



On deferred sequence spaces defined by invariant convergence

Ekrem Savaş^a

^aDepartment of Mathematics, Uşak University, Uşak, Turkey

Abstract. The main contribution of this paper is to define the notion of deferred Cesàro mean coming from modern summability methods by using invariant means and study some of its properties and also establish relations among these sequence spaces. These ideas and results are expected to be a source for researchers in the area of summability theory. In the end we can say that this concept can be generalized and applied for further studies.

1. Definitions

Let S denote the set of all real and complex sequences $s = (s_k)$. By m and c , we denote the Banach spaces of bounded and convergent sequences $s = (s_k)$ normed by $\|s\| = \sup_n |s_n|$, respectively.

Let σ be a mapping of the positive integers into themselves. A continuous linear functional φ on m is called to be an invariant mean or a σ -mean if and only if

1. $\sigma(s) \geq 0$ when the sequence $s = (s_n)$ has $s_n \geq 0$ for all n ;
2. $\sigma(e) = 1$ where $e = (1, 1, 1, \dots)$ and
3. $\sigma(s_{\sigma(n)}) = \varphi(s)$ for all $s \in m$.

The mapping σ is assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all $n, m \in \mathbb{Z}^+$, where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n .

For certain class of mapping σ every invariant mean φ extends the limit functional on space c , in the sense that $\varphi(s) = \lim s$ for all $s \in c$.

The space V_σ of the bounded sequences whose invariant means are equal may be defined, as follows:

$$V_\sigma = \left\{ \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n (s_{\sigma^k(m)} - \xi) = 0, \text{ uniformly in } m \right\}$$

Schaefer [21]) presented that a bounded sequence $s = (s_k)$ of real numbers is σ -convergent to ξ if and only if

$$\frac{1}{n} \sum_{k=1}^n (s_{\sigma^k(m)} - \xi) \rightarrow 0$$

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Email address: ekremsavas@yahoo.com (Ekrem Savaş)

ORCID iD: <https://orcid.org/0000-0003-2135-3094> (Ekrem Savaş)

as $n \rightarrow \infty$ uniformly in m . Consequently, $c \subset V_\sigma$.

Note that the space $[V_\sigma]$ is strongly σ -convergent sequence was introduced by Mursaleen [10] which is below: A sequence $s = (s_k)$ is said to be strongly σ -convergent if there exists a number ξ such that

$$[V_\sigma] = \left\{ \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |s_{\sigma^k(m)} - \xi| = 0, \text{ uniformly in } m \right\}$$

If we consider $\sigma(m) = m + 1$, then $[V_\sigma] = [\hat{c}]$, the set of all strongly almost convergent sequences which is defined by Maddox in [9].

Definition 1.1. (see, [8]).

Let X be a real or complex linear space, g be a function from X to the $\mathbb{R}$. The pair (X, g) is called a paranormed space and g is a paranorm for X , if the following properties are satisfied for all elements $x, y \in X$

1. $g(\theta) = 0$, where θ is the zero element of X ;
2. $g(x) \geq 0$;
3. $g(-x) = g(x)$;
4. $g(x + y) \leq g(x) + g(y)$ (triangle inequality);
5. If (λ_n) is a sequence of scalars with $\lambda_n \rightarrow \lambda$ as $n \rightarrow \infty$ and $(x_n)_{n=1}^\infty$ is a sequence in X with $g(x_n - x) \rightarrow 0$ as $n \rightarrow \infty$ then $g(\lambda_n x_n - \lambda x) \rightarrow 0$ as $n \rightarrow \infty$ (continuity of multiplication by scalars).

