



q -statistical convergence of double sequences of order $\tilde{\lambda}$

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Abstract.

The quantum calculus or a q -calculus is the generalization as well as modifications of the classical calculus. Over the last twenty years, the subject of q -calculus has served as a connection between mathematics and physics. In recent past there is a substantial increase of research in field of q -calculus because of its applications in many fields such as number theory, combinatorics, special functions basic hyper geometric functions mechanics, theory of relativity and other sciences-quantum theory. In this paper we introduce the concept of q -analogue of statistical convergence of double sequence of order $\tilde{\lambda}$ and studied some of its geometric properties. Further, we define q -cesàro summability of order $\tilde{\lambda}$ and gave the relationship between q -strong p -cesàro summability of order $\tilde{\lambda}$ and the q -statistical convergence of double sequences of order $\tilde{\mu}$ for $\tilde{\lambda} \leq \tilde{\mu}$.

1. Introduction

The concept of statistical convergence was introduced in problems related to series summation whose idea goes back to the first edition (published in Warsaw 1935). of the monograph of A.Zigmond [22]. The idea was introduced and investigated by Steinhaus [21] and Fast [7] and later on studied by Schoenberg [20].

Statistical convergence has now recently became an active area of research. Different mathematicians explored the idea in numerous areas like measure theory, trigonometric series, approximation theory etc. It was further studied in the area sequence space and summability methods by Fridy [8], Connor [6], Maddox [12], Mursaleen et al. [17].

The concept of statistical convergence was put forth to double sequences by Mursaleen and Edely [15], Karakaya [10] et al., Moricz [13] and many others.

The quantum calculus or a q -calculus is the generalization as well as modifications of classical calculus. Over the last twenty years, the subject of q -calculus had served as a link between mathematics and physics. In recent past there was a substantial increase of research in the field of q -calculus because of its applications in many other fields such as number theory, combinatorics, speed functions, basis hypergeometric functions,

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mechanics, theory of relativity and quantum theory.

The q -analogue of statistical convergence is applied in approximation theory and summability, later it was extended to double q -statistical convergence [18].

2. Definitions and Preliminaries

Definition 2.1. [17] The q -analogue of real numbers is given by

$$[r]_q = \begin{cases} \frac{1-q^r}{1-q}, & \text{if } q \in \mathbf{R}^+ \setminus \{1\}; \\ r & , \text{if } q = 1 \end{cases}$$

The formal definition of q -analogue is that “Analog of a theorem, identity or expression is a generalization including a new parameter q which refers back to the original theorem, identity or expression in the limit as $q \rightarrow 1$.”

This concludes a fact that q -analogue of something is not a unique expression as we just need to satisfy the limiting condition of 1 and also that above equation is not a definition but an example of many q -analogue of numbers. For example r^q can also be a q -analogue of numbers as many others. The prior q -analogue is the best one to the real numbers.

Definition 2.2. [17] Let $x = (x_k)$ be a number sequence. It is said to be Cesàro summable to L , if

$$\lim_n \frac{1}{n} \sum_{k=0}^n x_k = L.$$

Definition 2.3. [17] The q -analogue of Cesàro matrix is given by $C_1(q^k) = (c_{nk}^1(q^k))$, where

$$c_{nk}^1(q^k) = \begin{cases} \frac{q^k}{[n+1]_q}, & \text{if } k \leq n, \\ 0, & \text{otherwise} \end{cases}$$

Definition 2.4. [17] Let $K \subseteq \mathbf{N}$. For $q \geq 1$, the q -density of K is given by

$$\delta_q(K) = \delta_{C_1^q}(K) = \lim_{n \rightarrow \infty} \inf (C_1^q \chi_K)_n.$$

Definition 2.5. [17] A number sequence $x = (x_k)$ is said to be q -statistically convergent to L , if for every $\epsilon > 0$ and $n \in \mathbf{N}$, such that $\delta_q(K) = 0$, where, $K = \{k : k \leq n : |x_k - L| \geq \epsilon\}$.

The set of all q -statistically convergent sequences is indicated by S_q .

