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Characterization of ss-supplemented modules with respect to finitely generated factor modules in view of singularity

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Abstract. In this essay (amply) cofinitely δ_{ss}-supplemented modules are presented and fundamental algebraic features of these modules are examined. Privately, a ring characterization theorem is presented as follows. R is a δ_{ss}-perfect ring if and only if every (projective) left R-module is (amply) cofinitely δ_{ss}-supplemented. Moreover, the question when cofinitely δ_{ss}-supplemented modules are cofinitely ss-supplemented is checked. With this aim we define left Δ_{ss}-rings and the fact that a ring R is a left Δ_{ss}-ring if and only if each cofinitely δ_{ss}-supplemented R-module is cofinitely ss-supplemented is proven.

1. Introduction

In Module Theory, it is an important research area to investigate modules which have a decomposition by direct summands. Generalizing this, modules whose submodules have a supplement are of an importance too. In Figure 1 we emphasise the wide usage field of supplement submodules via keywords taken from Web of Science (WOS). Now let us introduce the basic concepts of this field that we need. At first, underline that R will indicate an associative ring with unit and W will indicate a unitary R-module in this essay.

By $A \leq W$ we indicate that A is a submodule of W. A module W is called simple if W has only trivial submodules. *Socle*(W) which points the sum of whole simple submodules of W, is denoted by $Soc(W)$. $A \leq W$ is called small (denoted by $A \ll W$) if $A + T \neq W$ for every proper $T \leq W$. The sum of whole small submodules of W is denoted by $Rad(W)$. For $A \leq W$, if there exists a submodule $T \leq W$ satisfying $A + T = W$ and $A \cap T \ll T$, then T is a supplement submodule of A in W. If $A \leq W$ is of a supplement contained in B whenever $A + B = W$ for any $B \leq W$, then A is of ample supplements in W. A module W is called (amply) supplemented if each submodule of W is of a (ample supplements) supplement in W. In [2], the authors characterized cofinitely supplemented modules as the modules whose maximal submodules are of supplements. $A \leq W$ is called cofinite if $\frac{W}{A}$ is finitely generated and W is called cofinitely supplemented if each cofinite submodule of W is of a supplement in W.

In the years of 2000 and 2007, [28] and [13] generalized small submodules and supplemented modules via singularity and contributed δ-small submodules and δ-supplemented modules in the literature. Afterwards, studies on the generalization of these modules were carried out and are still continuing [9, 17]. The sum of whole δ-small submodules of W will be indicated by $\delta(W)$.

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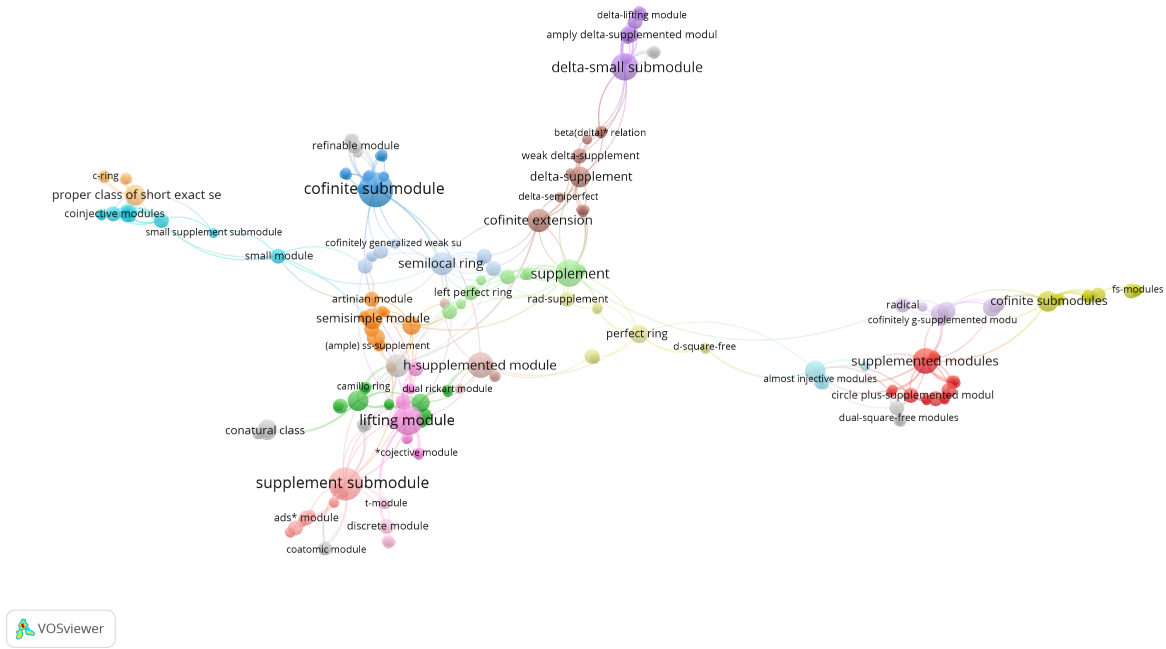

Figure 1: Bibliometric analyse of supplement submodules from WOS with respect to keywords

In [11], ss-supplemented modules are defined as follows. A module W is called ss-supplemented if there exists some $T \leq W$ such that $A + T = W$ and $A \cap T \leq Soc_s(T)$ for any $A \leq W$ where $Soc_s(T) = Soc(T) \cap Rad(T)$. Hence a new algebraic structure is constructed between semisimple submodules and supplemented modules. Motivetad by this, in [22] the authors introduced δ_{ss}-supplemented modules which takes place between ss-supplemented modules and δ-supplemented modules. A module W is called δ_{ss}-supplemented if there exists a submodule $T \leq W$ such that $A + T = W$ and $A \cap T \leq Soc_\delta(T)$ for any submodule of A of W where $Soc_\delta(T) = Soc(T) \cap \delta(T)$. And also see [6, 7, 15, 18, 19] for further and lifting property of these type of modules.

In the year of 2023 in [23], the authors generalized ss-supplemented modules with respect to finitely generated factor modules and so the definition of cofinitely supplemented modules was born in the literature. Combining [11, 22, 23], we have designed this study. This paper consists of three main parts. In the fist section of the study, we present a literature summary that includes the articles that motivate the basic definitions we constructed in the study with a bibliometric analyse support. In the second part of the study, we begin the original part of the work including three subsections. In the first subsection, cofinitely δ_{ss}-supplemented modules are defined, basic features as therotically are investigated and also necessary and sufficient conditions that will be equivalent to the definition are determined. In the second subsection of the main results we define amply cofinitely δ_{ss}-supplemented modules. And here we continue the process we followed in the previous section. Apart from this, we concretize with examples the relationship between the three emerging module types, which are generalizations of each other. Also, ring characterizations (see in Theorem 2.31) of these modules are investigated. By means of these concepts we reach the hierarchy given below for a module.

