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Ideal relatively equal convergence involving difference operators with associated approximation theorems

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Abstract. We introduce the notions of ideal relatively equal convergence and ideal relatively uniform convergence inconjunction with the difference operators of sequences of functions ($\mathcal{I}(\Delta^j_{r,equi})$-convergence and $\mathcal{I}(\Delta^j_{r,u})$-convergence, respectively, for short). Under some condition, we obtain an equavalence relation by means of aforesaid notions. The Korovkin-type result is obtained through our newly notion of $\mathcal{I}(\Delta^j_{r,equi})$-convergence and construct an example by taking λ-Bernstein operators to support this result. Moreover, we analyze the rate of $\mathcal{I}(\Delta^j_{r,equi})$-convergence by utilizing the modulus of continuity.

1. Introduction and preliminaries

Ideal convergence, simply write, $\mathcal{I}$-convergence, was given by Kostyrko et al. [28] and Nuray and Ruckle [35], independently, which is an interesting generalization of widely studied notion of statistical convergence (see [19, 48]). The authors of [35] called $\mathcal{I}$-convergence by the named generalized statistical convergence. Both of the aforesaid concepts have been extensively examined from multiple angles and utilized to address various problems that arise within the convergence theory (see [1, 2, 5, 22, 34, 44]).

Consider an arbitrary set X. A nontrivial ideal $\mathcal{I} \in X$ is admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$. In what follows, $\mathcal{I}$ is a nontrivial admissible ideal in $\mathbb{N}$ (the set of natural numbers). We use the symbol $\mathcal{I}_f$ to denote the class of all finite subsets of $\mathbb{N}$. A sequence $s = (s_m)$ $(m \in \mathbb{N})$ is called $\mathcal{I}$-convergence to $\xi \in \mathbb{R}$ (the set of real numbers), in symbols, write $\mathcal{I}\text{-}\lim_m s_m = \xi$ ($\lim_m$ means $\lim_{m\to\infty}$) or $s_m \xrightarrow{\mathcal{I}} \xi$, if for every $\varepsilon > 0$, the set $S(\varepsilon) = \{m \in \mathbb{N} : |s_m - \xi| \geq \varepsilon\} \in \mathcal{I}$.

Take $\mathcal{I} = \mathcal{I}_d = \{S \subset \mathbb{N} : d(S) = 0\}$, where $d(S)$ is the density of any subset S of $\mathbb{N}$ defined by

$$d(S) = \lim_m \frac{1}{m}|\{s_1 \leq m : s_1 \in S\}|$$

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and here $|\ .\ |$ means the cardinality of enclosed set. In this case, $\mathcal{I}_d$-convergence concides with statistical convergence while (s_m) is statistical convergent to ξ if $d(S(\varepsilon)) = 0$ for any $\varepsilon > 0$. We refer to [4, 16, 24, 30, 45, 46] for some recent work.

The space of all continuous real-valued functions defined on a compact subset Y of $\mathbb{R}$ is assumed to be denoted by the notation $C(Y)$. For $h \in C(Y)$, $\|h\| = \sup_{y\in Y} |h(y)|$. We take $(h_m), h \in C(Y)$. For a sequence (h_m) of functions, Császár and Laczkovich [11] presented the notion of equal convergence and an interesting generalization of this notion was given Filipów and Szuca [21] and Das et al. [12], called by $\mathcal{I}$-equally (or, say $\mathcal{I}$-equi) convergence, which is based on ideal $\mathcal{I}$ and further studied by the authors Filipów and Staniszewski [20] and Staniszewski [47] in various aspects. Recall as in [12] that (h_m) is $\mathcal{I}$-equally ($\mathcal{I}_{equi}$) convergent to h if there is a sequence $(\varepsilon_m) \xrightarrow{\mathcal{I}} 0$ of positive reals such that

$$\{m \in \mathbb{N} : |h_m(y) - h(y)| \geq \varepsilon_m\} \in \mathcal{I} \qquad (y \in Y)$$

while the difference in Filipów and Szuca [21] definition is that they considered usual limit, that is, $(\varepsilon_m) \to 0$ instead of $(\varepsilon_m) \xrightarrow{\mathcal{I}} 0$. By this fact, $\mathcal{I}_{equi}$ convergence introduced by Filipów and Szuca [21] implies $\mathcal{I}_{equi}$ convergence due to Das et al. [12].

Firstly, Moore [33] gave the notion of relative uniform convergence for (h_m) and later discussed in [8–10]. Recently, Demirci and Orhan [13] and Dirik and Şahin [14], by taking into their account aforesaid notion and statistical convergence, respectively, defined statistical relatively uniform (simply, write $S_{r,u}$) and statistical relatively equal (simply, write $S_{r,equi}$) convergence and investigated several results related to their notions. Later, the notion of ideal relatively uniform convergence was given by the authors of [31] and, as an application, they have established Korovkin as well as Voronovskaya theorems. Moreover, $S_{r,equi}$) convergence defined and studied for double sequence in [43] and an application of $S_{r,u}$ convergence considered in [49].