Definition 2.6. [15] The double density of the set $K \subseteq \mathbf{N} \times \mathbf{N}$ is defined as

$$\delta^2(K) = \lim_{n,m \rightarrow \infty} \frac{|K(n,m)|}{nm},$$

where the cardinality of the enclosed set $K(n, m)$ is indicated by vertical bars and $K(n, m) = \{(j, k) \in \mathbf{N} \times \mathbf{N} : j \leq n \text{ and } k \leq m; (j, k) \in K\}$.

Definition 2.7. [15] A double sequence $x = (x_{jk})$ is said to be statistically convergent to the number L , if for every $\epsilon > 0$, $\delta^2(K) = 0$, where,

$$K = \{(j, k) : j \leq n \text{ and } k \leq m : |x_{jk} - L| \geq \epsilon\}.$$

Definition 2.8. [18] A double sequence $x = (x_{jk})$ is said to be q -statistically convergent to L , if for every $\epsilon > 0$, $\delta_q^2(K) = 0$, where

$$K = \{(j, k) : j \leq n \text{ and } k \leq m : |x_{jk} - L| \geq \epsilon\}.$$

Unless otherwise noted, we use $s, t, u, v \in (0, 1]$ throughout the paper. To be concise, we use $\tilde{\lambda}$ in place of (s, t) and $\tilde{\mu}$ in place of (u, v) . We also define

$$\tilde{\lambda} \leq \tilde{\mu} \Leftrightarrow s \leq u \text{ and } t \leq v;$$

$$\tilde{\lambda} < \tilde{\mu} \Leftrightarrow s < u \text{ and } t < v;$$

$$\tilde{\lambda} \cong \tilde{\mu} \Leftrightarrow s = u \text{ and } t = v;$$

$$\tilde{\lambda} \in (0, 1] \Leftrightarrow s, t \in (0, 1];$$

$$\tilde{\mu} \in (0, 1] \Leftrightarrow u, v \in (0, 1];$$

$$\tilde{\lambda} \cong 1 \text{ in case } s = t = 1;$$

$$\tilde{\mu} \cong 1 \text{ in case } u = v = 1;$$

$$\tilde{\lambda} > 1 \text{ in case } s > 1 \text{ and } t > 1;$$

and furthermore, we write $S_{\tilde{\lambda}}^2$ to denote $S_{(s,t)}^2$ and $S_{\tilde{\mu}}^2$ to denote $S_{(u,v)}^2$ in the following. Now the $\tilde{\lambda}$ -double density of the set $K \subseteq \mathbf{N} \times \mathbf{N}$ is defined as

$$\delta_{\tilde{\lambda}}^2(K) = \lim_{n,m} \frac{K(n,m)}{n^s m^t}.$$

Remark 2.9. Note that for any set $K \subseteq \mathbf{N} \times \mathbf{N}$, $\delta_{\tilde{\lambda}}^2(K)$ may be greater than 1, even equal to ∞ , but $\delta^2(K) \leq 1$. Also, $\delta^2(K^c) = 1 - \delta^2(K)$ holds but $\delta_{\tilde{\lambda}}^2(K) = 1 - \delta_{\tilde{\lambda}}^2(K)$ does not hold in general.

Definition 2.10. [5] Let $x = (x_{jk}) \in w^2$ and $\tilde{\lambda} \in (0, 1]$ be given. The sequence (x_{jk}) is said to be statistically convergent of order $\tilde{\lambda}$ if there is a complex number l such that for every $\epsilon > 0$,

$$\lim_{n,m \rightarrow \infty} \frac{1}{n^s m^t} |\{(j, k) : j \leq n, k \leq m : |x_{jk} - l| \geq \epsilon\}| = 0.$$

In this case, we write $S_{\tilde{\lambda}}^2 - \lim_{j,k} x_{jk} = l$. Denote the set of all statistically convergent of double sequences of order $\tilde{\lambda}$ by $S_{\tilde{\lambda}}^2$.

Remark 2.11. The definition of statistical convergence of order $\tilde{\lambda}$ reduce back to the original definition of statistical convergence of double sequence as soon as $\tilde{\lambda}$ tends to 1.

