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Almost Yamabe solitons on general relativistic spacetimes

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Abstract. Examining the characteristics of almost Yamabe solitons and gradient Yamabe solitons on general relativistic spacetimes is the primary objective of this article.

1. Introduction

Hamilton's work [32–34] on the Ricci flow and Rick Schoen's solution [52] to the Yamabe problem on manifolds of positive conformal Yamabe invariant led to the introduction of the Yamabe flow. The Yamabe flow is an intrinsic geometric flow in differential geometry, meaning that it deforms the metric of a Riemannian manifold. Yamabe flow, first described by Hamilton [32], is the negative L^2-gradient flow of the (normalized) total scalar curvature for non-compact manifolds, limited to a specific conformal class. When this flow converges, it can be understood as deforming a Riemannian metric to a conformal metric of constant scalar curvature.

A Yamabe flow on an n-dimensional semi-Riemannian manifold M is governed by the following equation:

$$\frac{\partial}{\partial t} g(t) = -r g(t), \quad g_0 = g(0),$$

where r is the scalar curvature of the manifold M, t is the time and g the semi-Riemannian metric. Note that the fast diffusion case of the porous media equation (the plasma equation) can be resolved with the help of the Yamabe flow. The Yamabe soliton is widely recognized as a unique solution to the aforementioned partial differential equation. A semi-Riemannian metric g of an n-dimensional semi-Riemannian manifold M satisfying

$$\pounds_V g = (\lambda - r) g, \tag{1}$$

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for the real constant λ on M, is said to be a Yamabe soliton. The operator $\pounds_V$ is termed as the Lie derivative operator in the direction of the soliton vector field V. We symbolize the Yamabe soliton by (g, V, λ) all the way through the manuscript. Remark that (g, V, λ) is expanding, shrinking or steady if λ is negative, positive or zero respectively. If we choose λ as a smooth function in (g, V, λ), then we recover the notion of almost Yamabe soliton on M. Let D denotes the gradient operator and suppose $V = Df$ for some smooth function f on M. Then equation (1) reduces to

$$2Hess f = (\lambda - r)g. \tag{2}$$

Here $Hess f$ is the Hessian of f. Study of Yamabe solitons on classes of semi-Riemannian manifolds (specially on Lorentzian manifolds) plays a crucial role in the differential geometry and mathematical physics. For instance, we refer [4, 11, 12, 15, 21, 23–26, 47, 48, 60].

Consider M to be a semi-Riemannian manifold of n dimensions, equipped with a semi-Riemannian metric g. An n-dimensional semi-Riemannian manifold M furnished with a Lorentzian metric g of signature $(1, n-1)$ or $(n-1, 1)$ is called a Lorentzian manifold M of dimension n. A Lorentzian manifold M of dimension $n \geq 3$ that has $M = -I \times_{\mathfrak{h}^2} \mathcal{H}$, where I is an open interval of $\mathbb{R}$ (set of real numbers), $\mathcal{H}$ is an $(n-1)$-dimensional Riemannian manifold, and $\mathfrak{h}$ is a smooth function known as the warping function or scale factor, is referred to as a generalized Robertson-Walker (GRW) spacetime [1, 2]. It was first proposed by Alías, Romero, and Sánchez [1, 2]. The GRW spacetime is reduced to the Robertson-Walker (RW) spacetime if $\mathcal{H}$ ($dim \mathcal{H} = 3$) has a constant scalar curvature. As a result, GRW spacetime is an obvious extension of RW spacetime. Observably, the GRW spacetimes are inhomogeneous spacetimes admitting an isotropic radiation and have various applications, such as the Friedmann cosmological models, static Einstein spacetimes, Einstein-de Sitter spacetimes, Lorentz–Minkowski spacetime and de Sitter spacetimes.
Vector fields are essential for characterizing the spacetimes. For instance, Chen [19] has characterized the Lorentzian manifolds endowed with a timelike concircular vector field and established the following:

Theorem A. [19] A Lorentzian manifold M, $dim M \geq 3$, with a timelike concircular vector field is a generalized Robertson-Walker spacetime and vice versa.

Mantica and Molinari [43] have characterized Lorentzian manifolds equipped with a torse-forming vector field as:

Theorem B. [43] If and only if a Lorentzian manifold of dimension $n \geq 3$ admits a unit time-like torse forming vector field, denoted as $\nabla_k u_j = \alpha(g_{kj} + u_j u_k)$, which is also an eigenvector of the Ricci tensor, then it is a GRW spacetime.

Chaubey, Suh and De [17] have considered a semi-symmetric metric ξ-connection and proved the following:

Theorem C. [17] A Lorentzian manifold of dimension $n \geq 3$ endowed with a semi-symmetric metric ξ-connection is a GRW spacetime.