(Amply) cofinitely ss-supp. $\Rightarrow$ (Amply) cofinitely δ_{ss}-supp. $\Rightarrow$ (Amply) cofinitely δ-supp.

And we show that the relations given above are not reversible (see in Example 2.33 and Example 2.34).

In the last subpart of the study we examine the suitable conditions which makes cofinitely δ_{ss}-supplemented modules cofinitely ss-supplemented given in [23] (see Proposition 2.45). With this aim we also define left Δ_{ss}-rings and it is proven in Proposition 2.41 that each cofinitely δ_{ss}-supplemented R-module is cofinitely ss-supplemented over a Δ_{ss}-ring R. And in Conclusion, reference studies that will help us generalize the definitions we have created and contribute to the sustainability of the study are included.

2. Main results

2.1. Cofinitely δ_{ss}-Supplemented Modules

In this part of the study, cofinitely δ_{ss}-supplemented modules are defined and basic theoretic properties of these modules are presented with additional examples.

Definition 2.1. *If there exists a submodule $B \leq W$ satisfying $A + B = W$ and $A \cap B \leq Soc_\delta(B)$ for each cofinite submodule $A \leq W$ where $Soc_\delta(B) = Soc(B) \cap \delta(B)$ then W is called a cofinitely δ_{ss}-supplemented module.*

Obviously, each cofinitely δ_{ss}-supplemented module is cofinitely δ-supplemented. Moreover, δ_{ss}-supplemented modules are cofinitely δ_{ss}-supplemented. Moreover, δ_{ss}-supplemented modules are cofinitely δ_{ss}-supplemented. The converse is provided for finitely generated modules.

Proposition 2.2. *Any factor module of a cofinitely δ_{ss}-supplemented module is cofinitely δ_{ss}-supplemented.*

Proof. Let W be cofinitely δ_{ss}-supplemented and $A \leq W$. For each cofinite submodule $\frac{K}{A}$ of $\frac{W}{A}$, we have $\frac{W}{K} \cong \frac{\frac{W}{A}}{\frac{K}{A}}$ is finitely generated. So $K \leq W$ is cofinite. By hypothesis, there exists $T \leq W$ satisfying $K + T = W$ and $K \cap T \leq Soc_\delta(T)$. Thus, $\frac{W}{A} = \frac{K}{A} + \frac{T+A}{A}$ and $\frac{K}{A} \cap \frac{T+A}{A} = \frac{K\cap(T+A)}{A} = \frac{A+(T\cap K)}{A} \ll_\delta \frac{T+A}{A}$ [28, Lemma 1.3(2)]. Moreover, $\frac{A+(T\cap K)}{A} \cong \frac{T\cap K}{T\cap(K\cap A)} = \frac{T\cap K}{T\cap A}$ is semisimple as a factor module of the semisimple module $K \cap T$ by [10, Corollary 8.1.5]. So, $\frac{K}{A} \cap \frac{T+A}{A} \leq Soc_\delta(\frac{T+A}{A})$. Hence, $\frac{T+A}{A}$ is a δ_{ss}-supplement of $\frac{K}{A}$ in $\frac{W}{A}$, that is, the factor module $\frac{K}{A}$ is cofinitely δ_{ss}-supplemented. □

Corollary 2.3. *Any homomorphic image of a cofinitely δ_{ss}-supplemented module is cofinitely δ_{ss}-supplemented module.*

Corollary 2.4. *Any direct summand of a cofinitely δ_{ss}-supplemented module is cofinitely δ_{ss}-supplemented module.*

Lemma 2.5. *Let W be a module and $A, X \leq W$ such that $X \ll_\delta W$. If $A + X$ is of a δ_{ss}-supplement B in W, then a projective semisimple direct summand P of W exists satisfying that $B + P$ is a δ_{ss}-supplement of A in W.*

Proof. By hypothesis, we get $(A + X) + B = W$, $(A + X) \cap B \leq Soc_\delta(B)$. Thus, $(A + X) \cap B \ll_\delta B$ and $(A + X) \cap B$ is semisimple. As $W = X + (A + B)$ and $X \ll_\delta W$, there exists a projective semisimple submodule $P \leq W$ provided $P \leq X$ and $W = P \oplus (A+B)$. In this case, $W = A+(B+P)$. Now, it remains to show that $A\cap(B+P) \ll_\delta B+P$ and also it is semisimple. Since $A\cap(B+P) \leq [B\cap(A+P)]+[P\cap(A+B)] = [B\cap(A+P)] \leq [B\cap(A+X)] \ll_\delta B$ and so $A\cap(B+P) \ll_\delta B+P$ by [28, Lemma 1.3(1)] and it is also semisimple as a submodule of the semisimple submodule $B \cap (A + X)$ by [10, Corollary 8.1.5]. This completes the proof as $A \cap (B + P) \ll_\delta Soc_\delta(B + P)$. □

We give the following standart lemma as it is useful to prove the fact that an arbitrary sum of cofinitely δ_{ss}-supplemented modules is cofinitely δ_{ss}-supplemented.

Lemma 2.6. *Let $A, B \leq W$ such that A is cofinitely δ_{ss}-supplemented and B is cofinite. If $A + B$ is of a δ_{ss}-supplement in W, then B is of a δ_{ss}-supplement in W.*

Proof. Let S be a δ_{ss}-supplement of $A+B$ in W, that is $W = (A+B)+S$ and $(A+B)\cap S \leq Soc_\delta(S)$. Since $B \leq W$ is cofinite, then $\frac{\frac{W}{B}}{\frac{S+B}{B}} \cong \frac{W}{S+B} = \frac{(S+B)+A}{S+B} \cong \frac{A}{A\cap(S+B)}$ is finitely generated and so, $A \cap (S+B) \leq A$ is cofinite. From the assumption, there exists $T \leq A$ satisfying $A = [A \cap (S+B)] + T$ and $[A \cap (S+B)] \cap T = T \cap (S+B) \leq Soc_\delta(T)$. In the remaining part of the proof, it will be shown that $S+T$ is a δ_{ss}-supplement of B in W. Clearly, $W = A+(B+S) = \{[A\cap(S+B)]+T\}+(B+S) = B+(S+T)$. Moreover, $B\cap(S+T) \leq [S\cap(B+T)]+[T\cap(B+S)] \leq [S\cap(B+A)]+[T\cap(B+S)] \ll_\delta S+T$ and $[S\cap(B+A)]+[T\cap(B+S)]$ is also semisimple by [10, Corollary 8.1.5]. Therefore $B \cap (S+T)$ is also δ-small and semisimple in $S+T$ as a submodule of $[S\cap(B+A)]+[T\cap(B+S)]$ by [28, Lemma 1.3(a)] and [10, Corollary 8.1.5]. □

