right member of the inequality (6) denoting the left member by  $Ms$  and the right member by  $Md$ . We have

$$Ms = \left| \frac{h(1 - \frac{1}{4}) + h(1 - \frac{3}{4})}{2} - \frac{\Gamma(\alpha + 1)}{2(\frac{3}{4} - \frac{1}{4})^\alpha} [I_{(1-\frac{3}{4})^+}^\alpha h(1 - \frac{1}{4}) + I_{(1-\frac{1}{4})^-}^\alpha h(1 - \frac{3}{4})] \right|,$$

or

$$Ms = \left| \frac{h(\frac{3}{4}) + h(\frac{1}{4})}{2} - \frac{2^\alpha \Gamma(\alpha + 1)}{2} [I_{(\frac{1}{4})^+}^\alpha h(\frac{3}{4}) + I_{(\frac{3}{4})^-}^\alpha h(\frac{1}{4})] \right|$$

$$Md = \frac{\alpha(\frac{3}{4} - \frac{1}{4})^2}{2(\alpha + 1)(\alpha + 2)} \left( 1 - \frac{(\frac{1}{4})^{2q} + (\frac{3}{4})^{2q}}{2} \right)^{\frac{1}{q}},$$

or

$$Md = \frac{\alpha}{8(\alpha + 1)(\alpha + 2)} \left( 1 - \frac{1 + 9^q}{2 \cdot 16^q} \right)^{\frac{1}{q}}.$$

By calculus we get respectively,

$$Ms = \left| \frac{(\frac{3}{4})^4 + (\frac{1}{4})^4}{24} - 2^{\alpha-1} \alpha \left[ \int_{\frac{1}{4}}^{\frac{3}{4}} \left( \frac{3}{4} - \xi \right)^{\alpha-1} \frac{\xi^4}{12} d\xi + \int_{\frac{1}{4}}^{\frac{3}{4}} \left( \xi - \frac{1}{4} \right)^{\alpha-1} \frac{\xi^4}{12} d\xi \right] \right|,$$

$$Ms = \frac{\alpha}{24 \cdot 4^2} \left| \frac{13}{\alpha + 1} - \frac{15}{\alpha + 2} + \frac{4}{\alpha + 3} - \frac{2}{\alpha + 4} \right|,$$

if we take into account that the integrals,

$$J_1 = \int_{\frac{1}{4}}^{\frac{3}{4}} \left( \frac{3}{4} - \xi \right)^{\alpha-1} \frac{\xi^4}{12} d\xi = \frac{1}{12} \left( \frac{3}{4} \right)^4 \left( \frac{1}{2} \right)^\alpha \left[ \frac{1}{\alpha} - \frac{8}{3(\alpha + 1)} + \frac{8}{3(\alpha + 2)} - \frac{32}{27(\alpha + 3)} + \frac{16}{81(\alpha + 4)} \right]$$

and that

$$J_2 = \int_{\frac{1}{4}}^{\frac{3}{4}} \left( \xi - \frac{1}{4} \right)^{\alpha-1} \frac{\xi^4}{12} d\xi = \frac{1}{12} \left( \frac{1}{4} \right)^4 \left( \frac{1}{2} \right)^\alpha \left[ \frac{1}{\alpha} + \frac{8}{\alpha + 1} + \frac{24}{\alpha + 2} + \frac{32}{\alpha + 3} + \frac{16}{\alpha + 4} \right].$$

Now we compute in the same way the right-member of the inequality (7), Theorem 2.7. The left-member is the same as in Theorem 2.4, inequality (6).

$$Md_2 = \frac{\alpha}{8(\alpha + 1)} \left( \frac{q\alpha + 1}{q\alpha + 2q - 1} \right)^{\frac{q-1}{q}} \left( 1 - \frac{1 + 9^q}{2 \cdot 16^q} \right)^{\frac{1}{q}}.$$

The validity of the results from Theorem 2.4, inequality (6), and of Theorem 2.7, inequality (7), is checked in the special case where  $\alpha, q \in [5, 20]$ , as shown in Figures 1(a) and (b). In Figure 1(c), the common left-hand side of inequalities (6) and (7) is represented in red, while the right-hand sides of (6) and (7) are represented in blue and green respectively. Additionally, the analysis is extended to the case where  $\alpha, q \in [1.3, 20]$ .

In order to check the inequality (10) from Theorem 3.6 and then the inequality from Theorem 3.8 for the same function  $h$  like before,  $\hbar : [0, 1] \rightarrow \mathbf{R}$  being a mapping,  $\delta_1 = 0$ ,  $\delta_2 = 1$ ,  $s_1 = \frac{1}{4}$  and  $s_2 = \frac{3}{4}$ , but this time when  $\alpha, q \in [5, 70]$ , we need to compute the common left-member  $M_{s_2}$  and the right-members respectively, denoted by  $Md_3$  and  $Md_4$ .

