Filomat 39:21 (2025), 7223–7237
https://doi.org/10.2298/FIL2521223C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Numerical radius inequalities on C^*-algebras via generalized Cartesian decomposition

Yueyang Chen[a], Botao Long[a,*]

[a] *School of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 211106, Jiangsu, China*

Abstract. We introduce a Cartesian type decomposition of an element in the setting of C^*-algebras, and hence obtain several lower and upper bounds for both a-numerical radius and algebraic numerical radius on C^*-algebras, which improve some existing bounds by means of Cartesian decomposition.

1. Introduction

Let $\mathcal{A}$ be a unital C^*-algebra with identity $1_{\mathcal{A}}$, and let $\mathcal{S}(\mathcal{A})$ be the state space of $\mathcal{A}$. By the algebraic numerical range of $x \in \mathcal{A}$, we mean the set $V(x) = \{f(x) : f \in \mathcal{S}(\mathcal{A})\}$. This set generalizes the algebraic numerical range $V(T)$ of a bounded linear operator $T \in \mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$, which is the closure of its classical spatial numerical range $W(T) = \{\langle T\xi, \xi\rangle : \xi \in \mathcal{H}, \|\xi\| = 1\}$. The algebraic numerical radius of $x \in \mathcal{A}$ is defined as $v(x) = \sup\{|z| : z \in V(x)\}$. The numerical radius serves as a crucial instrument for studying bounded linear operators (see [4] and the references therein).

For any element $x \in \mathcal{A}$, it can be represented as

$$x = \mathrm{Re}(x) + i\mathrm{Im}(x),$$

where $\mathrm{Re}(x) = \frac{x+x^*}{2}$ and $\mathrm{Im}(x) = \frac{x-x^*}{2i}$ are the real and imaginary parts of x, respectively. This decomposition is well known as the Cartesian decomposition. Using Cartesian decomposition, Kittaneh proved a famous inequality for the spatial numerical radius of a bounded linear operator, that is,

$$\frac{1}{4}\|T^*T + TT^*\| \le w^2(T) \le \frac{1}{2}\|T^*T + TT^*\|, \tag{1}$$

where $w(T) = \sup\{|z| : z \in W(T)\}$ [10]. In [14], Zamani gave various refinements and some useful characterizations of the algebraic numerical radius of an element in a C^*-algebra by means of Cartesian decomposition. For more information about numerical radii on C^*-algebra and Banach algebra, we refer the readers to [1, 3, 6, 7, 11, 12, 16].

2020 *Mathematics Subject Classification*. Primary 47A12; Secondary 47A30, 46L05.
Keywords. Numerical radius, Cartesian decomposition, C^*-algebra.
Received: 16 January 2025; Revised: 23 March 2025; Accepted: 31 March 2025
Communicated by Dragan S. Djordjević
The research was supported by the National Natural Science Foundation of China (No. 11801177).
* Corresponding author: Botao Long
Email addresses: 15961567896@163.com (Yueyang Chen), longbt289@163.com; longbt289@nuaa.edu.cn (Botao Long)
ORCID iDs: https://orcid.org/0009-0008-2248-7266 (Yueyang Chen), https://orcid.org/0000-0002-2129-9465 (Botao Long)

Motivated by the concepts of A-numerical range and A-numerical radius of operators in $\mathcal{B}(\mathcal{H})$ for a given positive operator A on the Hilbert space $\mathcal{H}$, Bourhim and Mabrouk introduced the notions of a-numerical range and a-numerical radius of elements for a given positive element a in the setting of C^*-algebras [7]. Moreover, they defined a-adjoint $x^{\#_a}$ of an element x in $\mathcal{A}$, and hence obtained a new type of Cartesian decomposition of x by the formula

$$x = \mathfrak{R}(x) + i\mathfrak{I}(x),$$

where $\mathfrak{R}(x) = \frac{x+x^{\#_a}}{2}$ and $\mathfrak{I}(x) = \frac{x-x^{\#_a}}{2i}$, as the classical Cartesian decomposition. Through this new kind of decomposition, they earned an important and useful characterization of the a-numerical radius as follows:

$$v_a(x) = \sup_{\theta\in\mathbb{R}} \|\mathfrak{R}(e^{i\theta}x)\|_a. \tag{2}$$

In [11], Mabrouk and Zamani derived an upper and lower bound of the a-numerical radius, similar to the famous inequality(1), that is,

$$\frac{1}{4}\|x^{\#_a}x + xx^{\#_a}\|_a \le v_a^2(x) \le \frac{1}{2}\|x^{\#_a}x + xx^{\#_a}\|_a. \tag{3}$$

Recently, Bhunia, Sen, and Paul defined a generalized Cartesian decomposition of $T \in \mathcal{B}(\mathcal{H})$ as

$$T = \mathrm{Re}_\lambda(T) + i\mathrm{Im}_\lambda(T),$$

where $\mathrm{Re}_\lambda(T) = \frac{T+\lambda T^*}{2}$ and $\mathrm{Im}_\lambda(T) = \frac{T-\lambda T^*}{2i}$ for all λ in the unit circle [5]. Furthermore, they developed several interesting descriptions of the spatial numerical radius of bounded linear operators, and refined several classical inequalities. Therefore, it is a very natural question whether there is a proper generalized Cartesian decomposition of an element in the framework of C^*-algebras.

In this paper, we introduce a Cartesian type decomposition of elements in the setting of C^*-algebras, and develop several upper and lower bounds of numerical radius on C^*-algebras. The contents of the sections of this paper are as follows. In Section 2, we give some basic concepts on numerical ranges and numerical radii. In Section 3, we define generalized a-Cartesian decomposition of an element, and give several characterizations of a-numerical radius on C^*-algebras for a given positive element a based on (2). Moreover, we develop several lower bounds of a-numerical radius which refine the inequality (3), and present some conditions when these equalities hold. In Section 4, as an application of this Cartesian decomposition, we give some more general upper bounds for the algebraic numerical radius on C^*-algebras.

2. Numerical ranges and numerical radii

Let $\mathcal{A}$ be a unital C^*-algebra with the identity $1_{\mathcal{A}}$, and let $\mathcal{A}'$ be the dual space of $\mathcal{A}$, the family of all continuous linear functional on $\mathcal{A}$. A linear functional $f \in \mathcal{A}'$ is said to be *positive* if $f(x^*x) \ge 0$ for all $x \in \mathcal{A}$, and written as $f \ge 0$. We denote $\mathcal{S}(\mathcal{A})$ by the set of all states on $\mathcal{A}$, i.e.,

$$\mathcal{S}(\mathcal{A}) = \{f \in \mathcal{A}' : f \ge 0, f(1_{\mathcal{A}}) = 1\}.$$

The *algebraic numerical range* and *algebraic numerical radius* of an element $x \in \mathcal{A}$ are defined by

$$V(x) = \{f(x) : f \in \mathcal{S}(\mathcal{A})\}$$

and

$$v(x) = \sup\{|z| : z \in V(x)\},$$

respectively. It is known that $v(\cdot)$ is a norm on $\mathcal{A}$ with

$$\frac{1}{2}\|x\| \le v(x) \le \|x\|$$

for all $x \in \mathcal{A}$. In particular, let $T \in \mathcal{B}(\mathcal{H})$ be a bounded linear operator on a Hilbert space $\mathcal{H}$, and let $\mathcal{A} = \mathcal{B}(\mathcal{H})$, then the *spatial numerical range* and the *spatial numerical radius* of T are given by

$$W(T) = \{\langle T\xi, \xi\rangle : x \in \mathcal{H}, \|\xi\| = 1\}$$

and

$$w(T) = \sup\{|z| : z \in W(T)\},$$

respectively.