Proposition 2.7. *An arbitrary sum of cofinitely δ_{ss}-supplemented modules is cofinitely δ_{ss}-supplemented.*

Proof. Let $\{M_i\}_{i\in I}$ be a community of cofinitely δ_{ss}-supplemented modules and $W = \sum_{i\in I} M_i$. For any cofinite $K \leq W$, we have $W = M_{i_1}+M_{i_2}+...+M_{i_n}+K$ where $n \in \mathbb{N}$ and $i_1, i_2, ..., i_n \in I$. By hypothesis, M_{i_1} is cofinitely δ_{ss}-supplemented and 0 is the trivial δ_{ss}-supplement of W, $M_{i_2}+...+M_{i_n}+K$ is of a δ_{ss}-supplement in W via Lemma 2.6. Applying Lemma 2.6 again by induction, it can be seen that K is of a δ_{ss}-supplement in W. □

Corollary 2.8. *Any direct sum of cofinitely δ_{ss}-supplemented modules is cofinitely δ_{ss}-supplemented.*

Corollary 2.9. *Let W be a cofinitely δ_{ss}-supplemented module. Then so is any W-generated module.*

Proof. Let G be a W-generated module. Then, there exists an epimorphism $f : W^{(I)} = \oplus W \longrightarrow G$. Thus, $W^{(I)}$ cofinitely δ_{ss}-supplemented by Corollary 2.8. So G is cofinitely δ_{ss}-supplemented by Proposition 2.2. □

Definition 2.10. *A module W is called $\oplus$-cofinitely-δ_{ss}-supplemented if there exists a δ_{ss}-supplement which is a direct summand of W for each cofinite submodule of W.*

Proposition 2.11. *Let W be a cofinitely δ_{ss}-supplemented module. In this case, each cofinite submodule of $\frac{W}{Soc_\delta(W)}$ is a direct summand.*

Proof. Let $\frac{K}{Soc_\delta(W)} \leq \frac{W}{Soc_\delta(W)}$ be cofinite. Then $K \leq W$ is cofinite and by hypothesis there exists $T \leq W$ such that $K+T = W$ and $K \cap T \leq Soc_\delta(T)$. Clearly, $\frac{K}{Soc_\delta(W)} + \frac{T+Soc_\delta(W)}{Soc_\delta(W)} = \frac{W}{Soc_\delta(W)}$. Moreover $\frac{K}{Soc_\delta(W)} \cap \frac{T+Soc_\delta(W)}{Soc_\delta(W)} = \frac{(K\cap T)+Soc_\delta(W)}{Soc_\delta(W)} = 0$. Hence, $\frac{K}{Soc_\delta(W)} \leq_\oplus \frac{W}{Soc_\delta(W)}$. □

Corollary 2.12. *Let W be a cofinitely δ_{ss}-supplemented module. In this case, $\frac{W}{Soc_\delta(W)}$ is $\oplus$-cofinitely δ_{ss}-supplemented.*

Corollary 2.13. *Let W be a cofinitely δ_{ss}-supplemented module. If $\frac{W}{Soc_\delta(W)}$ is finitely generated, then $\frac{W}{Soc_\delta(W)}$ is δ_{ss}-supplemented and semisimple.*

Remember from [22] that a module W is called strongly δ-local, if it is δ-local and $\delta(W) \leq Soc(W)$.

Lemma 2.14. *Let W be a module and $T, X \leq W$ such that T is a δ_{ss}-supplement of a maximal submodule of W and $T+X$ is of a δ_{ss}-supplement in W. In this case, X has a δ_{ss}-supplement in W.*

Proof. As T is a δ_{ss}-supplement of a maximal submodule of W, T is strongly δ-local or projective semisimple by [22, Proposition 3.4]. If T is projective semisimple, it is δ_{ss}-supplemented obviously. Or if T is strongly δ-local, it is δ_{ss}-supplemented by [22, Lemma 4.1]. Taking into account both cases, it can be obtained that X is of a δ_{ss}-supplement in W by [22, Lemma 4.8]. □

Now we want to construct the conditions when a module W is cofinitely δ_{ss}-supplemented. With this aim we point by $Cof_{\delta_{ss}}(W)$ the sum of whole submodules of W that are δ_{ss}-supplements of maximal submodules of W. If there is no such a module, $Cof_{\delta_{ss}}(W) = 0$ is accepted. Also, we prefer the notations δ-$Loc_s(W)$ and $Soc_P(W)$ to indicate the sum of strongly δ-local and the sum of projective semisimple submodules of W, respectively.

Theorem 2.15. *The implications given above are equivalent for a module W.*

(1) W is cofinitely δ_{ss}-supplemented.
(2) A δ_{ss}-supplement is present in W for each maximal submodule of W.
(3) $\frac{W}{\delta\text{-}Loc_s(W)+Soc_P(W)}$ is of no maximal submodule.
(4) $\frac{W}{Cof_{\delta_{ss}}(W)}$ is of no maximal submodule.

Proof. (1) $\Rightarrow$ (2) : Let $P \leq W$ be maximal. Then P is cofinite in W as $\frac{W}{P}$ is simple. Therefore, P is of a δ_{ss}-supplement in W.

(2) $\Rightarrow$ (3) : Suppose that the submodule $\frac{P}{\delta\text{-}Loc_s(W)+Soc_P(W)} \leq \frac{W}{\delta\text{-}Loc_s(W)+Soc_P(W)}$ be maximal. In this case, P is a maximal submodule of W containing $\delta\text{-}Loc_s(W) + Soc_P(W)$. From assumption, there exists a submodule $T \leq W$ such that $P + T = W$ and $P \cap T \leq Soc_\delta(T)$. In this case, T is strongly δ-local or projective semisimple by [22, Proposition 3.4]. Then, we have $T \leq \delta\text{-}Loc_s(W) + Soc_P(W) \leq P$. Hence, we get the contradiction $T + P = P = W$.

