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The generalized Aluthge transformation of operators with closed range

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Abstract. Let $T \in \mathcal{B}(\mathcal{H})$ be a bounded linear operator on a Hilbert space $\mathcal{H}$, with its polar decomposition give by $T = U|T|$. The generalized Aluthge transformation of T for any $\alpha, \beta > 0$ is defined as

$$\Delta_{\alpha,\beta}(T) = |T|^{\alpha}U|T|^{\beta}.$$

This paper investigates properties of operators and their generalized Aluthge transformation, focusing on null subspaces, closed ranges, and EP conditions. For binormal operators with closed ranges, we show the generalized Aluthge transformation also has a closed range and derive a formula for its Moore-Penrose inverse. Additionally, we explore the spectrum, numerical radius, and quasinormality of the Moore-Penrose inverse.

1. Introduction and preliminaries

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. For any operator $T \in \mathcal{B}(\mathcal{H})$, we denote its range, null subspace and adjoint operator by $\mathcal{R}(T), \mathcal{N}(T)$ and T^*, respectively. For any closed subspace M of $\mathcal{H}$, let P_M denote the orthogonal projection onto M. The spectrum of T is denoted by $\sigma(T)$.The numerical range of T is defined as:

$$W(T) := \{\langle Tx, x\rangle : \|x\| = 1, x \in \mathcal{H}\}$$

where $W(T)$ represents the numerical range. The numerical radius of T is given by:

$$w(T) := sup\{|\lambda| : \ \lambda \in W(T)\}.$$

For $T \in \mathcal{B}(\mathcal{H})$, there exists a unique factorization $T = U|T|$, where $\mathcal{N}(U) = \mathcal{N}(T) = \mathcal{N}(|T|), U$ is a partial isometry satisfying $UU^*U = U$ and $|T| = (T^*T)^{\frac{1}{2}}$ is the modulus of T. This factorization is Known as the polar decomposition of T. As a result, the following properties hold:

$$U^*U = P_{\overline{\mathcal{R}(T^*)}} = P_{\overline{\mathcal{R}(|T|)}} \ \text{ and } \ UU^* = P_{\overline{\mathcal{R}(T)}} = P_{\overline{\mathcal{R}(|T^*|)}}.$$

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From the polar decomposition, Aluthge [1] introduced a transformation defined as

$$\Delta_{\frac{1}{2},\frac{1}{2}}(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}},$$

later referred to as the Aluthge transformation. In [11], Furuta introduced a more generally, for any $\alpha, \beta > 0$, the generalized Aluthge transformation is given by

$$\Delta_{\alpha,\beta}(T) = |T|^{\alpha}U|T|^{\beta},$$

Clearly, setting $\alpha = \beta = \frac{1}{2}$ we obtain the usual Aluthge transformation.
These transformations have been widely studied across various contexts by numerous researchers (see [1, 5, 6, 13–16, 18, 20, 21]). One of their notable features is their ability to preserve many properties of the original operator. For instance, T has a nontrivial invariant subspace if and only if $\Delta_{\frac{1}{2},\frac{1}{2}}(T)$ does (see [15]). Additionally, T and $\Delta_{\frac{1}{2},\frac{1}{2}}(T)$ share the same spectrum (see [15]).

An operator $T \in \mathcal{B}(\mathcal{H})$ is called normal if $TT^* = T^*T$, and quasinormal if T commutes with T^*T meaning $TT^*T = T^*T^2$. Notably, if T has a polar decomposition of $T = U|T|$, it is quasinormal if and only if U and $|T|$ commute. An operator T is called binormal if TT^* and T^*T commute, a concept introduced by Campbell in [2]. The relationships among these classes of operators are such that normal $\Longrightarrow$ quasinormal $\Longrightarrow$ binormal, but the reverse implications do not hold. Additionally, T called hyponormal if $T^*T \geq TT^*$.

Let $T \in \mathcal{B}(\mathcal{H})$. The Moore-Penrose inverse of T, denoted by $T^+ \in \mathcal{B}(\mathcal{H})$, is the unique operator that satisfies the following conditions:

$$TT^+T = T, \;\; T^+TT^+ = T^+, \;\; (TT^+)^* = TT^+, (T^+T)^* = T^+T.$$

It is well established that the Moore-Penrose inverse of T exists if and only if $\mathcal{R}(T)$ is closed. Furthermore, it can be readily observed that $\mathcal{R}(T^+) = \mathcal{R}(T^*)$, TT^+ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{R}(T)$, and that T^+T is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{R}(T^*)$. An operator T is called an EP operator if $\mathcal{R}(T)$ is closed and $TT^+ = T^+T$. It is evident that

$$T \text{ is EP} \iff \mathcal{R}(T) = \mathcal{R}(T^*) \iff \mathcal{N}(T) = \mathcal{N}(T^*).$$

Clearly, every normal operator with a closed range is EP, however, the converse does not hold, even in finite-dimensional spaces. For more details about on EP operators see [4, 9].
The paper is structured as follows:
In section 2, we first examine the conditions under which an operator and its generalized Aluthge transformation share the same null subspace. The generalized Aluthge transformation preserves many properties of the original operator. However, if an operator $T \in \mathcal{B}(\mathcal{H})$ has a closed range, the generalized Aluthge transformation is not guaranteed to have a closed range, as demonstrated in Example 2.4. Next, we provide a necessary and sufficient condition for the ranges of T and its generalized Aluthge transformation to both have closed ranges. We also investigate when an operator and its generalized Aluthge transformation are both EP operators.
In Section 3, we show that if T is a binormal operator with a closed range, then the range of $\Delta_{\alpha,\beta}(T)$ is also closed. We then derive a formula for the Moore-Penrose inverse of $\Delta_{\alpha,\beta}(T)$ when T is a binormal operator with a closed range. Furthermore, we briefly discuss some classical results related to the spectrum and numerical radius through the Moore-Penrose inverse and the generalized Aluthge transformation. Additionally, we obtain the polar decomposition of $(\Delta_{\alpha,\beta}(T))^+$. Finally, we present results concerning the quasinormality of T^+.
To establish the main results, we first present a series of Lemmas.

Lemma 1.1. *[21, Lemma 1] Let $T \in \mathcal{B}(\mathcal{H})$ be a positive. Then $\mathcal{N}(T) = \mathcal{N}(T^q)$ for all $q > 0$.*

Lemma 1.2. *[16, Lemma 3.12]Let $T = U|T|$ be the polar decomposition of $T \in \mathcal{B}(\mathcal{H})$. Then any $q > 0$, the following assertions hold.*

(*i*) $U|T|^q = |T^*|^q U$;

(*ii*) $|T^*|^q = U|T|^q U^*$;

(*iii*) $|T|^q = U^*|T^*|^q U$.

Lemma 1.3. *[6, Lemma 2.2] Let $T = U|T|$ be the polar decomposition of $T \in \mathcal{B}(\mathcal{H})$. Then any $q \geq 0$, the following assertions hold.*

(*i*) *$|T|^q = U^*U|T|^q$ is the polar decomposition of $|T|^q$.*

(*ii*) *$|T^*|^q = UU^*|T^*|^q$ is the polar decomposition of $|T^*|^q$.*

Lemma 1.4. *[10] Let $T_1 = U_1|T_1|$ and $T_2 = U_2|T_2|$ be the decompositions of T_1 and T_2 respectively. If T_1 doubly commutes with T_2 (i.e.$[T_1, T_2] = 0$ and $[T_1, T_2^*] = 0$), then $T_1T_2 = U_1U_2|T_1||T_2|$ is also the polar decomposition of T_1T_2, that is, U_1U_2 is a partial isometry with $\mathcal{N}(U_1U_2) = \mathcal{N}(|T_1||T_2|)$ and $|T_1||T_2| = |T_1T_2|$.*

Lemma 1.5. *[7, Theorem 2.8] If T_1 be an arbitrary operator and T_2 is normal. Then $\sigma(T_1T_2) = \sigma(T_2T_1)$.*

Lemma 1.6. *[22] Let $T \in \mathcal{B}(\mathcal{H})$ be positive and $q > 0$. Then*

$$\mathcal{R}(T) \text{ is closed } \iff \mathcal{R}(T^q) \text{ is closed.}$$

In this case $\mathcal{R}(T) = \mathcal{R}(T^q)$.