In this series, the characterization of Lorentzian manifolds with different vector fields are given in [10, 13, 14, 55] and it is proved that the manifolds under consideration are GRW spacetimes. We refer the reader to [6, 27, 54, 56, 61] for more details on GRW spacetimes.
A four-dimensional Lorentzian manifold M^4 is modelled as a general relativistic spacetime. In this manuscript, we characterize the general relativistic spacetimes with almost Yamabe solitons. In this paper, the four-dimensional general relativistic spacetimes allowing a specific unit timelike vector field ξ defined by $\nabla \xi = I + \eta \otimes \xi$ are denoted by (M^4, ξ). For more details about the special vector field ξ, see Section 2.

A perfect fluid spacetime is defined as an n-dimensional Lorentzian manifold M whose non-vanishing Ricci tensor S satisfies the relation

$$S = \alpha g + \beta \eta \otimes \eta \tag{3}$$

for certain smooth functions α and β on M, where $g(X, \xi) = \eta(X)$ for all X on M, and η is a 1-form associated with the unit timelike vector field ξ. This 1-form η is related to the Lorentzian metric g. When $\beta = 0$, perfect fluid spacetime simplifies to Einstein spacetime. The fact that any RW spacetime is a perfect fluid spacetime is widely known, although the opposite is usually untrue. A four-dimensional GRW spacetime is a perfect fluid spacetime if and only if it is a RW spacetime. The four-dimensional perfect fluid spacetime was examined by Shepley and Taub [53] by using the divergence-free Weyl conformal tensor. Since then, Sharma and Ghosh [51], Mantica et al. [5, 38], Chaubey and Suh [13, 14] and Coley [22] have studied it. The necessary and sufficient criteria for the perfect fluid spacetimes to be generalized Robertson-Walker spacetimes were provided by Mantica et al. in [44]. Chaubey investigated the characteristics of a perfect fluid spacetime with Einstein solitons and a gradient η-Ricci soliton in [7], demonstrating that it is a GRW spacetime. In [55], authors have investigated the properties of almost Ricci solitons on perfect fluid spacetimes in general setting. Perfect fluid spacetimes with Yamabe and gradient Yamabe solitons were studied by De et al. in [24]. The detailed study of almost Yamabe solitons within the framework of general relativistic spacetimes admitting a special vector field ξ are lacking. In order to fill this gap, our aim is to explore the properties of almost Yamabe solitons within the framework of general relativistic spacetimes. To achieve our goal, we prove the following theorem if the soliton vector field V of (g, V, λ) coincides with the special vector field ξ.

Theorem 1.1. *Let (M^4, ξ) be a general relativistic spacetime endowed with a special vector field ξ. Then there does not exist an almost Yamabe soliton (g, ξ, λ) on (M^4, ξ).*

Next theorem is established when the soliton vector field V of (g, V, λ) is pointwise collinear with the special vector field ξ.

Theorem 1.2. *Let (M^4, ξ) be a general relativistic spacetime endowed with a special vector field ξ defined by (4). Suppose (M^4, ξ) admits an almost Yamabe soliton (g, V, λ) and the soliton vector field V of (g, V, λ) is pointwise collinear with ξ. Then (M^4, ξ) is a GRW spacetime.*

It is well-known that a compact Riemannian n-manifold is always geodesically complete, but in case of semi-Riemannian manifold it does not hold in general. The condition for which a compact Lorentzian manifold to be geodesically complete is given firstly by Kamishima [37] in 1993. According to him, a compact Lorentzian manifold of constant curvature endowed with a timelike Killing vector field is geodesically complete. Romero and Sánchez [49] generalized the results of Kamishima, and proved that a compact Lorentzian manifold admitting a timelike conformal Killing vector field is geodesically complete. Conditions for which the compact Lorentzian manifolds to be geodesically complete are established in [13, 14, 50] and also by others. In this paper, we give another new condition for which a class of semi-Riemannian manifold to be geodesically complete. The result is as follows:

Theorem 1.3. *Let the general relativistic spacetimes (M^4, ξ) endowed with a special vector field ξ admit an almost Yamabe soliton (g, V, λ). If the soliton vector V of (g, V, λ) is pointwise collinear with ξ, then M^4 is geodesically complete.*

Motivated by [24], we study the properties of almost Yamabe solitons on general relativistic perfect fluid spacetimes, and prove the following:

Theorem 1.4. *Let (M^4, ξ) be a perfect fluid spacetime. If it admits an almost Yamabe soliton (g, V, λ), then the following conditions are equivalent:*

- *(i) r is constant,*
- *(ii) r is harmonic,*
- *(iii) (M^4, ξ) is an Einstein spacetime,*
- *(iv) soliton vector field V of (g, V, λ) is Killing.*

Remark that the ways of approach used in this manuscript and [24] to explore the properties of almost Yamabe solitons on general relativistic perfect fluid spacetimes are different.

