Filomat 39:22 (2025), 7697–7705
https://doi.org/10.2298/FIL2522697E

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Bertrand-like curves in Euclidean 3-space

Kemal Eren[a,*], Soley Ersoy[b], Mića S. Stanković[c]

[a] *Picode Software, Education Training Consultancy Research and Development and Trade Ltd. Co., Sakarya University, Sakarya, 54050, Turkey*
[b] *Department of Mathematics, Faculty of Science, Sakarya University, Sakarya, 54050, Turkey*
[c] *Department of Mathematics, Faculty of Sciences and Mathematics, University of Niš, Serbia*

Abstract. In this paper, the pair of Bertrand-like curves is introduced with a linear dependency between the normal-like vectors in the Frenet-like curve frames of two curves at their corresponding points. The necessary and sufficient conditions to be a Bertrand-like curve are obtained. The main characteristic property of any Bertrand curve is known as the existence of a linear relation between its curvature and torsion. Its analogue is found for a Bertrand-like curve as $\lambda d_1 - \mu d_3 = 1$ for non-zero $\lambda,\ \mu \in \mathbb{R}$. More clearly, the existence of a linear relation between the Frenet-like curvatures d_1 and d_3 of a curve is the necessary and sufficient condition for it to be a Bertand-like curve. We present some characterizations for the conjugate of any Bertrand-like curve. Besides, the relations between the curvatures of each of the pairs are found. An example is presented with a graphic of a Bertrand-like curve pair.

1. Introduction

The history of the most referenced fundamental results on the curves dates back to the research of Leibniz (1646-1716) and Newton (1643-1727). The relations between the Frenet vectors of any curve pairs are important in the characterization of the curves. Such an idea was presented by B. de Saint Venant in 1845 on whether the principal normal vector field of a curve can be the principal normal vector field of another curve and was clarified in an article published by J. Bertrand in 1850 [1, 2]. The necessary and sufficient condition for the existence of such a second curve is that the curvatures of the given curve satisfy the condition $\lambda k_1 + \mu k_2 = 1, 0 \neq \lambda,\ \ \lambda,\ \mu \in \mathbb{R}$ [15]. Since this date, such a curve that meets this condition is called the Bertrand curve and the second corresponding curve is called the Bertrand conjugate curve of this curve. These curve pairs are also called Bertrand curve pairs. Bertrand curves have been studied extensively, and many of their properties have been revealed [4, 5, 8, 14, 17, 23]. In 1935, L. R. Pears studied Bertrand curves in $n-$dimensional ($n > 3$) Euclidean space and proved that these curves are degenerate curves. That is,

2020 *Mathematics Subject Classification*. Primary 53A04.
Keywords. Frenet-like curve frame, Bertrand curve, Polynomial curve.
Received: 16 March 2025; Accepted: 24 March 2025
Communicated by Ljubica Velimirović
The third author acknowledge the grant of the Ministry of Science, Technological Development and Innovation of Serbia 451-03-137/2025-03/200124 for carrying out this research.
* Corresponding author: Kemal Eren
Email addresses: kemal.eren1@ogr.sakarya.edu.tr (Kemal Eren), sersoy@sakarya.edu.tr (Soley Ersoy), mica.stankovic@pmf.edu.rs (Mića S. Stanković)
ORCID iDs: https://orcid.org/0000-0001-5273-7897 (Kemal Eren), https://orcid.org/0000-0002-7183-7081 (Soley Ersoy), https://orcid.org/0000-0002-5632-0041 (Mića S. Stanković)