Lemma 1.7. *[20, Lemma 3.1] Let $T \in \mathcal{B}(\mathcal{H})$ with closed range and let $q > 0$. Then*

(*i*) $(|T|^+)^q = (|T|^q)^+$;

(*ii*) $(|T^*|^+)^q = (|T^*|^q)^+$.

Lemma 1.8. *[14, Lemma 2.1 and Proposition 2.12] Let $T \in \mathcal{B}(\mathcal{H})$ with closed range. Then the following assertions hold.*

(*i*) *$(|T|^+)^q = U^*(|T^*|^+)^q U$ for each $q > 0$.*

(*ii*) *T^+ is quasinormal if and only if $U^*|T^*|^+ = |T^*|^+U^*$.*

(*iii*) *$(T^*)^+ = U|T|^+$ is the polar decomposition of $(T^*)^+$.*

Lemma 1.9. *[13, Lemma 2.1] Let $T \in \mathcal{B}(\mathcal{H})$ with closed range. Then $|T^*|^+ = |T^+|$.*

Lemma 1.10. *[20, Theorem 4.4 and Corollary 5.3] Let $T \in \mathcal{B}(\mathcal{H})$ be binormal with closed range and let $p, q > 0$. Then*

(*i*) *$U(|T|^q)^+ = (|T^*|^q)^+U$.*

(*ii*) *If T is binormal, then $(|T^*|^q)^+(|T|^p)^+ = (|T|^p)^+(|T^*|^q)^+$.*

2. The generalized Aluthge transformation and closed range operators

In this section, we first demonstrate that an operator and generalized Aluthge transformation share the same null subspace. It was proved in the case $\alpha = \beta = \frac{1}{2}$, (see [5, Lemma 3.2]).

Lemma 2.1. *Let $T = U|T|$ be its polar decomposition of $T \in \mathcal{B}(\mathcal{H})$. Then the following assertions hold.*

(*i*) *If $\mathcal{N}(T) \subset \mathcal{N}(T^*)$, then $\mathcal{N}(\Delta_{\alpha,\beta}(T)) = \mathcal{N}(T)$.*

(*ii*) *If $\mathcal{N}(T^*) \subset \mathcal{N}(T)$, then $\mathcal{N}((\Delta_{\alpha,\beta}(T))^*) = \mathcal{N}(T)$.*

Proof. (i) If $Tx = 0$, then $|T|^{\beta}x = 0$ by Lemma 1.1. Hence $|T|^{\alpha}U|T|^{\beta}x = 0$ and so $x \in \mathcal{N}(\Delta_{\alpha,\beta}(T))$. Conversely, Suppose that $\Delta_{\alpha,\beta}(T)x = 0$. From the hypothese, we have

$$\mathcal{N}(U) \subset \mathcal{N}(U^*). \tag{1}$$

Hence

$$\begin{aligned}
|T|^{\alpha}U|T|^{\beta}x = 0 &\implies |T|U|T|^{\beta}x = 0 &&\text{by lemma 1.1}\\
&\implies UU|T|^{\beta}x = 0 &&\\
&\implies U^*U|T|^{\beta}x = 0 &&\text{by (1)}\\
&\implies |T|^{\beta}x = 0 &&\text{by lemma 1.3 } (i)\\
&\implies |T|x = 0 &&\text{by lemma 1.1}\\
&\implies Tx = 0.
\end{aligned}$$

Therefore, $x \in \mathcal{N}(T)$.
(ii) Again by Lemma 1.1, we have $\mathcal{N}(T) \subset \mathcal{N}(\Delta_{\alpha,\beta}(T)^*)$. To show the other inclusion, suppose that $x \in \mathcal{N}(\Delta_{\alpha,\beta}(T)^*)$. Since $\mathcal{N}(T^*) \subset \mathcal{N}(T)$, then we get

$$\mathcal{N}(|T^*|) \subset \mathcal{N}(|T|) = \mathcal{N}(U). \tag{2}$$

Hence

$$\begin{aligned}
\Delta_{\alpha,\beta}(T)^*x = 0 &\implies |T|^{\beta}U^*|T|^{\alpha}x = 0 &&\\
&\implies U|T|^{\beta}U^*|T|^{\alpha}x = 0 &&\\
&\implies |T^*|^{\beta}|T|^{\alpha}x = 0 &&\text{by Lemma 1.2 } (ii)\\
&\implies |T^*||T|^{\alpha}x = 0 &&\text{by lemma 1.1}\\
&\implies U|T|^{\alpha}x = 0 &&\text{by (2)}\\
&\implies U^*U|T|^{\alpha}x = 0 &&\\
&\implies |T|^{\alpha}x = 0 &&\text{by lemma 1.3 } (i)\\
&\implies |T|x = 0 &&\text{by lemma 1.1}\\
&\implies Tx = 0.
\end{aligned}$$

This completes the proof. □

Recall that the reduced minimum modulus of an operator $T \in \mathcal{B}(\mathcal{H})$ is defined by

$$\gamma(T) := \begin{cases} \inf\{\|Tx\|;\ \|x\| = 1, x \in \mathcal{N}(T)^{\perp}\} & \text{if } T \neq 0\\ +\infty & \text{if } T = 0. \end{cases}$$

It is well known that $\gamma(T) > 0$ if and only if T has a closed range [12]. In [5, Lemma 3.1], Chabbabi and Mbekhta provided a formula for the reduced minimum modulus of the Aluthge transformation. The following result extends this by presenting the reduced minimum modulus of the generalized Aluthge transformation.

Lemma 2.2. *Let $T = U|T|$ be the polar decomposition of $T \in \mathcal{B}(\mathcal{H})$ and let $\alpha, \beta > 0$. Then*

$$\gamma(\Delta_{\alpha,\beta}(T)) = \gamma(|T|^{\alpha}|T^*|^{\beta}) = \gamma(|T^*|^{\beta}|T|^{\alpha}).$$

Proof. By using [19, Corollaire 1.6], then we have

$$\begin{aligned}
\gamma(\Delta_{\alpha,\beta}(T))^2 &= \gamma(\Delta_{\alpha,\beta}(T)(\Delta_{\alpha,\beta}(T))^*) \\
&= \gamma(|T|^\alpha U|T|^{2\beta}U^*|T|^\alpha) \\
&= \gamma(|T|^\alpha|T^*|^{2\beta}|T|^\alpha) && \text{by Lemma 1.2 } (ii) \\
&= \gamma(|T|^\alpha|T^*|^\beta(|T|^\alpha|T^*|^\beta)^*) \\
&= \gamma(|T^*|^\beta|T|^\alpha)^2 = \gamma(|T|^\alpha|T^*|^\beta)^2.
\end{aligned}$$

□

Lemma 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$ with closed range and $T = U|T|$ be the polar decomposition. Then any $q > 0$, the following assertions hold.*

(i) $(|T^*|^+)^q = U(|T|^+)^q U^*$.