3. Main Results

Definition 3.1. A double sequence $x = (x_{jk})$ is said to be q -statistical convergent of order $\tilde{\lambda}$ if

$$\lim_{n,m \rightarrow \infty} \frac{1}{[n^s]_q [m^t]_q} |\{(i, j) : j \leq n, k \leq m : |x_{jk} - l| \geq \epsilon\}| = 0$$

or we can write the above definition as

$$[S_{\tilde{\lambda}}^2]_q - \lim_{j,k} x_{jk} = l.$$

Denoting the set of all q -statistically convergent of double sequences of order $\tilde{\lambda}$ as $[S_{\tilde{\lambda}}^2]_q$.

Remark 3.2. The definition of q -statistical convergence of order $\tilde{\lambda}$ reduce back to the original definition of q -statistical convergence of double sequence as soon as $\tilde{\lambda}$ tends to 1.

The limit of q -statistically convergent of order $\tilde{\lambda}$ is determined uniquely and clearly expressed for $\tilde{\lambda} \in (0, 1]$ but not for $\tilde{\lambda} > 1$ for instance, let $x = (x_{jk})$ be defined as :

$$x_{jk} = \begin{cases} 1, & \text{if } j+k \text{ even} \\ 0, & \text{if } j+k \text{ odd.} \end{cases}$$

Then,

$$\begin{aligned} & \lim_{n,m \rightarrow \infty} \frac{1}{[n^s]_q [m^t]_q} |\{(j, k) : j \leq n, k \leq m : |x_{jk} - 1| \geq \epsilon\}| \\ & \leq \lim_{n,m \rightarrow \infty} \frac{(n/2+1)(m/2+1)}{[n^s]_q [m^t]_q} \leq \lim_{n,m \rightarrow \infty} \frac{(n/2+1)(m/2+1)}{[\frac{1-n^{sq}}{1-q}] [\frac{1-m^{tq}}{1-q}]} \\ & \leq \lim_{n,m \rightarrow \infty} \frac{(n/2+1)(m/2+1)(1-q)^2}{[1-n^{sq}] [1-m^{tq}]} = 0. \end{aligned}$$

Applying L-Hospital rule on simultaneous limit.

$$\begin{aligned} & \lim_{n,m \rightarrow \infty} \frac{1}{[n^s]_q [m^t]_q} |\{(j, k) : j \leq n, k \leq m : |x_{ij} - 1| \geq \epsilon\}| \\ & \leq \lim_{n,m \rightarrow \infty} \frac{(n/2+1)(m/2+1)}{[n^s]_q [m^t]_q} \leq \lim_{n,m \rightarrow \infty} \frac{(n/2+1)(m/2+1)}{[\frac{1-n^{sq}}{1-q}] [\frac{1-m^{tq}}{1-q}]} \\ & \leq \lim_{n,m \rightarrow \infty} \frac{(n/2+1)(m/2+1)(1-q)^2}{[1-n^{sq}] [1-m^{tq}]} = 0. \end{aligned}$$

Applying L-Hospital rule on simultaneous limit. for $\tilde{\lambda} > 1$ that is $s > 1$ and $t > 1$, so that $x = (x_{jk})$ is q -statistically convergent of order $\tilde{\lambda}$ both to 1 and 0, that is, $[S_{\tilde{\lambda}}^2]_q - \lim_{j,k} x_{jk} = 1$ and $[S_{\tilde{\lambda}}^2]_q - \lim_{j,k} x_{jk} = 0$, which means that the q -statistical limit of order $\tilde{\lambda}$ of the double sequence (x_{jk}) is not unique. however, this is not possible.

Theorem 3.3. q -statistical convergence of order $\tilde{\lambda}$ is linear. That is,

(i) If $[S_{\tilde{\lambda}}^2]_q - \lim x_{jk} = l$ and $a \in \mathbf{C}$, then $[S_{\tilde{\lambda}}^2]_q - \lim ax_{jk} = al$.

(ii) If $[S_{\tilde{\lambda}}^2]_q - \lim x_{jk} = l_1$ and $[S_{\tilde{\lambda}}^2]_q - \lim y_{jk} = l_2$, then $[S_{\tilde{\lambda}}^2]_q - \lim (x_{jk} + y_{jk}) = l_1 + l_2$.