For any sequence space U, the difference operator Δ^j $(j \in \mathbb{N})$ [17] involving sequence spaces is defined by

$$\Delta^j(U) = \{s = (s_m) : (\Delta^j s_m) \in U\},$$

where

$$\Delta^0 s = (s_m), \qquad \Delta^j s = (\Delta^{j-1} s_m - \Delta^{j-1} s_{m+1})$$

and so

$$\Delta^j s_m = \sum_{j_1=0}^{j} (-1)^{j_1} \binom{j}{j_1} s_{m+j_1}.$$

Also, $\Delta s = (\Delta s_m) = (s_m - s_{m+1})$ due to [26]. Temizsu et al. [50] showed that if $s \in \Delta^j(U)$, there is only one $s' = (s'_m) \in U$ so that $s'_m = \Delta^j s_m$. The difference operators have been used to defined some sequence spaces (see [38, 39, 42]).

2. Ideal relatively equal convergence of difference sequence of functions

Definition 2.1. *We say that (h_m) is*

(D1) Δ^j-ideally relative equal convergent, shortly, $\mathcal{I}(\Delta^j_{r,equi})$-convergent, to h on Yif there is a sequence (ε_m) of positive reals satsfies

$$\mathcal{I}\text{-}\lim_m \varepsilon_m = 0, \tag{1}$$

and a scale function $\chi(y)$, $|\chi(y)| > 0$, such that

$$\left\{m \in \mathbb{N} : \left|\frac{\Delta^j h_m(y) - h(y)}{\chi(y)}\right| \geq \varepsilon_m\right\} \in \mathcal{I} \tag{2}$$

for any $y \in Y$, where

$$\Delta^j h_m(y) = \sum_{j_1=0}^{j} (-1)^{j_1} \binom{j}{j_1} h_{m+j_1}(y).$$

Denoted by

$$h_m \xrightarrow{\mathcal{I}(\Delta^{j,\chi}_{r,equi})} h \quad or \quad \mathcal{I}(\Delta^{j,\chi}_{r,equi})\text{-}\lim_m h_m = h,$$

in this case.

(D2) *Δ^j-ideally relative uniform convergent, shortly, $\mathcal{I}(\Delta^j_{r,u})$-convergent, to h on Y if $\chi(y)$ ($|\chi(y)| > 0$) such that for every $\varepsilon > 0$, we have*

$$\left\{ m \in \mathbb{N} : \sup_{y \in Y} \left| \frac{\Delta^j h_m(y) - h(y)}{\chi(y)} \right| \geq \varepsilon \right\} \in \mathcal{I}. \tag{3}$$

Denoted by

$$h_m \xrightarrow{\mathcal{I}(\Delta^{j,\chi}_{r,u})} h \quad or \quad \mathcal{I}(\Delta^{j,\chi}_{r,u})\text{-}\lim_m h_m = h.$$

The choice of $j = 0$ in (D_1) gives the notion of ideally relative equal convergent, shortly, $\mathcal{I}_{r,equi}$-convergent, to h on Y. In this case, (2) becomes

$$\left\{ m \in \mathbb{N} : \left| \frac{h_m(y) - h(y)}{\chi(y)} \right| \geq \varepsilon_m \right\} \in \mathcal{I}$$

and denoted by

$$h_m \xrightarrow{\mathcal{I}^{j,\chi}_{r,equi}} h \quad or \quad \mathcal{I}^{j,\chi}_{r,equi}\text{-}\lim_m h_m = h.$$

Remark 2.2. *(i) Take $\mathcal{I} = \mathcal{I}_d = \{S \subset \mathbb{N} : d(S) = 0\}$ in (D_1). Then, in this case,* (2) *becomes*

$$d\left(\left\{ m \in \mathbb{N} : \left| \frac{\Delta^j h_m(y) - h(y)}{\chi(y)} \right| \geq \varepsilon_m \right\} \right) = 0$$

and called by $S_{r,equi}(\Delta^j)$-convergent to h. Additionally, if $j = 0$ then $\mathcal{I}(\Delta^j_{r,equi})$-convergence coincides with $S_{r,equi}$-convergence [14].
(ii) If $\mathcal{I} = \mathcal{I}_d = \{S \subset \mathbb{N} : d(S) = 0\}$ and $j = 0$ in (D_2), then $\mathcal{I}(\Delta^j_{r,u})$-convergence coincides with $S_{r,u}$-convergence [13].
(iii) If $j = 0$ and $\chi(y)$ is constant in (D_1), then $\mathcal{I}(\Delta^j_{r,equi})$-convergence coincides with $\mathcal{I}_{equi}$-convergence defined in [12]. Additionally, by taking classical limit in (1), *we obtain $\mathcal{I}_{equi}$-convergence due to [21].*
(iv) If $j = 0$ in (D_2), then $\mathcal{I}(\Delta^j_{r,u})$-convergence coincides with $\mathcal{I}_{r,u}$-convergence [31].
(v) If $\mathcal{I} = \mathcal{I}_f$, $j = 0$, $\chi(y)$ is constant in (D_1), then $\mathcal{I}(\Delta^j_{r,equi})$-convergence coincides with equal convergence [11]

Theorem 2.3. *The implication*

$$h_m \xrightarrow{\mathcal{I}(\Delta^{j,\chi}_{r,u})} h \Longrightarrow h_m \xrightarrow{\mathcal{I}(\Delta^{j,\chi}_{r,equi})} h \tag{4}$$

holds.

