Filomat 39:21 (2025), 7287–7305
https://doi.org/10.2298/FIL2521287A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The L-fuzzy bi-ideal degrees and its induced convex structure

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Abstract. In this paper, considering L being a completely distributive lattice, we propose a degree approach to L-fuzzy bi-ideals in an ordered semigroup. Firstly, we introduce the concept of L-fuzzy bi-ideal degree with respect to an ordered semigroup, which can be used to describe the degree to which an L-fuzzy subset of the ordered semigroup becomes an L-fuzzy bi-ideal. Secondly, we characterize L-fuzzy bi-ideal degree by cut sets. Finally, we provide a natural way to construct an L-fuzzy convex structure on an ordered semigroup via the L-fuzzy bi-ideal degree, and show that the homomorphism between two ordered semigroups is an L-fuzzy convexity-preserving mapping and the monohomomorphism is an L-fuzzy convex-to-convex mapping.

1. Introduction

As a branch of order algebra, ordered semigroups play a very important role. For example, ordered semigroups are closely related to theoretical computer science, especially formal language and automata theory. So far, ordered semigroups have been studied from different aspects, including regular congruence theory of ordered semigroups [6, 32], decomposition of ordered semigroups [1, 18, 20], residual theory of ordered semigroups [17, 22] and ideals and filters of ordered semigroups [7, 8, 12–15, 23, 31], etc. For the development of ordered semigroups, ideal is a good tool to study the algebraic structure of ordered semigroups. With the development of fuzzy mathematics, fuzzy sets in ordered semigroups/ordered groupoids were first introduced by Kehayopulu and Tsingelis [13]. They also proposed fuzzy bi-ideals with the unit interval $[0, 1]$ as the truth value table and showed their important roles in ordered semigroups [14].

Convexity exits in many mathematical enviroments, such as vector spaces, metric spaces, lattices and so on [27]. Rosa [21] first proposed the concept of fuzzy convex structures, now known as L-convex structures. Later, Shi and Xiu proposed the notion of M-fuzzifying convex structures and further introduced the definition of (L, M)-fuzzy convex structures [25, 26], providing a more general framework for fuzzy convex structures [19, 34]. Note that L and M are usually required to be completely distributive lattices. In order to describe the degree to which subsystems of fuzzy algebraic systems maintain the properties of fuzzy

2020 *Mathematics Subject Classification*. Primary 06F05; Secondary 08A72, 52A01.
Keywords. Ordered semigroup, L-fuzzy bi-ideal, L-fuzzy convex structures, Cut sets
Received: 19 January 2025; Revised: 25 March 2025; Accepted: 31 March 2025
Communicated by Ljubiša D. R. Kočinac
Research supported by the Natural Science Foundation of China (Nos. 12471428, 12271036, 12071033).
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algebraic systems, Shi and Xin [24] proposed the concept of L-fuzzy subgroup degree, and used it to describe the degree to which an L-fuzzy subset in a group is an L-fuzzy subgroup. This approach has also been applied to other mathematical frameworks [2–4, 9, 16, 30]. Recently, Wang and Xu [29] have applied the idea of degrees to vector space and explored the relationship between L-fuzzy vector subspace degrees and L-fuzzy convex structures.

Through the above analysis, we first extend the concept of fuzzy bi-ideals on ordered semigroups to completely distributive lattices, and then consider whether all L-fuzzy bi-ideals on ordered semigroups can form an L-convex structure. Next we will introduce a degree approach to fuzzy bi-ideals in ordered semigroup and establish its relations with L-fuzzy convex structures.

The paper is organized as follows. In Section 2, we will give some necessary notations and definitions. In Section 3, we propose the concept of L-fuzzy bi-ideal degree with respect to an ordered semigroup, which can be used to describe the degree to which an L-fuzzy subset is an L-fuzzy bi-ideal. Moreover, we provide some characterizations of the L-fuzzy bi-ideal degree using four kinds of cut sets of L-fuzzy subsets. In Section 4, we use the L-fuzzy bi-ideal degree to construct an L-fuzzy convex structure on the ordered semigroup, and study the corresponding L-fuzzy convexity-preserving mappings and L-fuzzy convex-to-convex mappings.

2. Preliminaries

In this section, we provide some concepts and notations of L-fuzzy convex structures and ordered semigroups that will be used in this paper.

2.1. L-fuzzy convex structures

Let L be a complete lattice with the largest element $\top$ and the smallest element $\perp$. An element λ in a complete lattice L is said to be a prime element if $\mu \wedge \theta \leq \lambda$ implies $\mu \leq \lambda$ or $\theta \leq \lambda$. An element λ is said to be co-prime if $\lambda \leq \mu \vee \theta$ implies $\lambda \leq \mu$ or $\lambda \leq \theta$. The set of non-largest prime elements in L is denoted by $P(L)$. The set of non-smallest co-prime elements in L is denoted by $J(L)$.

The binary relation $\prec$ in a complete lattice L is defined as follows: for $\lambda, \mu \in L$, $\lambda \prec \mu$ if and only if for any subset $A \subseteq L$, $\mu \leq \bigvee A$ implies $\lambda \leq \theta$ for some $\theta \in A$. The set $\left\{ \lambda \,|\, \lambda \prec \mu \right\}$ is said to be the greatest minimal family of μ, denoted by $\beta(\mu)$ [28]. Dually, for $\lambda, \mu \in L$, $\mu \prec^{op} \lambda$ if and only if for any subset $A \subseteq L$, $\bigwedge A \leq \mu$ implies $\theta \leq \lambda$ for some $\theta \in A$. The set $\left\{ \lambda \,|\, \mu \prec^{op} \lambda \right\}$ is said to be the greatest maximal family of μ, denoted by $\alpha(\mu)$. A complete lattice L is a completely distributive lattice if and only if $\mu = \vee\beta(\mu) = \wedge\alpha(\mu)$ for all $\mu \in L$ [28].

In a completely distributive lattice L, for each $b \in L$, let $\beta^*(b) = \beta(b) \cap J(L)$ and $\alpha^*(b) = \alpha(b) \cap P(L)$. We know that α is an $\bigwedge - \bigcup$ mapping, i.e., $\alpha\left(\bigwedge_{i\in\Omega} a_i\right) = \bigcup_{i\in\Omega} \alpha\left(a_i\right)$ for all $\{a_i\}_{i\in I} \subseteq L, \beta$ is a union-preserving mapping, i.e., $\beta\left(\bigvee_{i\in\Omega} a_i\right) = \bigcup_{i\in\Omega} \beta\left(a_i\right)$ for all $\{a_i\}_{i\in I} \subseteq L$ and $b = \bigvee \beta(b) = \bigvee \beta^*(b) = \bigwedge \alpha(b) = \bigwedge \alpha^*(b)$.

In this paper, if not otherwise specified, we always assume that L is a completely distributive lattice. There exists an implication operation $\rightarrow: L \times L \longrightarrow L$ as the right adjoint for the meet operator $\wedge$, which is defined by

$$\lambda \rightarrow \mu = \bigvee \left\{ \theta \in L \,|\, \lambda \wedge \theta \leq \mu \right\},$$

for all $\lambda, \mu \in L$.

Lemma 2.1. ([10]) *Let L be a completely distributive lattice and the operation $\rightarrow$ be the implication operator corresponding to $\wedge$. For any $\lambda, \mu, \theta \in L$ and $\left\{\lambda_i\right\}_{i\in I} \subseteq L$, the following statements hold:*

(1) $\top \rightarrow \lambda = \lambda$;

(2) $\lambda \leq \theta \rightarrow \mu \Longleftrightarrow \lambda \wedge \theta \leq \mu$;

(3) $\lambda \rightarrow \mu = \top \Longleftrightarrow \lambda \leq \mu$;

(4) $\lambda \to \Big(\bigwedge_{i\in I} \lambda_i\Big) = \bigwedge_{i\in I} \Big(\lambda \to \lambda_i\Big)$, *hence* $\lambda \to \mu \le \lambda \to \theta$ *whenever* $\mu \le \theta$;

(5) $\Big(\bigvee_{i\in I} \lambda_i\Big) \to \mu = \bigwedge_{i\in I} \Big(\lambda_i \to \mu\Big)$, *hence* $\lambda \to \mu \le \theta \to \mu$ *whenever* $\theta \le \lambda$;

(6) $(\lambda \to \mu) \wedge (\mu \to \theta) \le \lambda \to \theta$.

Lemma 2.2. ([16]) *Let L be a completely distributive lattice and $\lambda, \mu \in L$. Then the following statements are equivalent:*

(1) $\lambda \le \mu$;

(2) *for any $\delta \in L$, $\delta \le \lambda$ implies $\delta \le \mu$;*

(3) *for any $\delta \in J(L)$, $\delta \le \lambda$ implies $\delta \le \mu$;*

(4) *for any $\delta \in P(L)$, $\lambda \nleq \delta$ implies $\mu \nleq \delta$;*

(5) *for any $\delta \in \alpha^*(\bot), \delta \notin \alpha^*(\lambda)$ implies $\delta \notin \alpha^*(\mu)$.*

An L-fuzzy subset of a set X is a mapping from X to L, and the family of all L-fuzzy subsets of X will be denoted by L^X, called the L-power set of X. $\top_X$ and $\bot_X$ denote the largest element and the smallest element in L^X, respectively.