Proof. Consider the two sequences $x = (x_{jk})$, $y = (y_{jk})$ and $\tilde{\lambda} \in (0, 1]$ the result is clear for $a = 0$. For $a \neq 0$ the proof of (i) follows from

$$\begin{aligned} & \frac{1}{[n^s]_q [m^t]_q} |\{(j, k) : j \leq n, k \leq m, |ax_{jk} - al| \geq \epsilon\}| \\ & = \frac{1}{[n^s]_q [m^t]_q} |\{(j, k) : j \leq n, k \leq m, |x_{jk} - l| \geq \frac{\epsilon}{|a|}\}| \end{aligned}$$

and that of (ii) follows from the following inequality:

$$\begin{aligned} & \frac{1}{[n^s]_q [m^t]_q} |\{(j, k) : j \leq n, k \leq m, |x_{jk} + y_{jk} - (l_1 + l_2)| \geq \epsilon\}| \\ & \leq \frac{1}{[n^s]_q [m^t]_q} |\{(j, k) : j \leq n, k \leq m, |x_{jk} - l_1| \geq \frac{\epsilon}{2}\}| + \frac{1}{[n^s]_q [m^t]_q} |\{(j, k) : j \leq n, k \leq m, |y_{jk} - l_2| \geq \frac{\epsilon}{2}\}|. \end{aligned}$$

It is quiet evident that every convergent double sequence is also q -statistically convergent of order $\tilde{\lambda}$, i.e. $[c^2]_q \subset [S_{\tilde{\lambda}}^2]_q$ for $\tilde{\lambda} \in (0, 1]$ but not vice versa. For instance let $x = (x_{jk})$ be defined by

$$x_{jk} = \begin{cases} jk, & \text{if } j = n^2, k = m^2 \\ 0, & \text{otherwise} \end{cases} \quad n = 1, 2, \dots; m = 1, 2, \dots$$

$$\lim_{n,m \rightarrow \infty} \frac{1}{[n^s]_q [m^t]_q} |\{(j,k) : j \leq n, k \leq m : |x_{jk} - 1|\}|$$

$$\leq \lim_{n,m \rightarrow \infty} \frac{jk}{[n^s]_q [m^t]_q} \longrightarrow 0.$$

So that $x = (x_{jk})$ is q -statistically convergent of order $\tilde{\lambda}$ with $[S_{\tilde{\lambda}}^2]_q - \lim x_{jk} = 0$ where $\tilde{\lambda} > 1/2$ that is $s > 1/2$, $t > 1/2$ but it is not q -convergent.

4. q -Cesàro Summability of order $\tilde{\lambda}$ of double sequence

Definition 4.1. [5] Let $\tilde{\lambda} \in (0, 1]$ be given. A sequence $x = x_{jk}$ is said to be Cesàro summable of order $\tilde{\lambda}$ if there is a complex number l such that

$$\lim_{n,m \rightarrow \infty} \frac{1}{n^s m^t} \sum_{j=1}^n \sum_{k=1}^m x_{jk} = l,$$

which is the case when we say that x is Cesàro summable of order $\tilde{\lambda}$ to l . For $\tilde{\lambda} \cong 1$, the Cesàro summability of order $\tilde{\lambda}$ reduces to the Cesàro summability that is given. The set of all Cesàro summable double sequences of order $\tilde{\lambda}$ will be indicated by $w_{\tilde{\lambda}}^2$.

The set of all Cesàro summable double sequences will be denoted by w^2 .

Definition 4.2. [18] The double q -Cesàro matrix is given by $C_{(1,1)}^q = (c_{stpr}(q^{s+t}))$ with

$$c_{stpr}(q) = \begin{cases} \frac{q^{s+t}}{[r+1]_q [p+1]_q}, & s \leq p \text{ and } t \leq r; \\ 0 & \text{otherwise} \end{cases}$$

The entire paper for $q \geq 1$, we take

$$\delta_q^{pr}(K) = \sum_{s,t \in K} C_{(1,1)}^q \chi_K(r, t) = \sum_{s,t \in K} \frac{q^{s+t-2}}{[p]_q [t]_q},$$

and the double q -density of K , $\delta_q^2(K)$ is given by

$$\delta_q^2(K) = P - \lim_{p,r} \delta_q^{pr}(K).$$

Now we define q -Cesàro summability of order $\tilde{\lambda}$.