(ii) $(|T|^+)^q = U^*U(|T|^+)^q$ *is the polar decomposition of* $(|T|^+)^q$.

(iii) $(|T^*|^+)^q = UU^*(|T^*|^+)^q$ *is the polar decomposition of* $(|T^*|^+)^q$.

Proof. (i) By Lemma 1.9 and Lemma 1.8 (iii), we get

$$\begin{aligned}
(|T^*|^+)^2 = (|T^+|)^2 &= (T^+)^*T^+ \\
&= (T^*)^+T^+ && \text{since } (T^+)^* = (T^*)^+ \\
&= U|T|^+|T|^+U^* \\
&= U(|T|^+)^2U^*.
\end{aligned}$$

Then for any $q > 0, g(t) = t^{\frac{q}{2}}$ is an operator function, we get $(|T^*|^+)^q = U(|T|^+)^q U^*$.
(ii) Clearly U^*U is a partial isometry. Now, we show that $(|T|^+)^q = U^*U(|T|^+)^q$.

$$\begin{aligned}
(|T|^+)^q &= U^*(|T^*|^+)^q U && \text{by Lemma 1.8}(i) \\
&= U^*(|T^*|^q)^+ U && \text{by Lemma 1.7 } (ii) \\
&= U^*U(|T|^q)^+ && \text{by Lemma 1.10 } (i) \\
&= U^*U(|T|^+)^q. && \text{by Lemma 1.7 } (i).
\end{aligned}$$

Next, we show $\mathcal{N}((|T|^+)^q) = \mathcal{N}(U^*U)$. By ($i$) of Lemma 1.7 and since $\mathcal{N}((|T|^q)^+) = \mathcal{N}((|T|^q)^*)$. Now, by Lemma 1.1 we have

$$\mathcal{N}((|T|^+)^q) = \mathcal{N}((|T|^q)^+) = \mathcal{N}((|T|^q)^*) = \mathcal{N}(|T|^q) = \mathcal{N}(|T|) = \mathcal{N}(T) = \mathcal{N}(U) = \mathcal{N}(U^*U).$$

(iii) Clearly UU^* is a partial isometry. Now, we show that $(|T^*|^+)^q = UU^*(|T^*|^+)^q$.

$$\begin{aligned}
(|T^*|^+)^q &= U(|T|^+)^q U^* && \text{by } (i) \\
&= U(|T|^q)^+U^* && \text{by Lemma 1.7 } (i) \\
&= UU^*(|T^*|^q)^+ && \text{by Lemma 1.10 } (i) \\
&= UU^*(|T^*|^+)^q. && \text{by Lemma 1.7 } (ii).
\end{aligned}$$

Next, we show $\mathcal{N}((|T^*|^+)^q) = \mathcal{N}(UU^*)$. By Lemma 1.9 and Lemma 1.1, we have

$$\mathcal{N}((|T^*|^+)^q) = \mathcal{N}((|T^+|)^q) = \mathcal{N}((|T^+|)) = \mathcal{N}(T^+) = \mathcal{N}(U^*) = \mathcal{N}(UU^*).$$

□

Now, the following example shows that if an operator $T \in \mathcal{B}(\mathcal{H})$ have a closed range, then the generalized Aluthge transformation having a closed range.

Example 2.4. *Let* $T = \begin{pmatrix} B & 0 \\ (I - B^*B)^{\frac{1}{2}} & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$*, where* B *is a contraction and* $\mathcal{R}(B)$ *is not closed. Then*

$$T^*T = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

is an orthogonal projection. It follows that $TT^*T = T$*, and so* T *is a partial isometry and also* $\mathcal{R}(T)$ *is closed. By [18, Lemma 3.4], we have the polar decomposition of* $T = T|T|$*. For any* $\alpha, \beta > 0$*, we have*

$$\begin{aligned} \Delta_{\alpha,\beta}(T) &= |T|^\alpha T |T|^\beta \\ &= (T^*T)^\alpha T (T^*T)^\beta \\ &= T^*TTT^*T \\ &= T^*TT \\ &= \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence, $\mathcal{R}(\Delta_{\alpha,\beta}(T))$ *is not closed.*

The following result provides a necessary and sufficient condition for the ranges of T and its generalized Aluthge transformation to be either both closed or both not closed. It was proven in the case where $\alpha = \beta = \frac{1}{2}$, (see [5, Theorem 3.4]).

Theorem 2.5. *Let* $T = U|T|$ *be the polar decomposition of* $T \in \mathcal{B}(\mathcal{H})$*. If* $\mathcal{N}(T) \subset \mathcal{N}(T^*)$*, then for* $\alpha, \beta > 0$*, we heve*

$$\mathcal{R}(T) \text{ is closed} \iff \mathcal{R}(\Delta_{\alpha,\beta}(T)) \text{ is closed}.$$

Proof. ($\Longrightarrow$). Suppose that $\mathcal{R}(T)$ is closed and $\mathcal{R}(\Delta_{\alpha,\beta}(T))$ is not closed. Then $\gamma(\Delta_{\alpha,\beta}(T)) = 0$. So there existe a sequence of unit vectors $x_n \in \mathcal{N}(\Delta_{\alpha,\beta}(T))^\perp$ such that $\Delta_{\alpha,\beta}(T)x_n \to 0$. Since $\mathcal{N}(T) \subset \mathcal{N}(T^*)$. Now, by using lemma 2.1, we have $\mathcal{N}(\Delta_{\alpha,\beta}(T)) = \mathcal{N}(T)$. Therefore, each $x_n \in \mathcal{N}(T)^\perp$ such that $Tx_n \to 0$, which is a contradiction with the fact that $\mathcal{R}(T)$ is closed.
($\Longleftarrow$). Suppose that $\mathcal{R}(\Delta_{\alpha,\beta}(T))$ is closed and $\mathcal{R}(T)$ is not closed. Then $\gamma(T) = 0$ and we can choose a sequence of unit vectors $x_n \in \mathcal{N}(T)^\perp$ such that $Tx_n \to 0$. By Lemma 1.1, we have

$$x_n \in \mathcal{N}(T)^\perp = \mathcal{N}(|T|)^\perp = \mathcal{N}(|T|^\beta)^\perp.$$

Therefore, each $x_n \in \mathcal{N}(|T|^\beta)^\perp$ such that $|T|^\beta x_n \to 0$. Then

$$\begin{aligned} |T|^\beta x_n &\to 0 \\ |T|^\alpha U |T|^\beta x_n &\to 0 \\ \Delta_{\alpha,\beta}(T)x_n &\to 0. \end{aligned}$$

For all n. Hence, $x_n \in \mathcal{N}(\Delta_{\alpha,\beta}(T))^\perp$. This contradicts with the fact that $\mathcal{R}(\Delta_{\alpha,\beta}(T))$ is closed. □

Remark 2.6. *The condition that* $\mathcal{N}(T) \subset \mathcal{N}(T^*)$ *is necessary in the previous Theorem 2.5. To see this, consider the example 2.4. Then* $\mathcal{R}(T)$ *is closed and* T *is not quasinormal, because*

$$T = TT^*T = \begin{pmatrix} B & 0 \\ (I - B^*B)^{\frac{1}{2}} & 0 \end{pmatrix} \neq \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} = T^*TT.$$