Thus we have,

$$M_{s_2} = |2^{2\alpha-1}\Gamma(\alpha+2)[I_{(\frac{1}{2})^+}^\alpha \hbar(\frac{3}{4}) + I_{(\frac{1}{2})^-}^\alpha \hbar(\frac{1}{4})] - (\alpha+1)\hbar(\frac{1}{2})|$$

or by calculus,

$$M_{s_2} = \frac{\alpha+1}{12.16} \left| \frac{25}{16} - \frac{52\alpha}{16(\alpha+1)} + \frac{30\alpha}{16(\alpha+2)} - \frac{4\alpha}{16(\alpha+3)} + \frac{\alpha}{16(\alpha+4)} \right|.$$

In addition, we get:

$$Md_3 = \frac{1}{32} \left( \frac{q-1}{q\alpha+2q-1} \right)^{\frac{q-1}{q}} \left[ \left( 1 - \frac{3+3^q}{4^{q+1}} \right)^{\frac{1}{q}} + \left( 1 - \frac{1+3^{q+1}}{4^{q+1}} \right)^{\frac{q-1}{q}} \right]$$

and

$$Md_4 = \frac{1}{32(\alpha+2)^{1-\frac{1}{q}}} \left[ \left( \frac{1}{\alpha+2} - \frac{1}{4^q} \left( \frac{1}{\alpha+2} - \frac{1}{2(\alpha+3)} \right) - \frac{1}{2(\alpha+3)} \frac{3^q}{4^q} \right)^{\frac{1}{q}} \right. \\ \left. + \left( \frac{1}{\alpha+2} - \frac{1}{4^q} \frac{1}{2(\alpha+3)} - \left( \frac{1}{\alpha+2} - \frac{1}{2(\alpha+3)} \right) \frac{3^q}{4^q} \right)^{\frac{1}{q}} \right].$$

The graphic representations for the left-member and the right members of the inequalities from Theorem 3.6 and 3.8 are given in Figure 2 (a), (b) and (c).

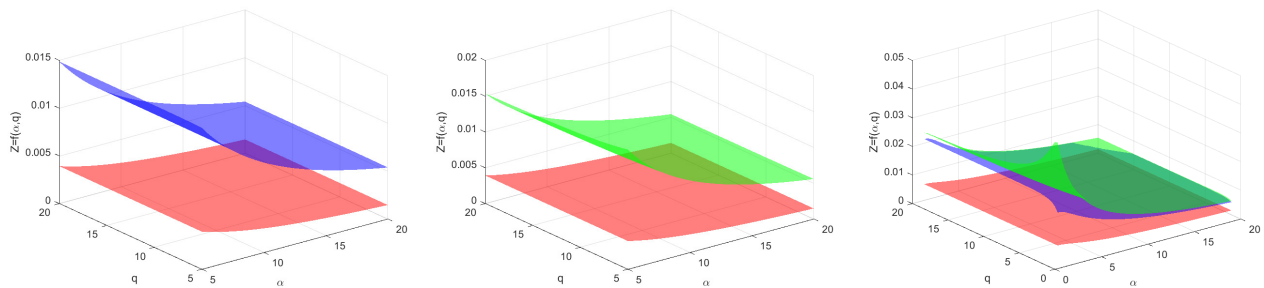


Figure 1: (a) The graph of the left- and the right-hand sides of inequality (6) from Theorem 2.4 when  $\hbar(x) = \frac{x^4}{12}$  and  $\delta_1 = 0$ ,  $\delta_2 = 1$ ,  $s_1 = \frac{1}{4}$ ,  $s_2 = \frac{3}{4}$ , and  $\alpha, q \in [5, 20]$ . The red surface represents the graphic of left member and the blue surface represents the graphic of the right member when the variables are considered to be  $\alpha$  and  $q$ ; (b) The graph of the left- and the right-hand sides of inequality (7) from Theorem 2.7 when  $\hbar(x) = \frac{x^4}{12}$  and  $\delta_1 = 0$ ,  $\delta_2 = 1$ ,  $s_1 = \frac{1}{4}$ ,  $s_2 = \frac{3}{4}$ , and  $\alpha, q \in [5, 20]$ . The red surface represents the graphic of left member and the green surface represents the graphic of the right member; (c) The graph of the left- and the right-hand sides of inequality (6) and (7) from Theorem 2.4 and Theorem 2.7 respectively, when  $\hbar(x) = \frac{x^4}{12}$  and  $\delta_1 = 0$ ,  $\delta_2 = 1$ ,  $s_1 = \frac{1}{4}$ ,  $s_2 = \frac{3}{4}$ , and  $\alpha, q \in [1.3, 20]$ . The red surface represents the graphic of left common member, the blue surface represents the graphic of the right member from inequality (6) Theorem 2.4, and the green surface represents the graphic of the right member from inequality (7), Theorem 2.7.