Let $f : X \longrightarrow Y$ be a mapping between two nonempty sets. Define $f_L^{\to} : L^X \longrightarrow L^Y$ and $f_L^{\leftarrow} : L^Y \longrightarrow L^X$ by

$$f_L^{\to}\Big(A\Big)(y) = \bigvee_{f(x)=y} A\Big(x\Big) \quad \text{and} \quad f_L^{\leftarrow}\Big(B\Big)(x) = B\Big(f(x)\Big),$$

for all $A \in L^X, B \in L^Y, x \in X$ and $y \in Y$. Then the L-fuzzy subset $f_L^{\to}(A)$ is called the image of A under f, and $f_L^{\leftarrow}(B)$ the preimage of B.

If L is a completely distributive lattice, then we can define

$$A_{[\lambda]} = \Big\{x \in X \,|\, A(x) \ge \lambda\Big\}, \qquad A^{(\lambda)} = \Big\{x \in X \,|\, A(x) \nleq \lambda\Big\},$$
$$A_{(\lambda)} = \Big\{x \in X \mid \lambda \in \beta\Big(A(x)\Big)\Big\}, \qquad A^{[\lambda]} = \Big\{x \in X \,|\, \lambda \notin \alpha\Big(A(x)\Big)\Big\},$$

for all $A \in L^X$ and $\lambda \in L$.

Definition 2.3. ([21]) A subset $\mathfrak{C}$ of L^X is called an L-convex structure on X if it satisfies:

(LC1) $\top_X, \bot_X \in \mathfrak{C}$;

(LC2) If $\Big\{A_i\Big\}_{i\in I} \subseteq \mathfrak{C}$, then $\bigwedge_{i\in I} A_i \in \mathfrak{C}$;

(LC3) If $\Big\{A_i\Big\}_{i\in I} \subseteq \mathfrak{C}$ is directed, then $\bigvee_{i\in I} A_i \in \mathfrak{C}$.

For an L-convex structure $\mathfrak{C}$ on X, the pair $(X, \mathfrak{C})$ is called an L-convex space.

Definition 2.4. ([26]) A mapping $C : L^X \longrightarrow L$ is said to be an L-fuzzy convex structure on X if it satisfies:

(C1) $C(\top_X) = C(\bot_X) = \top$;

(C2) If $\Big\{A_i\Big\}_{i\in I} \subseteq L^X$, then $C\Big(\bigwedge_{i\in I} A_i\Big) \ge \bigwedge_{i\in I} C(A_i)$;

(C3) If $\Big\{A_i\Big\}_{i\in I} \subseteq L^X$ is nonempty and directed, then $C\Big(\bigvee_{i\in I} A_i\Big) \ge \bigwedge_{i\in I} C(A_i)$.

For an L-fuzzy convex structure $\mathcal{C}$ on X, the pair $(X, \mathcal{C})$ is said to be an L-fuzzy convex space.

Definition 2.5. ([26]) Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be two L-fuzzy convex spaces. Then a mapping $f : X \longrightarrow Y$ is called

(1) an L-fuzzy convexity-preserving mapping if $\mathcal{C}_X(f_L^{\leftarrow}(B)) \geq \mathcal{C}_Y(B)$ for all $B \in L^Y$;

(2) an L-fuzzy convex-to-convex mapping if $\mathcal{C}_Y(f_L^{\rightarrow}(A)) \geq \mathcal{C}_X(A)$ for all $A \in L^X$.

Note that Shi and Xiu [26] introduced Definitions 2.4 and 2.5 in the framework of (L, M)-fuzzy convex structures. Herein we consider the special case that $M = L$.

2.2. Ordered semigroups

Definition 2.6. ([5]) An ordered semigroup is a system $(S, \cdot, \leq)$ if it satisfies:

(OS1) $(S, \cdot)$ is a semigroup;

(OS2) $(S, \leq)$ is a poset;

(OS3) $a \leq b \Longrightarrow ax \leq bx$ and $xa \leq xb$ for any $a, b, x \in S$.

In an ordered semigroup S, we usually use xy to represent $x \cdot y$ for any $x, y \in S$. For convenience, the following notations are frequently used:

$$SA = \{sa \mid s \in S, a \in A\} \text{ and } AS = \{as \mid s \in S, a \in A\}$$

for any $A \subseteq S$.

Let S be an ordered semigroup and $\emptyset \neq A \subseteq S$. Then A is called a subsemigroup of S if $A^2 \subseteq A$.

Definition 2.7. ([14]) A subsemigroup A of an ordered semigroup S is called a bi-ideal of S if

(1) $ASA \subseteq A$;

(2) $x \leq y \Longrightarrow x \in A$ for any $x \in S, y \in A$.

Definition 2.8. ([33]) Let $(S, \cdot, \leq_S)$ and $(T, *, \leq_T)$ be two ordered semigroups. A mapping $f : S \longrightarrow T$ is called a homomorphism provided that

(S1) $f(x \cdot y) = f(x) * f(y)$ for any $x, y \in S$;

(S2) $x \leq_s y \Rightarrow f(x) \leq_T f(y)$ for any $x, y \in S$.

3. *L*-fuzzy bi-ideal degrees

In this section, we will propose a degree approach to L-fuzzy bi-ideals in an ordered semigroup. In this approach, we can describe how a mapping from L^S to L becomes an L-fuzzy bi-ideal with respect to an ordered semigroup S in a degree sense. To this end, we first introduce the following definition.

Definition 3.1. A mapping $A : S \longrightarrow L$ is called an L-fuzzy bi-ideal of an ordered semigroup S provided that

(1) $A(y) \leq A(x)$ for any $x, y \in S$ with $x \leq y$;

(2) $A(z) \wedge A(w) \leq A(zw)$ for any $z, w \in S$;

(3) $A(z) \wedge A(n) \leq A(zwn)$ for any $z, w, n \in S$.

Remark 3.2. (1) When $L = \{0, 1\}$, an L-fuzzy bi-ideal will degenerate to a bi-ideal in Definition 2.7.

(2) When L=[0,1], an L-fuzzy bi-ideal will degenerate to a fuzzy bi-ideal in the sense of Kehayopulu and Tsingelis [14].

Using L-fuzzy bi-ideals of an ordered semigroup S, we can construct an L-convex structure on S in the following way.

Proposition 3.3. *Let S be an ordered semigroup and define $\mathfrak{C} \subseteq L^S$ as follows:*

$$\mathfrak{C} = \{A \in L^S \mid A \text{ is an } L\text{-fuzzy bi-ideal of } S\}.$$

Then $\mathfrak{C}$ is an L-convex structure on S.

Proof. (LC1) is obvious, we only need to prove that (LC2) and (LC3).

(LC2) Take $\{A_i\}_{i\in J} \subseteq \mathfrak{C}$, then for any $i \in J$, we have

$$A_i(x) \geq A_i(y), \forall x, y \in S \text{ with } x \leq y,$$

$$A_i(x) \wedge A_i(y) \leq A_i(xy), \forall x, y \in S,$$

$$A_i(z) \wedge A_i(w) \wedge A_i(n) \leq A_i(zwn), \forall z, w, n \in S.$$

This implies

$$\bigwedge_{i\in J} A_i(x) \geq \bigwedge_{i\in J} A_i(y), \forall x, y \in S \text{ with } x \leq y,$$

$$\bigwedge_{i\in J} A_i(x) \wedge \bigwedge_{i\in J} A_i(y) \leq \bigwedge_{i\in J} A_i(xy), \forall x, y \in S,$$

$$\bigwedge_{i\in J} A_i(z) \wedge \bigwedge_{i\in J} A_i(w) \wedge \bigwedge_{i\in J} A_i(n) \leq \bigwedge_{i\in J} A_i(zwn), \forall z, w, n \in S.$$

Therefore, $\bigwedge_{i\in J} A_i \in \mathfrak{C}$.

(LC3) Take $\{A_j\}_{j\in J} \subseteq^{\text{dir}} \mathfrak{C}$, we obtain

$$\begin{aligned}\bigvee_{j\in J} A_j(x) \wedge \bigvee_{j\in J} A_j(y) &= \bigvee_{i\in J} A_i(x) \wedge \bigvee_{j\in J} A_j(y)\\ &= \bigvee_{i\in J} \bigvee_{j\in J} \big(A_i(x) \wedge A_j(y)\big)\\ &\leq \bigvee_{k\in J} \big(A_k(x) \wedge A_k(y)\big)\\ &\leq \bigvee_{k\in J} A_k(xy).\end{aligned}$$

Similarly, we can prove that

$$\bigvee_{j\in J} A_j(x) \geq \bigvee_{j\in J} A_i(y),$$

for all $x, y \in S$ with $x \leq y$. And

$$\bigvee_{j\in J} A_j(z) \wedge \bigvee_{j\in J} A_j(w) \wedge \bigvee_{j\in J} A_j(n) \leq \bigvee_{k\in J} A_k(zwn).$$

for all $z, w, n \in S$. Hence $\bigvee_{j\in J} A_j \in \mathfrak{C}$. Therefore, $\mathfrak{C}$ is an L-convex structure on S. $\square$

By Definition 3.1, we can only examine that an L-subset A of an ordered semigroup S is an L-fuzzy bi-ideal or not. Now, we will propose the concept of L-fuzzy bi-ideal degrees of an ordered semigroup, which can be used to characterize the degree to which an L-subset becomes an L-fuzzy bi-ideal.

