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η-Ricci-Yamabe solitons in the framework of LP-Kenmotsu manifolds

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Abstract. This paper examines the geometric characteristics of η-Ricci-Yamabe solitons within the framework of LP-Kenmotsu manifolds. A key focus is on determining conditions under which these solitons satisfy the curvature relation $R \cdot S = 0$. Additionally, we explore their behavior in quasi-conformally flat settings. Further, we establish results for solitons on manifolds exhibiting quasi-conformal curvature tensor, φ-quasi-conformal semi-symmetry and φ-Ricci symmetry, alongside conditions involving the Codazzi-type Ricci tensor and the Cyclic parallel Ricci tensor. To justify our findings, we construct an explicit example demonstrating the existence of such solitons in LP-Kenmotsu manifolds.

1. Introduction

The concept of Ricci flow was introduced by Richard S. Hamilton[13, 14] in 1982 in order to discover a canonical metric on smooth manifolds, inspired by Eells and Sampon's work on the harmonic map flow in 1964. The fundamental idea behind Ricci flow is to deform the geometry of a Riemannian manifold in a way that the metric evolves over time, following the curvature tensor. A Riemannian manifold (M, g) is said to be a Ricci soliton if there exists a smooth vector field ξ on M satisfying

$$\frac{\partial}{\partial t} g = -2R, \tag{1.1}$$

A geometric flow known as the Yamabe flow in differential geometry seeks to convert a given Riemannian manifold into one with a constant scalar curvature. It bears the name of Hidehiko Yamabe, who popularized this flow in 1960. A given manifold is deformed by Yamabe flow by changing its metric in accordance with

$$\frac{\partial}{\partial t} g = -2Rg, \tag{1.2}$$

where R denotes the scalar curvature of the metric g. In the dimension n = 2, the Yamabe soliton is identical to the Ricci soliton. Guler and Crasmareamu [11] introduced the Ricci-Yamabe map, which is a scalar combination of Ricci and Yamabe flows. If a soliton moves only by one parameter group of diffeomorphism

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and scaling, it is said to be a Ricci-Yamabe soliton. In particular, a data $(g, V, \alpha, \lambda, \beta)$ satisfying a Ricci-Yamabe soliton on a Riemannain manifold (M, g)

$$£_V g + 2\alpha S + (2\lambda - \beta r)g = 0, \tag{1.3}$$

where S is the Ricci curvature tensor field of the metric g, r is the scalar curvature, $£_V$ is the Lie derivative with respect to the vector field. Depending on the value of λ, the manifold (M, g) is classified as follows: a Ricci-Yamabe shrinker when $\lambda > 0$, a Ricci-Yamabe expander when $\lambda < 0$, and a steady soliton when $\lambda = 0$. As a result, the preceding equation is known as a Ricci-Yamabe soliton of (α, β)-type, which is a generalization of Ricci and Yamabe solitons. We note that Ricci-Yamabe soliton of type $(\alpha, 0)$, $(0, \beta)$-type are α-Ricci soliton and β-Yamabe soliton respectively.

The conceptualization of η-Ricci soliton, developed by Cho and Kimura in 2009[9], is an enhanced extension of Ricci soliton. As a generalization of Ricci-Yamabe soliton, M. D. Siddiqi and M. A. Akyol developed the idea of η-Ricci-Yamabe soliton in 2020[18]. By perturbing the equation (1.3) that defines the type of soliton by a multiple of a certain $(0, 2)$-tensor field, we obtain a slightly more general notion, namely, η-Ricci-Yamabe soliton of type (α, β) defined as:

$$£_V g + 2\alpha S + (2\lambda - \beta r)g + 2\mu\eta \otimes \eta = 0. \tag{1.4}$$

Let us recall that η-Ricci-Yamabe soliton of type (α,0) or (1,0), (0,β) or (0,1)-type are α-η-Ricci soliton (or η-Ricci soliton) and β-η-Yamabe soliton (or η-Yamabe soliton) respectively. For further information on these specific cases, refer to [2], [11], [9], [12], [15], [19], [25], [26].

The paper has been formatted as follows:
In the preliminaries section, definition and some properties of LP-Kenmotsu manifolds are given. On LP-Kenmotsu manifolds, Ricci-Yamabe and η-Ricci-Yamabe solitons are studied in Section 3. The study of η-Ricci-Yamabe solitons on LP-Kenmotsu manifolds that satisfy $R \cdot S = 0$ and η-Ricci-Yamabe solitons on quasi-conformally flat LP-Kenmotsu manifolds has been explored in Sections 4 and 5. η-Ricci-Yamabe solitons on LP-Kenmotsu manifolds admitting Cyclic parallel Ricci tensor and Codazzi type of Ricci tensor are studied in Section 6. Studying φ-quasi-conformally semi-symmetric η-Ricci-Yamabe solitons on LP-Kenmotsu manifolds is the focus of Section 7. Beyond this, Section 8 focuses on the study of η-Ricci-Yamabe solitons in φ-Ricci symmetric LP-Kenmotsu manifolds. In the later sections, we further analyze η-Ricci-Yamabe solitons within the framework of quasi-conformal curvature tensor on LP-Kenmotsu manifolds.