Observe that, if in addition g is increasing and $g(x) = g(|x|)$, and the sequence (λ_n) converges to λ , then for large n 's

$$\begin{aligned} g(\lambda_n x_n - \lambda x) &\leq g((\lambda_n - \lambda)(x_n - x)) + g(\lambda(x_n - x)) + g((\lambda_n - \lambda)x) \\ &\leq g(|x_n - x|) + (1 + [|\lambda|])g(|x_n - x|) + g((\lambda_n - \lambda)x) \\ &= (2 + [|\lambda|])g(x_n - x) + g((\lambda_n - \lambda)x), \end{aligned}$$

where $[|\lambda|]$ is the integer part of $|\lambda|$. Hence to prove condition (5) it is enough to show that for all fixed $y \in X$ and $\lambda_n \rightarrow 0$, $g(\lambda_n y) \rightarrow 0$. Throughout this work we introduce the sequence spaces using the following type of transformation:

Definition 1.2. (see, [4], [14]). Let $T = (t_{n,k})$ denote a matrix transformation that maps complex sequences s into the sequence Ts where the n -th term of Ts is as follows:

$$(Ts)_n = \sum_{k=1}^{\infty} t_{n,k} s_k.$$

Such transformation is called to be nonnegative if $t_{n,k}$ is nonnegative.

The deferred Cesàro mean of sequence $s = (s_n)$ is defined by (see, [11], [2]) as follows:

$$(D_{p,q}s)_n = \frac{s_{p_{n+1}} + s_{p_{n+2}} + \cdots + s_{q_n}}{q_n - p_n}.$$

where $p = (p_n)$ and $q = (q_n)$ are sequences of non-negative integers (see, Agnew[1]) satisfying

- i) $p_n < q_n$ for all $n \in \mathbb{N}_0$
- ii) $\lim_{n \rightarrow \infty} q_n = \infty$.

In the notation of matrix transformation

$$(T_{p,q}s)_n = \sum_{k=0}^{\infty} t_{nk} s_k.$$

where

$$t_{nk} = \begin{cases} \frac{1}{q_n - p_n}, & (p_n < k \leq q_n) \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

It is obvious that (D_{p_n, q_n}) is regular with conditions (i) and (ii). Indeed, this mean generalizes various transforms such as $D_{n-1, n} = I$, the identity transform, $D_{0, n} = C(0, 1)$, the Cesàro transformation. Later on generalizations of deferred Cesàro mean of sequences are presenting in many articles by various authors (see, [3, 5–7, 12, 14–18]).

We now have

Definition 1.3. Suppose $T = (t_{n,k})$ is a non-negative summability matrix with the property

$$\sup_n \sum_{k=1}^{\infty} b_{n,k} < \infty.$$

and f is a modulus function. We consider the new deferred invariant sequence spaces as follows:

$$V_0^D(T, \sigma, f) = \left\{ s \in S : \lim_n \sum_{k=p_n+1}^{q_n} t_{n,k} f(|x_{\sigma^k(m)}|) = 0, \text{ uniformly in } m \right\},$$

$$V^D(T, \sigma, f) = \{ s \in S : s - \xi e \in V_0^D(T, \sigma, f) \text{ for some scalar } \xi \},$$

and

$$V_{\infty}^D(T, \sigma, f) = \left\{ s \in S : \sup_{n,m} \sum_{k=p_n+1}^{q_n} t_{n,k} f(|x_{\sigma^k(m)}|) < \infty \right\},$$

where $e = \{1, 1, 1, 1, \dots\}$ and $(p_n), (q_n)$ are sequences of non-negative integers satisfying the conditions (i) and (ii).

If $x \in V_0^D(T, \sigma, f)$, then we say that the sequence s is strong deferred T -invariant convergent to 0 with respect to a modulus f , where modulus function was defined by Ruckle [13] as follows.

Definition 1.4. A function $f : [0, \infty) \rightarrow [0, \infty)$ is called a modulus function provided that

1. $f(s) = 0$ if and only if $s = 0$,
2. $f(s+r) \leq f(s) + f(r)$ for all $s \geq 0$ and $r \geq 0$,
3. f is increasing, and
4. f is continuous from the right of 0.