Definition 3.4. Let S be an ordered semigroup and A be an L-fuzzy subset of S. Define a mapping $\mathfrak{B} : L^S \longrightarrow L$ as follows:

$$\mathfrak{B}(A) = \bigwedge_{\substack{x,y,z,w,n \in S \\ x \le y}} \Big(A(y) \to A(x)\Big) \wedge \Big(A(z) \wedge A(w) \to A(zw)\Big) \wedge \Big(A(z) \wedge A(n) \to A(zwn)\Big), \forall A \in L^S.$$

Then $\mathfrak{B}(A)$ is called the degree to which A is an L-fuzzy bi-ideal.

Remark 3.5. For an ordered semigroup S, take any L-fuzzy subset A of S. If $\mathfrak{B}(A) = \top$, then

$$\Big(A(y) \to A(x)\Big) \wedge \Big(A(z) \wedge A(w) \to A(zw)\Big) \wedge \Big(A(z) \wedge A(n) \to A(zwn)\Big) = \top,$$

for all $x, y, z, w, n \in S$ with $x \le y$. Then it follows that

$$A(y) \le A(x), A(z) \wedge A(w) \le A(zw) \text{ and } A(z) \wedge A(n) \le A(zwn),$$

for all $x, y, z, w, n \in S$ with $x \le y$. This means that A is an L-fuzzy bi-ideal of S. Hence, we could obtain that A is an L-fuzzy bi-ideal if and only if $\mathfrak{B}(A) = \top$. From a logical aspect, $\mathfrak{B}(A)$ can be considered as the degree to which A is an L-fuzzy bi-ideal.

In the following, we will give some examples of L-fuzzy bi-ideal degrees.

Example 3.6. Let $S = \{a, b, c, d\}$ be a semigroup with the multiplication table:

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	b
d	a	a	b	c

and $\le := \{(a, a), (b, b), (c, c), (d, d)\}$. It is easy to verify that $(S, \cdot, \le)$ is an ordered semigroup. Let $L = [0, 1]$. Consider the following L-fuzzy subsets:

(1) Define $A : S \longrightarrow [0, 1]$ by:

$$A = \frac{0.3}{a} + \frac{0.2}{b} + \frac{0.3}{c} + \frac{0.2}{d}.$$

It is easy to verify that $\mathfrak{B}(A) = 1$ with $A(y) \le A(x)$, $A(z) \wedge A(w) \le A(zw)$ and $A(z) \wedge A(n) \le A(zwn)$ for $x, y, z, w, n \in S$ with $x \le y$. Hence, the L-fuzzy subset A is an L-fuzzy bi-ideal.

(2) Define $A : S \longrightarrow [0, 1]$ by:

$$A = \frac{0.2}{a} + \frac{0.3}{b} + \frac{0.2}{c} + \frac{0.3}{d}.$$

It is easy to check that $\mathfrak{B}(A) = 0.2$ and $A(y) \le A(x)$ if $x \le y$. Let $z = b, w = b$, then $zw = a$, we have

$$A(z) \wedge A(w) = A(b) \wedge A(b) = 0.3 \nleq 0.2 = A(a) = A(zw).$$

Let $z = b, w = a, n = b$, then $zwn = a$, we have

$$A(z) \wedge A(n) = A(b) \wedge A(b) = 0.3 \nleq 0.2 = A(a) = A(zwn).$$

Hence, the L-fuzzy subset A is not an L-fuzzy bi-ideal.

(3) Define $A : S \longrightarrow [0, 1]$ by:

$$A = \frac{0.3}{a} + \frac{0.3}{b} + \frac{0.2}{c} + \frac{0.3}{d}.$$

It is easy to check that $\mathfrak{B}(A) = 0.2$ and $A(y) \leq A(x)$ and $A(z) \wedge A(n) \leq A(zwn)$ for $x, y, z, w, n \in S$ with $x \leq y$. Let $z = d, w = d$, then $zw = c$, we have

$$A(z) \wedge A(w) = A(d) \wedge A(d) = 0.3 \nleq 0.2 = A(c) = A(zw).$$

Hence, the L-fuzzy subset A is not an L-fuzzy bi-ideal.

(4) Define $A : S \longrightarrow [0, 1]$ by:

$$A = \frac{0.3}{a} + \frac{0.2}{b} + \frac{0.2}{c} + \frac{0.3}{d}.$$

It is easy to check that $\mathfrak{B}(A) = 0.2$ and $A(y) \leq A(x)$ and $A(z) \wedge A(w) \leq A(zw)$ for $x, y, z, w, n \in S$ with $x \leq y$. Let $z = d, w = d, n = d$, then $zwn = b$, we have

$$A(z) \wedge A(n) = A(d) \wedge A(d) = 0.3 \nleq 0.2 = A(b) = A(zwn).$$

Hence, the L-fuzzy subset A is not an L-fuzzy bi-ideal.

(5) Define $A : S \longrightarrow [0, 1]$ by:

$$\forall\, x \in S, A(x) = a \ (a \in [0, 1], a \text{ is constant}).$$

It is easy to check that $\mathfrak{B}(A) = 1$. Hence, the L-fuzzy subset A is an L-fuzzy bi-ideal of S.

Next, we will consider the characterizations of L-fuzzy bi-ideal degree. First, let us give the following lemmas.

Lemma 3.7. *Let S be an ordered semigroup and A be an L-fuzzy subset of S.*

(1) *If $A(x) = \top$ for all $x \in S$, then $\mathfrak{B}(A) = \top$;*

(2) *If $A(x) = \bot$ for all $x \in S$, then $\mathfrak{B}(A) = \top$.*

Proof. It follows immediately from Definition 3.4. □

Lemma 3.8. *Let S be an ordered semigroup and A be an L-fuzzy subset of S. For any $\lambda \in L$, $\lambda \leq \mathfrak{B}(A)$ if and only if for any $x, y, z, w, n \in S$ with $x \leq y$, then*

$$\lambda \wedge A(y) \leq A(x), \lambda \wedge A(z) \wedge A(w) \leq A(zw) \text{ and } \lambda \wedge A(z) \wedge A(n) \leq A(zwn).$$

Proof. Necessity: Take any $\lambda \in L$. If $\lambda \leq \mathfrak{B}(A)$, then

$$\lambda \leq \bigwedge_{\substack{x,y,z,w,n \in S \\ x \leq y}} \Big(A(y) \to A(x)\Big) \wedge \Big(A(z) \wedge A(w) \to A(zw)\Big) \wedge \Big(A(z) \wedge A(n) \to A(zwn)\Big).$$

It follows that

$$\lambda \leq \Big(A(y) \to A(x)\Big) \wedge \Big(A(z) \wedge A(w) \to A(zw)\Big) \wedge \Big(A(z) \wedge A(n) \to A(zwn)\Big),$$

which means

$$\lambda \leq A(y) \to A(x)\ ,\ \lambda \leq A(z) \wedge A(w) \to A(zw) \ \text{ and } \ \lambda \leq (A(z) \wedge A(n) \to A(zwn),$$

for all $x, y, z, w, n \in S$ with $x \leq y$. Hence, we obtain

$$\lambda \wedge A(y) \leq A(x), \lambda \wedge A(z) \wedge A(w) \leq A(zw) \text{ and } \lambda \wedge A(z) \wedge A(n) \leq A(zwn).$$

for all $x, y, z, w, n \in S$ with $x \leq y$.

Sufficiency is similar to Necessity. □

Next, we investigate the characterization of the L-fuzzy bi-ideal degrees.