Proof. Suppose $h_m \xrightarrow{\mathcal{I}(\Delta^{j,\chi}_{r,u})} h$. Therefore, for given $\varepsilon > 0$, one writes

$$B = \left\{ m \in \mathbb{N} : \sup_{y \in Y} \left| \frac{\Delta^j h_m(y) - h(y)}{\chi(y)} \right| \geq \varepsilon \right\} \in \mathcal{I}.$$

Let us define

$$\varepsilon_m = \begin{cases} \frac{1}{m} & (m \in B) \\ \sup_{y \in Y} \left| \frac{\Delta^j h_m(y) - h(y)}{\chi(y)} \right| + \frac{1}{m} & (m \notin B). \end{cases}$$

Thus, we have $(\varepsilon_m) \xrightarrow{\mathcal{I}} 0$, and

$$\left| \frac{\Delta^j h_m(y) - h(y)}{\chi(y)} \right| < \varepsilon_m \quad \forall m \notin B$$

which yields $h_m \xrightarrow{\mathcal{I}(\Delta^{j,\chi}_{r,equi})} h$. □

The example below shows that the converse of implication (4) doesn't hold true.

Example 2.4. *Suppose $y \in Y = [0,1]$ and $h(y) = 0$. For each m, define (h_m) of difference operators Δ^j $(j \in \mathbb{N})$ by*

$$\Delta^j h_m(y) = \begin{cases} y/2 & if \;\; m = k^2 \\ 0 & if \;\; m \neq k^2 \end{cases}, (k \in \mathbb{N}).$$

and

$$\chi(y) = \frac{1}{y+1} \quad (y \in [0,1]).$$

We also define a sequence (ε_m) by

$$\varepsilon_m = \begin{cases} 2m & if \;\; m = k^2 \\ 1/m & if \;\; m \neq k^2 \end{cases}, (k \in \mathbb{N}).$$

We see that $\varepsilon_m \xrightarrow{\mathcal{I}} 0$. Then, we obtain that

$$\left\{ m \in \mathbb{N} : \left| \frac{\Delta^j h_m(y) - 0}{\chi(y)} \right| \geq \varepsilon_m \right\} = \emptyset \in \mathcal{I}.$$

Consequently, we observe that

$$h_m \xrightarrow{\mathcal{I}(\Delta^{j,\chi}_{r,equi})} 0,$$

but

$$\mathcal{I}(\Delta^{j,\chi}_{r,u})\text{-}\lim_m h_m \neq 0$$

as well as (h_m) is not uniformly convergent on $[0,1]$. From here, we can observe that

$$h_m \xrightarrow{\mathcal{I}(\Delta^{j,\chi}_{r,equi})} h \not\Longrightarrow h_m \xrightarrow{\mathcal{I}(\Delta^{j,\chi}_{r,u})} h,$$

in general.

An ideal $\mathcal{I}$ is said to satisfy the chain condition (see [18]), in short, we shall write CC, if there is a sequence $(U_m) \subset \mathcal{I}$ having $U_1 \subset U_2 \subset \dots$ such that for any $V \in \mathcal{I}$ $\exists$ $m \in \mathbb{N}$ such that $V \subset U_m$.

Theorem 2.5. *Consider the ideal $\mathcal{I}$ that satisfy CC. Then, the following are equivalent.*

(C_1) $h_m \xrightarrow{\mathcal{I}(\Delta^{j,\chi}_{r,equi})} h$.

(C_2) There are sets $Y_i \subset Y$ such that

$$Y = \bigcup_{i\in\mathbb{N}} Y_i \quad \text{and} \quad h_m \xrightarrow{\mathcal{I}(\Delta^{j,\chi}_{r,u})} h \ \text{ on } Y_i \qquad \forall i = 1, 2, \dots.$$

(C_3) There are sets $Y_i \subset Y$ such that

$$Y = \bigcup_{i\in\mathbb{N}} Y_i, \quad Y_1 \subset Y_2 \subset \dots \quad \text{and} \quad h_m \xrightarrow{\mathcal{I}(\Delta^{j,\chi}_{r,u})} h \ \text{ on } Y_i \qquad \forall i = 1, 2, \dots.$$

Proof. $(C_1) \Rightarrow (C_3)$. Suppose $h_m \xrightarrow{\mathcal{I}(\Delta^{j,\chi}_{r,equi})} h$. So, the condition (1) holds, and $|\chi(t)| > 0$. For any $y \in Y$, there is a set $B_y \in \mathcal{I}$ such that

$$\left|\frac{\Delta^j h_m(y) - h(y)}{\chi(y)}\right| < \varepsilon_m \quad \forall m \notin B_y^c$$

Since $\mathcal{I}$ satisfies CC, there is $(U_i) \subset \mathcal{I}$ having $U_1 \subset U_2 \subset \dots$ such that for every $V \in \mathcal{I}$, there is some $U_i \in \mathcal{I}$, $i \in \mathbb{N}$ with $V \subset U_i$. We define

$$Y_i = \left\{ y \in Y : \left|\frac{\Delta^j h_m(y) - h(y)}{\chi(y)}\right| < \varepsilon_m \quad \forall m \notin U_i^c, i \in \mathbb{N} \right\}.$$

This means that $Y_1 \subset Y_2 \subset \dots$. We observe that if, as stated above, the set $B_y \in \mathcal{I}$ gives $h_m \xrightarrow{\mathcal{I}(\Delta^{j,\chi}_{r,equi})} h$, then, for some $i \in \mathbb{N}$, we have $B_y \subset U_i$. Thus, we get $y \in Y_i$. We therefore have

$$Y = \bigcup_{i\in\mathbb{N}} Y_i.$$

Consequently,

$$h_m \xrightarrow{\mathcal{I}(\Delta^{j,\chi}_{r,u})} h \ \text{ on } Y_i$$

which proves (C_3).