(3) $\Rightarrow$ (4) : Assume that $\frac{P}{Cof_{\delta_{ss}}(W)} \leq \frac{W}{Cof_{\delta_{ss}}(W)}$ is maximal. In this case, P is a maximal submodule of W including P. As $\delta\text{-}Loc_s(W) \leq Cof_{\delta_{ss}}(W)$ and $Soc_P(W) \leq Cof_{\delta_{ss}}(W)$, we get $\delta\text{-}Loc_s(W) + Soc_P(W) \leq Cof_{\delta_{ss}}(W) \leq P$. Consider the natural epimorphism $\pi : W \longrightarrow \frac{W}{\delta\text{-}Loc_s(W)+Soc_P(W)}$. Then $\frac{P}{\delta\text{-}Loc_s(W)+Soc_P(W)}$ is maximal in $\frac{W}{\delta\text{-}Loc_s(W)+Soc_P(W)}$ which is a contradiction.

(4) $\Rightarrow$ (1) : Let $K \leq W$ be cofinite. In this case, as $\frac{W}{K}$ is finitely generated for $K \leq K + Cof_{\delta_{ss}}(W) \leq W$, $\frac{\frac{W}{K}}{\frac{K+Cof_{\delta_{ss}}(W)}{K}} \cong \frac{W}{K+Cof_{\delta_{ss}}(W)}$ is finitely generated and so, $K + Cof_{\delta_{ss}}(W)$ is cofinite. Moreover, $K + Cof_{\delta_{ss}}(W)$ is not proper in W. Otherwise, if it were proper, then there would be a maximal submodule $\frac{P}{K+Cof_{\delta_{ss}}(W)} \leq \frac{W}{K+Cof_{\delta_{ss}}(W)}$ including every proper submodule of $\frac{W}{K+Cof_{\delta_{ss}}(W)}$ as it is finitely generated. Thus, $Cof_{\delta_{ss}}(W) \leq K + Cof_{\delta_{ss}}(W) \leq P \leq W$ where P is maximal in W. Thus, $\frac{P}{K+Cof_{\delta_{ss}}(W)} \leq \frac{W}{K+Cof_{\delta_{ss}}(W)}$ would be maximal which contradicts with (3). Then, $K + Cof_{\delta_{ss}}(W) = W$ is deduced. As $\frac{W}{K}$ is finitely generated, we have $W = K + A_1 + A_2 + ... + A_n$ for some positive integer $i = 1, 2, ..., n$ where each A_i is a δ_{ss}-supplement of a maximal submodule of W. As A_n is a δ_{ss}-supplement of a maximal submodule of W and $(K + A_1 + A_2 + ...A_{n-1}) + A_n$ is of the trivial δ_{ss}-supplement 0 in W, then $K + A_1 + A_2 + ...A_{n-1}$ is of a δ_{ss}-supplement in W by Lemma 2.14. Repeating Lemma 2.14 again and again, at the end it will be verified that K has a δ_{ss}-supplement in W. Hence, W is cofinitely δ_{ss}-supplemented. □

By means of the theorem given above we can give the following example verifying that a cofinitely δ_{ss}-supplemented module may not be δ_{ss}-supplemented.

Example 2.16. *Since the $\mathbb{Z}$-module $\mathbb{Q}$ does not have maximal submodules, then it is cofinitely δ_{ss}-supplemented by Theorem 2.15 as $\frac{\mathbb{Q}}{Cof_{\delta_{ss}}(\mathbb{Q})} = \frac{\mathbb{Q}}{0} \cong \mathbb{Q}$. However, it is not a δ_{ss}-supplemented module.*

Now we go on by giving anecdotes showing the conditions that make the relations reversible mentioned before.

The following proposition taken from [22] is given for the completeness.

Proposition 2.17. *Given implications below are equivalent for a finitely generated module W.*

(1) W is δ_{ss}-supplemented.
(2) W is cofinitely δ_{ss}-supplemented.
(3) $W = W_1 + W_2 + ... + W_n$ where each W_i is strongly δ-local or projective semisimple.

Theorem 2.18. *Let W be a cofinitely δ-supplemented module with $\delta(W) \leq Soc(W)$. In this case, W is cofinitely δ_{ss}-supplemented.*

Proof. Let the submodule $K \leq W$ be cofinite. Then, there exists a submodule S of W such that $K + S = W$ and $K \cap S \ll_\delta S$. Thus, $K \cap S \leq \delta(S) \leq \delta(W)$. Then $K \cap S$ is semisimple by hypothesis. Thus, S is a δ_{ss}-supplement of K in W. Consequently, W is cofinitely δ_{ss}-supplemented. □

2.2. Amply Cofinitely δ_{ss}-Supplemented Modules

Here we define cofinitely δ_{ss}-supplemented modules as a genaralization of the modules given in the previous subsection. We investigate basic features and ring characterization of these modules.

Definition 2.19. *A submodule A of W is of ample δ_{ss}-supplements in W if each submodule B of W with $W = A + B$ includes a δ_{ss}-supplement of A in W [22].*

Definition 2.20. *If each cofinite submodule of a module W is of ample δ_{ss}-supplements in W then W is called amply cofinitely δ_{ss}-supplemented.*

Proposition 2.21. *If each submodule of a module W is cofinitely δ_{ss}-supplemented, then W is an amply cofinitely δ_{ss}-supplemented module.*

Proof. Let $K \leq W$ be cofinite and $W = K + B$ for some $B \leq W$. As $\frac{W}{K} = \frac{K+B}{K} \cong \frac{B}{B\cap K}$ is finitely generated, $K \cap B \leq B$ is cofinite. then, there exists $T \leq W$ such that $(K \cap B) + T = B$ and $(K \cap B) \cap T = K \cap T \ll_\delta T$ and $K \cap T$ is semisimple. Also, we have $W = K + B = K + [(K \cap B) + T] = K + T$. Hence, T is a δ_{ss}-supplement of K contained in B. It means that W is amply cofinitely δ_{ss}-supplemented. □

Corollary 2.22. *The implications given below are equivalent for a ring R :*

(1) Each module is cofinitely δ_{ss}-supplemented.
(2) Each module is amply cofinitely δ_{ss}-supplemented.

Remember from [3] that a module W is called coatomic if each submodule of W is included in a maximal submodule of W and a ring R is called a left max ring if $Rad(W) \ll W$ for each left R-module W, which is equivalent to the fact that R is a left max ring iff each nonzero left R-module is coatomic.