Theorem 3.9. *Let S be an ordered semigroup and A be an L-fuzzy subset of S. Then*

$$\mathfrak{B}(A) = \bigvee \Big\{\lambda \in L | \lambda \wedge A(y) \le A(x), \lambda \wedge A(z) \wedge A(w) \le A(zw), \lambda \wedge A(z) \wedge A(n) \le A(zwn),\ \forall\, x \le y\Big\}.$$

Proof. On the one hand, take any $t \in L$ such that

$$t \le \mathfrak{B}(A) = \bigwedge_{\substack{x,y,z,w,n \in S \\ x \le y}} \Big(A(y) \to A(x)\Big) \wedge \Big(A(z) \wedge A(w) \to A(zw)\Big) \wedge \Big(A(z) \wedge A(n) \to A(zwn)\Big).$$

Then it follows from Lemma 3.8 that

$$t \wedge A(y) \le A(x), t \wedge A(z) \wedge A(w) \le A(zw) \text{ and } t \wedge A(z) \wedge A(n) \le A(zwn),$$

for all $x, y, z, w, n \in S$ with $x \le y$. This implies that

$$t \le \bigvee \Big\{\lambda \in L | \lambda \wedge A(y) \le A(x), \lambda \wedge A(z) \wedge A(w) \le A(zw), \lambda \wedge A(z) \wedge A(n) \le A(zwn),\ \forall\, x \le y\Big\}.$$

By the arbitrariness of t, we obtain

$$\mathfrak{B}(A) \le \bigvee \Big\{\lambda \in L | \lambda \wedge A(y) \le A(x), \lambda \wedge A(z) \wedge A(w) \le A(zw), \lambda \wedge A(z) \wedge A(n) \le A(zwn),\ \forall\, x \le y\Big\}.$$

On the other hand, take any $t \in L$ such that

$$t \prec \bigvee \Big\{\lambda \in L | \lambda \wedge A(y) \le A(x), \lambda \wedge A(z) \wedge A(w) \le A(zw), \lambda \wedge A(z) \wedge A(n) \le A(zwn),\ \forall\, x \le y\Big\}.$$

Then there exists $\lambda \in L$ such that

$$\lambda \wedge A(y) \le A(x),\ \lambda \wedge A(z) \wedge A(w) \le A(zw) \ \text{ and }\ \lambda \wedge A(z) \wedge A(n) \le A(zwn),$$

for all $x \le y$ and $t \le \lambda$. It follows that

$$t \le \lambda \le A(y) \to A(x),\ t \le \lambda \le (A(z) \wedge A(w)) \to A(zw) \ \text{ and }\ t \le \lambda \le (A(z) \wedge A(n)) \to A(zwn),$$

for all $x, y, z, w, n \in S$ with $x \le y$. This implies that

$$t \wedge A(y) \le A(x),\ t \wedge A(z) \wedge A(w) \le A(zw) \ \text{ and }\ t \wedge A(z) \wedge A(n) \le A(zwn),$$

for all $x, y, z, w, n \in S$ with $x \le y$. Then it follows from Lemma 3.8 that $t \le \mathfrak{B}(A)$. By the arbitrariness of t, we obtain

$$\mathfrak{B}(A) \ge \bigvee \Big\{\lambda \in L | \lambda \wedge A(y) \le A(x), \lambda \wedge A(z) \wedge A(w) \le A(zw), \lambda \wedge A(z) \wedge A(n) \le A(zwn),\ \forall\, x \le y\Big\}.$$

□

In the following, we will use cut sets to characterize L-fuzzy bi-ideal degrees. Since cut sets may be empty, we always assume that the empty set is a bi-ideal.

Theorem 3.10. *Let S be an ordered semigroup and A be an L-fuzzy subset of S. Then*

(1) $\mathfrak{B}(A) = \bigvee \Big\{\lambda \in L \mid \forall\, \mu \le \lambda,\ A_{[\mu]}$ *is a bi-ideal of* $S\Big\}$;

(2) $\mathfrak{B}(A) = \bigvee \Big\{\lambda \in L \mid \forall \mu \notin \alpha(\lambda),\ A^{[\mu]}$ *is a bi-ideal of* $S\Big\}$;

(3) $\mathfrak{B}(A) = \bigvee \Big\{\lambda \in L \mid \forall\, \mu \in P(L), \lambda \not\le \mu,\ A^{(\mu)}$ *is a bi-ideal of* $S\Big\}$.

Proof. (1) Assume that $A_{[\mu]}$ is a bi-ideal of S for each $\mu \leq \lambda$. Take any $z, w \in S$. Let $\theta = \lambda \wedge A(z) \wedge A(w)$. Then we have $\theta \leq \lambda, \theta \leq A(z)$ and $\theta \leq A(w)$, which imply $z, w \in A_{[\theta]}$. By the assumption, we know that $A_{[\theta]}$ is a bi-ideal of S. Then it shows that

$$zw \in A_{[\theta]},$$

which means $\theta \leq A(zw)$. That is,

$$\lambda \wedge A(z) \wedge A(w) \leq A(zw).$$

Similarly, for any $x, y, z, w, n \in S$ with $x \leq y$, we obtain

$$\lambda \wedge A(y) \leq A(x) \text{ and } \lambda \wedge A(z) \wedge A(n) \leq A(zwn).$$

Hence, it follows from Theorem 3.9 that

$$\begin{aligned}\mathfrak{B}(A) &= \bigvee \left\{\lambda \in L | \lambda \wedge A(y) \leq A(x), \lambda \wedge A(z) \wedge A(w) \leq A(zw), \lambda \wedge A(z) \wedge A(n) \leq A(zwn),\ \forall\, x \leq y\right\} \\ &\geq \bigvee \left\{\lambda \in L \mid \forall\, \mu \leq \lambda,\ A_{[\mu]} \text{ is a bi-ideal of } S\right\}.\end{aligned}$$

Conversely, assume that $\lambda \wedge A(y) \leq A(x), \lambda \wedge A(z) \wedge A(w) \leq A(zw)$ and $\lambda \wedge A(z) \wedge A(n) \leq A(zwn)$ for all $x, y, z, w, n \in S$ with $x \leq y$. For any $\mu \leq \lambda$, we need to prove $A_{[\mu]}$ is a bi-ideal of S.

If $y \in A_{[\mu]}$ with $x \leq y$, then $\mu \leq A(y)$. It implies that

$$\mu \leq \lambda \wedge A(y) \leq A(x).$$

Then it follows that $x \in A_{[\mu]}$.

If $z, w \in A_{[\mu]}$, then

$$\mu \leq \lambda \wedge A(z) \wedge A(w) \leq A(zw).$$

Hence, it follows that

$$zw \in A_{[\mu]}.$$

If $z, n \in A_{[\mu]}$, then

$$\mu \leq \lambda \wedge A(z) \wedge A(n) \leq A(zwn).$$

Hence, it follows that

$$zwn \in A_{[\mu]}.$$

That is to say, $A_{[\mu]}$ is a bi-ideal of S. This implies that

$$\begin{aligned}\mathfrak{B}(A) &= \bigvee \left\{\lambda \in L | \lambda \wedge A(y) \leq A(x), \lambda \wedge A(z) \wedge A(w) \leq A(zw), \lambda \wedge A(z) \wedge A(n) \leq A(zwn),\ \forall\, x \leq y\right\} \\ &\leq \bigvee \left\{\lambda \in L \mid \forall\, \mu \leq \lambda,\ A_{[\mu]} \text{ is a bi-ideal of } S\right\}.\end{aligned}$$

(2) Assume that $\lambda \in L$ with $\lambda \wedge A(y) \leq A(x), \lambda \wedge A(z) \wedge A(w) \leq A(zw)$ and $\lambda \wedge A(z) \wedge A(n) \leq A(zwn)$ for all $x, y, z, w, n \in S$ with $x \leq y$. For $\mu \notin \alpha(\lambda)$, we need to prove that $A^{[\mu]}$ is a bi-ideal of S.

If $x \leq y$ and $y \in A^{[\mu]}$, then $\mu \notin \alpha(A(y))$. It follows from

$$\lambda \wedge A(y) \leq A(x)$$

that

$$\alpha(A(x)) \subseteq \alpha(\lambda \wedge A(y)) = \alpha(\lambda) \cup \alpha(A(y)).$$

Since $\mu \notin \alpha(\lambda) \cup \alpha(A(y))$, it follows that

$$\mu \notin \alpha(A(x)).$$

Hence, we obtain $x \in A^{[\mu]}$.

If $z, w \in A^{[\mu]}$, then

$$\mu \notin \alpha(\lambda) \cup \alpha(A(z)) \cup \alpha(A(w)) = \alpha(\lambda \wedge A(z) \wedge A(w)).$$

It follows from

$$\lambda \wedge A(z) \wedge A(w) \leq A(zw)$$

that

$$\alpha(A(zw)) \subseteq \alpha(\lambda \wedge A(z) \wedge A(w)),$$

which means

$$\mu \notin \alpha(A(zw)).$$

Hence, we obtain $zw \in A^{[\mu]}$.