2. Preliminaries

An n-dimensional differentiable manifold M is referred to as a Lorentzian metric manifold if it possesses a $(1, 1)$-tensor field φ, a vector field ξ, a 1-form η, and a Lorentzian metric g all of which satisfy the conditions stated in [22].

$$\varphi^2 x = x + \eta(x)\xi, \quad \eta(\xi) = -1, \tag{2.1}$$

where x is the vector field on M, this leads to

$$(a)\varphi\xi = 0, \quad (b)\eta(\varphi x) = 0, \quad (c)rank(\varphi) = n - 1. \tag{2.2}$$

If M admits a Lorentzian metric g, then

$$g(\varphi x, \varphi y) = g(x, y) + \eta(x)\eta(y), \tag{2.3}$$

which implies that M admits a Lorentzian almost paracontact structure (φ, ξ, η, g).
Here, we have

$$(a)g(x, \xi) = \eta(x), \ (b)\nabla_x \xi = \varphi x, \tag{2.4}$$

$$(\nabla_x \varphi)y = g(x, y)\xi + \eta(y)x + 2\eta(x)\eta(y)\xi, \tag{2.5}$$

where, with regard to the Lorentzian metric g, ∇ signifies the operator of covariant differentiation.
If we put

$$\Omega(x,y) = g(x,\varphi y) = g(\varphi x,y) = \Omega(y,x), \tag{2.6}$$

For arbitrary vector fields x and y, the associated tensor field $\Omega(x,y)$ exhibits symmetry as a $(0,2)$ tensor field.

In Lorentzian geometry, the nature of a vector field is characterized by the sign of the metric evaluated on the vector, that is, by the value of $g(x,x)$. Let x be a non-zero vector field on a Lorentzian manifold. Then, x is said to be timelike if $g(x,x) < 0$. In this case, the vector lies within the so-called time cone, representing directions along which time flows. Timelike vectors are of particular importance in relativity, as they correspond to physically realizable trajectories of massive particles. If $g(x,x) > 0$, the vector is classified as spacelike. Such vectors lie outside the light cone and represent purely spatial directions, not corresponding to any possible particle path.

Definition 2.1. *An nearly paracontact manifold in Lorentzian manifold M is referred to as the Lorentzian Para-Kenmostu manifold (briefly, LP-Kenmotsu) if*

$$(\nabla_x\varphi)y = -g(\varphi x,y)\xi - \eta(y)\varphi x, \tag{2.7}$$

for any vector fields x,y on M.

Within the LP-Kenmostu manifold, there exits

$$\nabla_x\xi = -x - \eta(x)\xi, \tag{2.8}$$
$$(\nabla_x\eta)y = -g(x,y)\xi - \eta(x)\eta(y). \tag{2.9}$$

Similarly, the following relations hold in a LP-Kenmostu manifold M:

$$\eta(\nabla_x\xi) = 0, \nabla_\xi\xi = 0, \tag{2.10}$$
$$g(R(x,y)z,\xi) = \eta(R(x,y)z) = g(y,z)\eta(x) - g(x,z)\eta(y), \tag{2.11}$$
$$R(x,y)\xi = \eta(y)x - \eta(x)y, \tag{2.12}$$
$$R(\xi,x)y = g(x,y)\xi - \eta(y)x, \tag{2.13}$$
$$R(\xi,x)\xi = x + \eta(x)\xi, \tag{2.14}$$
$$S(x,\xi) = (n-1)\eta(x), \tag{2.15}$$
$$Q\xi = (n-1)\xi, \tag{2.16}$$

for any vector fields x, y, z on M.

Definition 2.2. *In any Riemannian or semi-Riemannian manifold M, the quasi-conformal curvature tensor w is characterized by*

$$\begin{aligned} w(x,y)z \;=\; & aR(x,y)z + b(S(y,z)x - S(x,z)y \\ & + g(y,z)Qx - g(x,z)Qy) \\ & - \frac{r}{n}\Big(\frac{a}{n-1} + 2b\Big)(g(y,z)x - g(x,z)y). \end{aligned} \tag{2.17}$$

given that a and b are constants with the condition $ab \neq 0$, it follows that neither a nor b is zero. Let R denote the Riemannian curvature tensor of type $(1,3)$, S represent the Ricci tensor of type $(0,2)$, Q signifies the Ricci operator defined by $g(Qx,y) = S(x,y)$ and r indicates the scalar curvature of the manifold, If $a = 1$ and $b = -\frac{1}{(n-2)}$, then it takes the form

$$\begin{aligned} w(x,y)z \quad &= R(x,y)z + \frac{1}{(n-2)}(S(y,z)x - S(x,z)y \\ &+g(y,z)Qx - g(x,z)Qy) \\ &-\frac{r}{(n-1)(n-2)}(g(y,z)x - g(x,z)y), \\ &= C(x,y)z. \end{aligned} \tag{2.18}$$

Let C denote the Conformal curvature tensor as described in [17]. It is known that the conformal curvature tensor C is a particular instance of the quasi-conformal curvature tensor w. A manifold is classified as quasi-conformally flat if the tensor w vanishes identically on M.