Definition 4.3. Let $x = (x_{jk})$ be sequence. Then x is called q -Cesàro summable of order $\tilde{\lambda}$ where $\tilde{\lambda} \in (0, 1]$ to $l \in \mathbb{C}$ if

$$\lim_{n,m \rightarrow \infty} \frac{1}{[n^s]_q [m^t]_q} \sum_{j=1}^n \sum_{k=1}^m x_{jk} = l,$$

The set of all q -Cesàro summable double sequences of order $\tilde{\lambda}$ will be denoted by $[w_{\tilde{\lambda}}^2]_q$. The set of all q -Cesàro summable double sequences will be denoted by $[w^2]_q$.

Remark 4.4. The definition of q -Cesàro summability of order $\tilde{\lambda}$ reduce back to the original definition of q -Cesàro summability as seen as $\tilde{\lambda}$ tends to 1.

Definition 4.5. Let $\tilde{\lambda} \in (0, 1]$ be given, and let p be positive real number. Then, a sequence $x = (x_{jk})$ is said to be q -strongly p -Cesàro summable of order $\tilde{\lambda}$ if there is a complex number l such that

$$\lim_{n,m \rightarrow \infty} \frac{1}{[n^s]_q [m^t]_q} \sum_{i=1}^n \sum_{j=1}^m |x_{ij} - l|^p = 0,$$

which is the case when we say that x is q -strongly p -Cesàro summable of order $\tilde{\lambda}$ to l .

Remark 4.6. The definition of q -strongly p -Cesàro summability of order $\tilde{\lambda}$ return to the definition of q -strongly p -Cesàro summability as soon as $\tilde{\lambda} \cong 1$.

This set is denoted by $[w_p^2]_q^{\tilde{\lambda}}$.

5. Relationship between the q -statistical convergence of order $\tilde{\lambda}$ and the q -statistical convergence of order $\tilde{\mu}$ for $\tilde{\lambda} \leq \tilde{\mu}$

Now we prove some important results of this paper, where we furnish the relationship between the q -statistical convergence of order $\tilde{\lambda}$ and the q -statistical convergence of order $\tilde{\mu}$ for double sequence spaces where $\tilde{\lambda} \leq \tilde{\mu}$. Further the relation between the q -statistical convergence of order $\tilde{\lambda}$ and the q -statistical convergence is also discussed.

Theorem 5.1. Let $\tilde{\lambda}, \tilde{\mu} \in (0, 1]$ such that $\tilde{\lambda} \leq \tilde{\mu}$, then, $[S_{\tilde{\lambda}}^2]_q \subseteq [S_{\tilde{\mu}}^2]_q$.

Proof. Let $\tilde{\lambda}, \tilde{\mu} \in (0, 1]$ be given. If $\tilde{\lambda} \leq \tilde{\mu}$ and so that $s \leq u$ and $t \leq v$, then

$$\frac{1}{[n^u]_q [m^v]_q} |\{(j, k) : j \leq n, k \leq m, |x_{jk} - l| \geq \epsilon\}| \leq \frac{1}{[n^s]_q [m^t]_q} |\{(j, k) : j \leq n, k \leq m, |x_{jk} - l| \geq \epsilon\}|,$$

for every $\epsilon > 0$ and this gives that $[S_{\tilde{\lambda}}^2]_q \subseteq [S_{\tilde{\mu}}^2]_q$.

Theorem 5.2. Let $\tilde{\lambda}, \tilde{\mu} \in (0, 1]$ be given such that $\tilde{\lambda} \leq \tilde{\mu}$, and let p be a positive real number. Then, $q - [w_p^2]_{\tilde{\lambda}} \subseteq q - [w_p^2]_{\tilde{\mu}}$ and the inclusion is strict for some $\tilde{\lambda}$ and $\tilde{\mu}$ such that $\tilde{\lambda} < \tilde{\mu}$.