$(C_2) \Rightarrow (C_1)$. To obtain this implication, suppose

$$Y = \bigcup_{i\in\mathbb{N}} Y_i \quad \text{and} \quad \left|\frac{\Delta^j h_m(y) - h(y)}{\chi(y)}\right| < \varepsilon_{km} \ \ \forall y \in Y_k,$$

when $m \notin A(k) \in \mathcal{I}$, where $(\varepsilon_{km}) \xrightarrow{\mathcal{I}} 0$ for fixed k, and a function $|\chi(y)|$ with $|\chi(y)| > 0$. Choosing the sets $A_i \in \mathcal{I}$ such that

$$A_1 \subset A_2 \subset \dots \subset A_i \subset \dots \quad \text{and} \quad \varepsilon_{im} < \frac{1}{i}$$

whenever $m \notin A_i$ $(i = 1, 2, \dots)$. Now, we define

$$\varepsilon_m = \begin{cases} 1 & (m \in A_2) \\ \frac{1}{i}, & (m \in A_{i+1} \setminus A_i) \\ \frac{1}{m}, & \left(m \notin \bigcup\limits_{i \in \mathbb{N}} A_i\right). \end{cases}$$

So, we obtain

$$(\varepsilon_m) \xrightarrow{\mathcal{I}} 0 \quad \text{and} \quad \left|\frac{\Delta^j h_m(y) - h(y)}{\chi(y)}\right| < \varepsilon_{km} < \varepsilon_m$$

for $y \in Y_k$ and if $m \notin A(k) \cup A_k \in \mathcal{I}$ which gives that $h_m \xrightarrow{\mathcal{I}(\Delta^{j,\chi}_{r,equi})} h$, so (C_1) holds. Hence $(C_2) \Rightarrow (C_1)$. Moreover, clearly $(C_3) \Rightarrow (C_2)$, which completes the proof. □

3. Approximation theorems

We now establish the Korovkin approximation theorem for the test function $h_k(y) = y^k$ $(k = 0, 1, 2)$ by $\mathcal{I}(\Delta^{j,\chi}_{r,equi})$-convergence while the classical, statistical and ideal version of this result were obtained in [27], [23] and [15], respectively. For related recent work, we refer to [6, 25, 29, 36, 37, 41].

Theorem 3.1. *Consider a sequence (H_m) $(m \in \mathbb{N})$ of positive linear operators acting from $C(Y)$ into itself. Then*

$$H_m(h; y) \xrightarrow{\mathcal{I}(\Delta^{j,\chi}_{r,equi})} h(y) \qquad (h \in C(Y)) \tag{5}$$

if and only if

$$H_m(h_0; y) \xrightarrow{\mathcal{I}(\Delta^{j,\chi_0}_{r,equi})} h_0(y), \tag{6}$$

$$H_m(h_1; y) \xrightarrow{\mathcal{I}(\Delta^{j,\chi_1}_{r,equi})} h_1(y), \tag{7}$$

$$H_m(h_2; y) \xrightarrow{\mathcal{I}(\Delta^{j,\chi_2}_{r,equi})} h_2(y), \tag{8}$$

where $\chi(y) = \max\{|\chi_i(y)| : |\chi_i(y)| > 0\}$ for $i = 0, 1, 2$.

Proof. The conditions (6)-(8) follows from (5) by using the fact that $h_0, h_1, h_3 \in C(Y)$. We now assume that (6)-(8) holds. Since h is continuous on Y, for all $r, y \in Y$, we obtain $|h(r) - h(y)| \leq 2D$, where $D = \|h\|$. By continuity of h on Y, for every $\varepsilon > 0$ $\exists$ $\delta > 0$ such that

$$|h(r) - h(y)| < \varepsilon \text{ whenever } |r - y| < \delta \quad \forall r, y \in Y.$$

Therefore

$$-\varepsilon - \frac{2D}{\delta^2}(r - y)^2 < h(r) - h(y) < \varepsilon + \frac{2D}{\delta^2}(r - y)^2.$$

Applying $\Delta^j H_m(h_0; y)$ to the last equation by using the fact that the operator $(\Delta^j H_m)$ is positive and linear, we get

$$\Delta^j H_m(h_0; y)\left(-\varepsilon - \frac{2D}{\delta^2}(r - y)^2\right) < \Delta^j H_m(h_0; y)(h(r) - h(y)) < \Delta^j H_m(h_0; y)\left(\varepsilon + \frac{2D}{\delta^2}(r - y)^2\right)$$

which yield that

$$\Delta^j H_m(h;y) - h(y)\Delta^j H_m(h_0;y) < \varepsilon\Delta^j H_m(h_0;y) + \frac{2D}{\delta^2}\Delta^j H_m((r-y)^2;y). \tag{9}$$

We can also write

$$\begin{aligned}\Delta^j H_m(h;y) - h(y) &= \Delta^j H_m(h;y) - h(y)\Delta^j H_m(h_0;y) + h(y)\Delta^j H_m(h_0;y) - h(y)\\ &= \Delta^j H_m(h;y) - h(y)\Delta^j H_m(h_0;y) + h(y)\{\Delta^j H_m(h_0;y) - h_0\}.\end{aligned}$$