Proposition 2.23. *For a ring R each left R-module is amply cofinitely δ_{ss}-supplemented iff each left R-module is the sum of strongly δ-local or projective semisimple submodules.*

Proof. Let W be an arbitrary left R-module. By assumption W is amply cofinitely δ_{ss}-supplemented and so it is cofinitely δ_{ss}-supplemented. It follows that R is a δ_{ss}-perfect ring by [22, Theorem 5.3] and also R is a left max ring by [22, Proposition 5.5]. Thus W is coatomic. Hence, W is the sum of strongly δ-local or projective semisimple submodules by [22, Proposition 4.10]. For the sufficiency, let W be an arbitrary module. By hypothesis W is the sum of strongly δ-local or projective semisimple submodules of W. Then W is coatomic and W is cofinitely δ_{ss}-supplemented by [22, Proposition 4.10]. Hence, W is amply cofinitely δ_{ss}-supplemented by Corollary 2.22. □

A module W is called π-projective if an $f \in End(W)$ exists such that $f(W) \leq A$ and $(I_W - f)(W) \leq B$ whenever $W = A + B$ for any $A, B \leq W$.

Proposition 2.24. *Let W be a π-projective cofinitely δ_{ss}-supplemented module. In this case, W is amply cofinitely δ_{ss}-supplemented.*

Proof. Let $K \leq W$ be cofinite and $B \leq W$ where $W = K + B$. From hypothesis there exists an endomorphism $f : W \longrightarrow W$ such that $f(W) \leq K$ and $(I_W - f)(W) \leq B$. It is clear that $(I_W - f)(K) \leq K$. Let T be a δ_{ss}-supplement of K in W. In this case, $K + T = W$, $K \cap T \ll_\delta T$ and $K \cap T$ is semisimple. Therefore, $W = f(W) + (I_W - f)(W) = f(W) + (I_W - f)(K + T) \leq K + (I_W - f)(K) + (I_W - f)(T) \leq K + (I_W - f)(T) \leq W$. Then we have $K + (I_W - f)(T) = W$. Note that $(I_W - f)(T) \leq (I_W - f)(W) \leq B$. Now it will be shown that $K \cap (I_W - f)(T) = (I_W - f)(K \cap T)$. Let $y \in K \cap (I_W - f)(T)$. Then $y \in K$ and $y = (I_W - f)(t) = t - f(t)$ for some $t \in T$. From here, $y + f(t) = t \in K \cap T$. This implies that $K \cap (I_W - f)(T) = (I_W - f)(K \cap T) \ll_\delta (I_W - f)(T)$ by [28, Lemma 1.5] and also it is semisimple by [10, Corollary 8.1.5]. □

Lemma 2.25. *Let L_i $(1 \leq i \leq n)$ be a finite community of submodules of a module W such that for each $1 \leq i \leq n$, L_i is either strongly δ-local or projective semisimple and let $A \leq W$ such that $A + L_1 + L_2 + \cdots + L_n$ is of a δ_{ss}-supplement T in W. In this case, there exists (possibly empty) $I \subseteq \{1, 2, ..., n\}$ satisfying that $T + \sum_{i\in I} X_i$ is a δ_{ss}-supplement of A in W such that $X_i = L_i$ or X_i is a projective semisimple direct summand of L_i.*

Proof. Let $n = 1$. By assumption, $(A + L_1) + T = A + (L_1 + T) = W$, $(A + L_1) \cap T \ll_\delta T$ and $(A + L_1) \cap T$ is semisimple. If $A \cap (L_1 + T) \leq Soc_\delta(L_1 + T)$ can be shown, it will be verified that $L_1 + T$ is a δ_{ss}-supplement of A in W where $X_1 = L_1$. It is a known fact that $A \cap (L_1 + T) \leq [L_1 \cap (A + T)] + [T \cap (A + L_1)]$. Let us consider the submodule $H = L_1 \cap (A + T)$.

Case 1 : Let $H = L_1 \cap (A + T) \ll_\delta L_1$. As $T \cap (A + L_1) \ll_\delta T$, then we have $A \cap (L_1 + T) \ll_\delta L_1 + T$. On the other side, $L_1 \cap (A + T) \leq \delta(L_1)$ and so H is semisimple whenever L_1 is strongly δ-local or projective semisimple. Thus, $A \cap (L_1 + T)$ is semisimple by [10, Corollary 8.1.5].

Case 2 : Suppose that H is not δ-small in L_1. Then, it is not possible that $L_1 \ll_\delta L_1$ and so L_1 is not projective semisimple by [21, Lemma 2.9]. As L_1 is strongly δ-local, $\delta(L_1) \ll_\delta L_1$ and also $\delta(L_1)$ is maximal in L_1. Thus, $H + \delta(L_1) = L_1$ and so there exists a projective semisimple submodule $X \leq \delta(L_1)$ provided $H \oplus X = L_1$. Then $W = A + (T + L_1) = A + (T + H + X) = A + (T + X)$ as $H = L_1 \cap (A + T) \leq A + T$. Moreover, we have $A \cap (T + X) = [T \cap (A + X)] + [X \cap (A + T)] \leq [T \cap (A + L_1)] + [X \cap (A + T)] \ll_\delta T + X$. Furthermore, since the submodules $T \cap (A + X) \leq T \cap (A + L_1)$ and $X \cap (A + T) \leq X$ are semisimple, then so is $A \cap (T + X)$. Hence, $T + X$ is a δ_{ss}-supplement of A in W where X is a projective semisimple direct summand of L_1. So the proof is completed for $n = 1$.

Let $n > 1$. With induction on n, a subset J of $\{2, ..., n\}$ and submodules $X_j \leq L_j$ exist where $j \in J$ such that $T + \sum_{j\in J} X_j$ is a δ_{ss}-supplement of $A + L_1$ in W and $X_j = L_j$ for each $j \in J$ or X_j is a projective semisimple direct summand of L_j. Then the case $n = 1$ shows the existince a submodule $X_1 \leq L_1$ such that $T + X_1 + \sum_{j\in J} X_j$ is a δ_{ss}-supplement of A in W where $X_1 = L_1$ or X_1 is a projective semisimple direct summand of L_1. □

Theorem 2.26. *The implications given below are equivalent for a module W :*

(1) W is amply cofinitely δ_{ss}-supplemented.
(2) Each maximal submodule of W has ample δ_{ss}-supplements in W.
(3) For each cofinite submodule K and a submodule $L \leq W$ such that $W = K + L$, an integer $n \in \mathbb{Z}^+$ exists such that $W = K + L_1 + L_2 + \cdots + L_n$ where each L_i is either strongly δ-local submodule of L or a projective semisimple submodule of L for each $1 \leq i \leq n$.

Proof. (1) ⇒ (2) : It is evident.

