Filomat 39:21 (2025), 7531–7538
https://doi.org/10.2298/FIL2521531A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On extending the domain of (co)convex polynomials: an update

Malik Saad Al-Muhja[a,b,*], Amer Himza Almyaly[a], Aws Nedal Dheyab[a,c]

[a] *Department of Mathematics and Computer Application, College of Sciences, University of Al-Muthanna, Samawa 66001, Iraq*
[b] *Department of Mathematics and Statistics, School of Quantitative Sciences, College of Arts and Sciences, Universiti Utara Malaysia, Sintok 06010, Kedah, Malaysia*
[c] *South Ural State University, Chelyabinsk, Russian Federation*

Abstract. In this study, we will employ a different methodology compared to the prevailing techniques in the literature. The literature mainly concentrates on the partitioning of convex sets using hyperplanes. The focus of our inquiry is the behaviour of a convex set that encompasses the domain of convex and coconvex polynomials, which we will refer to as (co)convex polynomials. The primary aim of this study is to investigate the following questions: Given that $\mathbb{D}$ is the domain of (co)convex polynomials of $\Delta^{(2)}(Y_s)$ for $s \geq 0$ and $x \in \mathbb{D}$, the question is whether x qualifies as an inflection point within $\mathbb{D}$.

1. Introduction

The domain of a polynomial is beneficial because it provides the best approximation to a given function. However, the domain of polynomials needs to be expanded in order to include more than just the properties of convex sets. For example, supporting hyperplanes and strongly $\mathfrak{h}$-hyperplane need to be added. The initial literature that discussed separation theorems in the best approximation involving discrete sets was found in 1979. Singer (1979) and Papini and Singer (1979) proposed some approximation characteristics to be used to describe the second separation theorem [6]; [15]. All these characteristics were limited to the approximation theory of an element to a convex set and to the best approximations of elements of convex sets. However, they have not yet looked into the theory that "simpler functions can approximate complex ones" in developing approximations.

Subsequently, numerous researchers have concentrated on building approximation theory with convex polynomials. For example, Leviatan introduced the best convex polynomial (CP) approximation of a continuous function, [4]. He showed that for any convex function $f \in \mathbb{C}(\mathbb{D})$, a sequence of CP p_n of degree $\leq n$ can be constructed such that p_n is the best approximation of f. Other researchers proved that any function $f \in \mathbb{C}(\mathbb{D})$ changes its convexity finitely many times in the interval, estimating the degree of approximation of f by polynomials of degree $\leq n$, which changes convexity exactly at the points Y_s. Moreover, Al-Muhja,

2020 *Mathematics Subject Classification*. Primary 41A10, 52A30; Secondary 52A27.
Keywords. Convex polynomial, Coconvex polynomial, Convex set, Inflection point.
Received: 30 January 2025; Revised: 31 May 2025; Accepted: 12 June 2025
Communicated by Miodrag Spalević
Research supported by University of Al-Muthanna.
* Corresponding author: Malik Saad Al-Muhja
Email addresses: malik@mu.edu.iq; dr.al-muhja@hotmail.com (Malik Saad Al-Muhja), amerhimzi@mu.edu.iq (Amer Himza Almyaly), Aws.nth@gmail.com (Aws Nedal Dheyab)
ORCID iDs: https://orcid.org/0000-0002-5508-6708 (Malik Saad Al-Muhja), https://orcid.org/0000-0003-2448-5121 (Amer Himza Almyaly), https://orcid.org/0000-0003-1460-5193 (Aws Nedal Dheyab)

Akhadkulov and Ahmad, in (2021a) and (2021b), discussed weighted constrained approximation on the interval $[-1, 1]$ by piecewise polynomial, [9]. Moreover, They examined the estimation of the best weighted approximation and convex function with varying degrees of polynomial order, [10]. Then Kareem, Kamel and Hussain, in (2022), have significant impact on the convex and coconvex multi-approximation. Furthermore, they discussed inverse inequality for the coconvex approximation of the function f using multi algebraic polynomials, see [12]. Bustamante has proven some estimates the best approximation of the function f, where f has differentiable function, [7]. In the sequel, Al-Muhja has discovered the principal benefit of the study "applications of the best approximation" conducted in (2024). This benefit is derived from the utilisation of a separation method based on two-best approximations, which are determined by convex polynomials. The study focuses on analysing the pupil deviation of the eye, as referenced in [11]. Leviatan and Shevchuk concluded to the Jackson type estimates of shape preserving approximation, [3].