3. On Ricci-Yamabe and η-Ricci-Yamabe Solitons in LP-Kenmotsu Manifolds

Let $(M, \varphi, \xi, \eta, g)$ be an paracontact metric manifold. Consider the equation

$$£_\xi g + 2S\alpha + (2\lambda - \beta r)g + 2\mu\eta \otimes \eta = 0, \tag{3.1}$$

Here, r represents the scalar curvature, and S denotes the Ricci curvature tensor field corresponding to the metric g. The symbols α, λ, β, and μ are real constants, while $£_\xi g$ stands for the Lie derivative of g along the vector field ξ. Expressing $£_\xi g$ in terms of the Levi-Civita connection ∇, hence we get:

$$2\alpha S(x,y) = -g(\nabla_x \xi, y) - g(x, \nabla_y \xi) - 2\lambda g(x,y) + \beta r g(x,y) - 2\mu\eta(x)\eta(y), \tag{3.2}$$

for each $x, y \in \chi(M)$, or equivalently

$$S(x,y) = \frac{1}{\alpha}\left[(1 - \lambda + \frac{\beta r}{2})g(x,y) + (1-\mu)\eta(x)\eta(y)\right], \tag{3.3}$$

for each $x, y \in \chi(M)$.
This equation results in

$$S(x,\xi) = \frac{1}{\alpha}[(\mu - \lambda + \frac{\beta r}{2})\eta(x)], \tag{3.4}$$

$$Qx = \frac{1}{\alpha}[(1 - \lambda + \frac{\beta r}{2})x + (1-\mu)\eta(x)], \tag{3.5}$$

From the equation (2.15) and (3.4), we conclude

$$(\mu - \lambda + \frac{\beta r}{2}) = \alpha(n-1). \tag{3.6}$$

An η-Ricci-Yamabe soliton is defined as a set of data $(g, \xi, \lambda, \beta, \alpha, \mu)$ that satisfy equation [9]. From this, we can deduce the following theorem:

Theorem 3.1. *Let M be a LP-Kenmotsu manifold of dimension n. The manifold is a η-Einstein manifold of form* (3.3) *if it admits a η-Ricci-Yamabe soliton $(g, \xi, \lambda, \beta, \alpha, \mu)$. The scalars λ, β, and μ have relations by $(\mu - \lambda + \frac{\beta r}{2}) = \alpha(n-1)$.*

In particular, by setting $\mu = 0$ in (3.3) and (3.6), we obtain $S(x,y) = \frac{1}{\alpha}[(1 - \lambda + \frac{\beta r}{2})g(x,y) + \eta(x)\eta(y)]$ and $(\lambda + \frac{\beta r}{2}) = \alpha(1-n)$.

Corollary 3.1. *Let M be a LP-Kenmotsu manifold of dimension n. When a η-Ricci-Yamabe soliton (g, ξ, λ, β) is present, the manifold assumes the characteristics of an η-Einstein manifold, with its expansion or contraction determined by the nature of the vector field as either spacelike or timelike.*

4. On η-Ricci-Yamabe Solitons in LP-Kenmotsu manifolds obeying Ricci semi-symmetric

In this section, we consider an n-dimensional LP-Kenmotsu manifold that admits an η-Ricci-Yamabe soliton satisfying $R \cdot S = 0$, which results in [10]

$$(R(x,y)\cdot S)(z,w) = 0. \tag{4.1}$$

By referring to (4.1), we conclude

$$S(R(x,y)z,w) + S(z,R(x,y)w) = 0. \tag{4.2}$$

Substituting $x = \xi$ into (4.2), we get

$$S(R(\xi,y)z,w) + S(z,R(\xi,y)w) = 0. \tag{4.3}$$

By replacing S as given in (3.3) and applying the symmetries of R, we get

$$\frac{(\mu-1)}{\alpha}[g(x,y)\eta(y) + g(x,y)\eta(z) + 2\eta(x)\eta(y)\eta(z)]. \tag{4.4}$$

Taking $z = \xi$ in the above equation, we get

$$\frac{(\mu-1)}{\alpha}[g(x,y) + \eta(x)\eta(y)] = 0 \tag{4.5}$$

We deduce that $\mu = 1$. Using the relation (3.6), we get $\lambda = \frac{\beta r}{2} - \alpha(n-1)$.
From this, we can deduce the following theorem:

Theorem 4.1. *Let M be a LP-Kenmotsu manifold of dimension n which admits a proper η-Ricci-Yamabe soliton and satisfies Ricci semi-symmetric. Then, it is implied that $\mu = 1$ and $\lambda = \frac{\beta r}{2} - \alpha(n-1)$.*

From the theorem mentioned above, we get

Corollary 4.1. *If Ricci semi-symmetric holds for a LP-Kenmotsu manifold M, then there is no η-Ricci-Yamabe soliton associated with the potential vector field ξ.*

5. On η-Ricci-Yamabe Solitons in quasi-conformally flat LP-Kenmotsu manifolds

Consider a manifold M admitting an η-Ricci-Yamabe soliton, and suppose it is quasi-conformally flat (i.e., $w = 0$). Then, from (2.17), we conclude that

$$\begin{aligned}R(x,y)z = &-\frac{b}{a}(S(y,z)x - S(x,z)y + g(y,z)Qx - g(x,z)Qy)\\ &-\frac{r}{an}(\frac{a}{n-1} + 2b)(g(y,z)x - g(x,z)y).\end{aligned} \tag{5.1}$$

Using equations (2.4), (3.3), and (3.4), and taking the inner product of equation (5.1) with the vector field ξ, we obtain:

$$\begin{aligned}\eta(R(x,y)z) = &\left[\frac{b\lambda}{a\alpha} - \frac{nb}{a\alpha} - \frac{\beta br}{2a\alpha} + \frac{r}{an}(\frac{a}{n-1} + 2b)\right]\\ &[g(y,z)\eta(x) - g(x,z)\eta(y)].\end{aligned} \tag{5.2}$$

By virtue of (2.9) and (5.3), we obtain

$$\left[\frac{b\lambda}{a\alpha} - \frac{nb}{a\alpha} - \frac{\beta br}{2a\alpha} + \frac{r}{an}(\frac{a}{n-1} + 2b) - 1\right][g(y,z)\eta(x) - g(x,z)\eta(y)] = 0. \tag{5.3}$$

Substituting $x = \xi$ into the last equation, we get

$$\left[\frac{b\lambda}{a\alpha} - \frac{nb}{a\alpha} - \frac{\beta br}{2a\alpha} + \frac{r}{an}(\frac{a}{n-1} + 2b) - 1\right][g(y,z) - \eta(z)\eta(y)] = 0, \tag{5.4}$$

which implies that $\lambda = \left[n + \frac{\beta r}{2} + \frac{a\alpha}{b} - \frac{r\alpha}{bn}(\frac{a}{n-1} + 2b)\right]$.

From the relation (3.6), we obtain $\mu = \left[\alpha(n-1) + n + \frac{a\alpha}{b} - \frac{r\alpha}{bn}(\frac{a}{n-1} + 2b)\right]$.

From this, we can deduce the following theorem:

Theorem 5.1. *A LP-Kenmotsu manifold that is quasi-conformally flat admits a proper η-Ricci-Yamabe soliton with parameters* $\lambda = \left[n + \frac{\beta r}{2} + \frac{a\alpha}{b} - \frac{r\alpha}{bn}\left(\frac{a}{n-1} + 2b\right)\right]$ *and* $\mu = \left[\alpha(n-1) + n + \frac{a\alpha}{b} - \frac{r\alpha}{bn}\left(\frac{a}{n-1} + 2b\right)\right]$.

From the theorem mentioned above, we get

Corollary 5.1. *A η-Ricci-Yamabe soliton with the potential vector field ξ does not exist for a LP-Kenmotsu manifold satisfying $w = 0$.*

6. On η-Ricci-Yamabe solitons in the framework of LP-Kenmotsu manifolds featuring Codazzi-type Ricci tensors and cyclic parallel Ricci tensors

In this section, we explore η-Ricci-Yamabe solitons within the context of LP-Kenmotsu manifolds exhibiting both cyclic parallel and Codazzi type Ricci tensors. The foundational concepts of Codazzi type Ricci tensor and cyclic parallel Ricci tensor were originally formulated by Gray in his 1978 work [3].

Definition 6.1. *In the context of LP-Kenmotsu geometry, a manifold is characterized as having a Codazzi-type Ricci tensor when its non-zero Ricci tensor S, which is of type $(0,2)$, fulfills a particular criterion. This classification is applied to manifolds where the following condition is satisfied:*

$$(\nabla_z S)(x,y) = (\nabla_x S)(y,z), \tag{6.1}$$

for all x, y, z on M.