In [24], authors investigated the properties of gradient Yamabe soliton on perfect fluid spacetimes. The question of existence of gradient Yamabe soliton on general relativistic spacetimes are prime and natural. We address this question in the following:

Theorem 1.5. *Let a general relativistic spacetime* (M^4, ξ) *endowed with a special vector field* ξ *admits a gradient Yamabe soliton. Then the gradient Yamabe soliton on* (M^4, ξ) *is trivial.*

2. General relativistic spacetimes

We symbolize (M^4, ξ) to denote the four-dimensional general relativistic spacetimes admitting a special unit timelike vector field ξ, that is, ξ satisfies

$$\nabla_X \xi = X + \eta(X)\xi, \tag{4}$$

for any vector field X on M^4, which straight forward shows that the curvature tensor R of M^4 holds the identity

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{5}$$

which gives

$$\eta(R(X, Y)Z) = \eta(X)g(Y, Z) - \eta(Y)g(X, Z) \tag{6}$$

and

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X \tag{7}$$

for all vector fields X, Y and Z on M^4. Here ∇ is symbolized as the Levi-Civita connection of g. The properties of the vector field ξ defined in equation (4) have been explored in [8, 9, 16, 18, 35, 36] and the references therein. Consider an orthonormal frame field on (M^4, ξ), and then contracting (5) over X, we find

$$S(Y, \xi) = 3\eta(Y) \Longleftrightarrow Q\xi = 3\xi, \tag{8}$$

where Q denotes the Ricci operator defined by $g(QX, Y) = S(X, Y)$ for the Ricci tensor S on (M^4, ξ). Equation (8) shows that the Ricci tensor S of (M^4, ξ) has eigenvalue 3 corresponding to the eigenvector ξ.

Next, we assume that the general relativistic spacetime M^4 is perfect fluid, that is, equation (3) holds on M^4. In consequence of (3) and (8), we notice that

$$\alpha = \frac{r}{3} - 1, \text{ and } \beta = \frac{r}{3} - 4.$$

Thus, equation (3) assumes the form

$$QX = \left(\frac{r}{3} - 1\right)X + \left(\frac{r}{3} - 4\right)\eta(X)\xi. \tag{9}$$

The covariant derivative of (9) gives

$$(\nabla_X Q)(Y) = \frac{X(r)}{3}(Y + \eta(Y)\xi) + \left(\frac{r}{3} - 4\right)[g(X, Y)\xi + \eta(Y)X + \eta(Y)\eta(X)\xi],$$

where $X(r) = g(X, Dr)$ for all X on M^4 and D is a gradient operator of g. Considering an orthonormal frame field on M^4 and contracting the above equation over X, we find

$$Y(r) = 2(r - 12)\eta(Y) \Longleftrightarrow Dr = 2(r - 12)\xi. \tag{10}$$

Taking covariant derivative of the above equation along the vector field X, we find

$$\nabla_X Dr = 2X(r)\xi + 2(r-12)\nabla_X \xi,$$

which, with the help of equation (4), gives

$$\triangle r = 2(r-12). \tag{11}$$

Here $\triangle$ is symbolized for Laplacian operator. An (M^4, ξ) is said to satisfy the Poisson equation if $\triangle \mathfrak{f} = \mathfrak{S}$ for some smooth functions $\mathfrak{f}$ and $\mathfrak{S}$ on (M^4, ξ). Particularly, if $\mathfrak{S} = 0$ then we recover the Laplace equation $\triangle \mathfrak{f} = 0$. This equation reveals the following:

Lemma 2.1. *Let (M^4, ξ) be a perfect fluid spacetime. Then the scalar curvature r of (M^4, ξ) satisfies the Poisson equation (11). Also, r satisfies the Laplace equation if and only if $r = 12$.*

Well known that if the general relativistic spacetime M possesses the constant scalar curvature r, then r is harmonic ($\triangle r = 0$). But the harmonicity of r does not imply that r is constant on M. Now, we state:

Corollary 2.2. *Let (M^4, ξ) be a perfect fluid spacetime. If the scalar curvature r of (M^4, ξ) is harmonic, then r is constant.*

A semi-Riemannian manifold M is said to be semisymmetric if and only if $R(X,Y) \cdot R = 0$ for all X and Y on M, where the linear endomorphism $R(X,Y)$ acts on R as a deviation [57, 58].
We have

$$\begin{aligned}(R(X,Y)\cdot R)(Z,U)W &= R(X,Y)R(Z,U)W - R(R(X,Y)Z,U)W \\ &-R(Z,R(X,Y)U)W - R(Z,U)R(X,Y)W, \ \forall\, X,Y,Z,U,W \in \mathfrak{X}(M).\end{aligned} \tag{12}$$

If possible, we suppose that (M^4, ξ) is semisymmetric. Equation (12) reduces to

$$R(X,Y)R(Z,U)W - R(R(X,Y)Z,U)W - R(Z,R(X,Y)U)W - R(Z,U)R(X,Y)W = 0.$$

Setting $X = \xi$ in the above equation and taking the inner product of foregoing equation with ξ gives

$$g(R(Z,U)W,Y) = g(Y,Z)g(U,W) - g(Y,U)g(Z,W) \Longleftrightarrow R(Z,U)W = g(U,W)Z - g(Z,W)U, \tag{13}$$

since equation (6) is used. Equation (13) infers that the spacetime (M^4, ξ) is a Robertson-Walker spacetime [20, 46]. Conversely, if (M^4, ξ) satisfies equation (13) then $R(X,Y) \cdot R = 0$. Thus, we state the following:

Theorem 2.3. *An (M^4, ξ) is a Robertson-Walker spacetime if and only if it is a semisymmetric spacetime.*

Let us suppose that (M^4, ξ) is Ricci semisymmetric ($R(X,Y)\cdot S = 0$). Then from equation (13) and Theorem 2.3, we conclude that (M^4, ξ) is a Ricci semisymmetric spacetime if and only if it is Einstein spacetime. We state our finding as:

Corollary 2.4. *An (M^4, ξ) is Ricci semisymmetric if and only if it is an Einstein spacetime.*

By the straight forward calculations, we notice that the conformal curvature tensor C [59] defined by $C(X,Y)Z = R(X,Y)Z - \frac{1}{2}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY - \frac{r}{3}[g(Y,Z)X - g(X,Z)Y]\}$ on M^4 is conformally flat ($C = 0$). Now, we state:

Corollary 2.5. *Let (M^4, ξ) be a semisymmetric spacetime. Then it is conformally flat and its scalar curvature is constant.*

3. Almost Yamabe solitons

Proof of Theorem 1.1: If a smooth vector field V on a general relativistic spacetime M^4 has conformal transformations in its flow, then it is considered a conformal vector field [64]. In other words, if V satisfies the equation

$$£_V g = 2\rho g \tag{14}$$

for some smooth function ρ (potential function of V), then it is said to be a conformal vector field. Equations (1) and (14) together give $\rho = \frac{\lambda - r}{2}$. Note that the conformal vector field V is Killing provided $\rho = 0$. For $\rho \neq 0$, V is said to be non-trivial.

Now, we discuss the existence of almost Yamabe soliton (g, ξ, λ) on (M^4, ξ). Equation (1) can be written as

$$g(\nabla_X V, Y) + g(X, \nabla_Y V) = (\lambda - r)g(X, Y). \tag{15}$$

Put $V = \xi$ in the above equation, and following equation (4) we find

$$2\{g(X, Y) + \eta(X)\eta(Y)\} = (\lambda - r)g(X, Y). \tag{16}$$

Setting $X = Y = \xi$ in the above equation, we infer that $\lambda = r$. Let us consider an orthonormal frame field on (M^4, ξ). Contraction of (16) over X and Y gives $3 = 2(\lambda - r) = 0$, which is inadmissible. This completes the proof. □

Proof of Theorem 1.2: We suppose that the soliton vector field V of almost Yamabe soliton (g, V, λ) is pointwise collinear with special vector field ξ, that is, $V = \mathfrak{a}\xi$, where $\mathfrak{a}(\neq 0)$ is a smooth function on M^4. Now, $g(V, V) = \mathfrak{a}^2 g(\xi, \xi) < 0$. This infers that V is timelike. The covariant derivative of $V = \mathfrak{a}\xi$ along the vector field X gives

$$\nabla_X V = X(\mathfrak{a})\xi + \mathfrak{a}(X + \eta(X)\xi), \tag{17}$$

which converts the equation (15) as

$$X(\mathfrak{a})\eta(Y) + Y(\mathfrak{a})\eta(X) + 2\mathfrak{a}\{g(X, Y) + \eta(X)\eta(Y)\} = (\lambda - r)g(X, Y). \tag{18}$$

Setting $Y = \xi$ in the above equation, we lead to

$$X(\mathfrak{a}) = [\xi(\mathfrak{a}) - (\lambda - r)]\eta(X) \Longrightarrow \xi(\mathfrak{a}) = \frac{\lambda - r}{2}. \tag{19}$$

In view of (19), equation (18) reduces to

$$[(\lambda - r) - 2\mathfrak{a}]\{g(X, Y) + \eta(X)\eta(Y)\} = 0,$$

which gives

$$2\mathfrak{a} = \lambda - r. \tag{20}$$

Using equations (19) and (20) in (17), we find

$$\nabla_X V = \mathfrak{a}X, \tag{21}$$

which shows that V is a timelike concircular vector field. Thus, equation (21) together with Theorem A complete the proof. □

Proof of Theorem 1.3: From (21) we have

$$(£_V g)(X, Y) = g(\nabla_X V, Y) + g(X, \nabla_Y V) = 2\mathfrak{a}g(X, Y).$$

This shows that V is a timelike conformal Killing vector field. This equation together with the results of Romero and Sánchez [49] complete the proof of the Theorem 1.3. □

Contracting equation (15) for X and Y, we find

$$divV = 2(\lambda - r). \tag{22}$$

This shows that the soliton vector field V of almost Yamabe soliton (g, V, λ) is conservative if and only if $\lambda = r$. Now, using (22) in (1) we lead

$$\pounds_V g = \frac{divV}{2} g.$$

Since a conservative vector field is always irrotational. Thus we state:

Proposition 3.1. *The soliton vector field V of an almost Yamabe soliton (g, V, λ) on (M^4, ξ) is irrotational if and only if it is Killing.*