Functional analysis and optimization are significantly influenced by convex sets in topological spaces. A subset A of a topological vector space is convex if the line segment connecting any two coordinates is entirely contained in A. The study of optimization problems, separation theorems, and fixed-point theorems is fundamentally dependent on convexity. Convex sets are frequently linked to supporting hyperplanes and duality principles in locally convex spaces. Generalizing results from finite-dimensional spaces to infinite-dimensional analysis is facilitated by comprehending convexity in various topological contexts, such as, the study of Banach and Hilbert spaces. We will adopt the following concepts in this work.

Definition 1.1. *[2] A subset X of $\mathbb{R}^n$ is a convex set if $(1-\lambda)x + \lambda y \in X$, for all $x, y \in X$ and $0 < \lambda < 1$.*

Definition 1.2. *[18] The epigraph of f is the set $\{(x, \mu) \in X \times \mathbb{R} : X \subseteq \mathbb{R}^n, \mu \in \mathbb{R}, \mu \geq f(x)\}$, denoted by epi(f). We define the function f on X as a convex function on X if epi(f) is a convex subset of $\mathbb{R}^{n+1}$.*

Theorem 1.3. *[17] The function $f : \mathbb{R}^n \to (-\infty, \infty)$ is convex if and only if*

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)\alpha + \lambda\beta, \quad 0 < \lambda < 1,$$

where $f(x) < \alpha$ and $f(y) < \beta$.

Theorem 1.4. *[1] Suppose $f \in \mathbb{C}^{2n}[-1, 1]$ is $(2n-1)$-convex function, and $n \in \{2, 3\}$. Then*

$$0 \leq \int_{-1}^{1} (f(t) - G_n[f])dt \leq \mathcal{L}_{n+1}[f] - \int_{-1}^{1} f(t)dt.$$

Definition 1.5. *[5] A function $f : \mathbb{D} \to \mathbb{R}$ is strongly $\mathfrak{h}$-convex with modulus u if*

$$f((1-\lambda)x + \lambda y) \leq \mathfrak{h}(1-\lambda)f(x) + \mathfrak{h}(\lambda)f(y) - u\lambda(1-\lambda)\|x-y\|^2,$$

for all $x, y \in \mathbb{D}$ and $0 < \lambda < 1$, where $\mathfrak{h} : (0, 1) \to (0, \infty)$ is given function.

Theorem 1.6. *[19] Let $\mathfrak{h} : [0, 1] \to \mathbb{R}$ be a multiplicative function such that $\mathfrak{h}(t) \leq t$ for all $t \in [0, 1]$. If a function $g : \mathbb{D} \to \mathbb{R}$ is ϵ-strongly $\mathfrak{h}$-convex with modulus u, then there exists a function $\phi : \mathbb{D} \to \mathbb{R}$ strongly $\mathfrak{h}$-convex with modulus u such that*

$$g(x) - \epsilon \leq \phi(x) \leq g(x), \quad x \in \mathbb{D}.$$

Definition 1.7. *[13] If X is a vector space that has a topology τ, then we say that X is a locally convex space if every point has a neighborhood base consisting of convex sets.*

Definition 1.8. *[4] Let Π_n be the space of all algebraic polynomials of degree $\leq n-1$, and $\Delta^{(2)}$ be the set of all convex functions on I, and*

$$E_n^{(2)}(f) = \inf_{p_n \in \Pi_n \cap \Delta^{(2)}} \|f - p_n\|$$

denote the degree of best uniform convex polynomial approximation of $f \in \mathbb{C}[-1, 1] \cap \Delta^{(2)}$.

Definition 1.9. *[14] Let Π_n be the space of all algebraic polynomials of degree $\leq n-1$, and $\Delta^{(2)}(Y_s)$ be the collection of all functions $f \in \mathbb{C}[-1,1]$ that change convexity at the points of the set Y_s and are convex in $[Y_s, 1]$. The degree of best uniform coconvex polynomial approximation of f is defined by*

$$E_n^{(2)}(f, Y_s) = \inf_{p_n \in \Pi_n \cap \Delta^{(2)}(Y_s)} \|f - p_n\|$$

where $Y_s = \{y_i\}_{i=1}^{s}$ such that $y_\circ = -1 < y_1 < \cdots < y_s < 1 = y_{s+1}$.