A positive element $x \in \mathcal{A}$ is denoted by $x \geq 0$. Denote $\mathcal{A}^+$ by the set of all positive elements in $\mathcal{A}$. Suppose that $a \in \mathcal{A}^+$ is a nonzero positive element in $\mathcal{A}$. As the notation in [7], we set

$$\mathcal{S}_a(\mathcal{A}) = \{f \in \mathcal{A}' : f \geq 0, f(a) = 1\},$$

the set of all positive linear functionals on $\mathcal{A}$ with $f(a) = 1$.

The *a-numerical range* and *a-numerical radius* of an element $x \in \mathcal{A}$ are defined by

$$V_a(x) = \{f(ax) : f \in \mathcal{S}_a(\mathcal{A})\}$$

and

$$v_a(x) = \sup\{|z| : z \in V_a(x)\},$$

respectively [7], which are extensions of the notions of *A-numerical range* and *A-numerical radius* of a bounded linear operator $T \in \mathcal{B}(\mathcal{H})$, given by

$$W_A(T) = \{\langle T\xi, \xi\rangle_A : \xi \in \mathcal{H}, \|\xi\|_A = 1\}$$

and

$$w_A(T) = \sup\{|z| : z \in W_A(T)\},$$

respectively [15]. Here A is a positive bounded linear operator on a Hilbert space $\mathcal{H}$ and $\|\xi\|_A = \sqrt{\langle A\xi, \xi\rangle}$ for $\xi \in \mathcal{H}$.

For any $x \in \mathcal{A}$, we define

$$\|x\|_a = \sup\left\{\sqrt{f(x^*ax)} : f \in \mathcal{S}_a(\mathcal{A})\right\}.$$

It follows from [7, Proposition 3.3] that $\|\cdot\|_a$ is actually a semi-norm on the set $\mathcal{A}^a = \{x \in \mathcal{A} : \|x\|_a < \infty\}$, and

$$\|xy\|_a \leq \|x\|_a\|y\|_a$$

for all $x, y \in \mathcal{A}^a$.

For any $x \in \mathcal{A}$, an element $x^{\#_a} \in \mathcal{A}$ is said to be *a-adjoint* of x if $ax^{\#_a} = x^*a$. Denote $\mathcal{A}_a$ by the collection of all elements in $\mathcal{A}$ that admits an a-adjoint. Moreover, $\mathcal{A}_a$ and $\mathcal{A}^a$ are unital subalgebras of $\mathcal{A}$ with $\mathcal{A}_a \subset \mathcal{A}^a$ [7]. Furthermore, for any $x \in \mathcal{A}_a$, it follows from [7, Corollary 4.9] that the following equation holds:

$$\|x\|_a^2 = \|xx^{\#_a}\|_a = \|x^{\#_a}x\|_a = \|x^{\#_a}\|_a^2. \tag{4}$$

In addition, we also have

$$\frac{1}{2}\|x\|_a \leq v_a(x) \leq \|x\|_a$$

for all $x \in \mathcal{A}_a$.

An element $x \in \mathcal{A}$ is said to be *a-hermitian* or *a-selfadjoint* (*a-positive*) if ax is hermitian or selfadjoint (positive), that is, $ax = x^*a$ ($ax \geq 0$). If $x^{\#_a}$ is an a-adjoint of x, then it is easy to check that

$$x = \Re(x) + i\Im(x), \tag{5}$$

the *a-Cartesian decomposition* of x, where $\Re(x) = \frac{x+x^{\#_a}}{2}$ and $\Im(x) = \frac{x-x^{\#_a}}{2i}$ are a-hermitian. However, this decomposition is not unique in general [7].

3. *a*-numerical radius

Let $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Motivated by a generalized Cartesian decomposition of a bounded linear operator on a Hilbert space defined in [5], we introduce the following Cartesian decomposition in the framework of unital C^*-algebras.

Definition 3.1. *The* generalized a-Cartesian decomposition *of an element $x \in \mathcal{A}_a$ is*

$$x = \mathfrak{R}_\lambda(x) + i\mathfrak{I}_\lambda(x),$$

where

$$\mathfrak{R}_\lambda(x) = \frac{x + \lambda x^{\#_a}}{2}, \quad \mathfrak{I}_\lambda(x) = \frac{x - \lambda x^{\#_a}}{2i},$$

for all $\lambda \in \mathbb{T}$.

In particular, if $\lambda = 1$, then this decomposition is just the usual a-Cartesian decomposition in [7].

The following proposition gives some new characterizations of a-numerical radius by means of generalized a-Cartesian decomposition.

Proposition 3.2. *Let $x \in \mathcal{A}_a$. Then*

(i) $v_a(x) = \sup_{\lambda\in\mathbb{T}} \|\mathfrak{R}_\lambda(x)\|_a = \sup_{\lambda\in\mathbb{T}} \|\mathfrak{I}_\lambda(x)\|_a$.

(ii) For any $\lambda \in \mathbb{T}$, we have

$$v_a(x) = \sup_{\alpha,\beta\in\mathbb{R},\alpha^2+\beta^2=1} \|\alpha\mathfrak{R}_\lambda(x) + \beta\mathfrak{I}_\lambda(x)\|_a.$$

(iii) For any $\phi \in \mathbb{R}$, we have

$$v_a(x) = \sup_{\theta\in\mathbb{R}} \left\| \frac{e^{i\theta}x + e^{i(\phi-\theta)}x^{\#_a}}{2} \right\|_a.$$

Proof. (i) From [7, Theorem 4.11], we see that

$$v_a(x) = \sup_{\theta\in\mathbb{R}} \|\mathfrak{R}(e^{i\theta}x)\|_a = \sup_{\theta\in\mathbb{R}} \|\mathfrak{I}(e^{i\theta}x)\|_a.$$

It follows that

$$\begin{aligned} v_a(x) = \sup_{\theta\in\mathbb{R}} \|\mathfrak{R}(e^{i\theta}x)\|_a &= \sup_{\theta\in\mathbb{R}} \left\| \frac{e^{i\theta}x + e^{-i\theta}x^{\#_a}}{2} \right\|_a \\ &= \sup_{\theta\in\mathbb{R}} \left\| \frac{x + e^{-2i\theta}x^{\#_a}}{2} \right\|_a \\ &= \sup_{\lambda\in\mathbb{T}} \left\| \frac{x + \lambda x^{\#_a}}{2} \right\|_a = \sup_{\lambda\in\mathbb{T}} \left\| \mathfrak{R}_\lambda(x) \right\|_a. \end{aligned}$$

Similarly, we have

$$\begin{aligned} v_a(x) = \sup_{\theta\in\mathbb{R}} \|\mathfrak{I}(e^{i\theta}x)\|_a &= \sup_{\theta\in\mathbb{R}} \left\| \frac{e^{i\theta}x - e^{-i\theta}x^{\#_a}}{2i} \right\|_a \\ &= \sup_{\theta\in\mathbb{R}} \left\| \frac{x - e^{-2i\theta}x^{\#_a}}{2i} \right\|_a \\ &= \sup_{\lambda\in\mathbb{T}} \left\| \frac{x - \lambda x^{\#_a}}{2i} \right\|_a = \sup_{\lambda\in\mathbb{T}} \left\| \mathfrak{I}_\lambda(x) \right\|_a. \end{aligned}$$