Using (9), the last equality becomes

$$\Delta^j H_m(h;y) - h(y) < \varepsilon\Delta^j H_m(h_0;y) + \frac{2D}{\delta^2}\Delta^j H_m((r-y)^2;y) + h(y)\{\Delta^j H_m(h_0;y) - h_0\}. \tag{10}$$

Let us compute $\Delta^j H_m((r-y)^2;y)$ as

$$\begin{aligned}\Delta^j H_m((r-y)^2;y) &= \{\Delta^j H_m(h_2;y) - h_2(y)\} - 2y\{\Delta^j H_m(h_1;y) - h_1(y)\}\\ &\quad + y^2\{\Delta^j H_m(h_0;y) - h_0(y)\}.\end{aligned}$$

We obtain by substituting the value of $\Delta^j H_m((r-y)^2;y)$ in (10) that

$$\begin{aligned}|\Delta^j H_m(h;y) - h(y)| &< |\varepsilon\Delta^j H_m(h_0;y) + \frac{2D}{\delta^2}[\{\Delta^j H_m(h_2;y) - h_2(y)\}\\ &\quad -2y\{\Delta^j H_m(h_1;y) - h_1(y)\} + y^2\{\Delta^j H_m(h_0;y) - h_0(y)\}]\\ &\quad +h(y)\{\Delta^j H_m(h_0;y) - h_0(y)\}|\\ &= |\varepsilon\{\Delta^j H_m(h_0;y) - h_0(y)\} + \varepsilon + \frac{2D}{\delta^2}[\{\Delta^j H_m(h_2;y) - h_2(y)\}\\ &\quad -2y\{\Delta^j H_m(h_1;y) - h_1(y)\} + y^2\{\Delta^j H_m(h_0;y) - h_0(y)\}]\\ &\quad +h(y)\{\Delta^j H_m(h_0;y) - h_0(y)\}|\\ &\le \varepsilon + \left(\varepsilon + D + \frac{2D}{\delta^2}\|h_2\|_{C(Y)}\right)|\Delta^j H_m(h_0;y) - h_0(y)|\\ &\quad +\frac{4D}{\delta^2}\|h_1\|_{C(Y)}|\Delta^j H_m(h_1;y) - h_1(y)|\\ &\quad +\frac{2D}{\delta^2}|\Delta^j H_m(h_2;y) - h_2(y)|.\end{aligned} \tag{11}$$

Let

$$D_0 = \max\left\{\varepsilon + D + \frac{2D}{\delta^2}\|h_2\|_{C(Y)}, \frac{2D}{\delta^2}\|h_1\|_{C(Y)}, \frac{2D}{\delta^2}\right\}.$$

Then, from (11), we have

$$\begin{aligned}|\Delta^j H_m(h;y) - h(y)| &\le \varepsilon + D_0\{|\Delta^j H_m(h_0;y) - h_0(y)| + |\Delta^j H_m(h_1;y) - h_1(y)|\\ &\quad +|\Delta^j H_m(h_2;y) - h_2(y)|\}.\end{aligned}$$

Since ε is arbitrary and $\chi(y) = \max\{|\chi_i(y)| : |\chi_i(y)| > 0\}$, $(i = 0, 1, 2)$, so, we have

$$\begin{aligned}\left|\frac{\Delta^j H_m(h;y) - h(y)}{\chi(y)}\right| &\le D_0\left\{\left|\frac{\Delta^j H_m(h_0;y) - h_0(y)}{\chi_0(y)}\right| + \left|\frac{\Delta^j H_m(h_1;y) - h_1(y)}{\chi_1(y)}\right|\right.\\ &\quad \left. + \left|\frac{\Delta^j H_m(h_2;y) - h_2(y)}{\chi_2(y)}\right|\right\}.\end{aligned} \tag{12}$$

The assumption (6), that is, $H_m(h_0; y) \xrightarrow{\mathcal{I}(\Delta^{j,\chi_0}_{r,equi})} h_0(y)$ gives that there is a sequence (ε'_m) of positive reals satisfies that $\mathcal{I}\text{-}\lim \varepsilon'_m = 0$ and $\chi_0(y)$, $|\chi_0(y)| > 0$ such that

$$\left\{m \in \mathbb{N} : \left|\frac{\Delta^j H_m(h_0, y) - h_0(y)}{\chi_0(y)}\right| \geq \varepsilon'_m\right\} \in \mathcal{I} \qquad (y \in Y).$$

From assumption $H_m(h_1; y) \xrightarrow{\mathcal{I}(\Delta^{j,\chi_1}_{r,equi})} h_1(y)$, we have that there is (ε''_m) satisfies that $\mathcal{I}\text{-}\lim \varepsilon''_m = 0$ and $\chi_1(y)$, $|\chi_1(y)| > 0$ such that

$$\left\{m \in \mathbb{N} : \left|\frac{\Delta^j H_m(h_1, y) - h_1(y)}{\chi_1(y)}\right| \geq \varepsilon''_m\right\} \in \mathcal{I} \qquad (y \in Y).$$