(2) ⇒ (3) : Let $K \leq W$ be cofinite and $W = K + L$ for some submodule $L \leq W$. It will be shown that the existence of finitely many strongly δ-local or projective semisimple submodules $X_1, X_2, ..., X_n$ of W satisfying $W = K + (X_1 + X_2 + ... + X_n)$. Let ϕ be the community of submodules X of L where $X = X_1 + X_2 + ... + X_n$; X_i is strongly δ-local or projective semisimple for $i = 1, 2, ..., n$. Let us assume that $W \neq K + X$ for every $X \in \phi$. Then it is possible to find a set of submodules U such that $W \neq U + X$. By Zorn's Lemma a maximal element V of this collection exists satisfying $W \neq V + X$ and also $K \leq V$ by [21, Lemma 3.5]. Here $V \leq W$ is cofinite obviously. As $\frac{W}{V}$ is finitely generated and V is proper, there exists a maximal submodule P of W containing V. In this case, $W = P + X$ and $V \leq P$ and V is the maximal one satisfying $W \neq V + X$. As $X \leq L$ we have $W = P + L$. Therefore, by (2) there exists a δ_{ss}-supplement T of P which is strongly δ-local or projective semisimple by [22, Proposition 3.4]. Note that $V \neq V + T$. If it were the contrast, we would get the contradiction $W = P + T = T$ since $T \leq V \leq P$ and T is the δ_{ss}-supplement of P where P is maximal in W. Then $V < V + T$ and $W = (V + T) + S = V + (T + S)$ is got where $T, S \in \phi$. As $T + S \in \phi$, $W = V + (T + S)$ is obtained and this contradicts with the maximality of V.

(3) ⇒ (1) : It is clear from Lemma 2.25. □

Theorem 2.27. *Let $K, L \leq W$ such that $W = K + L$. If L is δ_{ss}-supplemented, then L includes a δ_{ss}-supplement of K in W.*

Proof. By assumption, a submodule T of L exists such that $(K \cap L) + T = L$ and $(K \cap L) \cap T = K \cap T \leq Soc_\delta(T)$. Since $W = K + (K \cap L) + T = K + T$, then T is a δ_{ss}-supplement of K in W contained by L. □

Proposition 2.28. *Let W be a module such that each cyclic submodule of W is δ_{ss}-supplemented. Then W is amply cofinitely δ_{ss}-supplemented.*

Proof. Let $P \leq W$ be maximal and $W = P + L$ for some $L \leq W$. Then there exists $x \in L$ such that $x \notin P$ and so we have $W = P + Rx$ by the maximality of P. By hypothesis and Theorem 2.27, Rx contains a δ_{ss}-supplement of P in W. Hence, W is amply cofinitely δ_{ss}-supplemented by Theorem 2.26. □

Theorem 2.29. *The implications given below are equivalent for a finitely generated module W:*

(1) W is amply δ_{ss}-supplemented.
(2) Each maximal submodule of W has ample δ_{ss}-supplements in W.
(3) For each submodule A and a submodule B of W such that $W = A+B$, it can be written that $W = A+L_1+L_2+...+L_n$ provided each L_i is either strongly δ-local or projective semisimple submodule of L for $1 \leq i \leq n$.

Proof. It is a clear consequence of Theorem 2.26. □

Corollary 2.30. *If W is a finitely generated module such that each cyclic submodule of W is δ_{ss}-supplemented, then W is amply δ_{ss}-supplemented.*

Proof. It follows from Theorem 2.29 and Proposition 2.28. □

Theorem 2.31. *The implications given below are equivalent for a ring R:*

(1) R is δ_{ss}-perfect.
(2) ${}_RR$ is amply cofinitely δ_{ss}-supplemented.
(3) R is δ-semiperfect and $\delta(R) = Soc(R)$.
(4) Each projective left R-module is (amply) cofinitely δ_{ss}-supplemented.
(5) Each left R-module is (amply) cofinitely δ_{ss}-supplemented.
(6) Each left R-module is the sum of strongly δ-local or projective semisimple submodules.
(7) ${}_RR$ is the sum of strongly δ-local or projective semisimple submodules.
(8) Every maximal left ideal of R has ample δ_{ss}-supplements in R.

Proof. (1) ⇒ (2) : From (1) ${}_RR$ is cofinitely δ_{ss}-supplemented. Hence, ${}_RR$ is amply cofinitely δ_{ss}-supplemented by Proposition 2.21.

(2) ⇒ (3) : By hypothesis ${}_RR$ is δ_{ss}-supplemented. Then R is δ-semiperfect and $\delta(R) = Soc(R)$ by [22, Theorem 5.3].

(3) ⇒ (4) : By (3) each projective left R-module is δ_{ss}-supplemented and so cofinitely δ_{ss}-supplemented. Hence each projective left R-module is amply cofinitely δ_{ss}-supplemented by Proposition 2.21.

(4) ⇒ (5) : Because each left R-module is an epimorphic image of a projective R-module, then the proof completes from Corollary 2.3.

(5) ⇒ (6) : It follows from Proposition 2.23.

(6) ⇒ (7) : It is clear.

(7) ⇒ (8) : It follows from [22, Corollary 4.11].

(8) ⇒ (1) : It is clear from [22, Theorem 5.3]. □

In the next theorem the notations δ-$Loc_s(W)$, $Soc_P(W)$ and $\gamma(A)$ will represent the sum of whole strongly δ-local submodules of W; the sum of whole projective semisimple submodules of W; the (possibly empty) family of maximal submodules P of W with $A \leq P \leq W$, respectively.

Owing to these notations we will give a new characterization for modules whose maximal submodules have ample δ_{ss}-supplements.

Proposition 2.32. *The implications given below are equivalent for a module W:*

(1) Each maximal submodule of W has ample δ_{ss}-supplements in W.

(2) $\gamma(A) = \gamma[\delta\text{-}Loc_s(A) + Soc_P(A)]$ *for every submodule* $A \leq W$.
(3) $\gamma(Rx) = \gamma[\delta\text{-}Loc_s(Rx) + Soc_P(Rx)]$ *for every* $x \in W - Rad(W)$.

Proof. (1) ⇒ (2) : Let $A \leq W$ and P be a maximal submodule of W which does not include A. By maximality of P we have $W = P + A$. By hypothesis a δ_{ss}-supplement T of P exists contained in A. Then T is strongly δ-local or projective semisimple by [22, Proposition 3.4] and so we have $T \leq \delta\text{-}Loc_s(A) + Soc_P(A) \leq A \nleq P$. Hence, P does not contain $\delta\text{-}Loc_s(A) + Soc_P(A)$.

(2) ⇒ (3) : Evident by assumption.