(*ii*) For any $\lambda \in \mathbb{T}$, we have

$$\begin{aligned}
v_a(x) &= \sup_{\gamma\in\mathbb{T}}\|\mathfrak{R}_\gamma(x)\|_a = \sup_{\gamma\in\mathbb{T}}\left\|\frac{x+\gamma x^{\#_a}}{2}\right\|_a \\
&= \sup_{\theta\in\mathbb{R}}\left\|\frac{x+\lambda e^{i\theta}x^{\#_a}}{2}\right\|_a \\
&= \sup_{\theta\in\mathbb{R}}\left\|\frac{e^{\frac{1}{2}i\theta}(e^{-\frac{1}{2}i\theta}x+e^{\frac{1}{2}i\theta}\lambda x^{\#_a})}{2}\right\|_a \\
&= \sup_{\theta\in\mathbb{R}}\left\|\frac{e^{-\frac{1}{2}i\theta}x+e^{\frac{1}{2}i\theta}\lambda x^{\#_a}}{2}\right\|_a \\
&= \sup_{\theta\in\mathbb{R}}\left\|\frac{\cos(\frac{\theta}{2})x-i\sin(\frac{\theta}{2})x+\cos(\frac{\theta}{2})\lambda x^{\#_a}+i\sin(\frac{\theta}{2})\lambda x^{\#_a}}{2}\right\|_a \\
&= \sup_{\theta\in\mathbb{R}}\left\|\cos\left(\frac{\theta}{2}\right)\frac{x+\lambda x^{\#_a}}{2}+\sin\left(\frac{\theta}{2}\right)\frac{x-\lambda x^{\#_a}}{2i}\right\|_a \\
&= \sup_{\alpha,\beta\in\mathbb{R},\alpha^2+\beta^2=1}\|\alpha\mathfrak{R}_\lambda(x)+\beta\mathfrak{I}_\lambda(x)\|_a.
\end{aligned}$$

(*iii*) For any $\phi,\theta\in\mathbb{R}$, we take $\lambda=e^{i\phi}$, $\alpha=\cos\theta$ and $\beta=-\sin\theta$, respectively. It follows from the proof of (*ii*) that

$$\begin{aligned}
v_a(x) &= \sup_{\theta\in\mathbb{R}}\left\|\cos\theta\mathfrak{R}_\lambda(x)+(-\sin\theta)\mathfrak{I}_\lambda(x)\right\|_a \\
&= \sup_{\theta\in\mathbb{R}}\left\|\cos\theta\frac{x+\lambda x^{\#_a}}{2}+(-\sin\theta)\frac{x-\lambda x^{\#_a}}{2i}\right\|_a \\
&= \sup_{\theta\in\mathbb{R}}\left\|\frac{(\cos\theta+i\sin\theta)x+(\cos\theta-i\sin\theta)\lambda x^{\#_a}}{2}\right\|_a \\
&= \sup_{\theta\in\mathbb{R}}\left\|\frac{e^{i\theta}x+\lambda e^{-i\theta}x^{\#_a}}{2}\right\|_a \\
&= \sup_{\theta\in\mathbb{R}}\left\|\frac{e^{i\theta}x+e^{i(\phi-\theta)}x^{\#_a}}{2}\right\|_a.
\end{aligned}$$

□

Theorem 3.3. *If $x\in\mathcal{A}_a$, then*

$$v_a^2(x)\leq v_a(x^2)+\|\mathfrak{R}_\lambda(x)\|_a^2$$

for all $\lambda\in\mathbb{T}$.

Proof. For any $\phi,\theta\in\mathbb{R}$, by the equation (4) and Proposition 3.2, we have

$$\begin{aligned}
\|e^{i\theta}x+e^{i(\phi-\theta)}x^{\#_a}\|_a^2 &= \left\|\left(e^{i\theta}x+e^{i(\phi-\theta)}x^{\#_a}\right)^2\right\|_a \\
&= \|e^{2i\theta}x^2+e^{2i(\phi-\theta)}(x^{\#_a})^2+e^{i\phi}(x^{\#_a}x+xx^{\#_a})\|_a \\
&= \|(e^{2i\theta}-1)x^2+e^{2i\phi}(e^{-2i\theta}-1)(x^{\#_a})^2+(x+e^{i\phi}x^{\#_a})^2\|_a \\
&= \left\|2\sin\theta\left(e^{i(\theta+\pi/2)}x^2+e^{2i\phi}e^{-i(\theta+\pi/2)}(x^{\#_a})^2\right)+(x+e^{i\phi}x^{\#_a})^2\right\|_a \\
&\leq 2\|e^{i(\theta+\pi/2)}x^2+e^{2i\phi}e^{-i(\theta+\pi/2)}(x^{\#_a})^2\|_a+\|x+e^{i\phi}x^{\#_a}\|_a^2
\end{aligned}$$

$$\leq 4v_a(x^2) + \|x + e^{i\phi}x^{\#_a}\|_a^2.$$

Taking supremum over all $\theta \in \mathbb{R}$, we get

$$v_a^2(x) \leq v_a(x^2) + \left\|\frac{x + e^{i\phi}x^{\#_a}}{2}\right\|_a^2$$

by (i) of Proposition 3.2. This implies the desired inequality. □

Theorem 3.4. *Let $x \in \mathcal{A}_a$. Then*

$$v_a(x) \geq \frac{1}{2}\left\|x + \frac{\lambda + \mu}{2}x^{\#_a}\right\|_a,$$

for all $\lambda, \mu \in \mathbb{T}$.

Proof. For any $\lambda, \mu \in \mathbb{T}$, it follows from (i) of Proposition 3.2 that

$$\begin{aligned}
v_a(x) &\geq \max\left\{\left\|\frac{x + \lambda x^{\#_a}}{2}\right\|_a, \left\|\frac{x + \mu x^{\#_a}}{2i}\right\|_a\right\} \\
&= \frac{\left\|\frac{x+\lambda x^{\#_a}}{2}\right\|_a + \left\|\frac{x+\mu x^{\#_a}}{2i}\right\|_a}{2} + \left|\frac{\left\|\frac{x+\lambda x^{\#_a}}{2}\right\|_a - \left\|\frac{x+\mu x^{\#_a}}{2i}\right\|_a}{2}\right| \\
&\geq \frac{\left\|\frac{x+\lambda x^{\#_a}}{2} + i\frac{x+\mu x^{\#_a}}{2i}\right\|_a}{2} + \left|\frac{\left\|\frac{x+\lambda x^{\#_a}}{2}\right\|_a - \left\|\frac{x+\mu x^{\#_a}}{2i}\right\|_a}{2}\right| \\
&= \frac{1}{2}\left\|x + \frac{\lambda + \mu}{2}x^{\#_a}\right\|_a + \left|\frac{\left\|\frac{x+\lambda x^{\#_a}}{2}\right\|_a - \left\|\frac{x+\mu x^{\#_a}}{2i}\right\|_a}{2}\right| \\
&\geq \frac{1}{2}\left\|x + \frac{\lambda + \mu}{2}x^{\#_a}\right\|_a,
\end{aligned}$$

and hence the conclusion follows. □

Theorem 3.5. *Let $x \in \mathcal{A}_a$. Then for any $\lambda \in \mathbb{T}$, we have*

$$v_a(x) \geq \frac{\|x\|_a}{2} + \frac{|\|\mathfrak{R}_\lambda(x)\|_a - \|\mathfrak{I}_\lambda(x)\|_a|}{2}. \tag{6}$$

$$v_a(x) \geq \frac{\|x\|_a}{2} + \frac{|\frac{\|x\|_a}{2} - \|\mathfrak{R}_\lambda(x)\|_a|}{4} + \frac{|\frac{\|x\|_a}{2} - \|\mathfrak{I}_\lambda(x)\|_a|}{4}. \tag{7}$$

$$v_a(x) \geq \frac{\|x\|_a}{2} + \frac{|\|\mathfrak{R}_\lambda(x) + \mathfrak{I}_\lambda(x)\|_a - \|\mathfrak{R}_\lambda(x) - \mathfrak{I}_\lambda(x)\|_a|}{2\sqrt{2}}. \tag{8}$$