In the same way, from (8), that is, $H_m(h_2; y) \xrightarrow{\mathcal{I}(\Delta^{j,\chi_2}_{r,equi})} h_2(y)$, for any $y \in Y$, we get

$$\left\{m \in \mathbb{N} : \left|\frac{\Delta^j H_m(h_2, y) - h_2(y)}{\chi_2(y)}\right| \geq \varepsilon'''_m\right\} \in \mathcal{I}.$$

Then, upon setting, inequality (12) gives

$$X_0 = \left\{m \in \mathbb{N} : \left|\frac{\Delta^j H_m(h; y) - h(y)}{\chi(y)}\right| \geq 3D_0\varepsilon_m\right\},$$

$$X_1 = \left\{m \in \mathbb{N} : \left|\frac{\Delta^j H_m(h_0; y) - h_0(y)}{\chi_0(y)}\right| \geq \varepsilon'_m\right\},$$

$$X_2 = \left\{m \in \mathbb{N} : \left|\frac{\Delta^j H_m(h_1; y) - h_1(y)}{\chi_1(y)}\right| \geq \varepsilon''_m\right\},$$

$$X_3 = \left\{m \in \mathbb{N} : \left|\frac{\Delta^j H_m(h_2; y) - h_2(y)}{\chi_2(y)}\right| \geq \varepsilon'''_m\right\},$$

where $\varepsilon_m = \max\{\varepsilon'_m, \varepsilon''_m, \varepsilon'''_m\}$ and $\chi(y) = \max\{|\chi_i(y)| : |\chi_i(y)| > 0\}$ $(i = 0, 1, 2)$. From (12), we fairly have

$$X_0 \subseteq \bigcup_{i_1=1}^{3} X_{i_1}.$$

This yield from the hypotheses (6)-(8) that $X_0 \in \mathcal{I}$, that is,

$$H_m(h; y) \xrightarrow{\mathcal{I}(\Delta^{j,\chi}_{r,equi})} h(y)$$

which completes the proof. $\square$

For the validation of our last approximation theorem, we construct an example with the help of λ-Bernstein operators [7] in which Bézier bases [52] has been used. For more details on recently discussed these kind of operators, we refer to [3, 32, 40, 51].

Example 3.2. *For any function $h \in C(Y)$ $(Y = [0, 1])$, $y \in [0, 1]$ and $m \in \mathbb{N}$, consider the Bernstein operators with parameter $\lambda \in [-1, 1]$ as*

$$D_{m,\lambda}(h; y) = \sum_{n=0}^{m} \tilde{d}_{m,n}(\lambda; y) h\left(\frac{n}{m}\right), \tag{13}$$

where $\tilde{d}_{m,n}(\lambda; y)$ are Bézier bases with shape parameter λ defined by

$$\tilde{d}_{m,0}(\lambda; y) = d_{m,0}(y) - \frac{\lambda}{m+1} d_{m+1,1}(y),$$

$$\tilde{d}_{m,m_1}(\lambda; y) = d_{m,m_1}(y) + \lambda\left(\frac{m-2m_1+1}{m^2-1} d_{m+1,m_1}(y) - \frac{m-2m_1-1}{m^2-1} d_{m+1,m_1+1}(y)\right)$$

for $n = m_1$ $(1 \leq m_1 \leq m-1)$ and

$$\tilde{d}_{m,m}(\lambda; y) = d_{m,m}(y) - \frac{\lambda}{m+1} d_{m+1,m}(y).$$

In this case

$$d_{m,n}(y) = \binom{m}{n} t^n (1-y)^{m-n},$$

known as Bernstein basis functions. Now, we define the following sequence $(\Delta^j L_m)$ of PLO on $C(Y)$ by

$$\Delta^j L_m(h; y) = (1 + \Delta^j h_m(y)) D_{m,\lambda}(h; y) \quad (h \in C[0,1]), \tag{14}$$

where $(\Delta^j h_k(t))$ and a sequence (ε_m) are same as in Example 2.4 with $\chi(y) = \frac{1}{y+1}$. Thus, we obtain

$$\Delta^j L_m(h_0; y) = (1 + \Delta^j h_m(y)),$$

$$\Delta^j L_m(h_1; y) = \left(1 + \Delta^j h_m(y)\right)\left\{y + \frac{1 + 2y + y^{m+1} - (1-y)^{m+1}}{m(m-1)}\lambda\right\}$$

and

$$\begin{aligned}\Delta^j L_m(h_2; y) \quad &= \quad (1 + \Delta^j h_m(y))\left\{y^2 + \frac{y(1-y)}{m} + \lambda\left(\frac{2y - 4y^2 + 2y^{m+1}}{m(m-1)}\right.\right. \\ &\left.\left. + \frac{y^{m+1} + (1-y)^{m+1} - 1}{m^2(m-1)}\right)\right\}.\end{aligned}$$

Since

$$h_m \xrightarrow{\mathcal{I}(\Delta^{k,\chi}_{r,equi})} 0,$$

we thus obtain by using the fact $\mathcal{I}$ is an admissible ideal of $\mathbb{N}$ that

$$L_m(h_i; y) \xrightarrow{\mathcal{I}(\Delta^{k,\chi}_{r,equi})} h_i(y)$$

for each $i = 0, 1, 2$. This conclude that operators (13) *satisfy Theorem 3.1.*

For $j = 0$, we get $\mathcal{I}_{r,equi}$-convergence, so Theorem 3.1 gives the following:

Corollary 3.3. *Let (H_m) $(m \in \mathbb{N})$ and a function $\chi(y)$ are same as Theorem 3.1. Then*

$$H_m(h; y) \xrightarrow{\mathcal{I}^{j,\chi}_{r,equi}} h(y) \qquad (h \in C(Y))$$

if and only if

$$H_m(h_i; y) \xrightarrow{\mathcal{I}(\Delta^{j,\chi_i}_{r,equi})} h_i(y), \qquad (i = 0, 1, 2).$$

4. Rate of $\mathcal{I}(\Delta^{j}_{r,equi})$-convergence

We finally define and estimate the rate of $\mathcal{I}(\Delta^{j}_{r,equi})$-convergence.