he proved that some of the curvatures of the curve must be zero (see for more details on Bertrand curves [19]). To characterize many properties of a curve; Frenet's equations, which include the Frenet triplet and their variation along the curve, are used. However, the Frenet frame has some disadvantages in practice. For example; this frame is undefined where the second derivative of the curve is zero [3]. This has led mathematicians to search for frames that are more useful in applications. As a result of this effort, Dede defined a new frame called a Frenet-like curve (Flc) frame along a polynomial curve with singular points. In [6], Dede expressed the Flc frame extensively and investigated tube surfaces using this frame, at the same time, he compared the Frenet and the Flc frames and explained the advantages and disadvantages [7]. On the other hand, by using Flc frame, partner ruled surfaces, inextensible flows of polynomial curves, and spherical curves were investigated by Eren et al. [12, 13, 16, 20]. Şenyurt et al. investigated Smarandache curves and spherical indicatrix curves generated by Flc frame [21, 22].
We are inspired by these studies and we have defined the pair of Bertrand-like curves that has not been considered yet. We have found the Flc vectors and curvatures of the conjugate of a Bertrand-like curve. Thanks to these curvatures and Flc vectors, we have proved some characterizations of the Bertrand-like curves. We have given the condition ensuring that the normal-like vectors of both curves remain parallel, as the existence of a linear relation between the first and third curvatures of the given curve. Finally, we have exemplified these curves.

2. Preliminaries

In this section, we express some basic concepts for the Flc frame and Bertrand curves that are used throughout the paper. In Euclidean 3-space $\mathbb{E}^3$, Euclidean scalar product is given by

$$< x, y >= x_1y_1 + x_2y_2 + x_3y_3,$$

where $x = (x_1, x_2, x_3)\,,\ y = (y_1, y_2, y_3) \in \mathbb{E}^3$. The norm of $x \in \mathbb{E}^3$ is $\|x\| = \sqrt{< x, x >}$.
Let $\alpha = \alpha(s)$ be a differentiable curve. Then the Frenet vectors of $\alpha = \alpha(s)$ are defined by

$$T(s) = \frac{\alpha'(s)}{\|\alpha'(s)\|},\ B(s) = \frac{\alpha'(s) \times \alpha''(s)}{\|\alpha'(s) \times \alpha''(s)\|},\ N(s) = B(s) \times T(s)\,,$$

and also, the curvature and the torsion of the curve $\alpha = \alpha(s)$ are

$$\kappa(s) = \frac{\|\alpha'(s) \times \alpha''(s)\|}{\|\alpha'(s)\|^3},\ \tau(s) = \frac{\langle \alpha'(s) \times \alpha''(s)\,, \alpha'''(s)\rangle}{\|\alpha'(s) \times \alpha''(s)\|^2}.$$

Moreover, the Frenet formulas are given as follows:

$$T' = v\kappa N,\ N' = v(-\kappa T + \tau B)\,,\ B' = -v\tau N,$$

where $v = \|\alpha'(s)\|$ [18]. However, if the second or higher order derivatives of a space curve are zero at any point, then $\alpha'(s) \times \alpha''(s) = 0$. Therefore, the Frenet frame of the space curve cannot be defined at such points. To fix this issue, a different frame to be defined on a such curve is required. For this purpose, as stated below, Dede et al. defined a new frame called "Flc frame" along the polynomial curves [6, 7]. This Flc frame makes it feasible to study and, consequently, comment on this kind of curve.

Definition 2.1. *Let $\alpha = \alpha(s)$ be a polynomial curve. The tangent vector, the binormal-like vector, and the normal-like vector of the Flc frame along the curve α are defined by*

$$T(s) = \frac{\alpha'(s)}{\|\alpha'(s)\|},\ D_1(s) = \frac{\alpha'(s) \times \alpha^{(n)}(s)}{\left\|\alpha'(s) \times \alpha^{(n)}(s)\right\|},\ D_2(s) = D_1(s) \times T(s)\,,$$

respectively, where "$'$" and "$^{(n)}$" denote the first and n. order derivatives of α, respectively, in terms of s [6, 7].

Remark 2.2. *In fact, the Flc frame can be used for any curve (not need to be a polynomial curve) and solves the problem of singularities of a curve α that occur whenever*

$$\alpha'(s) \neq 0,\ \alpha''(s) = \alpha'''(s) = \ldots = \alpha^{(n-1)}(s) = 0,\ \alpha^{(n)}(s) \neq 0$$

at any point $\alpha(s)$. In the special case of $\alpha''(s) \neq 0$, the Flc frame coincides with the Frenet Frame.