If $f(s) = s$, then the above sequence spaces reduce to the following:

$$V_0^D(T, \sigma) = \left\{ s \in S : \lim_n \sum_{k=p_n+1}^{q_n} t_{n,k} |x_{\sigma^k(m)}| = 0, \text{ uniformly in } m \right\},$$

$$V^D(T, \sigma) = \{ s \in S : s - \xi e \in V_0^D(T, \sigma) \text{ for some } \xi \},$$

and

$$V_{\infty}^D(T, \sigma) = \left\{ s \in S : \sup_{n,m} \sum_{k=p_n+1}^{q_n} t_{n,k} |x_{\sigma^k(m)}| < \infty \right\}.$$

If T is the deferred matrix defined as in (1.1) with $q(n) = n$ and $p(n) = 0$ for all $n \in \mathbb{N}$. It is obvious that we get the generalizations of the classical strongly invariant summable sequences spaces $w_0^{\sigma}(f)$, $w_0^{\sigma}(f)$, and $w_{\infty}^{\sigma}(f)$ as defined in [19] that is:

$$w_0^\sigma(f) = \left\{ s \in S : \lim_n \frac{1}{n} \sum_{k=1}^n f(|x_{\sigma^k(m)}|) = 0, \text{ uniformly in } m \right\},$$

$$w^\sigma(f) = \{s \in S : s - \xi e \in w_0^\sigma(f) \text{ for some scalar } \xi\},$$

and

$$w_\infty^\sigma(f) = \left\{ s \in S : \sup_{n,m} \frac{1}{n} \sum_{k=1}^n f(|x_{\sigma^k(m)}|) < \infty \right\},$$

where $e = \{1, 1, 1, 1, \dots\}$

If T is the deferred matrix defined as in (1.1) with $q(n) = n$ and $p(n) = n - \lambda_n$ for all $n \in N$, we have (see, [20]),

$$w_0^\sigma(\lambda, f) = \left\{ s \in S : \lim_n \frac{1}{\lambda_n} \sum_{k=n-\lambda_n+1}^n f(|x_{\sigma^k(m)}|) = 0, \text{ uniformly in } m \right\},$$

$$w^\sigma(\lambda, f) = \{s \in S : s - \xi e \in w_0^\sigma(\lambda, f) \text{ for some scalar } \xi\},$$

and

$$w_\infty^\sigma(\lambda, f) = \left\{ s \in S : \sup_{n,m} \frac{1}{\lambda_n} \sum_{k=n-\lambda_n+1}^n f(|x_{\sigma^k(m)}|) < \infty \right\},$$

where $e = \{1, 1, 1, 1, \dots\}$ and (λ_n) being a non-decreasing sequence of positive numbers tending to ∞ and $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$.

If T is the deferred matrix defined as in (1.1) with $q(n) = k_n$ and $p(n) = k_{n-1}$ for all $n \in N$, and $\theta = (k_n)$, we have

$$w_0^\sigma(\theta, f) = \left\{ s \in S : \lim_n \frac{1}{h_n} \sum_{k \in I_n} f(|x_{\sigma^k(m)}|) = 0, \text{ uniformly in } m \right\},$$

$$w^\sigma(\theta, f) = \{s \in S : s - \xi e \in w_0^\sigma(\theta, f) \text{ for some scalar } \xi\},$$

and

$$w_\infty^\sigma(\theta, f) = \left\{ s \in S : \sup_{n,m} \frac{1}{h_n} \sum_{k \in I_n} f(|x_{\sigma^k(m)}|) < \infty \right\},$$

where $e = \{1, 1, 1, 1, \dots\}$ and a lacunary sequence being an increasing integer sequence $\theta = \{k_n\}_{n \in \mathbb{N} \cup \{0\}}$ such that $k_0 = 0$ and $h_n = k_n - k_{n-1} \rightarrow \infty$, as $n \rightarrow \infty$. Let $I_n = (k_{n-1}, k_n]$.