Hence $\mathcal{N}(T) \not\subseteq \mathcal{N}(T^*)$*, by [17, Proposition 2.1]. Then for* $\alpha, \beta > 0$*,* $\mathcal{R}(\Delta_{\alpha,\beta}(T))$ *is not closed.*

Corollary 2.7. *Let* $T \in \mathcal{B}(\mathcal{H})$*. If* T *is hyponormal, then*

$$\mathcal{R}(T) \text{ is closed} \iff \mathcal{R}(\Delta_{\alpha,\beta}(T)) \text{ is closed}.$$

Proof. Since T is hyponormal, then $\mathcal{N}(T) \subset \mathcal{N}(T^*)$ and by Theorem 2.5. This implies that $\mathcal{R}(T)$ is closed if and only if $\mathcal{R}(\Delta_{\alpha,\beta}(T))$ is closed. □

Corollary 2.8. *Let $T \in \mathcal{B}(\mathcal{H})$. If $\mathcal{N}(T) = \mathcal{N}(T^*)$. Then the following statements are equivalent.*

(*i*) *$\mathcal{R}(T)$ is closed.*

(*ii*) *$\mathcal{R}(\Delta_{\alpha,\beta}(T))$ is closed.*

(*iii*) *$\mathcal{R}(\Delta_{\alpha,\beta}(T)^*)$ is closed.*

(*iv*) *$\mathcal{R}(\Delta_{\alpha,\beta}(T^*))$ is closed.*

Theorem 2.9. *Let $T = U|T|$ be the polar decomposition of $T \in \mathcal{B}(\mathcal{H})$. If $\mathcal{R}(T)$ is closed, then for $\alpha, \beta > 0$, we have*

$$T \quad \text{is an EP operator} \quad \Longleftrightarrow \quad \Delta_{\alpha,\beta}(T) \;\text{ is EP and }\; \mathcal{R}(T) = \mathcal{R}(\Delta_{\alpha,\beta}(T)).$$

Proof. ($\Longrightarrow$). We suppose that T is EP. Then $\mathcal{R}(T)$ is closed and $\mathcal{N}(T) = \mathcal{N}(T^*)$. By using Theorem 2.5, we have $\mathcal{R}(\Delta_{\alpha,\beta}(T))$ is closed. Now we prove that $\Delta_{\alpha,\beta}(T)$ is EP. Since $\mathcal{N}(T) = \mathcal{N}(T^*)$. So by Lemma 2.1, we obtain

$$\mathcal{N}(\Delta_{\alpha,\beta}(T)^*) = \mathcal{N}(T) = \mathcal{N}(\Delta_{\alpha,\beta}(T)).$$

Consequently, $\Delta_{\alpha,\beta}(T)$ is also EP. Lastly, by taking the orthogonal complements in the relation $\mathcal{N}(T) = \mathcal{N}(\Delta_{\alpha,\beta}(T))$ and since T and $\Delta_{\alpha,\beta}(T)$ are EP, we conclude that $\mathcal{R}(T) = \mathcal{R}(\Delta_{\alpha,\beta}(T))$.
($\Longleftarrow$). We assume that $\Delta_{\alpha,\beta}(T)$ is EP and $\mathcal{R}(T) = \mathcal{R}(\Delta_{\alpha,\beta}(T))$. Then $\mathcal{R}(\Delta_{\alpha,\beta}(T))$ is closed and $\mathcal{R}(\Delta_{\alpha,\beta}(T)) = \mathcal{R}(\Delta_{\alpha,\beta}(T)^*)$. It is implies that

$$\mathcal{N}(T^*) = \mathcal{N}(\Delta_{\alpha,\beta}(T)) = \mathcal{N}(\Delta_{\alpha,\beta}(T)^*).$$

Since $\mathcal{N}(T) \subset \mathcal{N}(\Delta_{\alpha,\beta}(T))$. Then $\mathcal{N}(T) \subset \mathcal{N}(T^*)$ and by Lemma 2.1, we get

$$\mathcal{N}(T^*) = \mathcal{N}(\Delta_{\alpha,\beta}(T)) = \mathcal{N}(T).$$

Thus T is an EP operator. □

Remark 2.10. *Without the condition $\mathcal{R}(T) = \mathcal{R}(\Delta_{\alpha,\beta}(T))$, the reverse implication does not hold. Indeed, consider $T = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$. Then $\mathcal{R}(T)$ is closed. Since $T^2 = 0$. It follows that $\Delta_{\alpha,\beta}(T) = 0$ is EP and $\mathcal{R}(T) \neq \mathcal{R}(\Delta_{\alpha,\beta}(T))$, but T is not EP because*

$$T^+T = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \neq \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} = TT^+.$$

Corollary 2.11. *Let $T \in \mathcal{B}(\mathcal{H})$ with closed range. If T is EP operator, then for $\alpha, \beta > 0$, we have*

$$(T\Delta_{\alpha,\beta}(T))^+ = \Delta_{\alpha,\beta}(T)^+T^+ \quad \text{and} \quad (\Delta_{\alpha,\beta}(T)T)^+ = T^+\Delta_{\alpha,\beta}(T)^+.$$

Proof. Applying Theorem 2.9 and by [8, Theorem 5] conclude these results. □

Corollary 2.12. *Let $T = U|T|$ and $S = V|S|$ be the polar decompositions. If $T, S \in \mathcal{B}(\mathcal{H})$ are EP operators with $\mathcal{R}(T) = \mathcal{R}(S)$. Then for $\alpha, \beta > 0$, the following properties hold:*

(*i*) *$\Delta_{\alpha,\beta}(TS)$ is EP operator.*

(*ii*) *$\Delta_{\alpha,\beta}(T)\Delta_{\alpha,\beta}(S)$ is EP operator.*

Proof. Applying Theorem 2.9 and by [8, Theorem 5] conclude these results. □

3. Applications to Binormal Operators with Closed Range

We begin this section by demonstrating that the range of the generalized Aluthge transformation is closed when $T = U|T| \in \mathcal{B}(\mathcal{H})$ is a binormal with closed range.

Theorem 3.1. *Let $T = U|T|$ be the polar decomposition of a binormal operator $T \in \mathcal{B}(\mathcal{H})$ and let $\alpha, \beta > 0$. Then*

$$\mathcal{R}(T) \quad \text{is closed} \quad \Longrightarrow \quad \mathcal{R}(\Delta_{\alpha,\beta}(T)) \quad \text{is closed}.$$

Proof. Assume that $\mathcal{R}(T)$ is closed and $\mathcal{R}(\Delta_{\alpha,\beta}(T))$ is not closed. By Lemma 2.2, we have $\gamma(\Delta_{\alpha,\beta}(T)) = \gamma(|T|^{\alpha}|T^*|^{\beta}) = \gamma(|T^*|^{\beta}|T|^{\alpha}) = 0$. Then there exists a sequence of unit vectors $x_n \in \mathcal{N}(|T|^{\alpha}|T^*|^{\beta})^{\perp} = \mathcal{N}(|T^*|^{\beta}|T|^{\alpha})^{\perp}$ such that $|T^*|^{\beta}|T|^{\alpha}x_n \longrightarrow 0$ and $|T|^{\alpha}|T^*|^{\beta}x_n \longrightarrow 0$. On the other hand, by Lemma 1.3 and since T is binormal, then by applying Lemma 1.4 we have

$$|T|^{\alpha}|T^*|^{\beta} = U^*UUU^*||T|^{\alpha}|T^*|^{\beta}|$$

is the polar decomposition of $|T|^{\alpha}|T^*|^{\beta}$. This implies that $x_n \in \mathcal{N}(U^*UUU^*)^{\perp} \subset \mathcal{N}(U^*)^{\perp}$. Since $\mathcal{R}(U^*)$ is closed and each $x_n \in \mathcal{N}(U^*)^{\perp}$, there exists $\eta > 0$ such that $\|U^*x_n\| \geq \eta$ for all n. Put $z_n := \dfrac{U^*x_n}{\|U^*x_n\|}$ and note that

$$z_n \in \mathcal{R}(U^*) = \mathcal{R}(T^*) = \mathcal{R}(U^*|T^*|) \subset \mathcal{R}(|T^*|) = \mathcal{R}(T) = \mathcal{R}(U|T|) \subset \mathcal{R}(|T|) = \mathcal{N}(|T|)^{\perp}$$

for all n. It follows that

$$\||T|z_n\| \leq \frac{1}{\eta}\||T|U^*x_n\|$$

for all n, and thus $|T|z_n \longrightarrow 0$. Which is a contradiction with the fact that $\mathcal{R}(T)$ is closed. □