If $z, n \in A^{[\mu]}$, then

$$\mu \notin \alpha(\lambda) \cup \alpha(A(z)) \cup \alpha(A(n)) = \alpha(\lambda \wedge A(z) \wedge A(n)).$$

It follows from

$$\lambda \wedge A(z) \wedge A(n) \leq A(zwn)$$

that

$$\alpha(A(zwn)) \subseteq \alpha(\lambda \wedge A(z) \wedge A(n)).$$

Hence, we obtain

$$\mu \notin \alpha(A(zwn)).$$

Then it follows that $zwn \in A^{[\mu]}$, which means

$$\begin{aligned}\mathfrak{B}(A) &= \bigvee \left\{\lambda \in L | \lambda \wedge A(y) \leq A(x), \lambda \wedge A(z) \wedge A(w) \leq A(zw), \lambda \wedge A(z) \wedge A(n) \leq A(zwn),\ \forall\, x \leq y\right\} \\ &\leq \bigvee \left\{ \lambda \in L \mid \forall \mu \notin \alpha(\lambda),\ A^{[\mu]} \text{ is a bi-ideal of } S\right\}.\end{aligned}$$

Conversely, assume that $A^{[\mu]}$ is a bi-ideal of S for $\lambda \in L$ with $\mu \notin \alpha(\lambda)$. For any $x, y, z, w, n \in S$ with $x \leq y$, we need to prove

$$\lambda \wedge A(y) \leq A(x), \lambda \wedge A(z) \wedge A(w) \leq A(zw) \text{ and } \lambda \wedge A(z) \wedge A(n) \leq A(zwn).$$

Suppose that $\mu \notin \alpha(\lambda \wedge A(y))$. It follows from

$$\alpha(\lambda \wedge A(y)) = \alpha(\lambda) \cup \alpha(A(y))$$

that

$$\mu \notin \alpha(\lambda) \text{ and } \mu \notin \alpha(A(y)).$$

It implies that $y \in A^{[\mu]}$. By the assumption, we know that $A^{[\mu]}$ is a bi-ideal of S, which means $x \in A^{[\mu]}$. Then it follows that

$$\mu \notin \alpha(A(x)).$$

By the arbitrariness of μ, we have

$$\alpha(A(x)) \subseteq \alpha(\lambda \wedge A(y)).$$

Hence, we obtain

$$\lambda \wedge A(y) \leq A(x).$$

Similarly, for any $z, w, n \in S$, we obtain

$$\lambda \wedge A(z) \wedge A(w) \leq A(zw) \text{ and } \lambda \wedge A(z) \wedge A(n) \leq A(zwn).$$

Then it shows that

$$\begin{aligned}\mathfrak{B}(A) &= \bigvee \left\{\lambda \in L | \lambda \wedge A(y) \leq A(x), \lambda \wedge A(z) \wedge A(w) \leq A(zw), \lambda \wedge A(z) \wedge A(n) \leq A(zwn),\ \forall\, x \leq y\right\} \\ &\geq \bigvee \left\{ \lambda \in L \mid \forall \mu \notin \alpha(\lambda),\ A^{[\mu]} \text{ is a bi-ideal of } S\right\}.\end{aligned}$$

(3) Assume that $\lambda \in L$ with $\lambda \wedge A(y) \leq A(x)$, $\lambda \wedge A(z) \wedge A(w) \leq A(zw)$ and $\lambda \wedge A(z) \wedge A(n) \leq A(zwn)$ for all $x, y, z, w, n \in S$ with $x \leq y$. If $\mu \in P(L)$ and $\lambda \nleq \mu$, then we need to prove that $A^{(\mu)}$ is a bi-ideal of S.

Assume that $y \in A^{(\mu)}$. If $x \notin A^{(\mu)}$, then $A(x) \leq \mu$. It follows from

$$\lambda \wedge A(y) \leq A(x)$$

that

$$\lambda \wedge A(y) \leq \mu.$$

By $\mu \in P(L)$ and $y \in A^{(\mu)}$, i.e., $A(y) \nleq \mu$, we have $\lambda \leq \mu$. This is a contradiction. Hence, it follows that $x \in A^{(\mu)}$.

Similarly, for any $z, w, n \in S$, we obtain

$$z, w \in A^{(\mu)} \text{ implies } zw \in A^{(\mu)},$$

$$z, n \in A^{(\mu)} \text{ implies } zwn \in A^{(\mu)}.$$

Then it follows that

$$\begin{aligned}\mathfrak{B}(A) &= \bigvee \left\{\lambda \in L | \lambda \wedge A(y) \leq A(x), \lambda \wedge A(z) \wedge A(w) \leq A(zw), \lambda \wedge A(z) \wedge A(n) \leq A(zwn),\ \forall\, x \leq y\right\} \\ &\leq \bigvee \left\{ \lambda \in L \mid \forall\, \mu \in P(L), \lambda \nleq \mu,\ A^{(\mu)} \text{ is a bi-ideal of } S \right\}.\end{aligned}$$

Conversely, assume that $A^{(\mu)}$ is a bi-ideal of S for $\lambda \in L$ and $\mu \in P(L)$ with $\lambda \nleq \mu$. In what follows, for any $x, y, z, w, n \in S$ with $x \leq y$, we need to prove

$$\lambda \wedge A(y) \leq A(x), \lambda \wedge A(z) \wedge A(w) \leq A(zw) \text{ and } \lambda \wedge A(z) \wedge A(n) \leq A(zwn).$$

For any $x, y \in S$ with $x \leq y$, let $\mu \in P(L)$ and $\lambda \wedge A(y) \nleq \mu$. Then we have

$$\lambda \nleq \mu \text{ and } A(y) \nleq \mu.$$

It follows that $y \in A^{(\mu)}$. By the assumption, we know $A^{(\mu)}$ is a bi-ideal of S, then $x \in A^{(\mu)}$. Further, it implies that

$$A(x) \nleq \mu.$$

By the arbitrariness of μ, we have

$$\lambda \wedge A(y) \leq A(x).$$

Similarly, for any $z, w, n \in S$, we obtain

$$\lambda \wedge A(z) \wedge A(w) \leq A(zw) \text{ and } \lambda \wedge A(z) \wedge A(n) \leq A(zwn).$$

Then it follows that

$$\begin{aligned}\mathfrak{B}(A) &= \bigvee \left\{\lambda \in L | \lambda \wedge A(y) \leq A(x), \lambda \wedge A(z) \wedge A(w) \leq A(zw), \lambda \wedge A(z) \wedge A(n) \leq A(zwn),\ \forall\, x \leq y\right\} \\ &\geq \bigvee \left\{ \lambda \in L \mid \forall\, \mu \in P(L), \lambda \nleq \mu,\ A^{(\mu)} \text{ is a bi-ideal of } S \right\}.\end{aligned}$$

□

Proposition 3.11. *Let S be an ordered semigroup and A be an L-fuzzy subset of S. If $\beta(\lambda \wedge \mu) = \beta(\lambda) \cap \beta(\mu)$ for all $\lambda, \mu \in L$, then*

$$\mathfrak{B}(A) = \bigvee \left\{ \lambda \in L \mid \forall\, \mu \in \beta(\lambda),\ A_{(\mu)} \textit{ is a bi-ideal of } S\right\}.$$

Proof. Assume that $\lambda \in L$ with $\lambda \wedge A(y) \leq A(x)$, $\lambda \wedge A(z) \wedge A(w) \leq A(zw)$ and $\lambda \wedge A(z) \wedge A(n) \leq A(zwn)$ for all $x, y, z, w, n \in S$ with $x \leq y$. For any $\mu \in \beta(\lambda)$, we need to prove that $A_{(\mu)}$ is a bi-ideal of S.

If $y \in A_{(\mu)}$ and $x \leq y$, then

$$\mu \in \beta(A(y)) \cap \beta(\lambda) = \beta(A(y) \wedge \lambda) \subseteq \beta(A(x)).$$

This shows that $x \in A_{(\mu)}$.

If $z, w \in A_{(\mu)}$, then

$$\mu \in \beta(A(z)) \cap \beta(A(w)) \cap \beta(\lambda) = \beta(A(z) \wedge A(w) \wedge \lambda) \subseteq \beta(A(zw)).$$

It follows that $zw \in A_{(\mu)}$.

If $z, n \in A_{(\mu)}$, then

$$\mu \in \beta(A(z)) \cap \beta(A(n)) \cap \beta(\lambda) = \beta(A(z) \wedge A(n) \wedge \lambda) \subseteq \beta(A(zwn)).$$

Hence, we obtain $zwn \in A_{(\mu)}$.