2. Main Results

We now start our main discussions.

Theorem 2.1. $V_0^D(T, \sigma)f$, $VD(T, \sigma, f)$, and $V_\infty^D(T, \sigma, f)$ are linear space over the complex field $\mathbb{C}$.

Proof. Let us show that the space $V_0^D(T, \sigma, f)$ is linear. It is easy to prove that the linearity of $V^D(T, \sigma, f)$ and $V_\infty^D(T, \sigma, f)$ by proceeding in the similar line. Put $s, r \in V_0^D(T, \sigma, f)$. Let $\alpha, \beta \in \mathbb{C}$, and choose natural numbers

K_α and L_β such that $|\alpha| < K_\alpha$ and $|\beta| < L_\beta$. We know that f is monotonicity and subadditivity, so we write , for all m ,

$$\begin{aligned} \sum_{k=p_n+1}^{q_n} t_{n,k} f(|\alpha s_{\sigma^k(m)} + \beta s_{\sigma^k(m)}|) &\leq \sum_{k=p_n+1}^{q_n} t_{n,k} (f(|\alpha| |s_{\sigma^k(m)}|) + f(|\beta| |r_{\sigma^k(m)}|)) \\ &\leq \sum_{k=p_n+1}^{q_n} t_{n,k} f(K_\alpha |s_{\sigma^k(m)}|) + \sum_{k=p_n+1}^{q_n} t_{n,k} f(L_\beta |r_{\sigma^k(m)}|) \\ &\leq K_\alpha \sum_{k=p_n+1}^{q_n} t_{n,k} f(|s_{\sigma^k(m)}|) + L_\beta \sum_{k=p_n+1}^{q_n} t_{n,k} f(|r_{\sigma^k(m)}|). \end{aligned}$$

We know that $s, r \in V_0^D(T, \sigma, f)$, so it is clear that $\lim_{n \rightarrow \infty} \sum_{k=p_n+1}^{q_n} t_{n,k} f(|s_{\sigma^k(m)}|) = 0$ and $\lim_{n \rightarrow \infty} \sum_{k=p_n+1}^{q_n} t_{n,k} f(|r_{\sigma^k(m)}|) = 0$, uniformly in m . Finally $\alpha s + \beta r \in V_0^D(T, \sigma, f)$ and thus the space $V_0^D(B, f)$ is linear. $\square$

Theorem 2.2. For any non-negative matrix $T = (t_{nk})$ with the property $\sup_n \sum_{k=1}^{\infty} t_{n,k} < \infty$, and any modulus function f , the following inclusions hold.

1. $V^D(T, \sigma, f) \subset V_\infty^D(T, \sigma, f)$
2. $V_0^D(T, \sigma, f) \subset V^D(T, \sigma, f)$

Proof. The second part inclusion is obvious. Let $s \in V^D(T, \sigma, f)$. Now, by the definition of a modulus function (2) and (3), for all m we write

$$\begin{aligned} \sum_{k=p_n+1}^{q_n} t_{n,k} f(|s_{\sigma^k(m)}|) &= \sum_{k=p_n+1}^{q_n} t_{n,k} f(|s_{\sigma^k(m)} - \xi + \xi|) \\ &\leq \sum_{k=p_n+1}^{q_n} t_{n,k} f(|s_{\sigma^k(m)} - \xi|) + f(|\xi|) \sum_{k=p_n+1}^{q_n} t_{n,k}. \end{aligned}$$

Since $\sup_n \sum_{k=p_n+1}^{q_n} t_{n,k} < \infty$ and $x \in V^D(B, f)$ it is clear that $s \in V_\infty^D(T, \sigma, f)$. $\square$

Theorem 2.3. Let f be a modulus function and $T = (t_{nk})$ a non-negative summability matrix with the property $\sup_n \sum_{k=1}^{\infty} t_{n,k} < \infty$, then

1. $V^D(T, \sigma) \subset V^D(T, \sigma, f)$,
2. $V_0^D(T, \sigma) \subset V_0^D(T, \sigma, f)$, and
3. $V_\infty^D(T, \sigma) \subset V_\infty^D(T, \sigma, f)$.