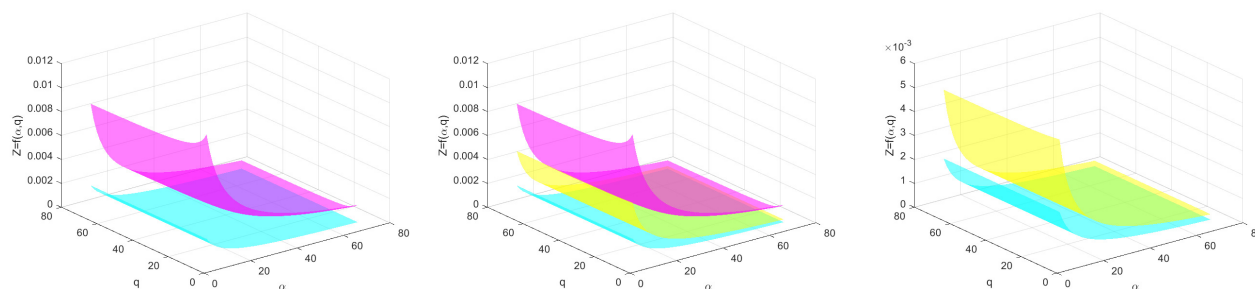


Figure 2: (a) The graph of the left- and the right-hand sides of inequality (10) from Theorem 3.6 when  $h(x) = \frac{x^4}{12}$  and  $\delta_1 = 0$ ,  $\delta_2 = 1$ ,  $s_1 = \frac{1}{4}$ ,  $s_2 = \frac{3}{4}$ , and  $\alpha, q \in [5, 70]$ . The cyan surface represents the graphic of left member and the magenta surface represents the graphic of the right member when the variables are considered to be  $\alpha$  and  $q$ ; (b) The graph of the left- and the right-hand sides of inequality (10) from Theorem 3.6 and of inequality from Theorem 3.8 when  $h(x) = \frac{x^4}{12}$  and  $\delta_1 = 0$ ,  $\delta_2 = 1$ ,  $s_1 = \frac{1}{4}$ ,  $s_2 = \frac{3}{4}$ , and  $\alpha, q \in [5, 70]$ . The cyan surface represents the graphic of the common left member of Theorem 3.6 and Theorem 3.8, the magenta surface represents the graphic of the right member of Theorem 3.6 and the yellow surface represents the graphic of the right member of Theorem 3.8; (c) The graph of the left- and the right-hand sides of inequality from Theorem 3.8 when  $h(x) = \frac{x^4}{12}$  and  $\delta_1 = 0$ ,  $\delta_2 = 1$ ,  $s_1 = \frac{1}{4}$ ,  $s_2 = \frac{3}{4}$ , and  $\alpha, q \in [5, 70]$ . The cyan surface represents the graphic of left member and the yellow surface represents the graphic of the right member when the variables are considered to be  $\alpha$  and  $q$ .

For graphical illustration from Figure 1 and Figure 2 it was used the MatlabR2023b software version and also some calculus can be done by using the same software. Similar validations can be done for particular cases for Theorem 3.4.

## 6. Conclusion

In this study, we presented some new generalizations of Hermite-Hadamard Mercer type inequalities involving Riemann-Liouville fractional integrals. We also demonstrated some novel trapezoidal type and midpoint type inequalities involving Riemann-Liouville fractional integrals for twice differentiable convex functions. The main advantage some of these inequalities is that we can use them to help us figure out the error bounds for the trapezoidal formula in both fractional calculus and classical calculus. Moreover, we prove that our results generalize the inequalities obtained in some earlier works. Next, we have considered some propositions in the context of special means; these confirm the efficiency of our findings.

Lastly, a couple of examples through graphical visualizations are presented in order to prove the validity of inequalities from Theorem 2.4, Theorem 2.7, Theorem 3.6 and Theorem 3.4 in a special case. In the future works, researchers can focus to generalize our findings by utilizing new fractional operators and other notions of convexity.

## Acknowledgments

The authors would like to express their sincere gratitude to the Editor and anonymous Reviewers for their valuable comments and constructive suggestions that helped improve the quality and clarity of this manuscript.

## References

- [1] J. A. Acevedo and J. E. Nápoles V., *On the Hermite-Hadamard Inequality. Some methodological remarks*, Phys. Astron. Int. J. **9** (2) (2025) 80–86.
- [2] M. A. Ali, T. Hussain, M. Z. Iqbal and F. Ejaz, *Inequalities of Hermite-Hadamard-Mercer Type for Convex Functions via K-Fractional Integrals*, Int. J. Math. Model. Comput. **10** (3) (2020) 227–238.
- [3] M. M. Ali and A. R. Khan, *Generalized integral Mercer vs inequality and integral means*, J. Inequal. Spec. Funct. **10**(1) (2019) 60–76.