Proof of Theorem 1.4: It is well-known that a conformal vector field V on a spacetime M satisfies the following identities:

$$(\pounds_V S)(X, Y) = -(n-2)g(\nabla_X D\rho, Y) + (\triangle\rho)g(X, Y), \tag{23}$$

$$\pounds_V r = -2\rho r + 2(n-1)\triangle\rho, \tag{24}$$

where $\triangle := -divD$, the Laplacian operator of g, $n = dim\, M$ and D is the gradient operator [62].
Let the perfect fluid spacetime (M^4, ξ) admit an almost Yamabe soliton (g, V, λ). Suppose that ξ is a unit timelike vector field of (M^4, ξ). Then the Lie derivative of $g(\xi, \xi) = \eta(\xi) = -1$ along the soliton vector field V gives

$$g(\pounds_V \xi, \xi) = \rho = \frac{\lambda - r}{2}, \tag{25}$$

and

$$(\pounds_V \eta)(\xi) = -\eta(\pounds_V \xi) = -\frac{\lambda - r}{2}, \tag{26}$$

since equation (14) is used.
We have

$$\rho = \frac{\lambda - r}{2} \implies \triangle\rho = -\frac{\triangle r}{2} = -(r-12), \quad D\rho = -\frac{Dr}{2} = -(r-12)\xi, \tag{27}$$

where equation (11) is used. From equations (4) and (27), we get

$$\nabla_X D\rho = -X(r)\xi - (r-12)(X + \eta(X)\xi). \tag{28}$$

In view of (27) and (28), equations (23) and (24) respectively assume the following form.

$$(\pounds_V S)(X, Y) = 2X(r)\eta(Y) + (r-12)\{g(X, Y) + 2\eta(X)\eta(Y)\} \tag{29}$$

and

$$\pounds_V r = r(r - \lambda) - 6(r - 12).$$

From (9) we have

$$S(X, Y) = \left(\frac{r}{3} - 1\right) g(X, Y) + \left(\frac{r}{3} - 4\right) \eta(X)\eta(Y). \tag{30}$$

The Lie derivative of the above equation gives

$$\begin{aligned}(\pounds_V S)(X,Y) &= \frac{\pounds_V r}{3}\left(g(X,Y)+\eta(X)\eta(Y)\right)+(\lambda-r)\left(\frac{r}{3}-1\right)g(X,Y)\\ &\quad+\left(\frac{r}{3}-4\right)\{(\pounds_V\eta)(X)\eta(Y)+\eta(X)(\pounds_V\eta)(Y)\}.\end{aligned}\tag{31}$$

In consequence of equations (29) and (31), we find

$$\begin{aligned}&2X(r)\eta(Y)+(r-12)\{g(X,Y)+2\eta(X)\eta(Y)\}\\ &=\frac{\pounds_V r}{3}\left(g(X,Y)+\eta(X)\eta(Y)\right)+(\lambda-r)\left(\frac{r}{3}-1\right)g(X,Y)\\ &\quad+\left(\frac{r}{3}-4\right)\{(\pounds_V\eta)(X)\eta(Y)+\eta(X)(\pounds_V\eta)(Y)\}.\end{aligned}\tag{32}$$

Take $Y=\xi$ in (32), we obtain

$$\left(\frac{r}{3}-4\right)(\pounds_V\eta)(X)=3(r-12)\eta(X)+(\lambda-r)\left(\frac{r}{6}+1\right)\eta(X),\tag{33}$$

where equations (10) and (26) are used. Substituting ξ in liu of X in equation (33) and following equation (26), we lead to

$$\lambda=12>0.\tag{34}$$

This shows that an almost Yamabe soliton on perfect fluid spacetimes to be a shrinking Yamabe soliton.

Now, using (34) in (33) we get

$$(r-12)\left[(\pounds_V\eta)(X)+\left(\frac{r}{2}-6\right)\eta(X)\right]=0.\tag{35}$$

Now we divide our study in two parts.

Case I. If possible we suppose that $r=12$. Then from equations (9), (27) and (29) we have the following equivalent conditions.

- $r=12$,
- $\triangle r=0$ (r is harmonic),
- $S=3g$ (Einstein spacetime),
- $\pounds_V g=0$ (V is Killing),
- $\pounds_V S=0$ (V is a Ricci inheritance vector field).