Definition 1.10. *[16] The weighted Ditzian-Totik modulus of smoothness (DTMS) of a function $f \in L_p[-1,1]$, when $0 < p < \infty$, is defined by*

$$\omega_{k,r}^{\phi}(f,t)_p = \sup_{0<h\leq t} \|\phi(x)^r \Delta_{h\phi}^k (f,x)\|_p$$

where $\phi(x) = \sqrt{1-x^2}$. If $r = 0$, then

$$\omega_k^{\phi}(f,t)_p = \omega_{k,0}^{\phi}(f,t)_p = \sup_{0<ht} \|\Delta_{h\phi}^k (f,x)\|_p$$

is the usual DTMS. Also, note that

$$\omega_{0,r}^{\phi}(f,t)_p = \|\phi^r f\|_p.$$

Definition 1.11. *[8] Let A be a subset of the topology space X, and $x \in X$. They say that A is a neighborhood of x, if A merely contains an open set containing x.*

2. The Main Results

Contributions to the domain of convex polynomials (DCP) and their properties are presented in this section. We will provide the following definition.

Definition 2.1. *A domain $\mathbb{D}$ of a convex polynomial p_n of $\Delta^{(2)}$ is a subset of X and $X \subseteq \mathbb{R}$, satisfying the following properties:*

1. *$\mathbb{D} \in \mathbf{K}$, where $\mathbf{K} = \{\mathbb{D} : \mathbb{D}$ is a compact subset of $X\}$ the class of all domain of convex polynomial,*
2. *there is the point $t \in X/\mathbb{D}$, such that $|p_n(t)| > \sup\{|p_n(x)| : x \in \mathbb{D}\}$, and*
3. *there is the function f of $\Delta^{(2)}$, such that $\|f - p_n\| \leq \frac{c}{n^2}\omega_{2,2}^{\phi}(f'', \frac{1}{2})$.*

Let $\mathbb{D}$ and X be as in Definition 2.1.

Definition 2.2. *If the compact set U is convex, there is a bounded neighborhood set $\Upsilon = \{\zeta \in X : |\zeta|^2 < c\}$ for c in a suitable position.*

Theorem 2.3. *If $\mathbb{D}$ is the DCP of p_n, and if $x_\circ \in \mathbb{D}$. Then there is a compact neighborhood Υ of the point $x_\circ$.*

Proof. Suppose that $\mathbb{D}$ is DCP, from Definition 2.1, then $\mathbb{D}$ is a compact subset of X, and $\mathbb{D} \in \mathbf{K}$. Then $\mathbb{D}$ is a compact and convex subset of X. From Definition 2.2, there is a bounded neighborhood Υ of the point $x_\circ$, such that $\Upsilon = \{x \in \mathbb{D} : |x|^2 < c\}$, for c suitably near and $\Upsilon \subseteq \mathbb{D}$. Since $x_\circ \in \mathbb{D}$, and $|x_\circ|^2 < c$, for c suitable near. Then, $\Upsilon = \mathbb{D}$. Therefore, Υ is a compact neighborhood of the point $x_\circ$, and Υ DCP of p_n. □

Lemma 2.4. *If $x_\circ \in \mathbb{D}$ is the DCP of p_n. Then $\mathbb{D}$ is a compact neighborhood of the point $x_\circ$.*

Proof. Clear. □

We operate within the standard topology of $\mathbb{R}$, as it constitutes a metric space. According to the Heine-Borel Theorem, "**In the Euclidean space $\mathbb{R}^n$, a subset $\mathbb{D}$ is a compact if and only if it is closed and bounded.**". Consequently, we can restrict our focus to all closed intervals of the form $[a, b] \subseteq \mathbb{R}$, where $a, b \in \mathbb{R}$. However, we can operate in the Euclidean space $\mathbb{R}^n$, where $n \geq 1$; so, $\mathbb{D}$ can be regarded as a closed rectangle.

Theorem 2.5. *If $p_n : \mathbb{D} \to \mathbb{D}$ is a convex polynomial of $\Delta^{(2)}$, and Y is a compact subset of $\mathbb{D}$. Then Y is DCP of p_n if and only if $p_n^{-1}(Y)$ is DCP of p_n.*

Proof. Suppose that Y is a compact subset of $\mathbb{D}$.

