



Corrigendum to "Analysis of Stability and Sensitivity of Deterministic and Stochastic Models for the Spread of the New Corona Virus SARS-CoV-2" [Filomat 35:3 (2021), 1045–1063]

Bojana Jovanović^a, Jasmina Đorđević^a, Jelena Manojlović^a, Nenad Šuvak^b

^aFaculty of Sciences and Mathematics, University of Niš, Niš, Serbia

^bDepartment of Mathematics, J.J. Strossmayer University of Osijek, Osijek, Croatia

Abstract. The authors correct part of the proof of Theorems 4.1 and 4.2 in the above-titled paper [1], in order to avoid the use of the estimate $E(t) \geq 1 \Leftrightarrow \frac{1}{E(t)} \leq 1$ which is not justified since $E(t) \in (0, \infty)$, while the idea remained the same. The authors apologize for any inconvenience that may have been caused.

Throughout the paper;

- Expression (10) on the page 1052, should be replaced by (10*)

$$R_0^S = \frac{\beta h_1 + \beta' h_2}{\kappa + \mu + \frac{\sigma_1^2}{2} h_1^2 + \frac{\sigma_2^2}{2} h_2^2}, \quad (10^*)$$

where $h_1 := \left(1 + l \frac{\gamma_a}{\omega_h}\right) \frac{\kappa \rho_1}{\omega_i} + l \frac{\gamma_a}{\omega_h} \frac{\kappa \rho_2}{\omega_p}$, $h_2 := \frac{\kappa \rho_2}{\omega_p}$.

- In the formulation of Theorem 4.1 on the page 1052, condition 1. should be replaced by the following:
 1. $\sigma_1^2 \leq \frac{\beta}{h_1}$, $\sigma_2^2 \leq \frac{\beta'}{h_2}$ and $R_0^S < 1$, where R_0^S is defined by (10*),
 and condition 2. should be deleted.

◦ The beginning of the proof of the Theorem 4.1 on the page 1052, that is first paragraph and relation (11), should be replaced by the following: 1. The proof of the first part of the theorem is derived following the idea from [24]. Let $\sigma_1^2 \leq \frac{\beta}{h_1}$, $\sigma_2^2 \leq \frac{\beta'}{h_2}$ and $R_0^S < 1$. By integrating the third equation of the system (1), we obtain

$$I(t) - I(0) = \kappa \rho_1 \int_0^t E(s) ds - (\gamma_a + k_1 \gamma_i + \delta_i) \int_0^t I(s) ds.$$

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Email addresses: bojana.jovanovic@pmf.edu.rs (Bojana Jovanović), jasmina.djordjevic@pmf.edu.rs (Jasmina Đorđević), jelenam@pmf.ni.ac.rs (Jelena Manojlović), nsuvak@mathos.hr (Nenad Šuvak)

ORCID iDs: <https://orcid.org/0000-0001-6788-307X> (Bojana Jovanović), <https://orcid.org/0000-0001-6204-1789> (Jasmina Đorđević), <https://orcid.org/0000-0003-0084-7458> (Jelena Manojlović)

Noting that $0 < I(t), I(0) \leq \frac{\Delta}{\mu}$, then

$$\begin{aligned} -\frac{\Delta}{\mu} &\leq \kappa\rho_1 \int_0^t E(s)ds - (\gamma_a + k_1\gamma_i + \delta_i) \int_0^t I(s)ds \leq \frac{\Delta}{\mu}, \\ 0 &\leq \kappa\rho_1 E(t) - (\gamma_a + k_1\gamma_i + \delta_i)I(t) \leq 0, \end{aligned}$$

which implies

$$\frac{I(t)}{E(t)} = \frac{\kappa\rho_1}{\gamma_a + k_1\gamma_i + \delta_i} = \frac{\kappa\rho_1}{\omega_i}. \quad (1^*)$$