Proof. We will give the proof of (3) only. The proofs of the remaining parts follow in similar manners. Let $s \in V_\infty^D(T, \sigma)$, that is $\sup_{nm} \sum_{k=p_n+1}^{q_n} t_{n,k} |s_{\sigma^k(m)}| < \infty$, and f be any modulus function. Further let $\epsilon > 0$ and $\delta \in (0, 1)$ such that $f(t) < \epsilon$ for $0 \leq t \leq \delta$. Take the following partition of $\sum_{k=p_n+1}^{q_n} t_{n,k} f(|s_{\sigma^k(m)}|)$ and denote each sum by Σ_1 and Σ_2 , respectively.

$$\sum_{k=p_n+1}^{q_n} t_{n,k} f(|s_{\sigma^k(m)}|) = \sum_{\{k=p_n+1 \text{ \& } |s_{\sigma^k(m)}| < \delta\}} t_{n,k} f(|s_{\sigma^k(m)}|) + \sum_{\{k=p_n+1 \text{ \& } |s_{\sigma^k(m)}| \geq \delta\}} t_{n,k} f(|s_{\sigma^k(m)}|). \quad (2)$$

We write

$$|s_{k+m}| < \frac{|s_{\sigma^k(m)}|}{\delta} < 1 + \left\lceil \frac{|s_{\sigma^k(m)}|}{\delta} \right\rceil,$$

where $[t]$ denotes the integer part of t . Thus by properties (2) and (3) of modulus function, for $|s_{\sigma^k(m)}| \geq \delta$ we write $f(|s_{\sigma^k(m)}|) \leq \left(1 + \left\lceil \frac{|s_{\sigma^k(m)}|}{\delta} \right\rceil\right) f(1) \leq 2f(1) \frac{|s_{\sigma^k(m)}|}{\delta}$. Therefore the second partition in (2.1) can be bounded as follows, for all m

$$\sum_2 \leq 2 \frac{f(1)}{\delta} \sum_{k=p_n+1}^{q_n} t_{n,k} |s_{\sigma^k(m)}|.$$

Also,

$$\sum_1 \leq \epsilon \sum_{k=p_n+1}^{q_n} t_{n,k},$$

and therefore, for all m

$$\sum_{k=p_n+1}^{q_n} t_{n,k} f(|s_{\sigma^k(m)}|) \leq \epsilon \sum_{k=p_n+1}^{q_n} t_{n,k} + 2 \frac{f(1)}{\delta} \sum_{k=p_n+1}^{q_n} t_{n,k} |s_{\sigma^k(m)}|.$$

Since

$$\sup_n \sum_{k=p_n+1}^{q_n} t_{n,k} < \infty$$

and $s \in V_\infty^D(T, \sigma)$ this gives that $s \in V_\infty^D(T, \sigma, f)$. $\square$

Theorem 2.4. Let f be a modulus function and T a non-negative summability matrix with the property

$$\sup_n \sum_{k=p_n+1}^{q_n} t_{n,k} < \infty.$$

If $\beta = \inf_{t>0} \frac{f(t)}{t} > 0$, then

$$V^D(T, \sigma) = V^D(T, \sigma, f).$$

Proof. Note that the previous theorem it remains to show that $V^D(T, \sigma, f) \subseteq V^D(T, \sigma)$. Since $f(t) \geq \beta t$ for all $t \geq 0$ and $\beta > 0$, we granted $t \leq \frac{1}{\beta} f(t)$ for all $t \geq 0$. Let $s \in V^D(T, \sigma, f)$, that is for some scalar ξ , $\lim_{n \rightarrow \infty} \sum_{k=p_n+1}^{q_n} f(|s_{\sigma^k(m)} - \xi|) = 0$. Since for every $n \in \mathbb{N}$, for all m ,

$$\sum_{k=p_n+1}^{q_n} t_{n,k} |s_{\sigma^k(m)} - \xi| \leq \frac{1}{\beta} \sum_{k=p_n+1}^{q_n} t_{n,k} f(|s_{\sigma^k(m)} - \xi|).$$