Proof. From (i) of Proposition 3.2, we have

$$v_a(x) \geq \|\mathfrak{R}_\lambda(x)\|_a, \quad v_a(x) \geq \|\mathfrak{I}_\lambda(x)\|_a$$

for all $\lambda \in \mathbb{T}$. Thus,

$$\begin{aligned}
v_a(x) &\geq \max\{\|\mathfrak{R}_\lambda(x)\|_a, \|\mathfrak{I}_\lambda(x)\|_a\} \\
&= \frac{\|\mathfrak{R}_\lambda(x)\|_a + \|\mathfrak{I}_\lambda(x)\|_a}{2} + \frac{|\|\mathfrak{R}_\lambda(x)\|_a - \|\mathfrak{I}_\lambda(x)\|_a|}{2}
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{\|\Re_\lambda(x) + i\Im_\lambda(x)\|_a}{2} + \frac{|\|\Re_\lambda(x)\|_a - \|\Im_\lambda(x)\|_a|}{2} \\
&= \frac{\|x\|_a}{2} + \frac{|\|\Re_\lambda(x)\|_a - \|\Im_\lambda(x)\|_a|}{2}.
\end{aligned}
$$

Take

$$
\alpha = \max\left\{\frac{\|x\|_a}{2}, \|\Re_\lambda(x)\|_a\right\}, \quad \beta = \max\left\{\frac{\|x\|_a}{2}, \|\Im_\lambda(x)\|_a\right\}.
$$

Hence

$$
\begin{aligned}
v_a(x) &\geq \frac{\alpha+\beta}{2} + \frac{|\alpha-\beta|}{2} \\
&= \frac{\frac{\|x\|_a}{2} + \|\Re_\lambda(x)\|_a}{4} + \frac{|\frac{\|x\|_a}{2} - \|\Re_\lambda(x)\|_a|}{4} \\
&\quad + \frac{\frac{\|x\|_a}{2} + \|\Im_\lambda(x)\|_a}{4} + \frac{|\frac{\|x\|_a}{2} - \|\Im_\lambda(x)\|_a|}{4} + \frac{|\alpha-\beta|}{2} \\
&= \frac{\|x\|_a}{4} + \frac{\|\Re_\lambda(x)\|_a + \|\Im_\lambda(x)\|_a}{4} + \frac{|\frac{\|x\|_a}{2} - \|\Re_\lambda(x)\|_a|}{4} \\
&\quad + \frac{|\frac{\|x\|_a}{2} - \|\Im_\lambda(x)\|_a|}{4} + \frac{|\alpha-\beta|}{2} \\
&\geq \frac{\|x\|_a}{4} + \frac{\|\Re_\lambda(x) + i\Im_\lambda(x)\|_a}{4} + \frac{|\frac{\|x\|_a}{2} - \|\Re_\lambda(x)\|_a|}{4} \\
&\quad + \frac{|\frac{\|x\|_a}{2} - \|\Im_\lambda(x)\|_a|}{4} + \frac{|\alpha-\beta|}{2} \\
&\geq \frac{\|x\|_a}{2} + \frac{|\frac{\|x\|_a}{2} - \|\Re_\lambda(x)\|_a|}{4} + \frac{|\frac{\|x\|_a}{2} - \|\Im_\lambda(x)\|_a|}{4}.
\end{aligned}
$$

Now we take $\alpha = \frac{\sqrt{2}}{2}$ and $\beta = \pm\frac{\sqrt{2}}{2}$. Then it follows from (*ii*) of Proposition 3.2 that

$$
v_a(x) \geq \frac{\sqrt{2}}{2}\|\Re_\lambda(x) \pm \Im_\lambda(x)\|_a
$$

for all $\lambda \in \mathbb{T}$. Therefore,

$$
\begin{aligned}
\sqrt{2}v_a(x) &\geq \max\{\|\Re_\lambda(x) + \Im_\lambda(x)\|_a, \|\Re_\lambda(x) - \Im_\lambda(x)\|_a\} \\
&= \frac{1}{2}(\|\Re_\lambda(x) + \Im_\lambda(x)\|_a + \|\Re_\lambda(x) - \Im_\lambda(x)\|_a) \\
&\quad + \frac{1}{2}|\|\Re_\lambda(x) + \Im_\lambda(x)\|_a - \|\Re_\lambda(x) - \Im_\lambda(x)\|_a| \\
&\geq \frac{1}{2}\|\Re_\lambda(x) + \Im_\lambda(x) + i\Re_\lambda(x) - i\Im_\lambda(x)\|_a \\
&\quad + \frac{1}{2}|\|\Re_\lambda(x) + \Im_\lambda(x)\|_a - \|\Re_\lambda(x) - \Im_\lambda(x)\|_a| \\
&= \frac{1}{2}\|(1+i)\lambda x^{\#_a}\|_a + \frac{1}{2}|\|\Re_\lambda(x) + \Im_\lambda(x)\|_a - \|\Re_\lambda(x) - \Im_\lambda(x)\|_a| \\
&= \frac{\sqrt{2}}{2}\|x\|_a + \frac{1}{2}|\|\Re_\lambda(x) + \Im_\lambda(x)\|_a - \|\Re_\lambda(x) - \Im_\lambda(x)\|_a|.
\end{aligned}
$$

This completes the proof of the theorem. □

Corollary 3.6. *Let $x \in \mathcal{A}_a$. Then the following statements are equivalent:*

(i) $v_a(x) = \frac{\|x\|_a}{2}$.

(ii) $\|\mathfrak{R}_\lambda(x)\|_a = \|\mathfrak{I}_\lambda(x)\|_a = \frac{\|x\|_a}{2}$ *for all* $\lambda \in \mathbb{T}$.

Proof. Suppose that $v_a(x) = \frac{\|x\|_a}{2}$. By the inequality (7) in Theorem 3.5, we conclude that

$$v_a(x) \geq \frac{\|x\|_a}{2} + \frac{\left|\frac{\|x\|_a}{2} - \|\mathfrak{R}_\lambda(x)\|_a\right|}{4} + \frac{\left|\frac{\|x\|_a}{2} - \|\mathfrak{I}_\lambda(x)\|_a\right|}{4} \geq \frac{\|x\|_a}{2},$$

for all $\lambda \in \mathbb{T}$. It is easy to see that $\|\mathfrak{R}_\lambda(x)\|_a = \|\mathfrak{I}_\lambda(x)\|_a = \frac{\|x\|_a}{2}$ for all $\lambda \in \mathbb{T}$.

From (*i*) of Proposition 3.2, we have

$$v_a(x) = \sup_{\lambda \in \mathbb{T}} \|\mathfrak{R}_\lambda(x)\|_a = \sup_{\lambda \in \mathbb{T}} \|\mathfrak{I}_\lambda(x)\|_a,$$

and hence the proof of the other direction is obvious. □

The following proposition gives another characterization of a-numerical radius through generalized a-Cartesian decomposition.