(3) ⇒ (1) : Let $P \leq W$ be maximal and $W = P + B$ for some $B \leq W$. Then, there exists $x \in B - P$ and so we have $W = P + Rx$ by maximality of P. As $x \notin Rad(W)$, Rx is not contained by P. Then $P \notin \gamma(Rx) = \gamma[\delta\text{-}Loc_s(Rx) + Soc_P(Rx)]$. Therefore, there exists a projective semisimple or strongly δ-local submodule L of W which is not contained by P. Using maximality of P we have $W = P + L$. Since $P + L$ has the trivial δ_{ss}-supplement 0 where L is projective semisimple or strongly δ-local, then by Lemma 2.25 P has a δ_{ss}-supplement T contained by L. Hence the proof is completed. □

A module W holds the following relationship.

(Amply) cof. ss-supplemented ⇒ (Amply) cof. δ_{ss}-supplemented ⇒ (Amply) cof. δ-supplemented

Now, we show that the relations given above are not reversible. In Example 2.33 we give a module proportion which is (amply) cofinitely δ_{ss}-supplemented but not (amply) cofinitely ss-supplemented; and in Example 2.33 we give a module proportion which is (amply) cofinitely δ-supplemented module but not (amply) cofinitely δ_{ss}-supplemented.

Example 2.33. *Let* $\mathbb{Q} = \prod_{i=1}^{\infty} \mathbb{Z}_2$ *and let* R *be the subring of* $\mathbb{Q}$ *generated by* $\bigoplus_{i=1}^{\infty} \mathbb{Z}_2$ *and* $1_{\mathbb{Q}}$. *Since* $\frac{R}{\bigoplus_{i=1}^{\infty} \mathbb{Z}_2}$ *is the only singular simple module, we have* $\delta(R) = \bigoplus_{i=1}^{\infty} \mathbb{Z}_2 = Soc(R)$. *Also* R *is a* δ*-semiperfect ring which is not semiperfect by [28, Example 4.1]. Hence* ${}_RR$ *is (amply) cofinitely* δ_{ss}*-supplemented by Theorem 2.31 as it is* δ_{ss}*-perfect. Furthermore,* ${}_RR$ *is not (amply) cofinitely ss-supplemented by [23, Theorem 3].*

Example 2.34. *Let*

$$R = \left\{(r_1, r_2, \ldots, r_n, r, r, ...) \mid n \in \mathbb{N},\ r_i \in M_2(F),\ r \in \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}\right\}.$$

where F *be a field. Here* R *is a ring with component-wise operations such that,*

$$\begin{aligned} Soc(R) &= \{(r_1, r_2, \ldots, r_n, 0, 0, ...) \mid n \in \mathbb{N},\ r_i \in M_2(F)\}, \\ \delta(R) &= \left\{(r_1, r_2, \ldots, r_n, r, r, ...) \mid n \in \mathbb{N},\ r_i \in M_2(F),\ r \in \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix} = Rad(R)\right\}. \end{aligned}$$

By [28, Lemma 4.3] R *is a* δ*-semiperfect ring but it is not* δ_{ss}*-perfect as* $\delta(R) \neq Soc(R)$ *by [22, Theorem 5.3]. Hence,* ${}_RR$ *is an (amply) cofinitely* δ*-supplemented module by [26, Theorem 4.3] but it is not (amply) cofinitely* δ_{ss}*-supplemented by Theorem 2.31.*

2.3. When Cofinitely δ_{ss}*-Supplemented Modules Are Cofinitely ss-Supplemented?*

It is a known fact that every cofinitely ss-supplemented module is cofinitely δ_{ss}-supplemented. In the remaining part of the study we examine the reversible relation between these modules.

Lemma 2.35. *Let* W *be a module with finitely generated socle and the submodule* $P \leq W$ *be maximal. If* P *has a* δ_{ss}*-supplement* S *in* W*, then* P *has an ss-supplement in* W *contained in* S*.*

Proof. By hypothesis, we have $P + S = W$ and $P \cap S \leq Soc_\delta(S)$. Since P is maximal and $\frac{W}{P} = \frac{P+S}{P} \cong \frac{S}{P \cap S}$, then $P \cap S$ is maximal in S. As $P \cap S \leq \delta(S) \leq S$, it must be true that $\delta(S) = S$ or $\delta(S) = P \cap S$ by the maximality of $P \cap S$.

Case 1 : Let $\delta(S) = S$. Then $\frac{S}{P \cap S}$ is not singular simple and so $P \cap S$ is not essential in S. So $S = (P \cap S) \oplus D$ for some $D \leq S$. It follows that $W = P + S = P + [(P \cap S) \oplus D] = P \oplus D$ and obviously D is an *ss*-supplement of P in W which is contained by S.

Case 2 : Let $\delta(S) = P \cap S$.

If $P \cap S \ll S$, then S is an *ss*-supplement of P in W.

Otherwise, for a proper submodule $T \leq S$ we have $(P \cap S) + T = S$. As $P \cap S \ll_\delta S$, a projective semisimple submodule Y of $P \cap S$ exists satisfying $Y \oplus T = S$. It follows that $\delta(S) = \delta(Y) \oplus \delta(T)$ and $T \cap P = T \cap P \cap S = T \cap \delta(S) = \delta(T) + [T \cap \delta(Y)] = \delta(T)$ by modularity. Moreover, $\delta(T) = T \cap P \leq P \cap S \ll_\delta S$ and as T is a direct summand of S, $\delta(T) = T \cap P \ll_\delta T$. Note that it is also semisimple as a submodule of $P \cap S$. Furthermore, $W = P + S = P + Y + T = P + T$, that is T is a δ_{ss}-supplement of P in W.