Definition 4.1. *We say that (h_m) is $\mathcal{I}(\Delta^{j}_{r,equi})$-convergent to h on Y with the rate $b \in (0,1)$ if there is a sequence (ε_m) of positive numbers satsfies $\mathcal{I}\text{-}\lim_m \varepsilon_m = 0$ and a function $\chi(y)$, $|\chi(y)| > 0$ such that*

$$\frac{\left\{m \in \mathbb{N} : \left|\frac{\Delta^j h_m(y) - h(y)}{\chi(y)}\right| \geq \varepsilon_m\right\}}{m^{1-b}} \in \mathcal{I} \qquad (y \in Y).$$

Denoted by

$$h_m - h = \mathcal{I}(\Delta^{j,\chi}_{r,equi}) - o(m^{-b})$$

on Y.

Lemma 4.2. *Consider the function sequences (h_m) and (h'_m) in $C(Y)$. Consider that $h_m - h = \mathcal{I}(\Delta^{j,\chi_1}_{r,equi}) - o(m^{-b_1})$ and $h'_m - h' = \mathcal{I}(\Delta^{j,\chi_2}_{r,equi}) - o(m^{-b_2})$. Then*

(i) $(h_m \pm h'_m) - (h \pm h') = \mathcal{I}(\Delta^{j,\chi}_{r,equi}) - o(m^{-b})$

(ii) $\alpha(h_m - h) = \mathcal{I}(\Delta^{j,\chi_1}_{r,equi}) - o(m^{-b_1})$

for any $\lambda \in \mathbb{R}$, where $b = \min\{b_1, b_2\}$ and $\chi(y) = \max\{|\chi_i(y)| : |\chi_i(y)| > 0, i = 1, 2\}$.

Proof. Since $h_m - h = \mathcal{I}(\Delta^{j,\chi_1}_{r,equi}) - o(m^{-b_1})$, there exists (ε'_m) satisfies $\mathcal{I}\text{-}\lim_m \varepsilon'_m = 0$ and $\chi_1(y)$, $|\chi_1(y)| > 0$, such that

$$\frac{\left\{m \in \mathbb{N} : \left|\frac{\Delta^j h_m(y) - h(y)}{\chi_1(y)}\right| \geq \varepsilon'_m\right\}}{m^{1-b_1}} \in \mathcal{I}$$

for any $y \in Y$. Again, since $h'_m - h' = \mathcal{I}(\Delta^{j,\chi_2}_{r,equi}) - o(m^{-b_2})$, there is (ε''_m) satisfies $\mathcal{I}\text{-}\lim_m \varepsilon''_m = 0$ and $\chi_2(y)$, $|\chi_2(y)| > 0$, such that

$$\frac{\left\{m \in \mathbb{N} : \left|\frac{\Delta^j h'_m(y) - h'(y)}{\chi_2(y)}\right| \geq \varepsilon''_m\right\}}{m^{1-b_2}} \in \mathcal{I} \qquad (y \in Y).$$

Upon setting, one gets

$$X'_0 = \left\{m \in \mathbb{N} : \left|\frac{(\Delta^j h_m \pm \Delta^j h'_m)(y) - (h \pm h')(y)}{\chi(y)}\right| \geq 2\varepsilon_m\right\},$$

$$X'_1 = \left\{m \in \mathbb{N} : \left|\frac{\Delta^j h_m(y) - h(y)}{\chi_1(y)}\right| \geq \varepsilon'_m\right\},$$

and

$$X'_2 = \left\{m \in \mathbb{N} : \left|\frac{\Delta^j h'_m(y) - h'(y)}{\chi_2(y)}\right| \geq \varepsilon''_m\right\},$$

where $\varepsilon_m = \max\{\varepsilon'_m, \varepsilon''_m\}$ and $\chi(y) = \max\{|\chi_i(y)| : |\chi_i(y)| > 0\}$ $(i = 1, 2)$. Consequently, we obtain

$$\frac{X'_0}{m^{1-b}} \subseteq \frac{X'_1}{m^{1-b_1}} \cup \frac{X'_2}{m^{1-b_2}}$$

where $b = \min\{b_1, b_2\}$. Hence

$$\frac{\left\{m \in \mathbb{N} : \left|\frac{(\Delta^j h_m \pm \Delta^j h'_m)(y) - (h \pm h')(y)}{\chi(y)}\right| \geq 2\varepsilon_m\right\}}{m^{1-b}} \in \mathcal{I}$$

which proves (i). Likewise, we can obtain (ii). □

For $h \in C(Y)$ and $\delta > 0$, the modulus of continuity is $\omega(h, \delta) = \sup_{|t-y| \leq \delta} |h(t) - h(y)|$ for $t, y \in Y$.