Case I. Suppose that Y is DCP of p_n. Since $p_n : \mathbb{D} \to \mathbb{D}$, and $Y \subseteq \mathbb{D} \subseteq X$, then Y is a compact subset of X, and $Y \in \mathbf{K}$. Therefore, p_n is continuous and $p_n^{-1}(Y) = \mathbb{D}$ is a compact subset of X. Let $t \notin p_n^{-1}(Y)$, then $t \in X/p_n^{-1}(Y)$. From Definition 2.1, we have

$$|p_n(t)| > \sup\{|p_n(x)| : x \in \mathbb{D}\},$$

and the function $f \in \Delta^{(2)}$, such that

$$\|f - p_n\| \leq \frac{c}{n^2}\omega_{2,2}^{\phi}(f'', \frac{1}{2}).$$

Therefore, $p_n^{-1}(Y)$ is the DCP of p_n.

Case II. Suppose that $p_n^{-1}(Y)$ is DCP of p_n. Since Y is a compact subset of $\mathbb{D}$. Let $y \notin \mathbb{D}$, then $y \in X/\mathbb{D}$, implies $y \in X/Y$. From Definition 2.1, we have

$$|p_n(y)| > \sup\{|p_n(x)| : x \in \mathbb{D}\},$$

then,

$$|p_n(y)| > \sup\{|p_n(x)| : x \in Y\},$$

and the function $f \in \Delta^{(2)}$, such that

$$\|f - p_n\| \leq \frac{c}{n^2}\omega_{2,2}^{\phi}(f'', \frac{1}{2}).$$

Therefore, Y is the DCP of p_n. □

Theorem 2.6. *If $\mathbb{D}$ is DCP of p_n, and $\mathcal{D} \subseteq \mathbb{D}$ is DCP of p_n. For every convex function f of $\Delta^{(2)}$, defined in a neighborhood of $\mathbb{D}$, then the set $\mathcal{D} \cup f^{-1}(0)$ is DCP of p_n.*

Proof. Suppose that $\mathcal{D} \subseteq \mathbb{D}$ is DCP of p_n. Let $x_\circ \in \mathbb{D}$, then from Theorem 2.3, there is a compact neighborhood Υ of the point $x_\circ$. If $f \in \Delta^{(2)}$, such that f is defined in Υ. From Theorem 2.5, then, $f^{-1}(\Upsilon)$ is DCP of p_n. Assume $x_\circ = 0$, then $\mathcal{D} \cup f^{-1}(0)$ is DCP of p_n. □

In the sequel, the presentation includes contributions to the domain of coconvex polynomials (DCCP) and their properties.

Definition 2.7. *A domain $\mathbb{D}$ of coconvex polynomial p_n of $\Delta^{(2)}(Y_s)$ is a subset of X and $X \subseteq \mathbb{R}$, satisfying the following properties:*

1. *$\mathbb{D} \in \mathbf{K}(Y_s)$, where $\mathbf{K}(Y_s) = \{\mathbb{D} : \mathbb{D}$ is a compact subset of X, and p_n changes convexity at $\mathbb{D}\}$ is the class of all domain of convex polynomial,*
2. *y_i's are inflection points, such that $|p_n(y_i)| \leq \frac{1}{2}, i = 1, \ldots, s$, and*
3. *there is the function f of $\Delta^{(2)}(Y_s)$, such that $\|f - p_n\| \leq \frac{c}{n^2}\omega_{k,2}^{\phi}(f'', \frac{1}{n})$.*

Let $\mathbb{D}$ and X be as in Definition 2.7.

Theorem 2.8. *If $p_n : \mathbb{D} \to \mathbb{D}$ is the coconvex polynomial of $\Delta^{(2)}(Y_s)$ and $\mathbb{D}$ is DCCP of p_n. Then Υ is the DCCP of p_n, if Υ is a compact neighborhood of the point $x_\circ$, where $p_n(x_\circ) = \frac{1}{2}$.*

Proof. Suppose that $p_n : \mathbb{D} \to \mathbb{D}$ is a coconvex polynomial of $\Delta^{(2)}(Y_s)$, such that $\mathbb{D}$ is a compact subset of X, and p_n changes the convexity at $\mathbb{D}$. Putting Υ is a compact neighborhood of $x_\circ$, implies $x_\circ \in \Upsilon$. Since $p_n(x_\circ) = \frac{1}{2}$, and $\mathbb{D}$ is DCCP of p_n. From Definition 2.7, then:

Case I. Either $x_\circ$ is an inflection point at $\mathbb{D}$. Therefore, $x_\circ \in \mathbb{D}$, and $\Upsilon \subseteq \mathbb{D}$. Since, $p_n(x_\circ) = \frac{1}{2}$. Then, $x_\circ$ is the inflection point at Υ.