The derivative formulas of the elements of the Flc frame are called the Frenet-like frame formulas and are as follows:

$$\begin{bmatrix} T' \\ D'_2 \\ D'_1 \end{bmatrix} = v \begin{bmatrix} 0 & d_1 & d_2 \\ -d_1 & 0 & d_3 \\ -d_2 & -d_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ D_2 \\ D_1 \end{bmatrix} \tag{1}$$

and the curvatures d_1, d_2, and d_3 of the polynomial curve α are

$$d_1 = \frac{\langle T', D_2 \rangle}{v},\ d_2 = \frac{\langle T', D_1 \rangle}{v},\ d_3 = \frac{\langle D'_2, D_1 \rangle}{v},$$

where $v = \|\alpha'(s)\|$ (see for more details [6, 7]).

Let φ be the angle between the principal normal vector N and the normal-like vector D_2 of a curve α, then the following relations exist:

$$\begin{bmatrix} T \\ D_2 \\ D_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2}$$

and the relations between the curvatures of the Frenet frame and the Flc frame of α are given by

$$d_1 = \kappa\cos\varphi,\ d_2 = -\kappa\sin\varphi,\ d_3 = \frac{d\varphi}{v} + \tau,$$

[6, 7]. It is obvious that in the case of $\varphi = 0°$, that is, these frames are coincident, the geometric meaning of the curvatures d_1, d_2, and d_3 becomes clear as

$$d_1 = \kappa,\ d_2 = 0,\ d_3 = \tau.$$

Theorem 2.3. *Let $\alpha = \alpha(s)$ be a polynomial curve with the curvatures d_1, d_2, and d_3, then the curve α is*

- *a straight line if $d_1 = d_2 = 0$,*
- *a planar curve if $d_2 = d_3 = 0$, [6, 7].*

3. Bertrand-like Curves via Frenet-Like Curve (Flc) Frame

In this section, the pairs of Bertrand-like curves are introduced according to the linear dependence of normal-like vectors of the curves correspondingly. After, the analogs of some characteristic theorems for Bertrand curves are investigated for Bertrand-like curves.

Definition 3.1. *Let $\alpha : I \to \mathbb{E}^3$ and $\alpha^* : I \to \mathbb{E}^3$ be two curves and the Flc frames of the curves α and α^* at each points $\alpha(s)$ and $\alpha^*(s)$ be $\{T(s),\ D_2(s),\ D_1(s)\}$ and $\{T^*(s),\ D_2{}^*(s),\ D_1{}^*(s)\}$, respectively.*

If $\{D_2{}^*(s), D_2(s)\}$ is linearly dependent for all $s \in I$ (in other words, the normal-like vectors of the curves α and α^* are in the same direction at each corresponding point), then the curves $\{\alpha,\ \alpha^*\}$ is called a Bertrand-like curve pair. Moreover, the curve α is called a Bertrand-like curve, and the curve α^* is called the Bertrand-like conjugate curve of α.

Also, according to the definition of Bertrand-like curve pairs, the relationship between the elements of the Flc frames of each curve can be written in the following form

$$\begin{aligned} &T^* = \cos\theta T - \sin\theta D_1, \\ &{D_2}^* = D_2, \\ &{D_1}^* = \sin\theta T + \cos\theta D_1, \end{aligned} \tag{3}$$

where θ is the angle between the tangent vectors T and T^* of the Bertrand-like partner curves α and α^*, respectively.

Theorem 3.2. *Let $\alpha^*, \alpha \subset \mathbb{E}^3$ be a pair of the Bertrand-like curves, then the relationship at the corresponding points of the curves α and α^* is presented by*

$$\alpha^*(s) = \alpha(s) + \lambda D_2(s), \tag{4}$$

where λ is a real value function, that is, the distance between the corresponding points on the curves α and α^* is constant.