Proposition 3.7. *Let* $x \in \mathcal{A}_a$. *Then*

$$v_a(x) = \frac{\sqrt{2}}{2} \sup_{\lambda \in \mathbb{T}} \|\mathfrak{R}_\lambda(x) \pm \mathfrak{I}_\lambda(x)\|_a.$$

Proof. From Proposition 3.2, by a simple computation, we have

$$\begin{aligned}
\frac{1}{\sqrt{2}} \sup_{\lambda \in \mathbb{T}} \|\mathfrak{R}_\lambda(x) \pm \mathfrak{I}_\lambda(x)\|_a &= \frac{1}{\sqrt{2}} \sup_{\lambda \in \mathbb{T}} \left\| \frac{x + \lambda x^{\#_a}}{2} \pm \frac{x - \lambda x^{\#_a}}{2i} \right\|_a \\
&= \frac{1}{\sqrt{2}} \sup_{\lambda \in \mathbb{T}} \left\| \frac{(1 \mp i)x + (1 \pm i)\lambda x^{\#_a}}{2} \right\|_a \\
&= \sup_{\lambda \in \mathbb{T}} \left\| \frac{x + \frac{1 \pm i}{1 \mp i} \lambda x^{\#_a}}{2} \right\|_a \\
&= v_a(x).
\end{aligned}$$

□

Corollary 3.8. *Let* $x \in \mathcal{A}_a$. *Then the following statements are equivalent:*

(i) $v_a(x) = \frac{\|x\|_a}{2}$.

(ii) $\|\mathfrak{R}_\lambda(x) \pm \mathfrak{I}_\lambda(x)\|_a = \frac{\sqrt{2}}{2}\|x\|_a$ *for all* $\lambda \in \mathbb{T}$.

Proof. Suppose that $v_a(x) = \frac{\|x\|_a}{2}$. As in the proof of the inequality (8) in Theorem 3.5, we have

$$\begin{aligned}
v_a(x) &\geq \frac{1}{\sqrt{2}} \max\{\|\mathfrak{R}_\lambda(x) + \mathfrak{I}_\lambda(x)\|_a, \|\mathfrak{R}_\lambda(x) - \mathfrak{I}_\lambda(x)\|_a\} \\
&\geq \frac{\|x\|_a}{2} + \frac{|\|\mathfrak{R}_\lambda(x) + \mathfrak{I}_\lambda(x)\|_a - \|\mathfrak{R}_\lambda(x) - \mathfrak{I}_\lambda(x)\|_a|}{2\sqrt{2}} \\
&\geq \frac{\|x\|_a}{2},
\end{aligned}$$

for all $\lambda \in \mathbb{T}$. Then it is straightforward that

$$\|\mathfrak{R}_\lambda(x) \pm \mathfrak{I}_\lambda(x)\|_a = \frac{\sqrt{2}}{2}\|x\|_a$$

for all $\lambda \in \mathbb{T}$.

Conversely, by Proposition 3.7 we can complete the proof of the other direction immediately. □

Theorem 3.9. *Let $x \in \mathcal{A}_a$. Then for any $\lambda \in \mathbb{T}$, we have*

$$v_a^2(x) \geq \frac{1}{4}\|xx^{\#_a} + x^{\#_a}x\|_a + \frac{1}{2}\big|\|\mathfrak{R}_\lambda(x)\|_a^2 - \|\mathfrak{I}_\lambda(x)\|_a^2\big|. \tag{9}$$

$$v_a^2(x) \geq \frac{1}{4}\|xx^{\#_a} + x^{\#_a}x\|_a + \frac{1}{4}\big|\|\mathfrak{R}_\lambda(x) + \mathfrak{I}_\lambda(x)\|_a^2 - \|\mathfrak{R}_\lambda(x) - \mathfrak{I}_\lambda(x)\|_a^2\big|. \tag{10}$$

$$v_a^2(x) \geq \frac{1}{4}\|xx^{\#_a} + x^{\#_a}x\|_a + \frac{1}{4}(\alpha + \beta), \tag{11}$$

where

$$\alpha = \left|\|\mathfrak{R}_\lambda(x)\|_a^2 - \frac{1}{4}\|xx^{\#_a} + x^{\#_a}x\|_a\right|$$

and

$$\beta = \left|\|\mathfrak{I}_\lambda(x)\|_a^2 - \frac{1}{4}\|xx^{\#_a} + x^{\#_a}x\|_a\right|.$$

Proof. From Proposition 3.2, we have

$$v_a(x) \geq \|\mathfrak{R}_\lambda(x)\|_a, \quad v_a(x) \geq \|\mathfrak{I}_\lambda(x)\|_a$$

for all $x \in \mathcal{A}_a$ and $\lambda \in \mathbb{T}$. Then it follows from the equation (4) that

$$\begin{aligned}
&\frac{1}{4}\|xx^{\#_a} + x^{\#_a}x\|_a + \frac{1}{2}\big|\|\mathfrak{R}_\lambda(x)\|_a^2 - \|\mathfrak{I}_\lambda(x)\|_a^2\big| \\
&\quad = \frac{\|(\mathfrak{R}_\lambda(x))^2 + (\mathfrak{I}_\lambda(x))^2\|_a}{2} + \frac{\big|\|\mathfrak{R}_\lambda(x)\|_a^2 - \|\mathfrak{I}_\lambda(x)\|_a^2\big|}{2} \\
&\quad \leq \frac{\|(\mathfrak{R}_\lambda(x))^2\|_a + \|(\mathfrak{I}_\lambda(x))^2\|_a}{2} + \frac{\big|\|\mathfrak{R}_\lambda(x)\|_a^2 - \|\mathfrak{I}_\lambda(x)\|_a^2\big|}{2} \\
&\quad = \frac{\|\mathfrak{R}_\lambda(x)\|_a^2 + \|\mathfrak{I}_\lambda(x)\|_a^2}{2} + \frac{\big|\|\mathfrak{R}_\lambda(x)\|_a^2 - \|\mathfrak{I}_\lambda(x)\|_a^2\big|}{2} \\
&\quad = \max\{\|\mathfrak{R}_\lambda(x)\|_a^2, \|\mathfrak{I}_\lambda(x)\|_a^2\} \\
&\quad \leq v_a^2(x),
\end{aligned}$$

for all $\lambda \in \mathbb{T}$.

By Proposition 3.7, for any $\lambda \in \mathbb{T}$, we have the inequality

$$\frac{\sqrt{2}}{2}\|\mathfrak{R}_\lambda(x) \pm \mathfrak{I}_\lambda(x)\|_a \leq v_a(x),$$

and so

$$\begin{aligned}
&\frac{1}{2}\|xx^{\#_a} + x^{\#_a}x\|_a + \frac{1}{2}\big|\|\mathfrak{R}_\lambda(x) + \mathfrak{I}_\lambda(x)\|_a^2 - \|\mathfrak{R}_\lambda(x) - \mathfrak{I}_\lambda(x)\|_a^2\big| \\
&\quad = \frac{1}{2}\|(\mathfrak{R}_\lambda(x) + \mathfrak{I}_\lambda(x))^2 + (\mathfrak{R}_\lambda(x) - \mathfrak{I}_\lambda(x))^2\|_a \\
&\qquad + \frac{1}{2}\big|\|\mathfrak{R}_\lambda(x) + \mathfrak{I}_\lambda(x)\|_a^2 - \|\mathfrak{R}_\lambda(x) - \mathfrak{I}_\lambda(x)\|_a^2\big| \\
&\quad \leq \frac{1}{2}\|(\mathfrak{R}_\lambda(x) + \mathfrak{I}_\lambda(x))^2\|_a + \frac{1}{2}\|(\mathfrak{R}_\lambda(x) - \mathfrak{I}_\lambda(x))^2\|_a \\
&\qquad + \frac{1}{2}\big|\|\mathfrak{R}_\lambda(x) + \mathfrak{I}_\lambda(x)\|_a^2 - \|\mathfrak{R}_\lambda(x) - \mathfrak{I}_\lambda(x)\|_a^2\big| \\
&\quad = \frac{1}{2}\|\mathfrak{R}_\lambda(x) + \mathfrak{I}_\lambda(x)\|_a^2 + \frac{1}{2}\|\mathfrak{R}_\lambda(x) - \mathfrak{I}_\lambda(x)\|_a^2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\big|\|\Re_\lambda(x)+\Im_\lambda(x)\|_a^2-\|\Re_\lambda(x)-\Im_\lambda(x)\|_a^2\big|\\
&=\max\{\|\Re_\lambda(x)+\Im_\lambda(x)\|_a^2,\|\Re_\lambda(x)-\Im_\lambda(x)\|_a^2\}\\
&\le 2v_a^2(x).
\end{aligned}
$$

This completes the proof of (10).