Theorem 4.3. *Consider a sequence $H_m : C(Y) \to C(Y)$ of PLO. Consider that*

$$H_m(h_0; y) - h_0 = \mathcal{I}(\Delta^{j,\chi_1}_{r,equi}) - o(m^{-b_1}) \tag{15}$$

$$\omega(h, \delta_m) = \mathcal{I}(\Delta^{j,\chi_2}_{r,equi}) - o(m^{-b_2}), \tag{16}$$

where $\delta_m(y) = \sqrt{H_m(g^2; y)}$ and $g(t) = (t - y)$. Then

$$H_m(h; y) - h = \mathcal{I}(\Delta^{j,\chi}_{r,equi})\text{-}o(m^{-b}) \tag{17}$$

for all g in $C(Y)$, where $h_k(y) = y^k$ $(k = 0, 1, 2)$, $b = \min\{b_1, b_2\}$ and $\chi(y) = \max\{|\chi_i(y) : |\chi_i(y) > 0|, i = 1, 2\}$.

Proof. Let $h \in C(Y)$, $y \in Y$. Then, by positivity and linearity of (H_m), we write

$$\begin{aligned}
|\Delta^j H_m(h; y) - h(y)| &\leq \Delta^j H_m(|h(t) - h(y)|; y) + |h(y)||\Delta^j H_m(h_0; y) - h_0(y)| \\
&\leq \Delta^j H_m\left(\left(1 + \frac{|g(t)|}{\delta}\right)\omega(h, \delta); y\right) + \|h\||\Delta^j H_m(h_0; y) - h_0(y)| \\
&= \omega(h, \delta)\Delta^j H_m(h_0; y) + \frac{\omega(h, \delta)}{\delta}\Delta^j H_m(|g(t)|; y) \\
&\quad + \|h\||\Delta^j H_m(h_0; y) - h_0(y)|.
\end{aligned}$$

The Cauchy-Schwarz inequality gives

$$\begin{aligned}
|\Delta^j H_m(h; y) - h(y)| &\leq \omega(h, \delta)\Delta^j H_m(h_0; y) + \frac{\omega(h, \delta)}{\delta}\sqrt{\Delta^j H_m(g^2; y)}\sqrt{\Delta^j H_m(h_0; y)} \\
&\quad + \|h\||\Delta^j H_m(h_0; y) - h_0(y)|.
\end{aligned}$$

Consequently,

$$\begin{aligned}
|\Delta^j H_m(h; y) - h(y)| &\leq \omega(h, \delta_m)\Delta^j H_m(h_0; y) \\
&\quad + \omega(h, \delta_m)\sqrt{\Delta^j H_m(h_0; y)} + \|h\||\Delta^j H_m(h_0; y) - h_0(y)|
\end{aligned}$$

for the choice of $\delta := \delta_m(y) = \sqrt{\Delta^j H_m(g^2; y)}$, and so gives

$$\begin{aligned}
|\Delta^j H_m(h; y) - h(y)| &\leq \omega(h, \delta_m)\Big\{|\Delta^j H_m(h_0; y) - h_0(y)| + 2h_0(y) \\
&\quad + \sqrt{\Delta^j H_m(h_0; y) - h_0(y)}\Big\} + \|h\||\Delta^j H_m(h_0; y) - h_0(y)|.
\end{aligned}$$

We thus get

$$\begin{aligned}\left|\frac{\Delta^j H_m(h;y)-h(y)}{\chi(y)}\right| &\leq ||h||\left|\frac{|\Delta^j H_m(h_0;y)-h_0(y)|}{\chi_1(y)}\right| + 2h_0\frac{\omega(h,\delta_m)}{|\chi_2(y)|}\\ &+\frac{\omega(h,\delta_m)}{|\chi_2(y)|}\left|\frac{\Delta^j H_m(h_0;y)-h_0(y)}{|\chi_1(y)|}\right|\\ &+\frac{\omega(h,\delta_m)}{|\chi_2(y)|}\sqrt{\left|\frac{\Delta^j H_m(h_0;y)-h_0(y)}{|\chi_1(y)|}\right|}\end{aligned}$$

which completes the proof by employing (15), (16) and Lemma 4.2. □

5. Conclusion

For the difference operators Δ^j, in our investigation, we defined the notions of ideal relatively equal convergence ($\mathcal{I}(\Delta^j_{r,equi})$-convergence) and ideal relatively uniform convergence ($\mathcal{I}(\Delta^j_{r,u})$-convergence) of sequences of functions. By assuming some conditions, we showed that these notions coincide with some previously defined and studied notions, and developed an example to view the implication of aforesaid notions. We also obtained an equavalence relation by means of $\mathcal{I}(\Delta^j_{r,equi})$-convergence and $\mathcal{I}(\Delta^j_{r,u})$-convergence inconjuction with chain condition. Moreover, we used our idea of $\mathcal{I}(\Delta^j_{r,equi})$-convergence to demonstrate the Korovkin-type theorem for the test funtion y^k ($k = 0, 1, 2$). An illustrative example is constructed in support of this approximation result by taking into our account λ-Bernstein operators, where $-1 \leq \lambda \leq 1$. Finally, we analyzed the rate of $\mathcal{I}(\Delta^j_{r,equi})$-convergence by utilizing the modulus of continuity. Besides this, one can try to define and studied above notions for double sequences.

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