- [4] M. A. Ali, T. Sitthiwiratham, E. Köbis and A. Hanif, *Hermite-Hadamard-Mercer Inequalities Associated with Twice-Differentiable Functions with Applications*, Axioms **13**(2) (2024) 114.
- [5] M. Alomari, M. Darus and S. S. Dragomir, *New Inequalities of Hermite-Hadamard Type for Functions Whose Second Derivatives Absolute Values Are Quasi-Convex*, Tamkang J. Math. **41** (4) (2010) 353–359.
- [6] A. Atangana, *Application of fractional calculus to epidemiology*, Fract. Dyn. (2015) 174–190.
- [7] D. Baleanu, Z. B. Güvenç and J. T. Machado, *New trends in nanotechnology and fractional calculus applications* 10 (2010).
- [8] F. Chen, *A note on Hermite-Hadamard inequalities for products of convex functions via Riemann-Liouville fractional integrals*, Ital. J. Pure Appl. Math **33** (2014) 299–306.
- [9] S. S. Dragomir, *New estimation of the remainder in Taylor's formula using Grüss type inequalities and applications*, Math. Inequal. Appl **2** (2) (1999), 183–194.
- [10] S. S. Dragomir and C. Pearce, *Selected topics on Hermite-Hadamard inequalities and applications*, Science direct working paper (S1574-0358) (2003) 04.
- [11] R. Hilfer, *Applications of fractional calculus in physics*, World Scientific 2000.
- [12] J. L. W. V. Jensen, *Sur les fonctions convexes et les inégalités entre les valeurs moyennes*, Acta Math. **30** (1) (1906) 175–193.
- [13] M. Kian and M. S. Moslehian, *Refinements of the operator Jensen-Mercer inequality*, Electron. J. Linear Algebra **26** (1) (2013) 742–753.
- [14] R. Magin, *Fractional calculus in bioengineering, part 1*, Critical Reviews™ in Biomedical Engineering **32**(1) (2004).
- [15] A. Matković, J. Pečarić and I. Perić, *A variant of Jensen's inequality of Mercer's type for operators with applications*, Linear Algebra Appl. **418** (2-3) (2006) 551–564.
- [16] A. M. Mercer, *A variant of Jensen's inequality*, J. Inequal. Pure and Appl. Math. **4** (4) (2003) 73.
- [17] C. A. Monje, B. M. Vinagre, V. Feliu and Y. Chen, *Tuning and auto-tuning of fractional order controllers for industry applications*, Control Eng. Pract. **16**(7) (2008) 798–812.
- [18] J. E. Nápoles Valdés, *A review of Hermite-Hadamard inequality*, Partners Univ. Int. Res. J., **1** (4) (2022) 98–101.
- [19] J. E. Nápoles Valdés, F. Rabossi and A. D. Samaniego, *Convex functions: Ariadne's thread or charlotte's spiderweb?*, Adv. Math. Models Appl. **5** (2) (2020) 176–191.
- [20] M. Niezgoda, *A generalization of mercer's result on convex functions*, Nonlinear Anal. Theory Methods Appl. **71** (7–8) (2009) 2771–2779.
- [21] H. Ögülmüş and M. Z. Sarıkaya, *Hermite-Hadamard-Mercer type inequalities for fractional integrals*, Filomat **35** (7) (2021) 2425–2436.
- [22] M. Z. Sarıkaya and M. E. Kiriş, *Some new inequalities of Hermite-Hadamard type for s-convex functions*, Miskolc Math. Notes **16** (1) (2015) 491–501.
- [23] M. Z. Sarıkaya, H. Yildirim, *On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals*, Miskolc Math. Notes **17** (2) (2016) 1049–1059.
- [24] D.-P. Shi, B.-Y. Xi and F. Qi, *Hermite-Hadamard type inequalities for  $(m, h_1, h_2)$ -convex functions via Riemann-Liouville fractional integrals*, Turkish J. Anal. Number Theory **2** (2014) 22–27.
- [25] M. Ç. Tatar and Ç. Yıldız, *Some New Estimates for Hermite-Hadamard-Jensen-Mercer Type Inequalities*, Turkish J. Sci. **9** (3) (2021) 225–240.
- [26] J. R. Wang, X. Li, M. Feckan and Y. Zhou, *Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity*, Appl. Anal. **92** (11) (2013) 2241–2253.
- [27] S. H. Wang and Feng Qi, *Hermite-Hadamard type inequalities for s-convex functions via Riemann-Liouville fractional integrals*, J. Comput. Anal. Appl **22**(6) (2017) 1124–1134.