Proof. Let $\alpha \subset \mathbb{E}^3$ be a unit speed Bertrand-like curve with arc-length parameter s without the loss of generality. Assume that s^* is the arc-length parameter of Bertrand-like conjugate curve α^*. By the definition of the Bertrand-like curve pair, we can write

$$\alpha^*(s) = \alpha(s) + u(s) D_2(s),$$

for $u(s) \neq 0$. Differentiating both sides of this last equation with respect to s and using the Flc frame, the following equality is found

$$v^* T^*(s) = T(s) + u'(s) D_2(s) + u(s)\left(-d_1 T(s) + d_3 D_1(s)\right),$$

where $v^* = \frac{ds^*}{ds}$ i.e., $s^*(s) = \int\limits_0^s \|\alpha^{*\prime}(t)\| dt$. Taking the inner product by ${D_2}^* = \pm D_2$ both sides of this equation, we get $u'(s) = 0$. So, we say that $u(s) = \lambda \in R$. Consequently, it is easy to see that

$$\|\alpha^*(s) - \alpha(s)\| = \|\lambda D_2(s)\| = |\lambda| = \text{constant}.$$

□

Theorem 3.3. *Let $\alpha^*, \alpha \subset \mathbb{E}^3$ be a pair of the Bertrand-like curves with the Flc frames $\left\{T^*, D_2^*, D_1^*\right\}$ and $\{T, D_2, D_1\}$ in Euclidean 3-space, respectively, then the angle between the tangent vectors at the corresponding points of the curves α and α^* is given by*

$$\theta = \int \left(d_2 - v^* d_2^*\right) ds.$$

Proof. Let θ be the angle between the tangent vectors at the corresponding points of the curves α and α^*. Then one can write

$$\cos\theta = \langle T^*, T\rangle.$$

Differentiating both sides of this last equation with respect to s and using Eqs. (1) and (3), we get

$$\begin{aligned} \frac{d}{ds}\left(\langle T, T^*\rangle\right) &= d_2 \langle D_1, T^*\rangle + v^* d_2^* \left\langle T, D_1^*\right\rangle, \\ &= \langle d_1 D_2 + d_2 D_1, T^*\rangle + v^* \left\langle T, d_1^* D_2^* + d_2^* D_1^*\right\rangle \\ &= \left\langle d_1 D_2^* + d_2 D_1, T^*\right\rangle + v^* \left\langle T, d_1^* D_2 + d_2^* D_1^*\right\rangle \\ &= d_2 \langle D_1, \cos\theta T - \sin\theta D_1\rangle + v^* d_2^* \langle T, \sin\theta T + \cos\theta D_1\rangle \\ &= -d_2 \sin\theta + v^* d_2^* \sin\theta \\ &= \left(v^* d_2^* - d_2\right) \sin\theta \end{aligned}$$

and also,

$$\frac{d}{ds}\left(\langle T, T^{*}\rangle\right) = \frac{d\theta}{ds}\frac{d}{d\theta}\left(\cos\theta\right) = -\sin\theta\frac{d\theta}{ds}.$$

Considering these last two relations together requires

$$\frac{d\theta}{ds} = \left(d_2 - v^{*}d_2^{*}\right).$$

This completes the proof. □

Corollary 3.4. *The angle θ between their tangent vectors of Bertrand-like partner curves $\{\alpha, \alpha^{*}\}$ is constant if and only if the second curvatures of these curves satisfy the relation $d_2 = v^{*}d_2^{*}$.*

Theorem 3.5. *Let $\alpha^{*}, \alpha \subset \mathbb{E}^3$ be a pair of the Bertrand-like curves. Then the following relations are provided between the Flc frames $\left\{T^{*}, D_2^{*}, D_1^{*}\right\}$ and $\{T, D_2, D_1\}$ of the Bertrand-like partner curves:*

$$\begin{bmatrix} T^{*} \\ D_2^{*} \\ D_1^{*} \end{bmatrix} = \begin{bmatrix} \frac{(1-\lambda d_1)}{v^{*}} & 0 & \frac{\lambda d_3}{v^{*}} \\ 0 & 1 & 0 \\ -\frac{\lambda d_3}{v^{*}} & 0 & \frac{(1-\lambda d_1)}{v^{*}} \end{bmatrix} \begin{bmatrix} T \\ D_2 \\ D_1 \end{bmatrix}, \tag{5}$$

where $v^{*} = \|\alpha^{*\prime}(s)\|$.