Case II. Consider that $r\neq 12$. Then from equation (35) we get

$$(\pounds_V\eta)(X)+\left(\frac{r}{2}-6\right)\eta(X)=0.\tag{36}$$

From (1) we have

$$\nabla_X\pounds_V g=-X(r)g.$$

Well-known that [62]

$$\begin{aligned}&(\pounds_V\nabla_X g-\nabla_X\pounds_V g-\nabla_{[V,X]}g)(Y,Z)\\ &\qquad=-g((\pounds_V\nabla)(X,Y),Z)-g((\pounds_V\nabla)(X,Z),Y)\\ &\iff(\nabla_X\pounds_V g)(Y,Z)=g((\pounds_V\nabla)(X,Y),Z)+g((\pounds_V\nabla)(Z,X),Y).\end{aligned}$$

Since the operator $\pounds_V\nabla$ is symmetric, therefore

$$(\pounds_V\nabla)(X,Y) = \frac{1}{2}g(X,Y)Dr - \frac{1}{2}X(r)Y - \frac{1}{2}Y(r)X,$$

which gives

$$(\nabla_Z\pounds_V\nabla)(X,Y) = \frac{1}{2}g(X,Y)\nabla_Z Dr - \frac{1}{2}g(X,\nabla_Z Dr)Y - \frac{1}{2}g(Y,\nabla_Z Dr)X.$$

This equation together with the curvature identity

$$(\pounds_V R)(X,Y)Z = (\nabla_X\pounds_V\nabla)(Y,Z) - (\nabla_Y\pounds_V\nabla)(X,Z)$$

assumes the form

$$\begin{aligned}(\pounds_V R)(X,Y)Z \quad &= \quad \frac{1}{2}g(Y,Z)\nabla_X Dr - \frac{1}{2}g(X,Z)\nabla_Y Dr \\ &+\frac{1}{2}\{g(X,\nabla_Y Dr) - g(Y,\nabla_X Dr)\}Z \\ &-\frac{1}{2}\{g(Z,\nabla_X Dr) - g(Z,\nabla_Y Dr)\}X,\end{aligned}$$

which, after contraction over X, gives

$$(\pounds_V S)(Y,Z) = \frac{1}{2}g(Y,Z)\triangle r + \frac{3}{2}g(\nabla_Y Dr, Z) - \frac{1}{2}g(Y,\nabla_Z Dr).$$

Covariant derivative of (10) provides

$$\nabla_X Dr = 6(r-12)\eta(X)\xi + 2(r-12)X \Longrightarrow \triangle r = 2(r-12),$$

where equations (4) and (10) are used. The last two equations infer that

$$(\pounds_V S)(Y,Z) = 3(r-12)[g(Y,Z) + 2\eta(Y)\eta(Z)].$$

Setting $Z = \xi$ in the above equation, we lead to

$$(\pounds_V S)(Y,\xi) = -3(r-12)\eta(Y). \tag{37}$$

The Lie derivative of equation (7) along the soliton vector field V gives

$$(\pounds_V S)(Y,\xi) + S(Y,\pounds_V\xi) = 3(\pounds_V\eta)(Y). \tag{38}$$

In consequence of equations (36) and (37), equation (38) assumes the form

$$S(Y,\pounds_V\xi) = \frac{3}{2}(r-12)\eta(Y). \tag{39}$$

From equations (26) and (30), we find

$$S(Y,\pounds_V\xi) = \left(\frac{r}{3}-1\right)g(Y,\pounds_V\xi) + \left(\frac{r}{3}-4\right)\left(\frac{12-r}{2}\right)\eta(Y). \tag{40}$$

From equations (39) and (40), we conclude that

$$\left(\frac{r}{3}-1\right)\pounds_V\xi = (r-12)\left(\frac{r}{3}-\frac{5}{2}\right)\xi,$$

which gives

$$\left(\frac{r}{3}-1\right)g(\pounds_V\xi,\xi) = -(r-12)\left(\frac{r}{3}-\frac{5}{2}\right).$$

This equation together with (25) and (34) infer that $2 = 5$, which contradicts our hypothesis $r \neq 12$. Thus, the only possibility is $r = 12$. Thus the proof of the Theorem 1.4 is completed. □

In consequence of equation (34), we state the following:

Proposition 3.2. *An almost Yamabe soliton on a perfect fluid spacetime (M^4, ξ) is a shrinking Yamabe soliton.*

Let the perfect fluid spacetime (M^4, ξ) admit an almost Yamabe soliton (g, V, λ). Then $S = 3g \implies \nabla_X S = 0$. That is, (M^4, ξ) is Ricci symmetric. If the divergence of the conformal curvature tensor C is denoted by $divC$, then

$$(divC)(X, Y)Z = \frac{1}{2}\{(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)\} + \frac{1}{6}\{dr(X)g(Y, Z) - dr(Y)g(X, Z)\}, \tag{41}$$

where

$$\begin{aligned} C(X, Y)Z &= R(X, Y)Z + \frac{r}{6}\{g(Y, Z)X - g(X, Z)Y\} \\ &\quad -\frac{1}{2}\{S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY\}. \end{aligned} \tag{42}$$

Equation (41) together with Theorem 1.4 infers that the conformal curvature tensor is divergence free, that is, $divC = 0$.
In [38], Mantica et al. proved that the n-dimensional perfect fluid spacetimes with divergence free conformal curvature tensor and constant scalar curvature are GRW spacetimes.
Combining the above facts together, and we can state our conclusion in the following:

Corollary 3.3. *Let (M^4, ξ) be a perfect fluid spacetime. Then (M^4, ξ) endowed with an almost Yamabe soliton is a GRW spacetime.*