Finally we get that $s \in V^D(T, \sigma)$. $\square$

Theorem 2.5. Let $T = (t_{nk})$ be a non-negative summability matrix with the property

$$\sup_n \sum_{k=p_n+1}^{q_n} t_{n,k} < \infty, \quad \inf_n \sum_{k=p_n+1}^{q_n} t_{n,k} > 0,$$

and f is a modulus function then $V_0^D(T, f)$ and $V^D(T, f)$ are linear topological spaces with paranorm

$$g(x) = \sup_{n,m} \sum_{k=p_n+1}^{q_n} t_{n,k} f(|s_{\sigma^k(m)}|).$$

Also, if for all $N \in \mathbb{N}$, $\cup_{n=1}^N [p_n + 1, q_n) = [1, q_N)$ then $V_0^D(T, \sigma, f)$ and $V^D(T, \sigma, f)$ are complete with respect to the paranorm g .

Proof. We will give the proof of $V^D(T, \sigma, f)$ only. The case of $V_0^D(T, \sigma, f)$ will follow analogously. For this, we will first show that $V^D(T, \sigma, f)$ with g is a paranormed linear topological space. By Theorem 2, for each $s \in V^D(T, f)$, $g(x)$ exists. Further, note $g(s) = 0$ if and only if $s = 0$, $g(-s) = g(s)$ and $g(s + r) \leq g(s) + g(r)$. Since by the nature of the function f , g is increasing and $g(s) = g(|s|)$, it remains to show that for fixed s taking a sequence (λ_m) of scalars converging to 0, we write that $g(\lambda_m s) \rightarrow 0$ (see the remark after definition paranorm).

Fix $\epsilon > 0$ and write $s \in V^D(T, f)$ and sequence (λ_m) converging to 0. Then there exists ξ and $N \in \mathbb{N}$ such that for all $n, m \geq N$

$$\sum_{k=p_n+1}^{q_n} t_{n,k} f(|s_{\sigma^k(m)} - \xi|) < \frac{\epsilon}{2}.$$

Denote by $K = \sup_{n,m} \sum_{k=p_n+1}^{q_n} t_{n,k}$ and $t = \max_{k \leq N} |s_{\sigma^k(m)} - \xi|$. Since f is continuous we can find $M \in \mathbb{N}$ such that $|\lambda_t| < 1$ and $f(|\lambda_t|) < \frac{\epsilon}{2K(2 + \lceil |t| \rceil + \lceil |\xi| \rceil)}$ for all $t \geq M$, where $[s]$ denotes the integer part of a positive scalar s . Then for all $n, m \in \mathbb{N}$ and $n \geq M$ we have for all m

$$\begin{aligned} \sum_{k=p_n+1}^{q_n} t_{n,k} f(|\lambda_m s_{\sigma^k(m)}|) &= \sum_{k=p_n+1}^{q_n} t_{n,k} f(|\lambda_m s_{\sigma^k(m)} + \lambda_t \xi - \lambda_m \xi|) \\ &\leq \sum_{k=p_n+1 \text{ \& } n \geq N}^{q_n} t_{n,k} f(|\lambda_m| |s_{\sigma^k(m)} - \xi|) + \sum_{k=p_n+1 \text{ \& } n < N}^{q_n} t_{n,k} f(|\lambda_t| |s_{\sigma^k(m)} - \xi|) + K f(|\lambda_m \xi|) \\ &\leq \sum_{k=p_n+1 \text{ \& } n \geq N}^{q_n} t_{n,k} f(|s_{\sigma^k(m)} - \xi|) + K f(|\lambda_m| t) + K f(|\lambda_t \xi|) \\ &\leq \sum_{k=p_n+1 \text{ \& } n \geq N}^{q_n} t_{n,k} f(|s_{\sigma^k(m)} - \xi|) + K(2 + \lceil |r| \rceil + \lceil |\xi| \rceil) f(|\lambda_m|) < \epsilon. \end{aligned}$$