Proof. Let $\alpha^{*}, \alpha \subset \mathbb{E}^3$ be a pair of the Bertrand-like curves with arc-length parameters s^{*} and s in terms of the Flc frame $\left\{T^{*}, D_2^{*}, D_1^{*}\right\}$ and $\{T, D_2, D_1\}$, respectively. Differentiating Eq. (4), we obtain

$$v^{*}T^{*}(s) = (v - v\lambda d_1)\, T(s) + v\lambda d_3 D_1.$$

Comparing this last equation with Eq. (3), it is seen that

$$\left(\cos\theta T - \sin\theta D_1\right) v^{*} = \left(1 - \lambda d_1\right) T + \lambda d_3 D.$$

Therefore, this equality allows us to say that

$$\cos\theta = \frac{(1-\lambda d_1)}{v^{*}} \text{ and } \sin\theta = -\frac{\lambda d_3}{v^{*}}. \tag{6}$$

Substituting these last equations into Eq. (3) completes the proof. □

Corollary 3.6. *Let θ denote the angle between the tangent vectors at the corresponding points of the Bertrand-like partner curves $\{\alpha, \alpha^{*}\}$. Then the following equation is satisfied*

$$tan\theta = \frac{\lambda d_3}{\lambda d_1 - 1} \tag{7}$$

for $d_1 \neq \frac{1}{\lambda}$.

Theorem 3.7. *Let $\alpha^{*}, \alpha \subset \mathbb{E}^3$ be any curves with arc-length parameters s^{*} and s in terms of the Flc frame $\left\{T^{*}, D_2^{*}, D_1^{*}\right\}$ and $\{T, D_2, D_1\}$ in Euclidean 3-space, respectively. The curve α is a Bertrand-like curve if and only if*

$$\lambda d_1 - \mu d_3 = 1$$

where $\lambda, \mu \in \mathbb{R} - \{0\}$.

Proof. (⇒:) Let $\alpha^*, \alpha \subset \mathbb{E}^3$ be a pair of the Bertrand-like curves. From Eq. (7), we have

$$\cot\theta = \frac{\lambda d_1 - 1}{\lambda d_3}.$$

For $\mu = \lambda\cot\theta$, the desired is found.
(⇐:) Conversely, let $\alpha(s)$ be a smooth curve and the linear equality $\lambda d_1 - \lambda\cot\theta d_3 = 1$ be satisfied for some constants λ and θ. The curve α^* is defined by $\alpha^*(s) = \alpha(s) + \lambda D_2(s)$. Under the given assumption, this equality leads to the principal normal-like vector D_2^* of the curve α^* being equal to the normal-like vector field D_2 of the curve α.
□

Theorem 3.8. *Let $\alpha^*, \alpha \subset \mathbb{E}^3$ be a pair of the Bertrand-like curves. Then, the relations between the Flc frame $\left\{T^*, D_2^*, D_1^*\right\}$ of the curve α^* and the Frenet frame $\{T, N, B\}$ of the curve α are given as:*

$$\begin{bmatrix} T^* \\ D_2^* \\ D_1^* \end{bmatrix} = \begin{bmatrix} \frac{(1-\lambda d_1)}{v^*} & \frac{-v\lambda d_3}{v^*}\sin\varphi & \frac{\lambda d_3}{v^*}\cos\varphi \\ 0 & \cos\varphi & \sin\varphi \\ -\frac{\lambda d_3}{v^*} & \frac{-v(1-\lambda d_1)}{v^*}\sin\varphi & \frac{(1-\lambda d_1)}{v^*}\cos\varphi \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

where φ is the angle between the principal normal vector N and the normal-like vector D_2 of α.

Proof. It is easily proved by substituting Eq.(2) into Eq. (5). □

Theorem 3.9. *Let $\alpha^*, \alpha \subset \mathbb{E}^3$ be a pair of the Bertrand-like curves. Then, there exist the following relations between the curvatures $\left\{d_1^*, d_2^*, d_3^*\right\}$ of the curve α^* and the curvatures $\{d_1, d_2, d_3\}$ of α:*

$$\begin{aligned} d_1^* &= \frac{d_1(1-\lambda d_1) - \lambda d_3^2}{(1-\lambda d_1) + \lambda^2 d_3^2}, \\ d_2^* &= \frac{d_2\left((1-\lambda d_1)^2 + \lambda^2 d_3^2\right) - (1-\lambda d_1)^2}{\left((1-\lambda d_1)^2 + \lambda^2 d_3^2\right)^{3/2}}\left(\frac{\lambda d_3}{\lambda d_1 - 1}\right)', \\ d_3^* &= \frac{d_3(1-\lambda d_1) + \lambda d_1 d_2}{(1-\lambda d_1)^2 + \lambda^2 d_3^2}, \end{aligned}$$

for $d_1 \neq \frac{1}{\lambda}$.