Similarly, by integrating the fourth equation of system (1), we obtain

$$\frac{P(t)}{E(t)} = \frac{\kappa\rho_2}{\omega_p} = h_2. \quad (2^*)$$

By integrating the fifth equation of the system (1), we obtain

$$\frac{H(t)}{I(t) + P(t)} = \frac{\gamma_a}{\omega_h}. \quad (3^*)$$

From (1*), (2*) and (3*), we conclude

$$\begin{aligned} \frac{(I(t)+IH(t))S(t)}{E(t)N(t)} &\leq \frac{I(t)+IH(t)}{E(t)} = \frac{\left(1+l\frac{\gamma_a}{\omega_h}\right)I(t)+l\frac{\gamma_a}{\omega_h}P(t)}{E(t)} = \left(1+l\frac{\gamma_a}{\omega_h}\right)\frac{\kappa\rho_1}{\omega_i} + l\frac{\gamma_a}{\omega_h}\frac{\kappa\rho_2}{\omega_p} = h_1, \\ \frac{P(t)S(t)}{E(t)N(t)} &\leq \frac{P(t)}{E(t)} = \frac{\kappa\rho_2}{\omega_p} = h_2. \end{aligned} \quad (4^*)$$

Applying the Itô formula (see e.g. [25]) to $\log E(t)$ we obtain

$$\begin{aligned} d(\log E(t)) &= \frac{1}{E(t)} \left(\frac{\beta}{N(t)} (I(t) + IH(t)) S(t) + \frac{\beta'}{N(t)} P(t) S(t) - (\kappa + \mu) E(t) \right) dt \\ &\quad - \frac{S^2(t)}{2E^2(t)N^2(t)} \left(\sigma_1^2 (I(t) + IH(t))^2 + \sigma_2^2 P^2(t) \right) dt \\ &\quad + \frac{S(t)}{E(t)N(t)} \left(\sigma_1 (I(t) + IH(t)) dB_1(t) + \sigma_2 P(t) dB_2(t) \right) \\ &\leq \left(\beta h_1 + \beta' h_2 - (\kappa + \mu) - \frac{\sigma_1^2}{2} h_1^2 - \frac{\sigma_2^2}{2} h_2^2 \right) dt \\ &\quad + \frac{S(t)}{E(t)N(t)} \left(\sigma_1 (I(t) + IH(t)) dB_1(t) + \sigma_2 P(t) dB_2(t) \right) \\ &=: R^* dt + \sigma_1 \frac{S(t)(I(t)+IH(t))}{E(t)N(t)} dB_1(t) + \sigma_2 \frac{S(t)P(t)}{E(t)N(t)} dB_2(t). \end{aligned} \quad (11^*)$$

Second line on page 1053 should be replaced by

$$M_1(t) = \sigma_1 \int_0^t \frac{S(s)(I(s) + IH(s))}{E(s)N(s)} dB_1(s), \quad M_2(t) = \sigma_2 \int_0^t \frac{S(s)P(s)}{E(s)N(s)} dB_2(s).$$

Inequality (14) on page 1053 should be replaced by (14*)

$$\limsup_{t \rightarrow \infty} \frac{\log E(t)}{t} \leq \beta h_1 + \beta' h_2 - (\kappa + \mu) - \frac{\sigma_1^2}{2} h_1^2 - \frac{\sigma_2^2}{2} h_2^2 = \left(\kappa + \mu + \frac{\sigma_1^2}{2} h_1^2 + \frac{\sigma_2^2}{2} h_2^2 \right) (R_0^S - 1) < 0, \quad a.s. \quad (14^*)$$

• In the formulation of Theorem 4.2 on the page 1054, condition 1. should be replaced by the following:

1. $\sigma_1^2 \leq \frac{2\beta}{h_1}, \sigma_2^2 \leq \frac{2\beta'}{h_2}$ and $R_0^S > 1$.

In addition, constant C should be defined as $C := \frac{\kappa + \mu + \frac{\sigma_1^2}{2} h_1^2 + \frac{\sigma_2^2}{2} h_2^2}{\left(\left(\beta - \frac{\sigma_1^2}{2} h_1 \right)^{\frac{1}{\lambda}} h_1 + \left(\beta' - \frac{\sigma_2^2}{2} h_2 \right)^{\frac{1}{\lambda}} h_2 \right) (\beta h_1 + \beta' h_2)}$, and condition 2. should be deleted.