Equation (42) together with Theorem 1.4 takes the form

$$C(X, Y)Z = R(X, Y)Z - g(Y, Z)X + g(X, Z)Y, \tag{43}$$

which reduces to

$$R \cdot C = R \cdot R.$$

In [31], Eriksson et al. classified the semisymmetric spacetimes and established some remarkable results. They have investigated that the semisymmetric spacetimes are of Petrov types **D**, **N**, or **O**.
A spacetime is said to be a conformally semisymmetric if and only if $R \cdot C = 0$.
These facts together with the above equation state the following:

Corollary 3.4. *Let (M^4, ξ) be a conformally semisymmetric perfect fluid spacetime. Then (M^4, ξ) endowed with an almost Yamabe soliton is of Petrov types* **D**, **N**, *or* **O**.

Let (M^4, ξ) be a perfect fluid spacetime and satisfies the following Einstein's field equations with cosmological constant Ω:

$$S - \frac{1}{2}rg + \Omega g = \kappa T, \tag{44}$$

where κ is the gravitational constant and T is the energy-momentum tensor, and it is given by

$$T = pg + (p + \sigma)\eta \otimes \eta \tag{45}$$

for isotropic pressure p and energy density σ. Let (M^4, ξ) admit an almost Yamabe soliton (g, V, λ). Then from Theorem 1.4, we observe that $S = 3g$. This equation together with (44) and (45) infers that

$$\kappa p = \Omega - 3, \quad \kappa\sigma = 3 - \Omega, \text{ and } p + \sigma = 0. \tag{46}$$

Well-known that a matter with $p + \sigma = 0$ represents the phantom barrier [45] and in cosmology, it gives the impetuous expansion of the spacetime, that is, inflation [3]. Thus, we state our result as:

Corollary 3.5. *Let a four-dimensional perfect fluid spacetime (M^4, ξ) satisfies the Einstein's field equations with cosmological constant Ω. If (M^4, ξ) admits an almost Yamabe soliton (g, V, λ), then the equation of state represents the phantom barrier.*

Also, equation (44) and Theorem 1.4 lead to

$$\kappa T = (\Omega - 3)g, \tag{47}$$

which reduces to

$$\kappa \pounds_V T = (\Omega - 3)\pounds_V g = 0. \tag{48}$$

A spacetime admits a matter collineation along the vector field X if it satisfies the relation $\pounds_X T = 0$.
This definition together with (48) state:

Corollary 3.6. *Let (M^4, ξ) be a perfect fluid spacetime obeying the Einstein's field equations with cosmological constant Ω. Then (M^4, ξ) endowed with an almost Yamabe soliton (g, V, λ) possesses the matter collineation along the soliton vector field V.*

Again, from (47) we notice that $\nabla T = 0$, since $\kappa \neq 0$. Thus we state:

Corollary 3.7. *Let a perfect fluid spacetime (M^4, ξ) admits an almost Yamabe soliton. If (M^4, ξ) obeys the Einstein's field equations with cosmological constant Ω, then the energy-momentum tensor is symmetric.*

Let (M^4, ξ) admit an almost Yamabe soliton. Then the Lie derivative of (42) along the soliton vector field V gives

$$\pounds_V R = \pounds_V C.$$

A spacetime M is said to be curvature collineation [30] (resp., conformal collineation [63]) in the direction of $\aleph$ if $\pounds_\aleph R = 0$ (resp., $\pounds_\aleph C = 0$). For more details, see [29] and the references therein.
These definitions together with the above result state the following:

Corollary 3.8. *Let a perfect fluid spacetime (M^4, ξ) admits an almost Yamabe soliton. Then (M^4, ξ) admits a curvature collineation if and only if it has conformal collineation.*

4. Gradient Yamabe solitons on general relativistic spacetimes

Proof of Theorem 1.5: Let the general relativistic spacetimes (M^4, ξ) admit a gradient Yamabe soliton. Then equation (2) can be rewritten as:

$$\nabla_X Df = \frac{1}{2}(\lambda - r)X. \tag{49}$$

Differentiating (49) covariantly along the vector field Y, we obtain

$$\nabla_Y \nabla_X Df = -\frac{Y(r)}{2}X + \frac{1}{2}(\lambda - r)\nabla_Y X. \tag{50}$$

Interchanging X and Y in the above equation and then using the foregoing equation, equations (49) and (50) in the curvature identity

$$R(X, Y)Df = \nabla_X \nabla_Y Df - \nabla_Y \nabla_X Df - \nabla_{[X,Y]} Df,$$

we find

$$R(X, Y)Df = \frac{1}{2}\{Y(r)X - X(r)Y\}, \tag{51}$$

which can be written as:

$$g(R(X, Y)Df, \xi) = \frac{1}{2}\{Y(r)\eta(X) - X(r)\eta(Y)\}. \tag{52}$$

Now, we write equation (5) as

$$g(R(X,Y)\xi, Df) = \eta(Y)X(f) - \eta(X)Y(f). \tag{53}$$

Equations (52) and (53) assume the form

$$Y(r)\eta(X) - X(r)\eta(Y) = 2\{\eta(X)Y(f) - \eta(Y)X(f)\}.$$

Setting $Y = \xi$ in the above equation, we have

$$X(r) + \xi(r)\eta(X) = 2[X(f) + \xi(f)\eta(X)]. \tag{54}$$

Contracting equation (51) over X, we get

$$\begin{aligned} S(Y, Df) &= \frac{3}{2}Y(r) \\ \implies \frac{3}{2}\xi(r) &= S(Df, \xi) = 3\eta(Df) \\ &= 3\eta(Df) = 3g(Df, \xi) \\ &= 3\xi(f) \\ \implies \xi(r) &= 2\xi(f), \end{aligned} \tag{55}$$

where equation (8) is used. Using (55) in (54), we conclude that

$$X(r) = 2X(f) \iff Dr = 2Df,$$

which gives

$$\nabla_X Dr = 2\nabla_X Df = (\lambda - r)X, \tag{56}$$

since equation (49) is used. Contracting the above equation over X we get

$$\triangle r = 4(\lambda - r).$$

This ensures that the scalar curvature of (M^4, ξ) is harmonic if and only if $\lambda = r$. From equation (8), we have

$$(\nabla_X Q)(\xi) + QX = 3X,$$

where equation (4) has been used. Contracting this equation over X, we get

$$\xi(r) = -2(r - 12). \tag{57}$$

Equations (55) and (57) give

$$g(\xi, Df) = -(r - 12). \tag{58}$$

Taking the covariant derivative of this equation and using equations (4), (55), (56) and (57), we lead

$$X(f) - (r - 12)\eta(X) + \frac{1}{2}(\lambda - r)\eta(X) = -X(r).$$

Take $X = \xi$ in the above equation, and then the foregoing equation with equations (57) and (58) gives

$$\lambda = -3(r - 16),$$

which shows that r is constant, and hence from (57) we note that $r = 12$. Also, the potential function f of the gradient Yamabe soliton is constant. Thus, the proof of the Theorem 1.5 is completed. □

Let (M^4, ξ) admit a gradient Yamabe soliton. Then from (9) and (42) we obtain $S = 3g$ and equation (43). A non-vanishing symmetric tensor $\mathfrak{H}$ on a semi-Riemannian manifold M is said to be Riemann compatible [39] if satisfies

$$\mathfrak{H}(R(X,Y)Z,U) + \mathfrak{H}(R(Y,Z)X,U) + \mathfrak{H}(R(Z,X)Y,U) = 0 \tag{59}$$

for all vector fields X, Y, Z and U on M. If we replace the Riemann curvature R with the conformal curvature tensor C in (59), then we recover the expression of conformal compatible tensor. Some deep results of Riemann compatible and conformal compatible tensors were studied in ([28, 40–42]).
Now equations (43) and (59) give

$$\begin{aligned}
&S(C(X,Y)Z,U) + S(C(Y,Z)X,U) + S(C(Z,X)Y,U)\\
&= S(R(X,Y)Z,U) + S(R(Y,Z)X,U) + S(R(Z,X)Y,U)\\
&= 0,
\end{aligned}$$

since the Bianchi's first identity and relation $S = 3g$ are used. The last equation state the following corollary.

Corollary 4.1. *Let (M^4, ξ) admit a gradient Yamabe soliton (or Yamabe soliton). Then the Ricci tensor of M^4 is*
(*i*) *Riemann compatible,*
(*ii*) *conformal compatible.*

Remark 4.2. *Let (M^4, ξ) admitting a gradient Yamabe soliton (or Yamabe soliton) obey the Einstein's field equations (44). Then we can find that the energy momentum tensor of the general relativistic spacetime is Riemannian as well as conformal compatible.*

Let (M^4, ξ) admit a gradient Yamabe soliton. Then from Theorem 1.5, we noticed that equation (43) is satisfied. Taking $Z = \xi$ in (43) and then following equation (5), we obtain $C(X,Y)\xi = 0$.
A spacetime is said to be a ξ-conformally flat spacetime if and only if $C(X,Y)\xi = 0$. Thus we conclude our result as:

Corollary 4.3. *An (M^4, ξ) admitting a gradient Yamabe soliton is ξ-conformally flat.*

Let (M^4, ξ) be a general relativistic spacetime endowed with a special vector field ξ. If (M^4, ξ) satisfies the Einstein's field equations with cosmological constant Ω, then it holds equation (44). This fact together with Theorem 1.5 infer that

$$T(X,Y) = (\Omega - 3)g(X,Y). \tag{60}$$

A Lorentzian manifold M obeying the Einstein's field equations is a vacuum spacetime if and only if $T = 0$. This fact together with equation (60) state the following:

Corollary 4.4. *Let (M^4, ξ) admits a gradient Yamabe soliton. If (M^4, ξ) obeys the Einstein's field equations with cosmological constant Ω, then it is a vacuum spacetime if and only if $\Omega = 3$.*

Data Availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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