Proof. Let $x = (x_k) \in q - [w_p^2]_{\tilde{\lambda}}$. Then, give $\tilde{\lambda}$ and $\tilde{\mu}$ such that $\tilde{\lambda} < \tilde{\mu}$ and a positive real number p , we may write

$$\frac{1}{[n^u]_q [m^v]_q} \sum_{i=1}^n \sum_{j=1}^m |x_{ij} - l|^p \leq \frac{1}{[n^s]_q [m^t]_q} \sum_{i=1}^n \sum_{j=1}^m |x_{ij} - l|^p,$$

and this gives that $q - [w_p^2]_{\tilde{\lambda}} \subseteq q - [w_p^2]_{\tilde{\mu}}$.

To show that this is a strict inclusion, consider the sequence $x = (x_{ij})$ defined by

$$x_{ij} = \begin{cases} ij, & i = n^2, j = m^2 \\ 0, & \text{otherwise} \end{cases} \quad m, n = 1, 2, 3, \dots$$

It is clear that

$$\frac{1}{[n^u]_q [m^v]_q} \sum_{i=1}^n \sum_{j=1}^m |x_{ij} - l|^p \leq \frac{\sqrt{n} \sqrt{m}}{[n^u]_q [m^v]_q} = \frac{(1-q)^2 \sqrt{nm}}{(1-n^{uq})(1-m^{vq})} \rightarrow 0 \text{ as } n, m \rightarrow \infty,$$

using L-Hospital rule. Hence $q - [w_p^2]_{\tilde{\mu}} - \lim x_{ij} = 0$; that is $x \in q - [w_p^2]_{\tilde{\mu}}$ for $\tilde{\mu} \in (1/2, 1]$ (i.e., for $1/2 < u \leq 1$ and $1/2 < v \leq 1$), but since

$$\frac{(\sqrt{n} - 1)(\sqrt{m} - 1)}{[n^s]_q [m^t]_q} \leq \frac{1}{[n^s]_q [m^t]_q} \sum_{i=1}^n \sum_{j=1}^m |x_{ij} - l|^p$$

and $([\sqrt{n} - 1]_q)/[n^s]_q \rightarrow \infty$ as $n \rightarrow \infty$, $m \rightarrow \infty$, then $x \notin q - [w_p^2]_{\tilde{\lambda}}$ for $\tilde{\lambda} \in (0, 1/2]$ (i.e., for $0 < s < 1/2$ and $0 < t < 1/2$). This completes the proof.

The following result is a consequence of Theorem 5.2.

Corollary. Let $\tilde{\lambda}, \tilde{\mu} \in (0, 1]$ be given such that $\tilde{\lambda} \leq \tilde{\mu}$, and let p be a positive real number. Then,

- (i) $q - [w_p^2]_{\tilde{\lambda}} = q - [w_p^2]_{\tilde{\mu}}$ if and only if $\tilde{\lambda} \cong \tilde{\mu}$;
- (ii) $q - [w_p^2]_{\tilde{\lambda}} \subseteq q - w_p^2$ for each $\tilde{\lambda}$ such that $\tilde{\lambda} \in (0, 1]$ and $0 < p < \infty$.

Theorem 5.3. Let $\tilde{\lambda} \in (0, 1]$, and let $0 < p < p' < \infty$. Then, $q - [w_{p'}^2]_{\tilde{\lambda}} \subset q - [w_p^2]_{\tilde{\lambda}}$.

Proof. The proof is a simple consequence of Hölder's inequality which is an extension of a result of Maddox [12].

Taking $\tilde{\lambda} \cong 1$ in Theorem 5.3, we obtain a result of Maddox [12]: if $0 < p < p' < \infty$, then $q - w_{p'}^2 \subset q - w_p^2$.

Theorem 5.4. Let $\tilde{\lambda}$ and $\tilde{\mu}$ be given such that $\tilde{\lambda} \leq \tilde{\mu}$, and let $0 < p < \infty$, where $\tilde{\lambda}, \tilde{\mu} \in (0, 1]$. If a sequence is q -strongly p -Cesàro summable of order $\tilde{\lambda}$ to l , then it is q -statistically convergent of order $\tilde{\mu}$ to l .