◦ In the proof of the Theorem 4.2, on the pages 1055 and 1056, we made the following changes:

◦ **Proof.** The proof is derived following the idea from [24]. Applying the Itô formula to the function $\log E(t)$, one can obtain

$$d(\log E(t)) = \frac{1}{E(t)} \left(\frac{\beta}{N(t)} (I(t) + IH(t)) S(t) + \frac{\beta'}{N(t)} P(t) S(t) - (\kappa + \mu) E(t) \right) dt - \frac{S^2(t)}{2E^2(t)N^2(t)} \left(\sigma_1^2 (I(t) + IH(t))^2 + \sigma_2^2 P^2(t) \right) dt + \frac{S(t)}{E(t)N(t)} (\sigma_1 (I(t) + IH(t)) dB_1(t) + \sigma_2 P(t) dB_2(t)).$$

Using (4*) and notations $h_1 = \left(1 + l \frac{\gamma_a}{\omega_h}\right) \frac{\kappa \rho_1}{\omega_i} + l \frac{\gamma_a}{\omega_h} \frac{\kappa \rho_2}{\omega_p}$, $h_2 = \frac{\kappa \rho_2}{\omega_p}$, we obtain

$$d(\log E(t)) \geq \left(\beta h_1 + \beta' h_2 - (\kappa + \mu) - \frac{\sigma_1^2}{2} h_1^2 - \frac{\sigma_2^2}{2} h_2^2 - \left(\beta - \frac{\sigma_1^2}{2} h_1 \right) \left(h_1 - \frac{(I(t)+IH(t))S(t)}{E(t)N(t)} \right) - \left(\beta' - \frac{\sigma_2^2}{2} h_2 \right) \left(h_2 - \frac{P(t)S(t)}{E(t)N(t)} \right) \right) dt + \frac{S(t)}{E(t)N(t)} (\sigma_1 (I(t) + IH(t)) dB_1(t) + \sigma_2 P(t) dB_2(t)). \quad (15^*)$$

From the first equation of system (1)

$$dS(t) \geq \left(\frac{\Delta}{h_2} \left(h_2 - \frac{P(t)S(t)}{E(t)N(t)} \right) - \beta h_1 E(t) - \beta' h_2 E(t) - \mu S(t) \left(1 - \frac{\Delta}{\mu h_2} \frac{P(t)}{E(t)N(t)} \right) \right) dt - \frac{S(t)}{E(t)N(t)} (\sigma_1 (I(t) + IH(t)) dB_1(t) + \sigma_2 P(t) dB_2(t)) \\ \geq \left(\frac{\Delta}{h_2} \left(h_2 - \frac{P(t)S(t)}{E(t)N(t)} \right) - \beta h_1 E(t) - \beta' h_2 E(t) \right) dt - \frac{S(t)}{E(t)N(t)} (\sigma_1 (I(t) + IH(t)) dB_1(t) + \sigma_2 P(t) dB_2(t)),$$

which implies

$$-\left(h_2 - \frac{P(t)S(t)}{E(t)N(t)} \right) dt \geq -\frac{h_2}{\Delta} (\beta h_1 + \beta' h_2) E(t) dt - \frac{h_2}{\Delta} dS(t) - \frac{h_2}{\Delta} \frac{S(t)}{E(t)N(t)} (\sigma_1 (I(t) + IH(t)) dB_1(t) + \sigma_2 P(t) dB_2(t)).$$