Differentiating equation (3.3) covariantly along z, we obtain:

$$(\nabla_z S)(x,y) = \frac{(1-\mu)}{\alpha}[(\nabla_z \eta)(x)\eta(y) + \eta(x)(\nabla_z \eta)(y)]. \tag{6.2}$$

By using (2.9) and (6.2), we get

$$(\nabla_z S)(x,y) = \frac{(1-\mu)}{\alpha}[g(z,x)\eta(y) + g(z,y)\eta(x) + 2\eta(x)\eta(y)\eta(z)] \tag{6.3}$$

$$(\nabla_z S)(x,y) = (\nabla_x S)(y,z) \tag{6.4}$$

Using (6.3), (6.4) has the following form:

$$\frac{\mu-1}{\alpha}[g(z,y)\eta(x) - g(x,y)\eta(z)] = 0 \tag{6.5}$$

With $z = \xi$ substituted in (6.5), we obtain

$$\frac{\mu-1}{\alpha}[\eta(y)\eta(x) + g(x,y)] = 0 \tag{6.6}$$

which implies that $\mu = 1$. We get $\lambda = \frac{\beta r}{2} + 1 - \alpha(n-1)$ from the relation (3.6). As a result, the following theorem can be stated:

Theorem 6.1. *Let (g, ξ, λ, μ) be a proper η-Ricci-Yamabe soliton in an n-dimensional LP-Kenmotsu manifold. If the manifold has a Ricci tensor of Codazzi type, then $\mu = 1$ and $\lambda = \frac{\beta r}{2} + 1 - \alpha(n-1)$.*

From this, we can deduce the following theorem:

Corollary 6.1. *An LP-Kenmotsu manifold with a Codazzi-type Ricci tensor cannot admit a Ricci-Yamabe soliton when the potential vector field is ξ.*

Definition 6.2. *An LP-Kenmotsu manifold is defined to have a cyclic parallel Ricci tensor if it fulfills the condition given by [14]*

$$(\nabla_x S)(y,z) + (\nabla_y S)(z,x) + (\nabla_z S)(x,y) = 0, \tag{6.7}$$

for all x, y, z on M.

Consider an n-dimensional LP-Kenmotsu manifold with an η-Ricci-Yamabe soliton given by (g, ξ, λ, μ), where the manifold possesses a cyclic parallel Ricci tensor. In this case, equation (6.7) is satisfied. Taking the covariant derivative of (3.3) and utilizing (2.9), we get

$$(\nabla_z S)(x,y) = \frac{(\mu-1)}{\alpha}[g(z,x)\eta(y) + g(z,y)\eta(x) + 2\eta(x)\eta(y)\eta(z)], \tag{6.8}$$

Similarly, we have

$$(\nabla_x S)(y,z) = \frac{(\mu-1)}{\alpha}[g(x,y)\eta(z) + g(x,z)\eta(y) + 2\eta(x)\eta(y)\eta(z)], \tag{6.9}$$

and

$$(\nabla_y S)(z,x) = \frac{(\mu-1)}{\alpha}[g(y,z)\eta(x) + g(y,x)\eta(z) + 2\eta(x)\eta(y)\eta(z)], \tag{6.10}$$

By applying (6.8)–(6.10) in (6.10), we obtain:

$$\frac{(\mu-1)}{\alpha}[g(x,y)\eta(z) + g(y,z)\eta(x) + g(z,x)\eta(y) + 3\eta(x)\eta(y)\eta(z)] = 0. \tag{6.11}$$

Now, substituting $z = \xi$ in (6.11), we get

$$2\frac{(\mu-1)}{\alpha}[g(x,y) + \eta(x)\eta(y)] = 0 \tag{6.12}$$

from this, it follows that $\mu = 1$. Using the relation (3.6), we get $\lambda = \frac{\beta r}{2} + 1 - \alpha(n-1)$. From this, we can deduce the following theorem:

Theorem 6.2. *Let (g, ξ, λ, μ) be a proper η-Ricci-Yamabe soliton in an n-dimensional LP-Kenmotsu manifold. If the manifold has cyclic parallel Ricci tensor, then $\lambda = \frac{\beta r}{2} + 1 - \alpha(n-1)$ and $\mu = 1$.*

From the theorem stated above, we conclude that

Corollary 6.2. *A Ricci-Yamabe soliton with a Ricci tensor of Codazzi type cannot be admitted by a LP-Kenmotsu manifold if it possesses a potential vector field ξ.*

7. φ-quasi-conformally semi-symmetric LP-kenmotsu manifolds with η-Ricci-Yamabe solitons

On LP-Kenmotsu manifolds, consider φ-quasi-conformally semi-symmetric η-Ricci-Yamabe solitons. Next, we have

$$w \cdot \varphi = 0, \tag{7.1}$$

which implies that

$$w(x, y)\varphi z - \varphi(w(x, y)z) = 0. \tag{7.2}$$

substituting $z = \xi$ in (7.2), we get

$$\varphi(w(x, y)\xi) = 0. \tag{7.3}$$

Applying (2.2), (2.12), (2.15), and (2.16) to (2.17) after replacing $z = \xi$, we now get

$$w(x, y)\xi = \left(a + nb - \frac{ra}{n(n-1)} - \frac{2rb}{n} - b\lambda\right)[\eta(y)x - \eta(x)y], \tag{7.4}$$