Set

$$\gamma=\max\left\{\|\Re_\lambda(x)\|_a^2,\frac{1}{4}\|xx^{\#_a}+x^{\#_a}x\|_a\right\}$$

and

$$\delta=\max\left\{\|\Im_\lambda(x)\|_a^2,\frac{1}{4}\|xx^{\#_a}+x^{\#_a}x\|_a\right\}.$$

One can easily conclude that for any $\lambda\in\mathbb{T}$, we get

$$
\begin{aligned}
&\frac{\alpha+\beta}{4}+\frac{1}{4}\|xx^{\#_a}+x^{\#_a}x\|_a\\
&\le\frac{1}{4}\|xx^{\#_a}+x^{\#_a}x\|_a+\frac{\alpha+\beta}{4}+\frac{|\gamma-\delta|}{2}\\
&=\frac{\|\Re_\lambda(x)^2+\Im_\lambda(x)^2\|_a}{4}+\frac{1}{8}\|xx^{\#_a}+x^{\#_a}x\|_a+\frac{\alpha+\beta}{4}+\frac{|\gamma-\delta|}{2}\\
&\le\frac{\|\Re_\lambda(x)\|_a^2+\|\Im_\lambda(x)\|_a^2}{4}+\frac{1}{8}\|xx^{\#_a}+x^{\#_a}x\|_a+\frac{\alpha+\beta}{4}+\frac{|\gamma-\delta|}{2}\\
&=\max\{\gamma,\delta\}=\frac{\gamma+\delta}{2}+\frac{|\gamma-\delta|}{2}\\
&\le v_a^2(x).
\end{aligned}
$$

This completes the proof of the theorem. □

Example 3.10. *Let $\mathcal{A}=M_2(\mathbb{C})$, and let $a=\begin{bmatrix}1&0\\0&1\end{bmatrix}$. Consider $x=\begin{bmatrix}1+i&0\\0&0\end{bmatrix}$ and $\lambda=i$. It is easy to see that*

$$\Re_\lambda(x)=\begin{bmatrix}1+i&0\\0&0\end{bmatrix},\quad \Im_\lambda(x)=0.$$

Then the three inequalities in Theorem 3.9 give the following bounds:

$$v_a^2(x)\ge 2,\quad v_a^2(x)\ge 1,\quad v_a^2(x)\ge\frac{3}{2},$$

respectively.

While for $\lambda=-1$, we have

$$\Re_\lambda(x)=\begin{bmatrix}i&0\\0&0\end{bmatrix},\quad \Im_\lambda(x)=\begin{bmatrix}-i&0\\0&0\end{bmatrix}.$$

It follows that the three inequalities in Theorem 3.9 give the following bounds:

$$v_a^2(x)\ge 1,\quad v_a^2(x)\ge 2,\quad v_a^2(x)\ge 1,$$

respectively.

Corollary 3.11. *Let $x\in\mathcal{A}_a$. Then the following statements are equivalent:*

(i) $v_a^2(x)=\frac{1}{4}\|xx^{\#_a}+x^{\#_a}x\|_a$.

(ii) $\|\Re_\lambda(x)\|_a^2=\|\Im_\lambda(x)\|_a^2=\frac{1}{4}\|xx^{\#_a}+x^{\#_a}x\|_a$ *for all $\lambda\in\mathbb{T}$.*

Proof. Suppose that $v_a^2(x) = \frac{1}{4}\|xx^{\#_a} + x^{\#_a}x\|_a$. From the inequality (11), we have

$$v_a^2(x) \geq \frac{1}{4}\|xx^{\#_a} + x^{\#_a}x\|_a + \frac{1}{4}(\alpha + \beta) \geq \frac{1}{4}\|xx^{\#_a} + x^{\#_a}x\|_a,$$

where

$$\alpha = \left|\|\mathfrak{R}_\lambda(x)\|_a^2 - \frac{1}{4}\|xx^{\#_a} + x^{\#_a}x\|_a\right|$$

and

$$\beta = \left|\|\mathfrak{I}_\lambda(x)\|_a^2 - \frac{1}{4}\|xx^{\#_a} + x^{\#_a}x\|_a\right|$$

for all $\lambda \in \mathbb{T}$. Then it is easy to see that

$$\|\mathfrak{R}_\lambda(x)\|_a^2 = \|\mathfrak{I}_\lambda(x)\|_a^2 = \frac{1}{4}\|xx^{\#_a} + x^{\#_a}x\|_a$$

for all $\lambda \in \mathbb{T}$.

From (*i*) of Proposition 3.2, we have

$$v_a(x) = \sup_{\lambda\in\mathbb{T}}\|\mathfrak{R}_\lambda(x)\|_a = \sup_{\lambda\in\mathbb{T}}\|\mathfrak{I}_\lambda(x)\|_a,$$

and hence the proof of the other direction is obvious. □

Corollary 3.12. *Let $x \in \mathcal{A}_a$. Then the following statements are equivalent:*

(i) $v_a^2(x) = \frac{1}{4}\|xx^{\#_a} + x^{\#_a}x\|_a$.
(ii) $\|\mathfrak{R}_\lambda(x) \pm \mathfrak{I}_\lambda(x)\|_a^2 = \frac{1}{2}\|xx^{\#_a} + x^{\#_a}x\|_a$ *for all* $\lambda \in \mathbb{T}$.

Proof. Suppose that $v_a^2(x) = \frac{1}{4}\|xx^{\#_a} + x^{\#_a}x\|_a$. From the proof of the inequality (10), we get

$$\begin{aligned} v_a^2(x) &\geq \frac{1}{2}\max\{\|\mathfrak{R}_\lambda(x) + \mathfrak{I}_\lambda(x)\|_a^2, \|\mathfrak{R}_\lambda(x) - \mathfrak{I}_\lambda(x)\|_a^2\} \\ &\geq \frac{1}{4}\|xx^{\#_a} + x^{\#_a}x\|_a + \frac{1}{4}\left|\|\mathfrak{R}_\lambda(x) + \mathfrak{I}_\lambda(x)\|_a^2 - \|\mathfrak{R}_\lambda(x) - \mathfrak{I}_\lambda(x)\|_a^2\right| \\ &\geq \frac{1}{4}\|xx^{\#_a} + x^{\#_a}x\|_a, \end{aligned}$$

for all $\lambda \in \mathbb{T}$. Then it is obvious that

$$\|\mathfrak{R}_\lambda(x) \pm \mathfrak{I}_\lambda(x)\|_a^2 = \frac{1}{2}\|xx^{\#_a} + x^{\#_a}x\|_a$$

for all $\lambda \in \mathbb{T}$.