Case II. Or $x_\circ$ is not an inflection point at $\mathbb{D}$. Now, we must prove that p_n changes convexity at Υ. Let $1 \leq s < \infty$, $y_{s-1}, y_s \in \mathbb{D}$, and y_{s-1}, y_s be inflection points at $\mathbb{D}$, such that $p_n(y_{s-1})) \leq p_n(x_\circ) \leq p_n(y_s)$. Since $p_n(x_\circ) = \frac{1}{2}$, and y_s is an inflection points at $\mathbb{D}$, implies $p_n(x_\circ) = p_n(y_s)$. This is a contradiction. Therefore, $x_\circ$ is an inflection points at Υ, and $\Upsilon \subseteq \mathbb{D}$. Thus, p_n changes convexity at Υ. To prove that we Υ have all inflection points $\leq \frac{1}{2}$, let y_j be an inflection point in Υ, such that $j < s$, and $|p_n(y_j)| > \frac{1}{2}$. We get a contradiction. Since $\Upsilon \subseteq \mathbb{D}$, then $f \in \Delta^{(2)}(Y_s)$, such that $\|f - p_n\| \leq \frac{c_1}{(n^2)}\omega^{\phi}_{k,2}(f'', \frac{1}{n})$. Thus, Υ is the DCCP of p_n. □

Definition 2.9. *$\gamma - H$ is said to support the hyperplane in the domain of the (co)convex polynomial p_n if at least one point $\hat{x}_\circ$ of $\mathbb{D}$ lies in $\gamma - H$, and $p_n(y) \geq \hat{\alpha}$, for all $y \in \mathbb{D}/x_\circ$, and $\alpha \in \mathbb{R}$.*

Definition 2.10. *If $\mathbb{D}$ is the domain of the (co)convex polynomial of p_n and $x \notin \mathbb{D}$. $\gamma - H$ is said to be strictly separate$\mathbb{D}$, if we choose $\mathfrak{b} \in \mathbb{R}$ such that*

$$\sup\{p_n(y) \in \mathbb{D}\} < \mathfrak{b} < p_n(y), \quad y \in Y.$$

Definition 2.11. *If $\mathfrak{h} : [0, 1] \to \mathbb{R}$ is a given function, $\mathbb{D}_1$ and $\mathbb{D}_2$ are domains of (co)convex polynomials of p_n and q_n respectively. $\gamma - H$ and $\gamma_{\mathfrak{h}} - H$ are said to be strongly hyperplane and strongly $\mathfrak{h}$-hyperplane respectively, if and only if*

$$\inf\{p_n(a) : a \in \mathbb{D}_1\} \geq \sup\{q_n(b) : b \in \mathbb{D}_2\}$$

and

$$\inf\{\mathfrak{h}(t)p_n(a) : a \in \mathbb{D}_1\} \geq \sup\{\mathfrak{h}(t)q_n(b) : b \in \mathbb{D}_2\},$$

where $t \in [0, 1]$.

3. Examples of the Study

We now state and provide examples for some of the objectives of this work.

Example 3.1. *Let $n = 3$, $p_3 : \mathbb{D} \to (-\infty, \infty)$ be polynomial of degree $\leq n - 1$, such that $\mathbb{D} = [-3, 3]$ and $p_3(x) = 0.5x^2 - x$.*

1) *Suppose that $x = 3$, $y = -3$ and $\lambda = 0.6 (0 < \lambda < 1)$. Then, $p_3(x = 3) = 1.5$ and $p_3(y = -3) = 7.5$. Now, $p_3((1 - \lambda)x + \lambda y) = p_3(-0.6) = -0.78$, also, $(1 - \lambda)p_3(3) + \lambda p_3(-3) = (0.4) \times (1.5) + (0.6) \times (7.5) = 5.1$. Therefore, $p_3((1 - \lambda) \times (3) + \lambda \times (-3)) \leq (1 - \lambda)p_3(3) + \lambda p_3(-3)$. Then, p_3 is convex polynomial, and $\mathbb{D} \in \mathbf{K}$.*