Then it follows that

$$\begin{aligned} \mathfrak{B}(A) &= \bigvee \left\{ \lambda \in L | \lambda \wedge A(y) \leq A(x), \lambda \wedge A(z) \wedge A(w) \leq A(zw), \lambda \wedge A(z) \wedge A(n) \leq A(zwn),\ \forall\, x \leq y \right\} \\ &\leq \bigvee \left\{ \lambda \in L \mid \forall\, \mu \in \beta(\lambda),\ A_{(\mu)} \text{ is a bi-ideal of } S \right\}. \end{aligned}$$

Conversely, assume that $A_{(\mu)}$ is a bi-ideal of S for $\lambda \in L$ with $\mu \in \beta(\lambda)$. For any $x, y, z, w, n \in S$ with $x \leq y$, we need to prove

$$\lambda \wedge A(y) \leq A(x), \lambda \wedge A(z) \wedge A(w) \leq A(zw) \text{ and } \lambda \wedge A(z) \wedge A(n) \leq A(zwn).$$

Let $\mu \in \beta(\lambda \wedge A(y))$. It follows from

$$\beta(\lambda \wedge A(y)) = \beta(\lambda) \cap \beta(A(y))$$

that

$$\mu \in \beta(\lambda) \text{ and } \mu \in \beta(A(y)).$$

This implies $y \in A_{(\mu)}$. By the assumption, we know that $A_{(\mu)}$ is a bi-ideal of S. Then it shows $x \in A_{(\mu)}$. Hence, we have

$$\mu \in \beta(A(x)).$$

By the arbitrariness of μ, we have

$$\beta(\lambda \wedge A(y)) \subseteq \beta(A(x)).$$

Then it follows that

$$\lambda \wedge A(y) \leq A(x).$$

Let $\mu \in \beta(\lambda \wedge A(z) \wedge A(w))$. It follows from

$$\beta(\lambda \wedge A(z) \wedge A(w)) = \beta(\lambda) \cap \beta(A(z)) \cap \beta(A(w))$$

that

$$\mu \in \beta(\lambda), \mu \in \beta(A(z)) \text{ and } \mu \in \beta(A(w)).$$

This means that $z, w \in A_{(\mu)}$. By the assumption, we know that $A_{(\mu)}$ is a bi-ideal of S. Then we obtain $zw \in A_{(\mu)}$, which means $\mu \in \beta(A(zw))$. By the arbitrariness of μ, we have

$$\beta(\lambda \wedge A(z) \wedge A(w)) \subseteq \beta(A(zw)).$$

This implies

$$\lambda \wedge A(z) \wedge A(w) \leq A(zw).$$

Similarly, for any $z, w, n \in S$, we obtain

$$\lambda \wedge A(z) \wedge A(n) \leq A(zwn).$$

Hence, we have

$$\begin{aligned}\mathfrak{B}(A) &= \bigvee \left\{\lambda \in L | \lambda \wedge A(y) \leq A(x), \lambda \wedge A(z) \wedge A(w) \leq A(zw), \lambda \wedge A(z) \wedge A(n) \leq A(zwn),\ \forall\, x \leq y\right\} \\ &\geq \bigvee \left\{\lambda \in L | \forall\, \mu \in \beta(\lambda),\ A_{(\mu)} \text{ is a bi-ideal of } S\right\}.\end{aligned}$$

□

In case that $L = [0, 1]$, Jun et al. [11] established the relationship between a fuzzy bi-ideal A and its cut sets $A_{[\lambda]}$ in ordered semigroups S. Now, we generalize the unit interval $[0, 1]$ to a completely distributive lattice L, and use the cut sets $A_{[\lambda]}$ to characterize an L-fuzzy bi-ideal. Moreover, since there are three new types of cut sets with respect to a completely distributive lattice L, that is, $A^{[\lambda]}$, $A^{(\lambda)}$ and $A_{(\lambda)}$, we will also use them to characterize L-fuzzy bi-ideals. In particular, we still assume that the empty set is a special bi-ideal of an ordered semigroup S.

Theorem 3.12. *Let S be an ordered semigroup and A be an L-fuzzy subset of S. Then the following statements are equivalent:*

(1) *A is an L-fuzzy bi-ideal of S;*

(2) *for every $\lambda \in L$, $A_{[\lambda]}$ is a bi-ideal;*

(3) *for every $\lambda \in J(L)$, $A_{[\lambda]}$ is a bi-ideal;*

(4) *for every $\lambda \in L$, $A^{[\lambda]}$ is a bi-ideal;*

(5) *for every $\lambda \in P(L)$, $A^{[\lambda]}$ is a bi-ideal;*

(6) *for every $\lambda \in P(L)$, $A^{(\lambda)}$ is a bi-ideal.*

Proof. (1) ⇒ (2) Assume that A is an L-fuzzy bi-ideal. For any $x, y \in S$, $\lambda \in L$, let $x \leq y$ and $y \in A_{[\lambda]}$. Since A is an L-fuzzy bi-ideal, we have

$$A(x) \geq A(y) \geq \lambda.$$

Hence $x \in A_{[\lambda]}$.

For any $z, w \in S$, let $z, w \in A_{[\lambda]}$, i.e., $A(z) \geq \lambda, A(w) \geq \lambda$. Since A is an L-fuzzy bi-ideal, we have

$$A(zw) \geq A(z) \wedge A(w) \geq \lambda.$$

Then $zw \in A_{[\lambda]}$.

For any $z, w, n \in S$, let $z, n \in A_{[\lambda]}$, i.e., $A(z) \geq \lambda, A(n) \geq \lambda$. Since A is an L-fuzzy bi-ideal, we have

$$A(zwn) \geq A(z) \wedge A(n) \geq \lambda,$$

i.e., $zwn \in A_{[\lambda]}$. Therefore, $A_{[\lambda]}$ is a bi-ideal.

(2) ⇒ (3) is obvious.

(3) ⇒ (1) For any $x, y \in S$ with $x \leq y$, in order to show $A(y) \leq A(x)$, take any $\lambda \in J(L)$ such that $\lambda \leq A(y)$. Then $y \in A_{[\lambda]}$. Since $A_{[\lambda]}$ is a bi-ideal of S, we obtain $x \in A_{[\lambda]}$. That is, $\lambda \leq A(x)$. By Lemma 2.2, we know $A(y) \leq A(x)$.

For any $z, w \in S$, take any $\lambda \in J(L)$ such that $\lambda \leq A(z) \wedge A(w)$, i.e., $z, w \in A_{[\lambda]}$. Since $A_{[\lambda]}$ is a bi-ideal of S, we have $zw \in A_{[\lambda]}$, i.e., $\lambda \leq A(zw)$. By Lemma 2.2, we know

$$A(z) \wedge A(w) \leq A(zw).$$

For any $z, w, n \in S$, take any $\lambda \in J(L)$ such that $\lambda \leq A(z) \wedge A(n)$, i.e., $z, n \in A_{[\lambda]}$. Since $A_{[\lambda]}$ is a bi-ideal of S, we have $zwn \in A_{[\lambda]}$, i.e., $\lambda \leq A(zwn)$. By Lemma 2.2, we know

$$A(z) \wedge A(w) \leq A(zwn).$$

Thus, A is an L-fuzzy bi-ideal.

(1) ⇒ (4) Assume that A is an L-fuzzy bi-ideal. For any $\lambda \in L$, and $x, y \in S$, let $x \leq y, y \in A^{[\lambda]}$, i.e., $\lambda \notin \alpha(A(y))$. Since A is an L-fuzzy bi-ideal, we have

$$\lambda \notin \alpha(A(x)).$$

Hence $x \in A^{[\lambda]}$.

Similarly, for any $\lambda \in L$ and $z, w, n \in S$, if $z, w \in A^{[\lambda]}$, then we have $zw \in A^{[\lambda]}$; if $z, n \in A^{[\lambda]}$, then we have $zwn \in A^{[\lambda]}$. Therefore, $A^{[\lambda]}$ is a bi-ideal.

(4) ⇒ (5) is obvious.

(5) ⇒ (1) For any $x, y \in S$ with $x \leq y$, in order to show $A(y) \leq A(x)$, take any $\lambda \in P(L)$ such that $\lambda \notin \alpha^*(A(y))$, then $\lambda \notin \alpha(A(y))$, i.e., $y \in A^{[\lambda]}$. Since $A^{[\lambda]}$ is a bi-ideal of S, we have $x \in A^{[\lambda]}$, i.e., $\lambda \notin \alpha(A(x))$. Further, we obtain

$$\lambda \notin \alpha^*(A(x)).$$

This implies

$$\alpha^*(A(y)) \supseteq \alpha^*(A(x)).$$

Hence

$$A(y) = \bigwedge \alpha^*(A(y)) \leq \bigwedge \alpha^*(A(x)) = A(x).$$

For any $z, w \in S$, take any $\lambda \in P(L)$ such that $\lambda \notin \alpha^*(A(z) \wedge A(w))$, then

$$\lambda \notin \alpha(A(z) \wedge A(w)).$$

By

$$\alpha(A(z) \wedge A(w)) = \alpha(A(z)) \cup \alpha(A(w)),$$

we have

$$\lambda \notin \alpha(A(z)) \text{ and } \lambda \notin \alpha(A(w)),$$

i.e., $z, w \in A^{[\lambda]}$. Since $A^{[\lambda]}$ is a bi-ideal of S, we have $zw \in A^{[\lambda]}$, i.e., $\lambda \notin \alpha(A(zw))$. Further, we obtain

$$\lambda \notin \alpha^*(A(zw)).$$

This implies

$$\alpha^*(A(z) \wedge A(w)) \supseteq \alpha^*(A(zw)).$$

Hence

$$A(z) \wedge A(w) = \bigwedge \alpha^*(A(z) \wedge A(w)) \leq \bigwedge \alpha^*(A(zw)) = A(zw).$$

Similarly, for any $z, w, n \in S$, take any $\lambda \in P(L)$ such that $\lambda \notin \alpha^*(A(z))$ and $\lambda \notin \alpha^*(A(n))$. Then we can obtain

$$A(z) \wedge A(n) \leq A(zwn).$$

Therefore, A is an L-fuzzy bi-ideal.