According to Proposition 3.7, the proof of the other direction is completed. □

Theorem 3.13. *If $x \in \mathcal{A}_a$, then*

$$v_a^2(x) \geq \frac{1}{2}(\|\mathfrak{R}_\lambda(x)\|_a^2 + \|\mathfrak{I}_\lambda(x)\|_a^2) \geq \frac{\|xx^{\#_a} + x^{\#_a}x\|_a}{4}$$

for all $\lambda \in \mathbb{T}$.

Proof. By a simple computation, one can easily find that

$$\mathfrak{R}_\lambda^2(x) + \mathfrak{I}_\lambda^2(x) = \frac{\lambda}{2}(xx^{\#_a} + x^{\#_a}x),$$

for all $\lambda \in \mathbb{T}$, and so

$$\begin{aligned}\frac{\|xx^{\sharp_a} + x^{\sharp_a}x\|_a}{4} &= \frac{\|\lambda(xx^{\sharp_a} + x^{\sharp_a}x)\|_a}{4} \\ &= \frac{1}{2}\|\mathfrak{R}_\lambda^2(x) + \mathfrak{I}_\lambda^2(x)\|_a \\ &\leq \frac{1}{2}(\|\mathfrak{R}_\lambda^2(x)\|_a + \|\mathfrak{I}_\lambda^2(x)\|_a) \leq v_a^2(x),\end{aligned}$$

for all $\lambda \in \mathbb{T}$. □

4. Algebraic numerical radius

It is obvious that if a is the identity $1_{\mathcal{A}}$ of a unital C^*-algebra $\mathcal{A}$, then $v_a(x) = v(x)$ for all $x \in \mathcal{A}$. Moreover, for any $x \in \mathcal{A}$, one can easily see that $x^{\sharp_a} = x^*$, and the generalized a-Cartesian decomposition of x is just given as follows:

$$x = \mathrm{Re}_\lambda(x) + i\mathrm{Im}_\lambda(x),$$

where

$$\mathrm{Re}_\lambda(x) = \frac{x + \lambda x^*}{2}, \quad \mathrm{Im}_\lambda(x) = \frac{x - \lambda x^*}{2i}.$$

It is easy to find that $\mathrm{Re}_\lambda(x)$ and $\mathrm{Im}_\lambda(x)$ are normal elements of $\mathcal{A}$ for all $x \in \mathcal{A}$ and $\lambda \in \mathbb{T}$.

Theorem 4.1 ([2], Theorem 1.6.3). *Let $f \in \mathcal{A}'$ with $f \geq 0$. Then there is a representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ and a vector $\xi \in \mathcal{H}$ such that $f(x) = \langle \pi(x)\xi, \xi\rangle$ for every $x \in \mathcal{A}$. Moreover, if $f \in \mathcal{S}(\mathcal{A})$, then $\|\xi\| = 1$.*

Lemma 4.2 ([8, 13]). *Let $A \in \mathcal{B}(\mathcal{H})$. Then the following inequalities hold:*

(i) $|\langle A\xi, \eta\rangle|^2 \leq \langle |A|\xi, \xi\rangle\langle |A^*|\eta, \eta\rangle$ for all $\xi, \eta \in \mathcal{H}$.

(ii) If $A \in \mathcal{B}(\mathcal{H})$ is positive and $\xi \in \mathcal{H}$ with $\|\xi\| = 1$, then

$$\langle A\xi, \xi\rangle^r \leq \langle A^r\xi, \xi\rangle$$

for all $r \geq 1$.

Theorem 4.3. *Let $x \in \mathcal{A}$, then*

$$v^2(x) \leq v^2(|\mathrm{Re}_\lambda(x)| + i|\mathrm{Im}_\lambda(x)|) \leq \frac{\|x^*x + xx^*\|}{2}$$

for all $\lambda \in \mathbb{T}$.

Proof. For any state $f \in \mathcal{S}(\mathcal{A})$, by Theorem 4.1 and Lemma 4.2, we have

$$\begin{aligned}|f(x)|^2 &= |\langle \pi(x)\xi, \xi\rangle|^2 \\ &= |\langle \mathrm{Re}_\lambda(\pi(x))\xi, \xi\rangle|^2 + |\langle \mathrm{Im}_\lambda(\pi(x))\xi, \xi\rangle|^2 \\ &\leq \langle |\mathrm{Re}_\lambda(\pi(x))|\xi, \xi\rangle^2 + \langle |\mathrm{Im}_\lambda(\pi(x))|\xi, \xi\rangle^2 \\ &= |\langle |\mathrm{Re}_\lambda(\pi(x))|\xi, \xi\rangle + i\langle |\mathrm{Im}_\lambda(\pi(x))|\xi, \xi\rangle|^2 \\ &= |\langle \pi(|\mathrm{Re}_\lambda(x)| + i|\mathrm{Im}_\lambda(x)|)\xi, \xi\rangle|^2 \\ &\leq \langle |\pi(\mathrm{Re}_\lambda(x))|^2\xi, \xi\rangle + \langle |\pi(\mathrm{Im}_\lambda(x))|^2\xi, \xi\rangle \\ &= \langle \pi(|\mathrm{Re}_\lambda(x)|^2 + |\mathrm{Im}_\lambda(x)|^2)\xi, \xi\rangle \\ &= \left\langle \pi\left(\frac{xx^* + x^*x}{2}\right)\xi, \xi\right\rangle\end{aligned}$$

$$\leq \frac{\|x^*x + xx^*\|}{2},$$

for all $\lambda \in \mathbb{T}$. Taking supremum over all states $f \in \mathcal{S}(\mathcal{A})$, we have

$$v^2(x) \leq v^2(|\mathrm{Re}_\lambda(x)| + i|\mathrm{Im}_\lambda(x)|) \leq \frac{\|x^*x + xx^*\|}{2}$$

for all $\lambda \in \mathbb{T}$. □

Example 4.4. *Let $\mathcal{A} = M_2(\mathbb{C})$. Consider $x = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $\lambda = i$. Then we have*

$$\mathrm{Re}_\lambda(x) = \frac{1}{2}\begin{bmatrix} 1+i & i \\ 1 & 1+i \end{bmatrix}, \quad \mathrm{Im}_\lambda(x) = \frac{1}{2}\begin{bmatrix} -1-i & -1 \\ -i & -1-i \end{bmatrix},$$

and so

$$|\mathrm{Re}_\lambda(x)| = \frac{1}{2}\begin{bmatrix} \sqrt{2} & \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \\ \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i & \sqrt{2} \end{bmatrix}, \quad |\mathrm{Im}_\lambda(x)| = \frac{1}{2}\begin{bmatrix} \sqrt{2} & \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i \\ \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i & \sqrt{2} \end{bmatrix}.$$

Therefore,

$$v(|\mathrm{Re}_\lambda(x)| + i|\mathrm{Im}_\lambda(x)|) = \frac{3}{2},$$

and hence the inequality in Theorem 4.3 gives

$$v^2(x) \leq v^2(|\mathrm{Re}_\lambda(x)| + i|\mathrm{Im}_\lambda(x)|) = \frac{9}{4} < \frac{5}{2} = \frac{\|x^*x + xx^*\|}{2}.$$

Lemma 4.5 ([9]). *If $A, B \in \mathcal{B}(\mathcal{H})$ are positive operators, then*

$$\|(A+B)^r\| \leq \|A^r + B^r\|$$

for $r \in [0, 1]$.

Theorem 4.6. *Let $x \in \mathcal{A}$. Then for any $r \in [0, 2]$, we have*

$$v^r(x) \leq \||\mathrm{Re}_\lambda(x)|^r + |\mathrm{Im}_\lambda(x)|^r\|$$

for all $\lambda \in \mathbb{T}$.