2) *Let $t \in \mathbb{R}/\mathbb{D}$, and $t = 6$ then $p_3(t = 6) = 12$, and $\sup\{|p_3(x)| : x \in D\} = |p_3(x = -3)| = 7.5$.*

3) *Let $p_3 : \mathbb{D} \to (-\infty, \infty)$ such that*

$$f(x) = \begin{cases} \frac{1}{2}x^4 - (x - 1)^3 - 2x^2 & ; \quad if\ 0 \leq x \leq 3 \\ x & ; \quad if\ -3 \leq x < 0 \end{cases}$$

$$f^{'}(x) = \begin{cases} 2x^3 - 3(x-1)^2 - 4x & ; \quad if\ 0 \leq x \leq 3 \\ 1 & ; \quad if\ -3 \leq x < 0 \end{cases}$$

and

$$f^{''}(x) = \begin{cases} 6x^2 - 6x^2 + 2 & ; \quad if\ 0 \leq x \leq 3 \\ 0 & ; \quad if\ -3 \leq x < 0 \end{cases}$$

Let $x_\circ = 1$, $y_\circ = 2$ *and* $\lambda = 0.5$ $(0 < \lambda < 1)$. *Then,* $f(x_\circ = 1) = -1.5$ *and* $f(y_\circ = 2) = -1$. *So,*

$$f((1-\lambda)(1) + \lambda(2)) = f(1.5) = -2.093,$$

also,

$$(1-\lambda)f(1) + \lambda f(2) = -1.25.$$

Therefore,

$$f((1-\lambda)\times(1) + \lambda(2)) \leq (1-\lambda)f(1) + \lambda f(2).$$

Hence, f *is a convex function, and it has* $f^{''}$. *Now,*

$$\|f(3) - p_3(3)\| = \|(\frac{1}{2}x^4 - (x-1)^3 - 2x^2) - (0.5x^2 - x)\| = 13,$$

and

$$\Delta^2_{0.4\times\phi}(f^{''}, x) = \sum_{i=0}^{2} \tbinom{2}{i}(-1)^{2-i} f^{''}(x - \frac{2\times(0.4)}{2} + i\times(0.4)) = \tbinom{2}{0}(-1)^2 \times f^{''}(x-(0.4)) + \tbinom{2}{1}(-1)\times f^{''}(x) + \tbinom{2}{2}\times f^{''}(x+(0.4))$$

$$\Delta^2_{0.4\times\phi}(f^{''}, 3) = f^{''}(2.6) - 2f^{''}(3) + f^{''}(3.4) = 1.92.$$

Therefore,

$$\omega^{\phi}_{2,2}(f^{''}(x) = 6x^2 - 6x + 2, \frac{1}{2}) = \sup_{0<h\leq\frac{1}{2}} \|(1-x^2)\times\Delta^2_h(f^{''}, x)\| = |(-8)\times(1.92)| = 15.36.$$

Thus,

$$\|f - p_3\| \leq \frac{c_1}{9}\omega^{\phi}_{2,2}(f^{''}(x) = 6x^2 - 6x + 2, \frac{1}{2}),$$

where $c_1 = 7.62$.

Example 3.2. *Let* $n = 5$, $p_5 : \mathbb{D} \to (-\infty, \infty)$ *be a polynomial of degree* $\leq n-1$, *such that* $\mathbb{D} = [-3, 3]$ *and* $p_5(x) = (x+2)(x+1)(x-1)(x-2)$.

1)*1Suppose that* $x = 1.5$, $y = 1$ *and* $\lambda = 0.5$ $(0 < \lambda < 1)$. *Then,* $p_5(x = 1.5) = -2.1875$ *and* $p_5(y = 1) = 0$. *Now,*

$$p_5((1-\lambda)x + \lambda y) = p_5(1.25) = -1.37,$$

also,

$$(1-\lambda)p_5(1.5) + \lambda p_5(1) = (0.5)\times(-2.1875) + (0.5)\lambda(0) = -1.09.$$

Therefore,

$$p_5((1-\lambda)\times(1.5)+\lambda(1))\leq(1-\lambda)p_5(1.5)+\lambda p_5(1).$$

Then, p_5 is changes convexity at $\mathbb{D}\in\mathbf{K}$.