Remark 3.2. (i) *The condition that "T is binormal " is necessary in the previous Theorem 3.1. Indeed if we take the example 2.4, we obtain $\mathcal{R}(T)$ is closed and T is not binormal. Then $\mathcal{R}(\Delta_{\alpha,\beta}(T))$ is not closed.*
(ii) *The reverse implication of the previous Theorem 3.1 is false, as the following example shows*

Example 3.3. *Let $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the unilateral weighted shift defined as*

$$Te_n = \lambda_n e_{n+1} \ \text{ for all } n \in \mathbb{N},$$

where

$$\lambda_n = \begin{cases} 0 & \text{if } n \text{ is even}, \\ \\ \frac{1}{n} & \text{if } n \text{ is odd}. \end{cases}$$

Then $T^2 = 0$ and so $(T^*)^2 = (T^2)^* = 0$. Hence $\Delta_{\frac{1}{2},\frac{1}{2}}(T) = 0$ has closed range and T is binormal, but $\mathcal{R}(T)$ is not closed.

Let $T \in \mathcal{B}(\mathcal{H})$ be binormal with closed range. By using [10, Theorem 2], we have

$$P_{\mathcal{R}(T)}P_{\mathcal{R}(T^*)} = P_{\mathcal{R}(T^*)}P_{\mathcal{R}(T)}.$$

In [13, Theorem 2.5], Jabbarzadeh and Bakhshkandi obtained $(\Delta_{\frac{1}{2},\frac{1}{2}}(T))^+ = (|T|^+)^{\frac{1}{2}}U^*(|T|^+)^{\frac{1}{2}}$, when $T = U|T| \in \mathcal{B}(\mathcal{H})$ be binormal with closed range. The following result we give a formula for the Moore-Penrose inverse of $\Delta_{\alpha,\beta}(T)$ under the same conditions.

Proposition 3.4. *Let $T = U|T| \in \mathcal{B}(\mathcal{H})$ be binormal with closed range, then $(\Delta_{\alpha,\beta}(T))^+ = (|T|^+)^{\beta}U^*(|T|^+)^{\alpha}$ for all $\alpha, \beta > 0$.*

Proof. Since T is binormal and $\mathcal{R}(T)$ is closed, then $\mathcal{R}(\Delta_{\alpha,\beta}(T))$ is closed, by Theorem 3.1. Now, for $\alpha, \beta > 0$, we put $S = (|T|^+)^\beta U^*(|T|^+)^\alpha$. By (i) of Lemma 1.7 and Lemma 1.6, Then we have

$$(|T|^+)^\alpha |T|^\alpha = (|T|^\alpha)^+ |T|^\alpha = P_{\mathcal{R}(|T|^\alpha)} = P_{\mathcal{R}(|T|)} = P_{\mathcal{R}(T^*)},$$

and

$$|T|^\beta (|T|^+)^\beta = |T|^\beta (|T|^\beta)^+ = P_{\mathcal{R}(|T|^\beta)} = P_{\mathcal{R}(|T|)} = P_{\mathcal{R}(T^*)}.$$

Consequently

$$\begin{aligned}
S\Delta_{\alpha,\beta}(T)S &= (|T|^+)^\beta U^*(|T|^+)^\alpha |T|^\alpha U|T|^\beta (|T|^+)^\beta U^*(|T|^+)^\alpha \\
&= (|T|^+)^\beta U^* P_{\mathcal{R}(T^*)} U P_{\mathcal{R}(T^*)} U^*(|T|^+)^\alpha \\
&= (|T|^+)^\beta U^* P_{\mathcal{R}(T^*)} U U^*(|T|^+)^\alpha \\
&= (|T|^+)^\beta U^* P_{\mathcal{R}(T^*)} P_{\mathcal{R}(T)} (|T|^+)^\alpha \\
&= (|T|^+)^\beta U^* P_{\mathcal{R}(T)} P_{\mathcal{R}(T^*)} (|T|^+)^\alpha \qquad \text{since } T \text{ is binormal} \\
&= (|T|^+)^\beta U^* P_{\mathcal{R}(T^*)} (|T|^+)^\alpha \\
&= (|T|^+)^\beta U^* (|T|^+)^\alpha = S,
\end{aligned}$$

$$\begin{aligned}
\Delta_{\alpha,\beta}(T)S\Delta_{\alpha,\beta}(T) &= |T|^\alpha U|T|^\beta (|T|^+)^\beta U^*(|T|^+)^\alpha |T|^\alpha U|T|^\beta \\
&= |T|^\alpha U P_{\mathcal{R}(T^*)} U^* P_{\mathcal{R}(T^*)} U|T|^\beta \\
&= |T|^\alpha U U^* P_{\mathcal{R}(T^*)} U|T|^\beta \\
&= |T|^\alpha P_{\mathcal{R}(T)} P_{\mathcal{R}(T^*)} U|T|^\beta \\
&= |T|^\alpha P_{\mathcal{R}(T^*)} P_{\mathcal{R}(T)} U|T|^\beta \qquad \text{since } T \text{ is binormal} \\
&= |T|^\alpha P_{\mathcal{R}(T^*)} U|T|^\beta \\
&= |T|^\alpha U|T|^\beta = S,
\end{aligned}$$

and

$$\begin{aligned}
S\Delta_{\alpha,\beta}(T) &= (|T|^+)^\beta U^*(|T|^+)^\alpha |T|^\alpha U|T|^\beta \\
&= (|T|^+)^\beta U^* P_{\mathcal{R}(T^*)} U|T|^\beta \\
&= (|T|^+)^\beta U^* P_{\mathcal{R}(T^*)} |T^*|^\beta U \qquad \text{by Lemma 1.2}(i) \\
&= (|T|^+)^\beta U^* |T^*|^\beta P_{\mathcal{R}(T^*)} U \\
&= (|T|^+)^\beta (|T^*|^\beta U)^* P_{\mathcal{R}(T^*)} U \\
&= (|T|^+)^\beta (U|T|^\beta)^* P_{\mathcal{R}(T^*)} U \qquad \text{by Lemma 1.2}(i) \\
&= (|T|^+)^\beta |T|^\beta U^* P_{\mathcal{R}(T^*)} U \\
&= P_{\mathcal{R}(T^*)} U^* P_{\mathcal{R}(T^*)} U \\
&= U^* P_{\mathcal{R}(T^*)} U.
\end{aligned}$$

By similar computation we have $\Delta_{\alpha,\beta}(T)S = P_{\mathcal{R}(T)} P_{\mathcal{R}(T^*)}$.Hence $S\Delta_{\alpha,\beta}(T)$ and $\Delta_{\alpha,\beta}(T)S$ are self-adjoint operator. From the uniqueness of Moore-Penrose inverse we have $(\Delta_{\alpha,\beta}(T))^+ = S$. □

The next Lemma we focus on generalized Aluthge transformation is EP when $T = T^*$.