(1) ⇒ (6) Assume that A is an L-fuzzy bi-ideal. For any $\lambda \in P(L)$, let $x \leq y, y \in A^{(\lambda)}$, i.e., $A(y) \nleq \lambda$. Since A is an L-fuzzy bi-ideal, we have

$$A(y) \leq A(x).$$

This implies $A(x) \nleq \lambda$, i.e., $x \in A^{(\lambda)}$.

Similarly, for any $z, w, n \in S, \lambda \in P(L)$, if $z, w \in A^{(\lambda)}$, then we have $zw \in A^{(\lambda)}$; if $z, n \in A^{(\lambda)}$, then we have $zwn \in A^{(\lambda)}$. Therefore, $A^{(\lambda)}$ is a bi-ideal.

(6) ⇒ (1) For any $x, y \in S$ with $x \leq y$, in order to show $A(y) \leq A(x)$, take any $\lambda \in P(L)$ such that $A(y) \nleq \lambda$. Then $y \in A^{(\lambda)}$. Since $A^{(\lambda)}$ is a bi-ideal, we obtain $x \in A^{(\lambda)}$. That is, $A(x) \nleq \lambda$. By Lemma 2.2, we know

$$A(y) \leq A(x).$$

For any $z, w \in S$, take any $\lambda \in P(L)$ such that $A(z) \wedge A(w) \nleq \lambda$. Then

$$A(z) \nleq \lambda \text{ and } A(w) \nleq \lambda,$$

i.e., $z, w \in A^{(\lambda)}$. Since $A^{(\lambda)}$ is a bi-ideal, we have $zw \in A^{(\lambda)}$, i.e., $A(zw) \nleq \lambda$. By Lemma 2.2, we know

$$A(z) \wedge A(w) \leq A(zw).$$

Similarly, for any $z, w, n \in S$, take any $\lambda \in P(L)$ such that $A(z) \wedge A(n) \nleq \lambda$. Then we can obtain

$$A(z) \wedge A(n) \leq A(zwn).$$

Thus, A is an L-fuzzy bi-ideal. □

Theorem 3.13. *Let S be an ordered semigroup and A be an L-fuzzy subset of S. Suppose that $\beta(\lambda \wedge \mu) = \beta(\lambda) \cap \beta(\mu)$ for all $\lambda, \mu \in L$. Then the following statements are equivalent.*

(1) *A is an L-fuzzy bi-ideal of S;*

(2) *for every $\lambda \in L$, $A_{(\lambda)}$ is a bi-ideal;*

(3) *for every $\lambda \in J(L)$, $A_{(\lambda)}$ is a bi-ideal.*

Proof. Adopting the proof of Theorem 3.12, it is easy to prove that. □

4. L-fuzzy bi-ideal degree as L-fuzzy structures

In this section, we will investigate the relationship between the L-fuzzy bi-ideal degree with respect to an ordered semigroup and an L-fuzzy convex structure on an ordered semigroup. Further, we will study the relations between the homomorphisms between two ordered semigroups and the L-fuzzy convexity-preserving mappings as well as the L-fuzzy convex-to-convex mappings between L-fuzzy convexities.

It is easily seen that the L-fuzzy bi-ideal degree $\mathfrak{B}$ with repect to S is a mapping $\mathfrak{B} : L^S \longrightarrow L$ defined by $A \longmapsto \mathfrak{B}(A)$. The following theorem will show that $\mathfrak{B}$ is an L-fuzzy structure on an ordered semigroup S.

Theorem 4.1. *Let S be an ordered semigroup and $\mathfrak{B}$ be an L-fuzzy bi-ideal degree with repect to S. Then $\mathfrak{B}$ is an L-fuzzy structure on S.*

Proof. By Lemma 3.7, we only need to prove (C2) and (C3).

(C2) Take any subfamily $\{A_i\}_{i\in I}$ of L-fuzzy subsets of S. Then it follows that

$$\begin{aligned}
\mathfrak{B}\Big(\bigwedge_{i\in I} A_i\Big) &= \bigwedge_{\substack{x,y,z,w,n\in S\\ x\le y}} \Big(\bigwedge_{i\in I} A_i(y) \to \bigwedge_{i\in I} A_i(x)\Big) \wedge \Big(\bigwedge_{i\in I} A_i(z) \wedge \bigwedge_{i\in I} A_i(w) \to \bigwedge_{i\in I} A_i(zw)\Big) \wedge \\
&\quad \Big(\bigwedge_{i\in I} A_i(z) \wedge \bigwedge_{i\in I} A_i(n) \to \bigwedge_{i\in I} A_i(zwn)\Big) \\
&= \bigwedge_{\substack{x,y,z,w,n\in S\\ x\le y}} \bigwedge_{i\in I} \Big(\bigwedge_{j\in I} A_j(y) \to A_i(x)\Big) \wedge \bigwedge_{i\in I} \Big(\bigwedge_{j\in I} A_j(z) \wedge \bigwedge_{j\in I} A_j(w) \to A_i(zw)\Big) \wedge \\
&\quad \bigwedge_{i\in I} \Big(\bigwedge_{j\in I} A_j(z) \wedge \bigwedge_{j\in I} A_j(n) \to A_i(zwn)\Big) \\
&= \bigwedge_{\substack{x,y,z,w,n\in S\\ x\le y}} \bigwedge_{i\in I} \Big(\Big(\bigwedge_{j\in I} A_j(y) \to A_i(x)\Big) \wedge \Big(\bigwedge_{j\in I} A_j(z) \wedge \bigwedge_{j\in I} A_j(w) \to A_i(zw)\Big) \wedge \\
&\quad \Big(\bigwedge_{j\in I} A_j(z) \wedge \bigwedge_{j\in I} A_j(n) \to A_i(zwn)\Big)\Big) \\
&\ge \bigwedge_{i\in I} \bigwedge_{\substack{x,y,z,w,n\in S\\ x\le y}} \Big(A_i(y) \to A_i(x)\Big) \wedge \Big(A_i(z) \wedge A_i(w) \to A_i(zw)\Big) \wedge \Big(A_i(z) \wedge A_i(n) \to A_i(zwn)\Big) \\
&= \bigwedge_{i\in I} \mathfrak{B}\Big(A_i\Big).
\end{aligned}$$

Hence, we can obtain $\mathfrak{B}(\bigwedge_{i\in I} A_i) \ge \bigwedge_{i\in I} \mathfrak{B}(A_i)$.

(C3) Take any directed subfamily $\{A_i\}_{i\in I}$ of L-fuzzy subsets of S. In order to show

$$\mathfrak{B}\Big(\bigvee_{i\in I} A_i\Big) \ge \bigwedge_{i\in I} \mathfrak{B}\Big(A_i\Big),$$

take any $\lambda \in L$ with $\lambda \le \bigwedge_{i\in I} \mathfrak{B}\big(A_i\big)$. Then it follows that $\lambda \le \mathfrak{B}\big(A_i\big)$ for all $i \in I$. By Lemma 3.8, we know

$$\lambda \wedge A_i(y) \le A_i(x), \lambda \wedge A_i(z) \wedge A_i(w) \le A_i(zw) \text{ and } \lambda \wedge A_i(z) \wedge A_i(n) \le A_i(zwn),$$

for all $x, y, z, w, n \in S$ with $x \le y$ and $i \in I$. Next we show

$$\lambda \wedge \Big(\bigvee_{i\in I} A_i(y)\Big) \le \bigvee_{i\in I} A_i(x),$$

$$\lambda \wedge \Big(\bigvee_{i\in I} A_i(z)\Big) \wedge \Big(\bigvee_{i\in I} A_i(w)\Big) \le \bigvee_{i\in I} A_i(zw),$$

$$\lambda \wedge \Big(\bigvee_{i\in I} A_i(z)\Big) \wedge \Big(\bigvee_{i\in I} A_i(n)\Big) \le \bigvee_{i\in I} A_i(zwn),$$

for all $x, y, z, w, n \in S$ with $x \le y$.