2) *Let $Y_s = y_{i}{}_{i=1}^{s=4}$ such that $y_0=-3<y_1=-2<y_2=-1<y_3=1<y_4=2<y_{s+1}=3$ and are convex in $[y_4,3]$. Then, $|p_5(y_i)|=0\leq\frac{1}{2}$, $i=1,\ldots,4$.*

3) *Let $f:\mathbb{D}\to(-\infty,\infty)$ such that*

$$f(x)=\begin{cases}|x^2-4|+x & ;\quad if\quad -3\leq x\leq 0\\ \\ |2x-4|-x & ;\quad if\quad 0<x\leq 3,\end{cases}$$

$$f'(x)=\begin{cases}\frac{(2x^3-8x)}{|x^2-4|}+1 & ;\quad if\quad -3\leq x\leq 0\\ \frac{(4x-8)}{|2x-4|}-1 & ;\quad if\quad 0<x\leq 3,\end{cases}$$

and

$$f''(x)=\begin{cases}\frac{(|x^2-4|)^2\times(6x-8)-(2x^3-8x)^2)}{(|x^2-4|)^3} & ;\quad if\quad -3\leq x\leq 0\\ 0 & ;\quad if\quad 0<x\leq 3.\end{cases}$$

Let $x_\circ=0$, $y_\circ=0.5$ and $\lambda=0.5$ $(0<\lambda<1)$. Then, $f(x_\circ=0)=4$ and $f(y_\circ=0.5)=2.5$. So,

$$f((1-\lambda)\times(0)+\lambda(2.5))=f(1.25)=0.25,$$

also,

$$(1-\lambda)f(0)+\lambda f(0.5)=3.25.$$

Therefore,

$$f((1-\lambda)\times(0)+\lambda(0.5))\leq(1-\lambda)f(0)+\lambda f(0.5).$$

Hence, f it changes convexity at $\mathbb{D}$, and it has f''. Now,

$$\|f(-3)-p_5(-3)\|=\|(|x^2-4|+x)-(x^4-5x^2+4)\|=38,$$

and

$$\Delta^4_{(0.1)\phi}(f'',x)=\sum_{i=0}^{4}\binom{4}{i}(-1)^{4-i}f''(x-\frac{4\times(0.1)}{2}+i\times(0.1))=\binom{4}{0}(-1)^4\times f''(x-0.2)+\binom{4}{1}(-1)^3\times f''(x-0.1)+\binom{4}{2}(-1)^2$$

$$\times f''(x)+\binom{4}{3}(-1)\times f''(x+0.1)+\binom{4}{4}\times f''(x+0.2)=f''(x-0.2)-4f''(x-0.1)+6f''(x)-4f''(x+0.1)+f''(x+0.2).$$

Then, $\Delta^4_{(0.1)\phi}(f'',-3)=124.678$. Therefore,

$$\omega^\phi_{4,2}(f''(x)=\frac{(|x^2-4|)^2\times(6x-8)-(2x^3-8x)^2}{(|x^2-4|)^3},\frac{1}{5})=\sup_{0<h\leq\frac{1}{2}}\|(1-x^2)\times\Delta^4_{(0.1)}(f'',x)\|=|(-8)\times(124.678)|=997.424.$$

Thus,

$$\|f-p_5\|\leq\frac{c_2}{25}\omega^\phi_{4,2}(f''(x)=\frac{(|x^2-4|)^2\times(6x-8)-(2x^3-8x)^2}{(|x^2-4|)^3},\frac{1}{5}),$$

where $c_2=0.953$.

4. Conclusion

This work presents an innovative way for broadening the scope of (co)convex polynomials, transcending conventional techniques that depend on the division of convex sets. Using separation theorems, supporting hyperplanes, and advanced approximation methods, it creates a more extensive framework for polynomial analysis. The results indicate increased versatility in the use of convex and coconvex polynomials, thereby augmenting their significance in optimization and real-world problem solving. This paper establishes a theoretical framework with significant implications for optimization, machine learning, and numerical analysis.