Lemma 3.5. *Let $T \in \mathcal{B}(\mathcal{H})$ with closed range and $T = U|T|$ be the polar decomposition. Then for $\alpha, \beta > 0$, we have*

$$T = T^* \implies \Delta_{\alpha,\beta}(T) \text{ is EP}.$$

Proof. Since $T = T^*$ and $\mathcal{R}(T)$ is closed. Then T is EP and so by Theorem 2.9, we have $\Delta_{\alpha,\beta}(T)$ is EP.

Here we give a new proof. Suppose that $T = T^*$, then T is binormal and since $\mathcal{R}(T)$ is closed. Then by Theorem 3.1 and Proposition 3.4, we have $\mathcal{R}(\Delta_{\alpha,\beta}(T))$ is closed and $(\Delta_{\alpha,\beta}(T))^+ = (|T|^+)^\beta U^*(|T|^+)^\alpha$ for $\alpha, \beta > 0$. Thus

$$\begin{aligned}
\Delta_{\alpha,\beta}(T)(\Delta_{\alpha,\beta}(T))^+ &= |T|^\alpha U|T|^\beta(|T|^+)^\beta U^*(|T|^+)^\alpha \\
&= |T|^\alpha U P_{\mathcal{R}(T^*)} U^*(|T|^+)^\alpha \\
&= |T|^\alpha UU^*UU^*(|T|^+)^\alpha \\
&= |T|^\alpha UU^*(|T|^+)^\alpha \\
&= |T|^\alpha P_{\mathcal{R}(T)}(|T|^+)^\alpha \\
&= |T|^\alpha P_{\mathcal{R}(T^*)}(|T|^+)^\alpha \\
&= |T|^\alpha U^*U(|T|^+)^\alpha \\
&= |T|^\alpha(|T|^+)^\alpha \qquad \text{by Lemma 1.3 } (i) \\
&= P_{\mathcal{R}(T^*)} = U^*U.
\end{aligned}$$

and

$$\begin{aligned}
(\Delta_{\alpha,\beta}(T))^+\Delta_{\alpha,\beta}(T) &= (|T|^+)^\beta U^*(|T|^+)^\alpha|T|^\alpha U|T|^\beta \\
&= (|T|^+)^\beta U^* P_{\mathcal{R}(T^*)} U|T|^\beta \\
&= (|T|^+)^\beta U^* P_{\mathcal{R}(T)} U|T|^\beta \qquad \text{since } T^* = T \\
&= (|T|^+)^\beta U^*UU^*U|T|^\beta \\
&= (|T|^+)^\beta U^*U|T|^\beta \qquad \text{by Lemma 1.3 } (i) \\
&= (|T|^+)^\beta|T|^\beta \\
&= P_{\mathcal{R}(T^*)} = U^*U.
\end{aligned}$$

Hence $\Delta_{\alpha,\beta}(T)$ is EP. □

Proposition 3.6. *Let $T = U|T| \in \mathcal{B}(\mathcal{H})$ is binormal with closed range and let $\alpha, \beta > 0$ such that $\alpha + \beta = 1$. Then*

$$\sigma(T^+) = \sigma(\Delta_{\alpha,\beta}(T^+)) = \sigma((\Delta_{\alpha,\beta}(T))^+).$$

Proof. Since $T^+ = |T|^+U^*$, $(|T|^+)^\alpha \geq 0$ and hence it is normal. Then by Lemma 1.5 and Proposition 3.4, we have

$$\sigma(T^+) = \sigma(|T|^+U^*) = \sigma((|T|^+)^\alpha(|T|^+)^\beta U^*) = \sigma((|T|^+)^\beta U^*(|T|^+)^\alpha) = \sigma((\Delta_{\alpha,\beta}(T))^+).$$

On the other hand, by [13, Proposition 2.2], $T^+ = U^*|T^+|$ is the polar decomposition of T^+. Since $|T^+|^\alpha \geq 0$, then $|T^+|^\alpha$ is normal and again by Lemma 1.5, we get

$$\sigma(T^+) = \sigma(U^*|T^+|) = \sigma(U^*|T^+|^\beta|T^+|^\alpha) = \sigma(|T^+|^\alpha U^*|T^+|^\beta) = \sigma(\Delta_{\alpha,\beta}(T^+)).$$

□

Proposition 3.7. *Let $T = U|T| \in \mathcal{B}(\mathcal{H})$ with closed range and let $\alpha, \beta > 0$ such that $\alpha + \beta = 1$. If T^+ is quasinormal, then $w(T^+) = w(\Delta_{\alpha,\beta}(T^+))$.*

Proof. Since T^+ is quasinormal. Then by (ii) of Lemma 1.8 and by Lemma 1.9 , we have $U^*|T^+| = |T^+|U^*$. It follows that $U^*(|T^+|)^q = (|T^+|)^q U^*$ for each $q > 0$. Then we have

$$\begin{aligned}
w(T^+) &= \sup_{\|x\|=1} |\langle U^*|T^+|x, x\rangle| \\
&= \sup_{\|x\|=1} |\langle U^*|T^+|^\alpha|T^+|^\beta x, x\rangle| \\
&= \sup_{\|x\|=1} |\langle |T^+|^\alpha U^*|T^+|^\beta x, x\rangle| \\
&= w(\Delta_{\alpha,\beta}(T^+)).
\end{aligned}$$

□

In [14, Theorem 2.6], the authors obtained the polar decomposition of $(\Delta_{\frac{1}{2},\frac{1}{2}}(T))^+ = U^*V|(\Delta_{\frac{1}{2},\frac{1}{2}}(T))^+|$, when $T = U|T| \in \mathcal{B}(\mathcal{H})$ be binormal with closed range and $(|T|^+)^{\frac{1}{2}}(|T^*|^+)^{\frac{1}{2}} = V|(|T|^+)^{\frac{1}{2}}(|T^*|^+)^{\frac{1}{2}}|$. In the following theorem we obtain the polar decomposition of $(\Delta_{\alpha,\beta}(T))^+$.

Theorem 3.8. *Let $\alpha,\beta > 0$. Let $T = U|T| \in \mathcal{B}(\mathcal{H})$ with closed range and $(|T|^+)^\alpha(|T^*|^+)^\beta = V|(|T|^+)^\alpha(|T^*|^+)^\beta|$ be the polar decompositions. If T is binormal, then $(\Delta_{\alpha,\beta}(T))^+ = U^*V|(\Delta_{\alpha,\beta}(T))^+|$ is also the polar decomposition.*

Proof. Let $T = U|T|$ be the polar decomposition. Since T is binormal and $\mathcal{R}(T)$ is closed, by Proposition 3.4, $(\Delta_{\alpha,\beta}(T))^+ = (|T|^+)^\beta U^*(|T|^+)^\alpha$ for $\alpha,\beta > 0$. Then
(i) Proof of $((\Delta_{\alpha,\beta}(T))^+ = U^*V|(\Delta_{\alpha,\beta}(T))^+|$.