Similarly,

$$dS(t) \geq \left(\frac{\Delta}{h_1} \left(h_1 - \frac{(I(t)+IH(t))S(t)}{E(t)N(t)} \right) - \beta h_1 E(t) - \beta' h_2 E(t) - \mu S(t) \left(1 - \frac{\Delta}{\mu h_1} \frac{I(t)+IH(t)}{E(t)N(t)} \right) \right) dt - \frac{S(t)}{E(t)N(t)} (\sigma_1 (I(t) + IH(t)) dB_1(t) + \sigma_2 P(t) dB_2(t)) \\ \geq \left(\frac{\Delta}{h_1} \left(h_1 - \frac{(I(t)+IH(t))S(t)}{E(t)N(t)} \right) - \beta h_1 E(t) - \beta' h_2 E(t) \right) dt - \frac{S(t)}{E(t)N(t)} (\sigma_1 (I(t) + IH(t)) dB_1(t) + \sigma_2 P(t) dB_2(t)),$$

which implies

$$-\left(h_1 - \frac{(I(t)+IH(t))S(t)}{E(t)N(t)} \right) dt \geq -\frac{h_1}{\Delta} (\beta h_1 + \beta' h_2) E(t) dt - \frac{h_1}{\Delta} dS(t) - \frac{h_1}{\Delta} \frac{S(t)}{E(t)N(t)} (\sigma_1 (I(t) + IH(t)) dB_1(t) + \sigma_2 P(t) dB_2(t)),$$

where we used (3*). Substituting these inequalities into (15*), we obtain

$$d(\log E(t)) \geq \left(\beta h_1 + \beta' h_2 - (\kappa + \mu) - \frac{\sigma_1^2}{2} h_1^2 - \frac{\sigma_2^2}{2} h_2^2 - \left(\beta - \frac{\sigma_1^2}{2} h_1 \right) \frac{h_1}{\Delta} (\beta h_1 + \beta' h_2) E(t) - \left(\beta' - \frac{\sigma_2^2}{2} h_2 \right) \frac{h_2}{\Delta} (\beta h_1 + \beta' h_2) E(t) \right) dt \\ - \left(\beta - \frac{\sigma_1^2}{2} h_1 \right) \left(\frac{h_1}{\Delta} dS(t) + \sigma_1 \frac{h_1}{\Delta} \frac{(I(t)+IH(t))S(t)}{E(t)N(t)} dB_1(t) + \sigma_2 \frac{h_1}{\Delta} \frac{P(t)S(t)}{E(t)N(t)} dB_2(t) \right) \\ - \left(\beta' - \frac{\sigma_2^2}{2} h_2 \right) \left(\frac{h_2}{\Delta} dS(t) + \sigma_1 \frac{h_2}{\Delta} \frac{(I(t)+IH(t))S(t)}{E(t)N(t)} dB_1(t) + \sigma_2 \frac{h_2}{\Delta} \frac{P(t)S(t)}{E(t)N(t)} dB_2(t) \right) \\ + \frac{S(t)}{E(t)N(t)} (\sigma_1 (I(t) + IH(t)) dB_1(t) + \sigma_2 P(t) dB_2(t)).$$

Integrating the last inequality from 0 to t and dividing by t , we obtain

$$\frac{\log E(t) - \log E(0)}{t} \geq \beta h_1 + \beta' h_2 - (\kappa + \mu) - \frac{\sigma_1^2}{2} h_1^2 - \frac{\sigma_2^2}{2} h_2^2 - \left(\left(\beta - \frac{\sigma_1^2}{2} h_1 \right) \frac{h_1}{\Delta} + \left(\beta' - \frac{\sigma_2^2}{2} h_2 \right) \frac{h_2}{\Delta} \right) (\beta h_1 + \beta' h_2) [E(t)] + \frac{1}{t} \varphi(t), \quad (16^*)$$

where

$$\begin{aligned}\varphi(t) = & \sigma_1 \left(1 - \frac{h_1}{\Lambda} \left(\beta - \frac{\sigma_1^2}{2} h_1 \right) - \frac{h_2}{\Lambda} \left(\beta' - \frac{\sigma_2^2}{2} h_2 \right) \right) \int_0^t \frac{(I(s) + IH(s))S(s)}{E(s)N(s)} dB_1(s) \\ & + \sigma_2 \left(1 - \frac{h_1}{\Lambda} \left(\beta - \frac{\sigma_1^2}{2} h_1 \right) - \frac{h_2}{\Lambda} \left(\beta' - \frac{\sigma_2^2}{2} h_2 \right) \right) \int_0^t \frac{P(s)S(s)}{E(s)N(s)} dB_2(s) \\ & - \left(\left(\beta - \frac{\sigma_1^2}{2} h_1 \right) \frac{h_1}{\Lambda} + \left(\beta' - \frac{\sigma_2^2}{2} h_2 \right) \frac{h_2}{\Lambda} \right) (S(t) - S(0)).\end{aligned}$$