To partition the domain of convex polynomials p_n and q_n topologically, we can apply the separation axioms. Our continued operation within the metric space, which preserves its boundaries and closure, has no effect on the compactness of the domains $\mathbb{D}_{p_n}$ and $\mathbb{D}_{q_n}$. The theorem of separation for convex sets holds that if two convex sets in topological space are disjoint and closed, then there exists a continuous between them. This topological result lends credence to this viewpoint. The study can be improved by iteratively switching between the separation axioms τ_i, $i = 1, \ldots, 5$, which is particularly useful given that we are addressing compactness.

Finally, if p_n and q_n are two convex polynomials of Δ^2. If $\mathbb{D}_{p_n}$ is a nonempty set and compact (and $\mathbb{D}_{q_n}$ is a nonempty set and closed), such that $\mathbb{D}_{p_n}$ and $\mathbb{D}_{q_n}$ are disjoint. Are p_n they q_n strongly separated by a hyperplane?

Acknowledgements

Authors express their gratitude to the University of Al-Muthanna.

References

[1] A. Komisarski, S. Wasowicz, Inequalities between remainders of quadratures, *Aequationes mathematicae* **6** (2017), 1103–1114.
[2] B. Mordukhovich, N. Nam, *An easy path to convex analysis and applications*, Wiliston, W: Morgan and Claypool Publishers, (2014).
[3] D. Leviatan, I. Shevchuk, *Shape preserving approximation of periodic functions – Conclusion*, **to appear** (2024).
[4] D. Leviatan, Pointwise estimates for convex polynomial approximation, *Proceedings of the American Mathematical Society* **98** (1986), 471–474.
[5] H. Angulo, J. Gimenez, A. Moros, K. Nikodem, On strongly $\mathfrak{h}$-convex functions, *Annals of Functional Analysis* **2** (2011), 85–91.
[6] I. Singer, Generalizations of methods of best approximation to convex optimization in locally convex spaces. II: Hyperplane theorems, *Journal of Mahtematical Analysis and Applications* **69** (1979), 571–584.
[7] J. Bustamante, New estimates related with the best polynomial approximation, *Journal of Numerical Analysis and Approximation Theory* **52(1)** (2023), 75–81. https://doi.org/10.33993/jnaat521-1313
[8] J. Munkres,*Topology James Munkres Second Edition*, George Lobeli: Printed in the United States of America, (1975).
[9] M. Al-Muhja, H. Akhadkulov, N. Ahmad, Estimates for constrained approximation in $\mathbb{L}_{p,r}^{\alpha,\beta}$ space: Piecewise polynomial, *International Journal of Mathematics and Computer Science* **16(1)** (2021a), 389–406.
[10] M. Al-Muhja, H. Akhadkulov, N. Ahmad, On weighted approximation of (co)convex functions with polynomials of varying degrees, *AIP Conference Proceedings* **2365** (2021b), 050009. https://doi.org/10.1063/5.0059023
[11] M. Al-Muhja, Separation theorems and best approximation: hybrid, *Filomat* **38(25)**, (2024), 8999–9008.
[12] M. Kareem, R. Kamel, A. Hussain, Coconvex multi approximation in the uniform norm, *Journal of Interdisciplinary Mathematics* **25(5)** (2022), 1435–1443. https://www.tandfonline.com/loi/tjim20
[13] M. Osborne, *Locally convex spaces*, New York, NY: Graduate Texts in Mathematics **269** Springer International Publishing Switzerland, (2014). https://link.springer.com/book/10.1007/978-3-319-02045-7
[14] M. S. Al-Muhja, *Best weighted approximation and Its degree for (co)convex and unconstrained polynomials*, (Published doctoral dissertation), Universiti Utara Malaysia, Kedah, Malaysia, (2022).
[15] P. Papini, I. Singer, Best coapproximation in normed linear spaces, *Monatshefte Mathematik* **88** (1979), 27–44.
[16] R. DeVore, G. Lorentz, *Constructive approximation*, New York: A Series of Comprehensive Studies in Mathematics **303** Springer-Verlag, (1993).
[17] R. Rockafellar, *Convex analysis*, New Jersey: Princeton Univ. Press, Princeton, (1970).
[18] T. Hoang, *Convex analysis and global optimization*, Springer optimization and its applications: **110** New York: Springer International Publishing Switzerland, (2016).
[19] T. Lara, N. Merentes, K. Nikodem, Strong $\mathfrak{h}$-convexity and separation theorems, *International Journal of Analysis* **2016** (2016), 1–5.