$$\begin{aligned}
U^*V|(\Delta_{\alpha,\beta}(T))^+| &= U^*V(((\Delta_{\alpha,\beta}(T))^+)^*(\Delta_{\alpha,\beta}(T))^+)^{\frac{1}{2}} \\
&= U^*V((|T|^+)^\alpha U(|T|^+)^\beta(|T|^+)^\beta U^*(|T|^+)^\alpha)^{\frac{1}{2}} \\
&= U^*V((|T|^+)^\alpha U(|T|^+)^{2\beta}U^*(|T|^+)^\alpha)^{\frac{1}{2}} \\
&= U^*V((|T|^+)^\alpha UU^*(|T^*|^+)^{2\beta}UU^*(|T|^+)^\alpha)^{\frac{1}{2}} && \text{by Lemma 1.8 } (i) \\
&= U^*V((|T|^+)^\alpha(|T^*|^+)^{2\beta}(|T|^+)^\alpha)^{\frac{1}{2}} && \text{by Lemma 2.3 } (iii) \\
&= U^*V((|T|^\alpha)^+(|T^*|^\beta)^+(|T^*|^\beta)^+(|T|^\alpha)^+)^{\frac{1}{2}} && \text{by Lemma 1.7 } (i), (ii) \\
&= U^*V((|T^*|^\beta)^+(|T|^\alpha)^+(|T|^\alpha)^+(|T^*|^\beta)^+)^{\frac{1}{2}} && \text{by Lemma 1.10} \\
&= U^*V((|T^*|^+)^\beta(|T|^+)^\alpha(|T|^+)^\alpha(|T^*|^+)^\beta)^{\frac{1}{2}} && \text{by Lemma 1.7 } (i), (ii) \\
&= U^*V|(|T|^+)^\alpha(|T^*|^+)^\beta| \\
&= U^*(|T|^+)^\alpha(|T^*|^+)^\beta \\
&= U^*(|T|^\alpha)^+(|T^*|^\beta)^+ && \text{by Lemma 1.7 } (i), (ii) \\
&= U^*(|T^*|^\beta)^+(|T|^\alpha)^+ && \text{by Lemma 1.10} \\
&= U^*(|T^*|^+)^\beta(|T|^+)^\alpha && \text{by Lemma 1.7 } (i), (ii) \\
&= U^*(|T^*|^+)^\beta UU^*(|T|^+)^\alpha \\
&= (|T|^+)^\beta U^*(|T|^+)^\alpha && \text{by Lemma 1.8 } (i) \\
&= (\Delta_{\alpha,\beta}(T))^+.
\end{aligned}$$

(ii) We will show $\mathcal{N}((\Delta_{\alpha,\beta}(T))^+) = \mathcal{N}(U^*V)$.

$$\begin{aligned}
U^*Vx = 0 &\Leftrightarrow U^*(|T|^+)^\alpha(|T^*|^+)^\beta x = 0 && \text{since } \mathcal{N}(V) = \mathcal{N}((|T|^+)^\alpha(|T^*|^+)^\beta) \\
&\Leftrightarrow U^*(|T^*|^+)^\beta(|T|^+)^\alpha x = 0 && \text{by Lemma 1.7 and Lemma 1.10} \\
&\Leftrightarrow U^*(|T^*|^+)^\beta UU^*(|T|^+)^\alpha x = 0 \\
&\Leftrightarrow (|T|^+)^\beta U^*(|T|^+)^\alpha x = 0 && \text{by Lemma 1.8} \\
&\Leftrightarrow (\Delta_{\alpha,\beta}(T))^+x = 0.
\end{aligned}$$

that is, $\mathcal{N}(U^*V) = \mathcal{N}((\Delta_{\alpha,\beta}(T))^+)$.
(iii) We claim that U^*V is a partial isometry. By (ii) and (iii) of Lemma 2.3, we have $(|T|^+)^\alpha = U^*U(|T|^+)^\alpha$ and $(|T^*|^+)^\beta = UU^*(|T^*|^+)^\beta$ are the polar decompositions of $(|T|^+)^q$ and $(|T^*|^+)^\beta$, respectively. Then by Lemma 1.4, we have

$$(|T|^+)^\alpha(|T^*|^+)^\beta = U^*UUU^*|(|T|^+)^\alpha(|T^*|^+)^\beta|$$

is unique polar decomposition of $(|T|^+)^\alpha(|T^*|^+)^\beta$. This implies that $V = U^*UUU^*$ and it follows that

$$\begin{aligned} U^*V(U^*V)^*U^*V &= (U^*U^*UUU^*)(UU^*U^*UU)(U^*U^*UUU^*) \\ &= U^*U^*UUU^*U^*UUU^*U^*UUU^* \\ &= U^*P_{\mathcal{R}(T^*)}P_{\mathcal{R}(T)}P_{\mathcal{R}(T^*)}P_{\mathcal{R}(T)}P_{\mathcal{R}(T^*)}P_{\mathcal{R}(T)} \\ &= U^*P_{\mathcal{R}(T^*)}P_{\mathcal{R}(T)} \qquad \text{since } T \text{ is binormal} \\ &= U^*U^*UUU^* \\ &= U^*V. \end{aligned}$$

that is, U^*V is a partial isometry. Hence the proof is complete. □

Let $T \in \mathcal{B}(\mathcal{H})$ with closed range. By using [20, Theorem 6.2] and [13, Theorem 2.8], we have T is binormal if and only if $\Delta_{\alpha,\beta}(T^+) = U^*|\Delta_{\alpha,\beta}(T^+)|$ for $\alpha, \beta > 0$. We show a characterization of binormal operators via $(\Delta_{\alpha,\beta}(T))^+$.

Theorem 3.9. *Let $T \in \mathcal{B}(\mathcal{H})$ be binormal with closed range and $T = U|T|$ be the polar decomposition. Then $(\Delta_{\alpha,\beta}(T))^+ = U^*|(\Delta_{\alpha,\beta}(T))^+|$ for all $\alpha, \beta > 0$.*

Proof. Since $T \in \mathcal{B}(\mathcal{H})$ be binormal with closed range. Then by Proposition 3.4 for all $\alpha, \beta > 0$, we have

$$\begin{aligned} U^*|(\Delta_{\alpha,\beta}(T))^+| &= U^*((|T|^+)^\alpha U(|T|^+)^{2\beta}U^*(|T|^+)^\alpha)^{\frac{1}{2}} \\ &= U^*((|T|^+)^\alpha U(|T|^{2\beta})^+U^*(|T|^+)^\alpha)^{\frac{1}{2}} \qquad \text{by Lemma 1.7 } (i) \\ &= U^*((|T|^+)^\alpha (|T^*|^{2\beta})^+UU^*(|T|^+)^\alpha)^{\frac{1}{2}} \qquad \text{by Lemma 1.10 } (i) \\ &= U^*((|T|^+)^\alpha (|T^*|^+)^{2\beta}UU^*(|T|^+)^\alpha)^{\frac{1}{2}} \qquad \text{by Lemma 1.7 } (ii) \\ &= U^*((|T|^+)^\alpha (|T^*|^+)^{2\beta}(|T|^+)^\alpha)^{\frac{1}{2}} \qquad \text{by Lemma 2.3 } (iii) \\ &= U^*(((|T^*|^+)^\beta(|T|^+)^\alpha)^2)^{\frac{1}{2}} \qquad \text{by Lemma 1.10 } (ii) \\ &= U^*(|T^*|^+)^\beta(|T|^+)^\alpha \\ &= U^*(|T^*|^\beta)^+(|T|^+)^\alpha \qquad \text{by Lemma 1.7 } (i) \\ &= ((|T^*|^\beta)^+U)^*(|T|^+)^\alpha \\ &= (U(|T|^\beta)^+)^*(|T|^+)^\alpha \qquad \text{by Lemma 1.10 } (i) \\ &= (|T|^\beta)^+U^*(|T|^+)^\alpha \\ &= (|T|^+)^\beta U^*(|T|^+)^\alpha \qquad \text{by Lemma 1.7 } (i) \\ &= (\Delta_{\alpha,\beta}(T))^+. \end{aligned}$$