From the large number theorem for local martingales and the boundedness of the solution, we conclude

$$\lim_{t \rightarrow \infty} \frac{1}{t} \varphi(t) = 0, \quad \lim_{t \rightarrow \infty} \frac{\log E(t) - \log E(0)}{t} = 0.$$

Taking the inferior limit on both sides of (16*), we obtain

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t E(s) ds \geq C(R_0^S - 1) > 0, \text{ a.s.}, \quad (19^*)$$

where C is defined in the theorem, since the condition $R_0^S > 1$ holds.

The rest of the proof is the same with exceptions:

- Relation (21) should be replaced by (21*)

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t I(t) dt \geq \frac{\kappa \rho_1}{\gamma_a + k_1 \gamma_i + \delta_i} C(R_0^S - 1) > 0, \text{ a.s.} \quad (21^*)$$

- Relation (23) should be replaced by (23*)

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t P(t) dt \geq \frac{\kappa \rho_2}{\gamma_a + k_2 \gamma_i + \delta_p} C(R_0^S - 1) > 0, \text{ a.s.} \quad (23^*)$$

- Relation (25) should be replaced by (25*)

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t A(t) dt \geq \frac{\kappa(1-\rho_1-\rho_2)}{\gamma_i + \mu} C(R_0^S - 1) > 0, \text{ a.s.} \quad (25^*)$$

- Relation (27) should be replaced by (27*)

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t H(t) dt \geq \left(\frac{\kappa \rho_1}{\gamma_a + k_1 \gamma_i + \delta_i} + \frac{\kappa \rho_2}{\gamma_a + k_2 \gamma_i + \delta_p} \right) \frac{\gamma_a}{\gamma_r + \delta_h} C(R_0^S - 1) > 0, \text{ a.s.} \quad (27^*)$$

• First sentence of the remark 4.3 should be replaced by:

Regarding Theorems 4.1 and 4.2, we know that: if $\sigma_1^2 \leq \frac{\beta}{h_1}$, $\sigma_2^2 \leq \frac{\beta'}{h_2}$ and $R_0^S < 1$, the disease $I(t)$ will go to extinction a.s; while if $\sigma_1^2 \leq \frac{2\beta}{h_1}$, $\sigma_2^2 \leq \frac{2\beta'}{h_2}$ and $R_0^S > 1$, the disease $I(t)$ will be persistent in mean with probability one.

• In the Table 2, on the page 1058, values of the noise intensities should be replaced by 0.05 and 0.1 (that is, $\sigma_1 = 0.05$ and $\sigma_2 = 0.1$).

• Table 3, on the page 1058, should be replaced by Table 3*.

Parameter	R_0^D - value of sensitivity index	R_0^S - value of sensitivity index
Λ	---	---
β	0.9986	0.9986
l	0.7289	0.7275
β'	0.0014	0.0014
κ	0.0001	-0.0009
ρ_1	0.9974	0.9955
ρ_2	0.0026	0.0026
γ_a	-0.0459	-0.0459
γ_r	-0.6706	-0.6693
γ_i	-0.1892	-0.1889
k_1	-0.1887	-0.1883
k_2	-0.0005	-0.0005
δ_i	-0.0358	-0.0357
δ_p	-0.0001	-0.0001
δ_h	-0.0583	-0.0582
μ	-0.0001	-0.0001
σ_1	---	-0.002
σ_2	---	$-1.62425 \cdot 10^{-9}$

Table 3*: Sensitivity of R_0^D and R_0^S for parameter values given in Tables 3* and 4*.