With reference to (7.3) and (7.4), we obtain

$$\begin{aligned}\varphi(w(x, y)\xi) &= \left(a + nb - \frac{ra}{n(n-1)} - \frac{2rb}{n} - b\lambda\right)[\eta(Y)\varphi x - \eta(x)\varphi y] \\ &= 0.\end{aligned} \tag{7.5}$$

Now, by replacing x in (7.5) with φx, we find

$$\left(a + nb - \frac{ra}{n(n-1)} - \frac{2rb}{n} - b\lambda\right)\eta(y)\varphi^2 x = 0. \tag{7.6}$$

Using (2.1) and (7.7), we derive

$$\left(a + nb - \frac{ra}{n(n-1)} - \frac{2rb}{n} - b\lambda\right)[x + \eta(x)\xi] = 0. \tag{7.7}$$

Comparing (7.7)'s inner product with u, we find

$$\left(a + nb - \frac{ra}{n(n-1)} - \frac{2rb}{n} - b\lambda\right)[g(x, U) + \eta(x)\eta(u)] = 0, \tag{7.8}$$

which implies that $\lambda = \frac{a}{b}(1 - \frac{r}{n(n-1)}) + (n - \frac{2r}{n})$. From the relation (3.6) we obtain $\mu = \frac{a}{b}(1 - \frac{r}{n(n-1)}) + (\alpha(n - 1) + n - \frac{2r}{n})$.
From this, we can deduce the following theorem:

Theorem 7.1. *A φ-conformally semi-symmetric LP-Kenmotsu manifold admits a proper η-Ricci-Yamabe soliton with* $\lambda = \frac{a}{b}\left(1 - \frac{r}{n(n-1)}\right) + \left(n - \frac{2r}{n}\right)$ *and* $\mu = \frac{a}{b}\left(1 - \frac{r}{n(n-1)}\right) + \left(\alpha(n-1) + n - \frac{2r}{n}\right)$.

From the theorem stated above, we conclude that

Corollary 7.1. *When $w \cdot \varphi = 0$ in an LP-Kenmotsu manifold, there is no Ricci-Yamabe soliton with the potential vector field ξ.*

8. η-Ricci-Yamabe solitons on φ-Ricci symmetric LP-Kenmotsu manifolds

Definition 8.1. *A LP-Kenmotsu manifold is said to be φ-Ricci symmetric if*

$$\varphi^2(\nabla_x Q)y = 0, \tag{8.1}$$

holds for vector fields x, y.

If x, y are orthogonal to ξ, then the manifold is said to be locally φ-Ricci symmetric. It is well-known that φ-symmetric implies φ-Ricci symmetric, but the convese is not true. φ-Ricci symmetric have been studied by De and Sarkar [32].

The Ricci tensor corresponding to an η-Ricci-Yamabe soliton on a LP-Kenmotsu manifold is given by

$$S(x, y) = \frac{1}{\alpha}\Big[(1 - \lambda + \frac{\beta r}{2})g(x, y) + (1 - \mu)\eta(x)\eta(y)\Big], \tag{8.2}$$

Consequently, it follows that

$$Qy = \frac{1}{\alpha}\Big[(1 - \lambda + \frac{\beta r}{2})y + (1 - \mu)\eta(y)\xi\Big], \tag{8.3}$$

$$\begin{aligned}(\nabla_x Q)y &= \nabla_x Qy - Q(\nabla_x y)\\ &= \frac{1-\mu}{\alpha}\big[(\nabla_x\eta)(y)\xi - \eta(y)\nabla_x\xi\big].\end{aligned} \tag{8.4}$$

Using (2.8) and (2.9), we obtain

$$(\nabla_x Q)y = \frac{\mu - 1}{\alpha}\big[g(x, y)\xi + \eta(y)x + 2\eta(x)\eta(y)\xi\big]. \tag{8.5}$$

Using φ^2 on both sides of the equation above, we find

$$\varphi^2(\nabla_x Q)y = \frac{(\mu - 1)}{\alpha}\eta(y)\varphi^2 x. \tag{8.6}$$

From (8.1) and (8.6), we have

$$\frac{(\mu - 1)}{\alpha}\eta(y)\varphi^2 x = 0. \tag{8.7}$$

Using (2.1) and (8.7), we find

$$\frac{(\mu - 1)}{\alpha}[x + \eta(x)\xi] = 0. \tag{8.8}$$

Taking (8.8) inner product with respect to u, we obtain

$$\frac{(\mu - 1)}{\alpha}\big[g(x, u) + \eta(x)\eta(u)\big] = 0, \tag{8.9}$$

This implies that $\mu = 1$. We get $\lambda = \frac{\beta r}{2} + 1 - \alpha(n - 1)$ from the relation (3.6). From this, we can deduce the following theorem:

Theorem 8.1. *The LP-Kenmotsu manifold with φ-Ricci symmetry permits an adequate η-Ricci-Yamabe soliton with $\lambda = \frac{\beta r}{2} + 1 - \alpha(n - 1)$ and $\mu = 1$.*

As a corollary of this theorem, we have

Corollary 8.1. *A Ricci-Yamabe soliton with the potential vector field ξ does not exist on a φ-Ricci symmetric LP-Kenmotsu manifold.*

9. η-Ricci-Yamabe Solitons on LP-Kenmotsu Manifolds with Quasi-Conformal curvature tensor

Let η-Ricci-Yamabe solitons be considered on Quasi-Conformal curvature tensor LP-Kenmotsu manifolds. After that, we have

$$w \cdot S = 0, \tag{9.1}$$

which implies

$$(w(x,y) \cdot S)(z,u) = 0. \tag{9.2}$$

From (9.2), we get

$$S\big(w(x,y)z,u\big) + S\big(z,w(x,y)u\big) = 0. \tag{9.3}$$

In (9.3), we have (3.3).

$$\begin{aligned} &\frac{1}{\alpha}\big(1-\alpha+\frac{\beta r}{2}\big)g(w(x,y)z,u) + \frac{(1-\mu)}{\alpha}\eta(w(x,y)z)\eta(u) \\ &+\frac{1}{\alpha}\big(1-\alpha+\frac{\beta r}{2}\big)g(z,w(x,y)u) + \frac{(1-\mu)}{\alpha}\eta(w(x,y)u)\eta(z) \\ &= 0. \end{aligned} \tag{9.4}$$

Using $x = u = \xi$ in (9.4), we obtain

$$\begin{aligned} &\frac{1}{\alpha}\big(1-\alpha+\frac{\beta r}{2}\big)g(w(\xi,y)z,\xi) + \frac{(1-\mu)}{\alpha}\eta(w(\xi,y)z)\eta(u) \\ &+\frac{1}{\alpha}\big(1-\alpha+\frac{\beta r}{2}\big)g(z,w(\xi,y)\xi) + \frac{(1-\mu)}{\alpha}\eta(w(\xi,y)\xi)\eta(z) \\ &= 0. \end{aligned} \tag{9.5}$$

Using (2.18), we obtain

$$w(\xi,y)\xi = \left[[a+nb-b\lambda-\frac{r}{n}(\frac{a}{(n-1)}+2b)](y+\eta(y)\xi)\right]. \tag{9.6}$$

Using (9.6), we obtain

$$\begin{aligned} \eta(w(\xi,y)z) &= g(w(\xi,y)z,\xi) \\ &= -g(w(\xi,y)\xi,z). \\ &= -\left[a+nb-b\lambda-\frac{r}{n}(\frac{a}{(n-1)}+2b)\right][g(y,z)+\eta(y)\eta(z)]. \end{aligned} \tag{9.7}$$

Also, from (9.7), we find

$$\eta(w(\xi,y)\xi) = 0. \tag{9.8}$$

Combining (9.6), (9.7), and (9.8) in (2.5), we obtain

$$\frac{(1-\mu)}{\alpha}\left[a+nb-b\lambda-\frac{r}{n}(\frac{a}{(n-1)}+2b)\right][g(y,z)+\eta(y)\eta(z)] = 0. \tag{9.9}$$

Thus either $(1-\mu) = 0$ or $[a+nb-b\lambda-\frac{r}{n}(\frac{a}{(n-1)}+2b)] = 0$.

Case I: if $\mu = 1$ then from (3.6), we get $\lambda = \frac{\beta r}{2} + 1 - \alpha(n-1)$.

Case II: if $\left[a + nb - b\lambda - \frac{r}{n}(\frac{a}{(n-1)} + 2b)\right] = 0$, implies that $\lambda = \frac{a}{b}\left(1 - \frac{r}{n(n-1)}\right) + \left(n - \frac{2r}{n}\right)$ and $\mu = \frac{a}{b}\left(1 - \frac{r}{n(n-1)}\right) + \left(\alpha(n-1) + n - \frac{2r}{n}\right)$.