Proof. For any sequences $x = (x_{ij})$ and $\epsilon > 0$, we have

$$\sum_{i=1}^n \sum_{j=1}^m |x_{ij} - l|^p \geq |\{(i, j) : i \leq n, j \leq m, |x_{ij} - l| \geq \epsilon\}| \cdot \epsilon^p,$$

so that since $\tilde{\lambda} \leq \tilde{\mu}$,

$$\begin{aligned} & \frac{1}{[n^s]_q [m^t]_q} \sum_{i=1}^n \sum_{j=1}^m |x_{ij} - l|^p \\ & \geq \frac{1}{[n^s]_q [m^t]_q} |\{(i, j) : i \leq n, j \leq m, |x_{ij} - l| \geq \epsilon\}| \cdot \epsilon^p \geq \frac{1}{[n^u]_q [m^v]_q} |\{(i, j) : i \leq n, j \leq m, |x_{ij} - l| \geq \epsilon\}| \cdot \epsilon^p, \end{aligned}$$

From this, it follows that if $x = (x_{ij})$ is q -strongly p -Cesàro summable of order $\tilde{\lambda}$ to l , then it is q -statistically convergent of order $\tilde{\mu}$ to l .

If we take $\tilde{\mu} \cong \tilde{\lambda}$ in Theorem 5.4, we obtain the following result.

Corollary. Let $\tilde{\lambda} \in (0, 1]$ be given, and let $0 < p < \infty$. If a double sequence is q -strongly p -Cesàro summable of order $\tilde{\lambda}$ to l , then it is q -statistically convergent of order $\tilde{\lambda}$ to l .

Remark 5.5. Note that the converse of Theorem is not valid in a general context. We see that a bounded and q -statistically convergent double sequence of order $\tilde{\lambda}$ need not be q -strongly p -Cesàro summable of order $\tilde{\lambda}$ in general for $\tilde{\lambda} \in (0, 1]$.

The sequence $x = (x_{ij})$ defined by

$$x_{ij} = \begin{cases} \frac{1}{\sqrt{i}\sqrt{j}}, & i \neq n^3, j \neq m^3 \\ 1, & \text{otherwise} \end{cases} \quad m, n = 1, 2, 3, \dots$$

is an example for this case. We see that $x \in l_\infty^2$ and $x \in S_\lambda^2$ for each $\tilde{\lambda} \in (1/3, 1]$.

Consider the inequality

$$\sum_{i=1}^n \sum_{j=1}^m \frac{1}{\sqrt{i}\sqrt{j}} > \sqrt{m}\sqrt{n}$$

which holds for every positive integer $m \geq 2$ and $n \geq 2$. Define the sequence spaces $A_n = \{i \leq n : i \neq k^2, k = 1, 2, 3, \dots\}$, $B_m = \{j \leq m : j \neq k^2, k = 1, 2, 3, \dots\}$ and take $p = 1$.

Now since

$$\sum_{i=1}^n \sum_{j=1}^m |x_{ij}|^p > \sqrt{m}\sqrt{n},$$

we have (see [5]);

$$\begin{aligned} & \frac{1}{[n^s]_q[m^t]_q} \sum_{i=1}^n \sum_{j=1}^m |x_{ij}|^p \\ &= \frac{1}{[n^s]_q[m^t]_q} \sum_{i=1}^n \sum_{j=1}^m |x_{ij}| > \frac{1}{[n^s]_q[m^t]_q} \sum_{i=1}^n \sum_{j=1}^m \frac{1}{\sqrt{i} \sqrt{j}} > \frac{\sqrt{n} \sqrt{m}}{[n^s]_q[m^t]_q} \rightarrow 0 \text{ as } n, m \rightarrow \infty, \end{aligned}$$

so that $x \notin q - [w_p^2]_{\tilde{\alpha}}$ for $\tilde{\alpha} \in (0, 1/2]$ if $p = 1$. Therefore, $x \in q - S_{\tilde{\alpha}}^2 - [w_p^2]_{\tilde{\alpha}}$ for $\tilde{\alpha} \in (1/3, 1/2]$ if $p = 1$.

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