Finally, $g(\lambda_m s) = \sup_{n,m} \sum_{k=p_n+1}^{q_n} t_{n,k} f(|\lambda_m s_{\sigma^k(m)}|) \rightarrow 0$ as $n \rightarrow \infty$, uniformly in m .

We assume next that

$$\cup_{n=1}^N [p_n + 1, q_n] = [1, q_N], \quad (3)$$

that is $p_1 = 0$ and the sums $\sum_{k=p_n+1}^{q_n} r_{\sigma^k(m)}$ taken over all natural numbers n will run through each coordinate r_k of given r at least once. We will establish that under the above condition $V^D(T, f)$ is complete with respect to its paranorm g . Let (s_k^γ) be a Cauchy sequence in $(V^D(T, f, g))$. Then the convergence $\lim_{\gamma, \eta \rightarrow \infty} g(s^\gamma - s^\eta) = 0$ means that, as $\gamma, \eta \rightarrow \infty$ for all n, m

$$\sum_{k=p_n+1}^{q_n} t_{n,k} f(|x_{\sigma^k(m)}^\gamma - s_{\sigma^k(m)}^\eta|) \rightarrow 0. \quad (4)$$

Thus by assumption (3) for each fixed $k \in \mathbb{N}$, it follows that $t_{n,k} f(|s_{\sigma^k(m)}^\gamma - s_{\sigma^k(m)}^\eta|) \rightarrow 0$, with $\gamma, \eta \rightarrow \infty$, and since $T = (t_{n,k})$ is a non-negative and f continuous, (s_k^γ) is a Cauchy sequence in $\mathbb{C}$ for each fixed k . Since $\mathbb{C}$ is complete there exists a sequence $s = (s_k) \subseteq \mathbb{C}$ such that as $\gamma \rightarrow \infty$, $s_k^\gamma \rightarrow s_k$ for each k .

Now from (4), given $\epsilon > 0$ there exists a natural number N such that for all $\gamma, \eta \geq N$ and $n \in \mathbb{N}$, for all m ,

$$\sum_{k=p_n+1}^{q_n} t_{n,k} f(|s_{\sigma^k(m)}^\gamma - s_{\sigma^k(m)}^\eta|) < \frac{\epsilon}{2}. \quad (5)$$

Since by the continuity for f we also get that for fixed k , $f(|s_{\sigma^k(m)}^\eta - s_{\sigma^k(m)}|) \rightarrow 0$ as $\eta \rightarrow \infty$, for every $k \in \mathbb{N}$ there is $M_k \in \mathbb{N}$ such that if $\eta \geq M_k$ then $f(|s_{\sigma^k(m)}^\eta - s_{\sigma^k(m)}|) < \frac{\epsilon}{2C}$, where $C = \sup_n \sum_{k=p_n+1}^{q_n} t_{n,k}$. Write $N_n = \max_{p_n+1 \leq k \leq q_n} \{M_k, N\}$. We have for $\eta \geq N_n$

$$\sum_{k=p_n+1}^{q_n} t_{n,k} f(|s_{\sigma^k(m)}^\eta - s_{\sigma^k(m)}|) \leq \sum_{k=p_n+1}^{q_n} t_{n,k} \frac{\epsilon}{2C} \leq C \cdot \frac{\epsilon}{2C} = \frac{\epsilon}{2}, \quad (6)$$

for all n, m .