Proof. As in the proof of Theorem 4.3, we have

$$\begin{aligned} |f(x)|^2 = |\langle \pi(x)\xi, \xi\rangle|^2 &\leq \langle \pi(|\mathrm{Re}_\lambda(x)|^2 + |\mathrm{Im}_\lambda(x)|^2)\xi, \xi\rangle \\ &\leq \||\mathrm{Re}_\lambda(x)|^2 + |\mathrm{Im}_\lambda(x)|^2\| \end{aligned}$$

for all $\lambda \in \mathbb{T}$. Thus,

$$v^2(x) \leq \||\mathrm{Re}_\lambda(x)|^2 + |\mathrm{Im}_\lambda(x)|^2\|$$

for all $\lambda \in \mathbb{T}$. It follows from Lemma 4.5 that for any $r \in [0, 2]$, we get

$$\begin{aligned} v^r(x) &\leq \||\mathrm{Re}_\lambda(x)|^2 + |\mathrm{Im}_\lambda(x)|^2\|^{\frac{r}{2}} \\ &= \|(|\mathrm{Re}_\lambda(x)|^2 + |\mathrm{Im}_\lambda(x)|^2)^{\frac{r}{2}}\| \\ &\leq \||\mathrm{Re}_\lambda(x)|^r + |\mathrm{Im}_\lambda(x)|^r\|, \end{aligned}$$

for all $\lambda \in \mathbb{T}$. □

Theorem 4.7. *Let $x \in \mathcal{A}$. Then for any $r \geq 2$, we have*

$$\frac{1}{2}\||\mathrm{Re}_\lambda(x)|^r + |\mathrm{Im}_\lambda(x)|^r\| \leq v^r(x) \leq 2^{\frac{r}{2}-1}\||\mathrm{Re}_\lambda(x)|^r + |\mathrm{Im}_\lambda(x)|^r\|$$

for all $\lambda \in \mathbb{T}$.

Proof. By Proposition 3.2, we have

$$v^2(x) \geq \frac{1}{4}\|x \pm \lambda x^*\|^2$$

for all $\lambda \in \mathbb{T}$. Therefore,

$$v^r(x) \geq \frac{1}{2^r}\|(x \pm \lambda x^*)^2\|^{\frac{r}{2}} = \frac{1}{2^r}\||x \pm \lambda x^*|^r\|.$$

This implies

$$\begin{aligned} v^r(x) &\geq \frac{1}{2}(\||\mathrm{Re}_\lambda(x)|^r\| + \||\mathrm{Im}_\lambda(x)|^r\|) \\ &\geq \frac{1}{2}\||\mathrm{Re}_\lambda(x)|^r + |\mathrm{Im}_\lambda(x)|^r\|. \end{aligned}$$

For any $f \in \mathcal{S}(\mathcal{A})$, by Theorem 4.1 and Lemma 4.2, we get

$$\begin{aligned} |f(x)| &= |\langle \pi(x)\xi, \xi\rangle| \\ &= 2^{\frac{1}{2}}\left(\frac{|\langle \mathrm{Re}_\lambda(\pi(x))\xi, \xi\rangle|^2 + |\langle \mathrm{Im}_\lambda(\pi(x))\xi, \xi\rangle|^2}{2}\right)^{\frac{1}{2}} \\ &\leq 2^{\frac{1}{2}}\left(\frac{|\langle \mathrm{Re}_\lambda(\pi(x))\xi, \xi\rangle|^r + |\langle \mathrm{Im}_\lambda(\pi(x))\xi, \xi\rangle|^r}{2}\right)^{\frac{1}{r}} \\ &\leq 2^{\frac{1}{2}-\frac{1}{r}}(\langle|\mathrm{Re}_\lambda(\pi(x))|\xi, \xi\rangle^r + \langle|\mathrm{Im}_\lambda(\pi(x))|\xi, \xi\rangle^r)^{\frac{1}{r}} \\ &\leq 2^{\frac{1}{2}-\frac{1}{r}}(\langle|\mathrm{Re}_\lambda(\pi(x))|^r\xi, \xi\rangle + \langle|\mathrm{Im}_\lambda(\pi(x))|^r\xi, \xi\rangle)^{\frac{1}{r}} \\ &= 2^{\frac{1}{2}-\frac{1}{r}}\langle \pi(|\mathrm{Re}_\lambda(x)|^r + |\mathrm{Im}_\lambda(x)|^r)\xi, \xi\rangle^{1/r} \\ &\leq 2^{\frac{r}{2}-1}\||\mathrm{Re}_\lambda(x)|^r + |\mathrm{Im}_\lambda(x)|^r\|. \end{aligned}$$

Taking supremum over all $f \in \mathcal{S}(\mathcal{A})$, we obtain

$$v^r(x) \leq 2^{\frac{r}{2}-1}\||\mathrm{Re}_\lambda(x)|^r + |\mathrm{Im}_\lambda(x)|^r\|.$$

This completes the proof. □

Acknowledgement

The authors would like to thank the referee for several valuable suggestions and comments.

References

[1] A. Alahmari, M. Mabrouk, A. Zamani, *Further results on the a-numerical range in C*-algebras*, Banach J. Math. Anal. **16** (2022), no. 2, Paper No. 25, 16 pp.

[2] W. Arveson, *An invitation to C*-algebras*, Springer-Verlag, New York, 1976.

[3] P. Bhunia. *Inequalities for linear functionals and numerical radii on C*-algebras*, Acta Math. Hungar. **176** (2025), no. 1, 111-138.

[4] P. Bhunia, S. S. Dragomir, M. S. Moslehian, K. Paul, *Lectures on numerical radius inequalities*, Infosys Science Foundation Series in Mathematical Sciences, Springer, New York, 2022.

[5] P. Bhunia, A. Sen, K. Paul, *Generalized Cartesian decomposition and numerical radius inequalities*, Rend. Circ. Mat. Palermo (2) **73** (2024), no. 3, 887–897.

[6] H. Blazhko, D. Homza, F. L. Schwenninger, J. Vries, M. Wojtylak, *The algebraic numerical range as a spectral set in Banach algebras*, Canad. J. Math. 2025, https://doi.org/10.4153/S0008414X25000124.
[7] A. Bourhim, M. Mabrouk, *a-numerical range on C^*-algebras*, Positivity **25** (2021), 1489–1510.
[8] T. Kato, *Notes on some inequalities for linear operators*, Math. Ann. **125** (1952), 208–212.
[9] F. Kittaneh, *Norm inequalities for positive operators*, Lett. Math. Phys. **43** (1998), 225–231.
[10] F. Kittaneh, *Numerical radius inequalities for Hilbert space operators*, Studia Math. **168** (2005), 73–80.
[11] M. Mabrouk, A. Zamani, *An extension of the a-numerical radius on C^*-algebras*, Banach J. Math. Anal. **17** (2023), no. 3, Paper No. 42, 23 pp.
[12] S. Mahapatra, R. Birbonshi, A. Patra, *Upper bounds of numerical radius and anumerical radius in C^*-algebra setting using Orlicz functions*, arXiv: 2411.04875 (2024).
[13] B. Simon, *Trace ideals and their applications*, Cambridge University Press, Cambridge, 1979.
[14] A. Zamani, *Characterization of numerical radius parallelism in C^*-algebras*, Positivity **23** (2019), 397–411.
[15] A. Zamani, *A-numerical radius inequalities for semi-Hilbertian space operators*, Linear Algebra Appl. **578** (2019), 159–183.
[16] A. Zamani, *New estimates for numerical radius in C^*-algebras*, Adv. Oper. Theory. **10** (2025), no. 2, Paper No. 46, 15 pp. https://doi.org/10.1007/s43036-025-00434-0.