Therefore, the theorem can be stated as follows:

Theorem 9.1. *If a Quasi-Conformal curvature tensor on LP-Kenmotsu manifold admits a proper η-Ricci-Yamabe soliton, then $\mu = 1$ and $\lambda = \frac{\beta r}{2} + 1 - \alpha(n-1)$, or $\lambda = \frac{a}{b}\left(1 - \frac{r}{n(n-1)}\right) + \left(n - \frac{2r}{n}\right)$ and $\mu = \frac{a}{b}\left(1 - \frac{r}{n(n-1)}\right) + \left(\alpha(n-1) + n - \frac{2r}{n}\right)$.*

From this, we can deduce the following theorem:

Corollary 9.1. *On a LP-Kenmotsu manifold that is Quasi-Conformal curvature tensor, a Ricci-Yamabe soliton cannot exist with the potential vector field ξ.*

10. Example

Consider the three-dimensional manifold $M = \{(x, y, z) \in R^3 : z \neq 0\}$,where x, y, and z are the usual coordinate functions in R^3.
The vector fields

$$a_1 = z\frac{\partial}{\partial x},\ a_2 = z\frac{\partial}{\partial y} \text{ and } a_3 = z\frac{\partial}{\partial z} = \xi, \tag{10.1}$$

At every point M, the coordinate functions x, y, and z are linearly independent. We introduce the metric g as

$$g(a_1, a_3) = g(a_2, a_3) = g(a_1, a_2) = 0$$
$$g(a_1, a_1) = g(a_2, a_2) = g(a_3, a_3) = 1,$$

Consider the 1-form η given by

$$\eta(z) = g(z, a_3) = g(z, \xi) \tag{10.2}$$

for any vector field z on M.
Let φ be the (1,1)-tensor field defined by

$$\varphi(a_1) = a_2, \quad \varphi(a_2) = a_1, \quad \varphi(a_3) = 0. \tag{10.3}$$

Thus, by applying the linearity of φ and g, we obtain

$$\eta(a_3) = 1, \quad \varphi^2 z = -z + \eta(z)\xi, \tag{10.4}$$
$$g(\varphi z, \varphi w) = g(z, w) - \eta(z)\eta(w), \tag{10.5}$$

where $\xi = a_3$ and z, w are arbitrary vector fields on M. It is easy to see that

$$\eta(a_1) = 0, \quad \eta(a_2) = 0, \quad \eta(a_3) = 1.$$

Denote ∇ as the Levi-Civita connection with respect to the Lorentzian metric g. Thus, when $a_3 = \xi$, the structure (φ, ξ, η, g) creates a Lorentzian nearly paracontact metric structure on M. As a result, we get

$$[a_1, a_2] = 0, [a_1, a_3] = a_1, [a_2, a_3] = a_2. \tag{10.6}$$

By applying Koszul's formula for the Levi-Civita connection ∇ associated with the metric g, i.e.,

$$2g(\nabla_x y,z) = xg(y,z) + yg(z,x) - zg(x,y) \\ -g(x,[y,z]) - g(y,[x,z]) + g(z,[x,y]), \tag{10.7}$$

One can easily calculate

$$\nabla_{a_1}a_1 = -a_3, \nabla_{a_1}a_2 = 0, \nabla_{a_1}a_3 = a_1, \tag{10.8}$$
$$\nabla_{a_2}a_1 = 0, \nabla_{a_2}a_2 = -a_3, \nabla_{a_2}a_3 = a_2, \tag{10.9}$$
$$\nabla_{a_3}a_1 = 0, \nabla_{a_3}a_2 = 0, \nabla_{a_3}a_3 = 0 \tag{10.10}$$

Based on the above computations, it follows that the given manifold satisfies

$$\nabla_z\xi = -z - \eta(z)\xi, \ and \ (\nabla_z\varphi)w = -g(\varphi z,w)\xi - \eta(w)\varphi Z. \tag{10.11}$$

It is also possible to find the Riemannian curvature tensor R by

$$R(x,y)z = \nabla_x\nabla_y z - \nabla_y\nabla_x z - \nabla_{[x,y]}z. \tag{10.12}$$

Then

$$R(a_1,a_2)a_2 = -a_1, R(a_1,a_3)a_3 = -a_1, R(a_1,a_2)a_1 = a_2 \tag{10.13}$$
$$R(a_2,a_3)a_1 = 0, R(a_1,a_3)a_2 = 0, R(a_1,a_2)a_3 = 0 \tag{10.14}$$
$$R(a_2,a_3)a_3 = -a_2, R(a_1,a_3)a_1 = a_3, R(a_2,a_3)a_2 = a_3. \tag{10.15}$$

And the Ricci tensor S is given by

$$S(a_i,a_i) = -2, \quad for \quad i = 1,2,3 \tag{10.16}$$

and by contracting the Ricci tensor we have $r = \sum_{i=1}^{3}$=-6.
From (3.4),we get

$$S(a_1,a_1) = S(a_2,a_2) = \frac{1}{\alpha}(1-\lambda-3\beta) \tag{10.17}$$
$$S(a_3,a_3) = \frac{1}{\alpha}(1-\lambda-3\beta) + \frac{1}{\alpha}(1-\mu) \tag{10.18}$$

therefore $\lambda = 1 - 2\alpha - 3\alpha\beta$ and $\mu = 1$. Consequently, on the LP-Kenmotsu manifold M, g defines a η-Ricci-Yamabe soliton.

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