Proof. Let $\alpha^*, \alpha \subset \mathbb{E}^3$ be a pair of the Bertrand-like curves. From Eq. (6) and the equality $\cos^2\theta + \sin^2\theta = 1$, we get

$$\left(\frac{1}{v^*}\right)^2\left((1-\lambda d_1)^2 + \lambda^2 d_3^2\right) = 1.$$

Hence, we have

$$v^* = \sqrt{\left((1-\lambda d_1)^2 + \lambda^2 d_3^2\right)}. \tag{8}$$

Differentiating Eq. (5) with respect to s and by using the Flc frame, we obtain

$$\begin{aligned} T^{*\prime} &= \left(\frac{\lambda d_3(1-\lambda d_1)^2}{v^*\left((1-\lambda d_1)^2 + \lambda^2 d_3^2\right)}\left(\frac{\lambda d_3}{\lambda d_1 - 1}\right)' - \frac{\lambda d_3 d_2}{v^*}\right)T + \left(\frac{d_1(1-\lambda d_1)}{v^*} - \frac{\lambda d_3^2}{v^*}\right)D_2 \\ &\quad + \left(\frac{d_2(1-\lambda d_1)}{v^*} - \frac{(1-\lambda d_1)^3}{v^*\left((1-\lambda d_1)^2 + \lambda^2 d_3^2\right)}\left(\frac{\lambda d_3}{\lambda d_1 - 1}\right)'\right)D_1, \\ D_2^{*\prime} &= \left(-d_1 T + d_3 D_1\right). \end{aligned}$$

Considering these last equations and Eq. (5) together, the curvature d_1^* is calculated as

$$d_1^* = \frac{\left\langle T^{*\prime}, D_2^* \right\rangle}{v^*} = \frac{1}{v^*}\left(\frac{v d_1 (1-\lambda d_1)}{v^*} - \frac{\lambda d_3^2}{v^*}\right).$$

Also, with the help of Eq. (8), it is found

$$d_1^* = \frac{d_1 (1-\lambda d_1) - \lambda d_3^2}{(1-\lambda d_1)^2 + \lambda^2 d_3^2}.$$

Similarly, the curvatures d_2^* and d_3^* are

$$d_2^* = \frac{\left\langle T^{*\prime}, D_1^* \right\rangle}{v^*} = \frac{d_2\left((1-\lambda d_1)^2 + \lambda^2 d_3^2\right) - (1-\lambda d_1)^2}{\left((1-\lambda d_1)^2 + \lambda^2 d_3^2\right)^{3/2}}\left(\frac{\lambda d_3}{\lambda d_1 - 1}\right)',$$

and

$$d_3^* = \frac{\left\langle D_2^{*\prime}, D_1^* \right\rangle}{v^*} = \frac{d_3 (1-\lambda d_1) + \lambda d_1 d_2}{(1-\lambda d_1)^2 + \lambda^2 d_3^2}.$$

□

Corollary 3.10. *Let $\alpha^*, \alpha \subset \mathbb{E}^3$ be a pair of the Bertrand-like curves with $\left\{d_1^*, d_2^*, d_3^*\right\}$ and $\{d_1, d_2, d_3\}$, then the following geometric interpretations are given:*

i. If α is a straight line such that $d_1 = d_2 = 0$ and $d_3 \neq 0$, then the curvatures of Bertrand-like conjugate curve are

$$d_1^* = \frac{-\lambda d_3^2}{1+\lambda^2 d_3^2},\ d_2^* = \frac{(\lambda d_3)'}{\lambda^3 d_3^3},\ \text{and } d_3^* = \frac{d_3}{1+\lambda^2 d_3^2}.$$