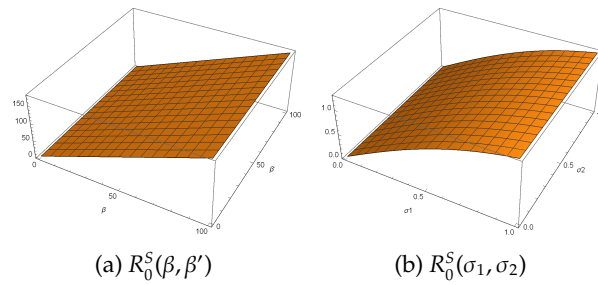
- The seventh line from below, on the page 1058 should be replaced by " $R_0^S = 4.5162$." The last six lines on the page 1058 should be deleted.
- Section Sensitivity analysis of the threshold R_0^S on model parameters, on the pages 1060 and 1061, should be replaced by the following:
Furthermore, from Table 3* we see that the threshold R_0^S related to the stochastic model is the most sensitive to the change in the values of parameters β , ρ_1 , l , γ_i and γ_r , as well as the threshold R_0^S . Here we point out the structure of the sensitivity index with respect to transmission rates β and β' , $\Upsilon_\beta^S + \Upsilon_{\beta'}^S = 1$, regardless of the value of parameters β and β' . The impact of the 10% increase in one parameter value, keeping all other parameters fixed, to the value of R_0^S is given in Table 4*.

Parameter	Value of R_0^S	Relative change in R_0^S (%)
β	4.9672	+9.98
ρ_1	4.9656	+9.95
l	4.8447	+7.27
γ_i	4.4324	-1.85
γ_r	4.2393	-6.13

Table 4*: Change of R_0^S under the 10% increase in values of parameters β , ρ_1 , l , γ_i and γ_r .

The behavior of threshold R_0^S with respect to parameter values with high sensitivity index is the same as behavior of threshold R_0^D , since they have approximately equal values of sensitivity indexes.

The behavior of the basic reproduction number R_0^S as a function of parameters σ_1 and σ_2 , as well as parameters β and β' , is shown in Figure 4*.

Figure 4*: R_0^S as a function of parameters

Comparing R_0^S with the basic reproduction number R_0^D of the deterministic model, we conclude

$$R_0^S = \frac{\kappa + \mu}{\kappa + \mu + \frac{\sigma_1^2}{2}h_1^2 + \frac{\sigma_2^2}{2}h_2^2} R_0^D \leq R_0^D.$$

That is, when the noise intensities $\sigma_1 = 0, \sigma_2 = 0$ then $R_0^S = R_0^D$, and when $\sigma_1 > 0$ or $\sigma_2 > 0$ then $R_0^S < R_0^D$. This means that the conditions for extinction of the disease in stochastic system are much weaker than in the corresponding deterministic system. Besides, since $R_0^S \leq R_0^D$, it may happen that $R_0^S < 1 < R_0^D$, which means that the disease is extincted in the stochastic model, while it is persistent in its deterministic counterpart. This implies that the presence of noise can suppress the disease.

Example 1. Let us observe another strain of the virus with a low transmission rate $\beta = 0.5615$. Values of the deterministic basic reproduction number and the corresponding stochastic threshold for $\beta = 0.5615$ and other parameter values given in Tables 3* and 4* are $R_0^D = 1.00023$ and $R_0^S = 0.99925$. Based on the considerations in Sections 2 and 3, it follows that in the deterministic model the disease will be persistent, while in the stochastic model the disease will be extincted. However, a moderate decrease in either noise intensity σ_1 or σ_2 , will lead to an increase in the value of R_0^S . Per example, if $\sigma_1 = 0.005$ or $\sigma_2 = 0.05$, then $R_0^D = 1.00023$ and $R_0^S = 1.00022$. Based on the results in Sections 2 and 3, it follows that the disease will be persistent in both, deterministic and stochastic models.

References

- [1] B. Jovanović, J. Đorđević, J. Manojlović, N. Šuvak, *Analysis of stability and sensitivity of deterministic and stochastic models for the spread of the new corona virus SARS-CoV-2*, Filomat, **35** (3) (2021), 1045–1063.