Take any $\eta \prec \lambda \wedge \big(\bigvee_{i\in I} A_i(z)\big) \wedge \big(\bigvee_{i\in I} A_i(w)\big)$. Then there exist $i \in I$ and $j \in I$ such that

$$\eta \le A_i(z),\ \eta \le A_j(w) \text{ and } \eta \le \lambda.$$

Since $\{A_i\}_{i\in I}$ is directed, there exists $k \in I$ such that $A_i \le A_k$ and $A_j \le A_k$. Then it follows that

$$A_i(z) \le A_k(z) \text{ and } A_j(w) \le A_k(w),$$

which means that

$$\eta \le \lambda \wedge A_k(z) \wedge A_k(w) \le A_k(zw) \le \bigvee_{i\in I} A_i(zw),$$

for all $z, w \in S$. Hence, we obtain

$$\lambda \wedge \Big(\bigvee_{i\in I} A_i(z)\Big) \wedge \Big(\bigvee_{i\in I} A_i(w)\Big) \le \bigvee_{i\in I} A_i(zw),$$

for all $z, w \in S$. Similarly, we can obtain

$$\lambda \wedge \Big(\bigvee_{i\in I} A_i(z)\Big) \wedge \Big(\bigvee_{i\in I} A_i(n)\Big) \le \bigvee_{i\in I} A_i(zwn),$$

$$\lambda \wedge \Big(\bigvee_{i\in I} A_i(y)\Big) \le \bigvee_{i\in I} A_i(x),$$

for all $x, y, z, w, n \in S$ with $x \le y$. Then it follows from Lemma 3.8 that

$$\lambda \le \mathfrak{B}\Big(\bigvee_{i\in I} A_i\Big).$$

By the arbitrariness of λ, we have $\mathfrak{B}\Big(\bigvee_{i\in I} A_i\Big) \ge \bigwedge_{i\in I} \mathfrak{B}\Big(A_i\Big)$. Hence, we obtain that $\mathfrak{B}$ is an L-fuzzy structure on S. □

Homomorphisms serve as the links between two ordered semigroups, while L-fuzzy convexity-preserving mappings and L-fuzzy convex-to-convex mappings serve as the links between two L-fuzzy convex spaces. Hence, we will study their relations herein.

Theorem 4.2. *Let $f : S \longrightarrow T$ be a homomorphism between ordered semigroups, and $\mathfrak{B}_S$, $\mathfrak{B}_T$ be the L-fuzzy bi-ideal degrees of S and T, respectively. Then $f : (S, \mathfrak{B}_S) \longrightarrow (T, \mathfrak{B}_T)$ is an L-fuzzy convexity-preserving mapping.*

Proof. In order to prove $f : (S, \mathfrak{B}_S) \longrightarrow (T, \mathfrak{B}_T)$ is an L-fuzzy convexity-preserving mapping, we just need to prove for any $B \in L^T$, $\mathfrak{B}_S\left(f_L^{\leftarrow}(B)\right) \ge \mathfrak{B}_T(B)$.

Let $\lambda \in L$ with $\lambda \le \mathfrak{B}_T(B)$. By Lemma 3.8 we obtain

$$\lambda \wedge B(y_1) \le B(x_1), \lambda \wedge B(z_1) \wedge B(w_1) \le B(z_1w_1) \text{ and } \lambda \wedge B(z_1) \wedge B(n_1) \le B(z_1w_1n_1),$$

for all $x_1, y_1, z_1, w_1, n_1 \in T$ satisfying $x_1 \le y_1$. Thus, for any $x, y, z, w, n \in S$ satisfying $x \le y$,

$$\begin{aligned} \lambda \wedge f_L^{\leftarrow}(B)(z) \wedge f_L^{\leftarrow}(B)(w) &= \lambda \wedge B(f(z)) \wedge B(f(w)) \\ &\le B(f(z)f(w)) \\ &= B(f(zw)) \\ &= f_L^{\leftarrow}(B)(zw). \end{aligned}$$

Similarly, we can obtain

$$\lambda \wedge f_L^{\leftarrow}(B)(z) \wedge f_L^{\leftarrow}(B)(n) \le f_L^{\leftarrow}(B)(zwn) \text{ and } \lambda \wedge f_L^{\leftarrow}(B)(y) \le f_L^{\leftarrow}(B)(x).$$

This shows that $\lambda \le \mathfrak{B}_S\left(f_L^{\leftarrow}(B)\right)$. Hence, $\mathfrak{B}_S\left(f_L^{\leftarrow}(B)\right) \ge \mathfrak{B}_T(B)$. Therefore, we obtain that f is an L-fuzzy convexity-preserving mapping. □

In order to investigate L-fuzzy convex-to-convex mappings, we first give following definition.

Definition 4.3. Let S and T be two ordered semigroups. A homomorphism $f : S \longrightarrow T$ is called a monohomomorphism provided that $f(x) \le f(y)$ implies $x \le y$ for all $x, y \in S$.

Theorem 4.4. *Let $f : S \longrightarrow T$ be a monohomomorphism between ordered semigroups, and $\mathfrak{B}_S$, $\mathfrak{B}_T$ be the L-fuzzy bi-ideal degrees of S and T, respectively. Then $f : (S, \mathfrak{B}_S) \longrightarrow (T, \mathfrak{B}_T)$ is an L-fuzzy convex-to-convex mapping.*

Proof. In order to prove $f : (S, \mathfrak{B}_S) \longrightarrow (T, \mathfrak{B}_T)$ is an L-fuzzy convex-to-convex mapping, we just need to prove for any $A \in L^S$, $\mathfrak{B}_S(A) \leq \mathfrak{B}_T\left(f_L^{\rightarrow}(A)\right)$.

Let $\lambda \in L$ with $\lambda \leq \mathfrak{B}_S(A)$. It follows that

$$\lambda \wedge A(y) \leq A(x), \lambda \wedge A(z) \wedge A(w) \leq A(zw) \text{ and } \lambda \wedge A(z) \wedge A(n) \leq A(zwn),$$

for all $x, y, z, w, n \in S$ satisfying $x \leq y$. Then we have for $x_1, y_1, z_1, w_1, n_1 \in T$ satisfying $x_1 \leq y_1$,

$$\begin{aligned}
\lambda \wedge \left(f_L^{\rightarrow}(A)\right)(z_1) \wedge \left(f_L^{\rightarrow}(A)\right)(w_1) &= \lambda \wedge \bigvee_{f(z)=z_1} A(z) \wedge \bigvee_{f(w)=w_1} A(w) \\
&= \bigvee \{\lambda \wedge A(z) \wedge A(w) \mid f(z) = z_1, f(w) = w_1\} \\
&\leq \bigvee \{A(zw) \mid f(z) = z_1, f(w) = w_1\} \\
&\leq \bigvee \{A(zw) \mid f(zw) = z_1 w_1\} \\
&\leq \bigvee \{A(m) \mid f(m) = z_1 w_1\} \\
&= \left(f_L^{\rightarrow}(A)\right)(z_1 w_1).
\end{aligned}$$

Similarly, we can prove that

$$\lambda \wedge \left(f_L^{\rightarrow}(A)\right)(y_1) \leq \left(f_L^{\rightarrow}(A)\right)(x_1) \text{ and } \lambda \wedge \left(f_L^{\rightarrow}(A)\right)(z_1) \wedge \left(f_L^{\rightarrow}(A)\right)(n_1) \leq \left(f_L^{\rightarrow}(A)\right)(z_1 w_1 n_1).$$

This implies that $\lambda \leq \mathfrak{B}_T\left(f_L^{\rightarrow}(A)\right)$. Hence, $\mathfrak{B}_S(A) \leq \mathfrak{B}_T\left(f_L^{\rightarrow}(A)\right)$. Therefore, $f : (S, \mathfrak{B}_S) \longrightarrow (T, \mathfrak{B}_T)$ is an L-fuzzy convex-to-convex mapping. □

As the applications of the above two theorems, we will discuss the relations between L-fuzzy bi-ideals and their images (inverse images) by L-fuzzy bi-ideal degrees.

Theorem 4.5. *Let S and T be two ordered semigroups and $f : S \longrightarrow T$ be a homomorphism. If B is an L-fuzzy bi-ideal of T, then $f_L^{\leftarrow}(B)$ is an L-fuzzy bi-ideal of S.*

Proof. Suppose that $\mathfrak{B}_S$ and $\mathfrak{B}_T$ are the L-fuzzy bi-ideal degrees with respect to S and T, respectively. Since B is an L-fuzzy bi-ideal of T, it follows from Theorem 4.2 that

$$\top = \mathfrak{B}_T(B) \leq \mathfrak{B}_S\left(f_L^{\leftarrow}(B)\right).$$

This means that $f_L^{\leftarrow}(B)$ is an L-fuzzy bi-ideal of S. □

Theorem 4.6. *Let S and T be two ordered semigroups and $f : S \longrightarrow T$ be a monohomomorphism. If A is an L-fuzzy bi-ideal of S, then $f_L^{\rightarrow}(A)$ is an L-fuzzy bi-ideal of T.*

Proof. The proof is similar to Theorem 4.5. □

5. Conclusions

In this paper, we first propose the concept of L-fuzzy bi-ideal degree with repect to an ordered semigroup, which can be used to describe the degree to which an L-fuzzy subset is an L-fuzzy bi-ideal. Then we provide an equivalent characterization of the L-fuzzy bi-ideal degree and its related properties using four cut sets of L-fuzzy sets. Finally, we show that an L-fuzzy convex structure can be constructed from the L-fuzzy bi-ideal degree on the ordered semigroup, and some of its properties are studied.

6. Acknowledgements

The authors would like to express their sincere thanks to the anonymous reviewers for their careful reading and constructive comments.

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