Thus, the Bertrand-like conjugate curve of a straight line cannot be a straight line, and also a planar curve.

ii. If α is a planar curve such that $d_2 = d_3 = 0$ and $d_1 \neq \frac{1}{\lambda}$, then the curvatures of its Bertrand-like conjugate curve are

$$d_1^* = \frac{d_1}{1-\lambda d_1},\ d_2^* = \frac{-1}{(1-\lambda d_1)},\ \text{and } d_3^* = 0.$$

So the Bertrand-like conjugate curve cannot be a straight line, and also a planar curve, too.

Example 3.11. *Consider a Bertrand-like curve α given by $\alpha(s) = \left(s, \frac{s^2}{2}, \frac{s^3}{6}\right)$. By straightforward calculations, we get the following the Flc frame apparatus:*

$$T(s) = \left(\frac{2}{2+s^2}, \frac{2s}{2+s^2}, \frac{s^2}{2+s^2}\right),$$
$$D_2(s) = \left(\frac{-s^2}{(2+s^2)\sqrt{1+s^2}}, \frac{-s^3}{(2+s^2)\sqrt{1+s^2}}, \frac{2\sqrt{1+s^2}}{2+s^2}\right),$$
$$D_1(s) = \left(\frac{s}{\sqrt{1+s^2}}, \frac{-1}{\sqrt{1+s^2}}, 0\right),$$

and it is easy to see that the curvatures of the Bertrand-like curve α are obtained as

$$d_1 = \frac{4s}{\sqrt{1+s^2}(2+s^2)^2},$$
$$d_2 = \frac{-4}{\sqrt{1+s^2}(2+s^2)^2},$$
$$d_3 = \frac{2s^2}{\sqrt{1+s^2}(2+s^2)^2}.$$

Therefore, for $\lambda = 10$, the Bertrand-like conjugate curve of α can be given as:

$$\alpha^*(s) = \left(s - \frac{10s^2}{\sqrt{1+s^2}\,(2+s^2)}, \frac{s^2}{2} - \frac{10s^3}{\sqrt{1+s^2}\,(2+s^2)}, \frac{s^3}{6} + \frac{20\sqrt{1+s^2}}{2+s^2} \right)$$

with the help of the equality $\alpha^*(s) = \alpha(s) + \lambda D_2(s)$, see Figure 1.

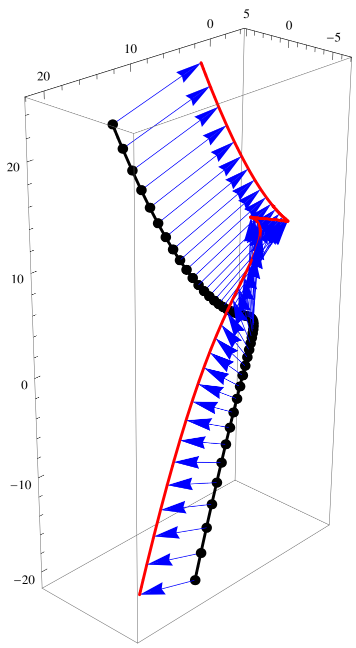

Figure 1: Bertrand-like curve α (black) and its Bertrand-like conjugate curve α^* (red) with $s \in (-5, 5)$.

4. Conclusion

In this paper, the pairs of Bertrand-like curves, which are associated curves with common normal-like vectors, are characterized by their curvatures of the curves with respect to their Flc frames. Unlike other existing literature, the problem is fixed for constructing a pair of curves $\{\alpha,\ \alpha^*\}$ that are sharing common principal normal vectors even if there exist singularities on a curve α in the case of

$$\alpha'(s) \neq 0,\ \alpha''(s) = \alpha'''(s) = \ldots = \alpha^{(n-1)}(s) = 0,\ \alpha^{(n)}(s) \neq 0$$

at any point $\alpha(s)$. The definition and characterizations of any Bertrand-like curve and its conjugate are given. Furthermore, the relations between the curvatures of these curves are obtained.

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

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