Let $n, m \in \mathbb{N}$, choose $\gamma \geq N$ and $\eta \geq N_n$. Then from (5) and (6),

$$\sum_{k=p_n+1}^{q_n} t_{n,k} f(|s_{\sigma^k(m)}^\gamma - s_{\sigma^k(m)}|) \leq \sum_{k=p_n+1}^{q_n} t_{n,k} f(|s_{\sigma^k(m)}^\gamma - s_{\sigma^k(m)}^\eta|) + \sum_{k=p_n+1}^{q_n} t_{n,k} f(|s_{\sigma^k(m)}^\eta - s_{\sigma^k(m)}|) < \epsilon.$$

Finally, $g(s^\gamma - \xi) = \sup_{n,m} \sum_{k=p_n+1}^{q_n} t_{n,k} f(|s_{\sigma^k(m)}^\gamma - s_{\sigma^k(m)}|) < \epsilon$, for all $\gamma \geq N$ and $g(s^\gamma - s) \rightarrow 0$ as $n \rightarrow \infty$, uniformly in m . Since all $s^\gamma \in V^D(T, f)$, for each γ there exists ξ^γ with

$$\lim_{n \rightarrow \infty} \sum_{k=p_n+1}^{q_n} t_{n,k} f(|s_{\sigma^k(m)}^\gamma - \xi^\gamma|) = 0,$$

uniformly in m . Write by $c = \inf_n \sum_{k=p_n+1}^{q_n} t_{n,k}$, which is assumed to be positive. Write that for all n, m and γ, η ,

$$\begin{aligned} cf(|\xi^\eta - \xi^\gamma|) &\leq \sum_{k=p_n+1}^{q_n} t_{n,k} f(|\xi^\eta - \xi^\gamma|) \leq \sum_{k=p_n+1}^{q_n} t_{n,k} f(|\xi^\eta - s_{\sigma^k(m)}^\eta|) \\ &\quad + \sum_{k=p_n+1}^{q_n} t_{n,k} f(|s_{\sigma^k(m)}^\eta - s_{\sigma^k(m)}^\gamma|) + \sum_{k=p_n+1}^{q_n} t_{n,k} f(|s_{\sigma^k(m)}^\gamma - \xi^\gamma|) \\ &\leq \sum_{k=p_n+1}^{q_n} t_{n,k} f(|\xi^\eta - s_{\sigma^k(m)}^\eta|) + g(x^\eta - x^\gamma) + \sum_{k=p_n+1}^{q_n} t_{n,k} f(|s_{\sigma^k(m)}^\gamma - \xi^\gamma|). \end{aligned}$$

Since the outer sums for fixed η and γ converge to 0 as $n \rightarrow \infty$, uniformly in m and $g(s^\gamma - s^\eta) \rightarrow 0$ as $\gamma, \eta \rightarrow \infty$, we have $f(|\xi^\gamma - \xi^\eta|) \rightarrow 0$ as $\gamma, \eta \rightarrow \infty$. By the continuity of f , (ξ^γ) is Cauchy in $\mathbb{C}$, and so its limit ξ exists. Using the assumption that $C = \sup_n \sum_{k=p_n+1}^{q_n} t_{n,k} < \infty$ and the estimate, for all m ,

$$\sum_{k=p_n+1}^{q_n} t_{n,k} f(|s - \xi|) \leq g(s - s^\gamma) + \sum_{k=p_n+1}^{q_n} t_{n,k} f(|s_{\sigma^k(m)}^\gamma - \xi^\gamma|) + Cf(|\xi^\gamma - \xi|),$$

it is clear that

$$\lim_{n \rightarrow \infty} \sum_{k=p_n+1}^{q_n} t_{n,k} f(|s_{\sigma^k(m)} - \xi|) = 0,$$

uniformly in m . Thus we get $s \in V^D(T, \sigma, f)$. $\square$

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