Hence, $(\Delta_{\alpha,\beta}(T))^+ = U^*|(\Delta_{\alpha,\beta}(T))^+|$. □

Remark 3.10. *Usually, $(\Delta_{\alpha,\beta}(T))^+ = U^*|(\Delta_{\alpha,\beta}(T))^+|$ in Theorem 3.9 is not the polar decomposition since $\mathcal{R}(U^*) = \mathcal{R}((\Delta_{\alpha,\beta}(T))^+)$ does not hold (see Proposition 3.11).*

Proposition 3.11. *Let $\alpha, \beta > 0$. Let $T \in \mathcal{B}(\mathcal{H})$ be binormal with closed range and $T = U|T|$ be the polar decomposition. Then*

$$(\Delta_{\alpha,\beta}(T))^+ = U^*U^*U|(\Delta_{\alpha,\beta}(T))^+|$$

is also polar decomposition of $(\Delta_{\alpha,\beta}(T))^+$.

Proof. Since $\mathcal{R}(T)$ is closed, so by (*ii*) and (*iii*) of Lemma 2.3, we have $(|T|^+)^\alpha = U^*U(|T|^+)^\alpha$ and $(|T^*|^+)^\beta = UU^*(|T^*|^+)^\beta$ are the polar decompositions of $(|T|^+)^q$ and $(|T^*|^+)^\beta$, respectively. Since T is binormal, then by (*ii*) of Lemma 1.10 and Lemma 1.4, we have

$$(|T|^+)^\alpha(|T^*|^+)^\beta = (|T^*|^+)^\beta(|T|^+)^\alpha = UU^*U^*U|(|T^*|^+)^\beta(|T|^+)^\alpha|$$

Thus, by Theorem 3.8, we obtain

$$(\Delta_{\alpha,\beta}(T))^+ = U^*UU^*U^*U|(\Delta_{\alpha,\beta}(T))^+| = U^*U^*U|(\Delta_{\alpha,\beta}(T))^+|.$$

is also polar decomposition of $(\Delta_{\alpha,\beta}(T))^+$. $\square$

Proposition 3.12. *Let $T = U|T| \in \mathcal{B}(\mathcal{H})$ with closed range. Then the following assertions are equivalent.*

(i) *T^+ is quasinormal;*

(ii) $U^*|T^*|^+ = |T^*|^+U^*$;

(iii) $T^+|T^*|^+ = |T^*|^+T^+$;

Proof. $(i) \iff (ii)$. The proof follows from (ii) of Lemma 1.8.
$(ii) \Longrightarrow (iii)$. Since $T^+ = U^*|T^*|^+$ is the polar decomposition of T^+. Then we have

$$T^+|T^*|^+ = U^*|T^*|^+|T^*|^+ = |T^*|^+U^*|T^*|^+ = |T^*|^+T^+.$$

$(iii) \Longrightarrow (ii)$. We have

$$\begin{aligned} T^+|T^*|^+ = |T^*|^+T^+ \quad &\Longrightarrow \quad U^*|T^*|^+|T^*|^+ = |T^*|^+U^*|T^*|^+ \\ &\Longrightarrow \quad (U^*|T^*|^+ - |T^*|^+U^*)|T^*|^+ = 0. \end{aligned}$$

So $U^*|T^*|^+ - |T^*|^+U^* = 0$ on $\overline{\mathcal{R}(|T^*|^+)}$. On the other hand, $U^*|T^*|^+ = |T^*|^+U^*$ on $\mathcal{N}(U^*) = \mathcal{N}(|T^*|^+)$. Thus $U^*|T^*|^+ = |T^*|^+U^*$ on $\mathcal{H}$. $\square$

Theorem 3.13. *Let $T = U|T| \in \mathcal{B}(\mathcal{H})$ with closed range and T^+ is quasinormal. Suppose that for each $n \in \mathbb{N}$, $|T^+|^\alpha(T^+)^n(|T|^+)^\beta = W_n||T^+|^\alpha(T^+)^n(|T|^+)^\beta|$ be the polar decomposition. Then $W_n = (U^*)^n$.*

Proof. Since T^+ is quasinormal, so by Proposition 3.12 and by Lemma 1.9, we get $T^+|T^+| = |T^+|T^+$ and by functional calculus, we obtain $T^+(|T^+|)^q = (|T^+|)^qT^+$ for all $q > 0$. Then we have

$$\begin{aligned} ||T^+|^\alpha(T^+)^n(|T|^+)^\beta|^2 &= (|T|^+)^\beta((T^+)^n)^*|T^+|^{2\alpha}(T^+)^n(|T|^+)^\beta \\ &= (|T|^+)^\beta((T^+)^n)^*(T^+)^n|T^+|^{2\alpha}(|T|^+)^\beta \\ &= (|T|^+)^\beta((T^+)^*T^+)^n|T^+|^{2\alpha}(|T|^+)^\beta \\ &= (|T|^+)^\beta|T^+|^{2n}|T^+|^{2\alpha}(|T|^+)^\beta \\ &= (|T|^+)^\beta|T^+|^{2n+2\alpha}(|T|^+)^\beta \\ &= (|T|^+)^\beta|T^+|^{2(n+\alpha)}(|T|^+)^\beta \\ &= ||T^+|^{n+\alpha}(|T|^+)^\beta|^2 \\ &= (|T^+|^{n+\alpha}(|T|^+)^\beta)^2. \end{aligned}$$

Hence

$$||T^+|^\alpha(T^+)^n(|T|^+)^\beta| = |T^+|^{n+\alpha}(|T|^+)^\beta.$$

Now, if $|T^+|^\alpha(T^+)^n(|T|^+)^\beta = W_n||T^+|^\alpha(T^+)^n(|T|^+)^\beta|$ be the polar decomposition, then

$$W_n||T^+|^\alpha(T^+)^n(|T|^+)^\beta| = W_n|T^+|^{n+\alpha}(|T|^+)^\beta. \tag{3}$$

On the other hand, again since T^+ is quasinormal, we have

$$\begin{aligned} |T^+|^\alpha(T^+)^n(|T|^+)^\beta &= |T^+|^\alpha(U^*)^n|T^+|^n(|T|^+)^\beta \\ &= (U^*)^n|T^+|^\alpha|T^+|^n(|T|^+)^\beta \\ &= (U^*)^n|T^+|^{n+\alpha}(|T|^+)^\beta. \end{aligned}$$

Hence

$$|T^+|^\alpha (T^+)^n (|T|^+)^\beta = (U^*)^n |T^+|^{n+\alpha} (|T|^+)^\beta. \tag{4}$$

By (3) and (4), we have

$$((U^*)^n - W_n)|T^+|^{n+\alpha}(|T|^+)^\beta = 0.$$

Hence, $(U^*)^n = W_n$ on $\mathcal{R}(|T^+|^{n+\alpha}(|T|^+)^\beta)$ and since $\mathcal{N}((U^*)^n) = \mathcal{N}(|T^+|^{n+\alpha}(|T|^+)^\beta) = \mathcal{N}(W_n)$, we conclude that $(U^*)^n = W_n$ for all $n \in \mathbb{N}$. □

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