Here, if $\delta(T) = T \cap P \ll T$, then T is an *ss*-supplement of the maximal submodule P of W contained in S and the proof is completed. Suppose the contrary. Since T is proper in S and Y is nonzero semisimple in S, $Soc(T) \nleq Soc(S)$ is got. Whenever $\delta(T)$ is not small in T, repeating the steps given above we get $Soc(T) \gneqq Soc(T^{'})$ for a proper submodule $T^{'} \lneqq T$ as $T = \delta(T) + T^{'}$. Repeating the process, we get the descending chain $Soc(T) \gneqq Soc(T^{'}) \gneqq \cdots$ of $Soc(W)$ throughout none of the $T, T^{'}, \ldots$ is an *ss*-supplement of P. We run into a contradiction as $Soc(W)$ is finitely generated. Hence, $\delta(T) \ll T$ and P has an *ss*-supplement T contained in S. □

Corollary 2.36. *Let W be a finitely generated module with a finitely generated socle. W is ss-supplemented if and only if W is δ_{ss}-supplemented.*

Corollary 2.37. *Let W be a module with a finitely generated socle. Then, W is cofinitely ss-supplemented iff W is cofinitely δ_{ss}-supplemented.*

Proof. It is clear by Lemma 2.35 and [23, Theorem 1]. □

Lemma 2.38. *Let W be a strongly δ-local module. Then $W = A \oplus B$ such that A is cyclic strongly δ-local and B is projective semisimple.*

Proof. By hypothesis, as $\delta(W)$ is maximal, for an element $x \in W - \delta(W)$, $\delta(W) + Rx = W$. As $\delta(W) \ll_\delta W$, a projective semisimple submodule Y of $\delta(W)$ exists such that $Y \oplus Rx = W$ by [28, Lemma 1.2] and $\delta(Y) \oplus \delta(Rx) = Y \oplus \delta(Rx) = \delta(W)$ by [28, Lemma 1.5(3)] and [20, Corollary 2.2(2)]. As $\frac{W}{\delta(W)} = \frac{Y \oplus Rx}{Y \oplus \delta(Rx)} \cong \frac{Rx}{\delta(Rx)}$, $\delta(Rx) \leq Rx$ is maximal and semisimple. Moreover, as $\delta(Rx) \leq \delta(W) \ll_\delta W$ and Rx is a direct summand of W, $\delta(Rx) \ll_\delta Rx$ by [1, Lemma 1.1(4)]. □

Lemma 2.39. *Let W be a strongly δ-local module. In this case, W is ss-supplemented if and only if $W = L \oplus P$ such that L is strongly local and P is projective semisimple.*

Proof. For the necessity, note that $\delta(W) \leq W$ is maximal, δ-small and semisimple in W from the assumption. As W is *ss*-supplemented a submodule $L \leq W$ exists for $\delta(W) \leq W$ such that $\delta(W) + L = W$ and $\delta(L) = \delta(W) \cap L \leq Soc_s(L)$. Clearly, $\frac{W}{\delta(W)} = \frac{L+\delta(W)}{\delta(W)} \cong \frac{L}{\delta(W) \cap L}$ is simple and $\delta(W) \cap L \leq L$ is maximal. Now we want to show that L is hollow. Let $X \lneqq L$. For the inclusion $\delta(W) \leq \delta(W) + X \leq W$, $\delta(W) + X = \delta(W)$ or $\delta(W) + X = W$ is got by the maximality of $\delta(W)$. As L is the minimal one of the submodules of W provided $\delta(W) + L = W$, then it is not possible $\delta(W) + X = W$. It means that $\delta(W) + X$ is proper in W. Thus $\delta(W) + X = \delta(W)$ is satisfied and so $X \leq \delta(W)$ is obtained. It follows that $X = X \cap L \leq \delta(W) \cap L \ll L$ as required. On the other side, as the submodule $\delta(L) \leq L$ is maximal and semisimple, L is strongly local. In addition to these, as $\delta(W) \ll W$ and $\delta(W) + L = W$, a projective semisimple submodule P of $\delta(W)$ exists satisfying $P \oplus L = W$. Also, the sufficiency is clear by [11, Proposition 15 and Corollary 24]. □

Definition 2.40. *A ring R is called a left Δ_{ss}-ring if every finitely generated δ_{ss}-supplemented left R-module is ss-supplemented.*

Proposition 2.41. *The implications given below are equivalent for a ring R :*

(1) R is a left Δ_{ss}-ring.
(2) Each cyclic δ_{ss}-supplemented R-module is ss-supplemented.
(3) Each cyclic strongly δ-local R-module is ss-supplemented.
(4) Each strongly δ-local R-module is ss-supplemented.
(5) Each cofinitely δ_{ss}-supplemented R-module is cofinitely ss-supplemented.

Proof. (1) $\Rightarrow$ (2) : Evident.

(2) $\Rightarrow$ (3) : Clear from [22, Lemma 4.1].

(3) $\Rightarrow$ (4) : Let W be a strongly δ-local module. In this case, W has a decomposition $W = A \oplus B$ provided that A is cyclic strongly δ-local and B is semisimple projective by Lemma 2.38. By (3), A is *ss*-supplemented and clearly B is *ss*-supplemented. Thus, W is *ss*-supplemented by [11, Corollary 24].

(4) $\Rightarrow$ (5) : Let W be a cofinitely δ_{ss}-supplemented module and $P \leq W$ be maximal. A submodule S of W exists such that $P + S = W$ and $P \cap S \leq Soc_\delta(S)$. It follows that $P \cap S \leq \delta(S)$. On the other side, as $\frac{W}{P} = \frac{P+S}{P} \cong \frac{S}{P\cap S}$ is simple, $P \cap S \leq S$ is maximal and so $\delta(S) = P \cap S$ or $\delta(S) = S$. Additionally, for every $x \in S - (P \cap S)$ we have $xS + (P \cap S) = S$. It follows that $xS \oplus D = S$ for a projective semisimple submodule D of $P \cap S$ because $P \cap S \ll_\delta S$ by [28, Lemma 1.2].

Case 1 : Let $\delta(S) = P \cap S$. Since $P \cap S$ is maximal, semisimple and δ-small in S, S is strongly δ-local and so it is *ss*-supplemented from (4). Thus, S is a direct sum of a strongly local submodule and a projective semisimple submodule. Hence S is *ss*-supplemented and so cofinitely *ss*-supplemented by [23].

Case 2 : Let $\delta(S) = S$. Then $\delta(xS) \oplus \delta(D) = xS \oplus D$ by [28, Lemma 1.5(3)]. Clearly, $\delta(D) = D$ as D is projective semisimple. Using [28, Lemma 1.5(4)] it must be true that $\delta(xS) = xS \ll_\delta xS$ since xS is cyclic. Thus, xS is projective semisimple and so $S = xS \oplus D$ is semisimple by [10, Corollary 8.1.5]. Hence S is cofinitely *ss*-supplemented.

Taking into account two cases handled above, since $W = P + S$ has a trivial *ss*-supplement 0, $P \leq W$ is cofinite and S is cofinitely *ss*-supplemented, then P also has an *ss*-supplement in W by [23, Lemma 3]. Hence, W is cofinitely *ss*-supplemented by [23, Theorem 1].

(5) $\Rightarrow$ (1) : It is obvious. □

Remark 2.42. *By [22, Corollary 5.10] and [23, Theorem 3] every left δ_{ss}-perfect ring with finitely generated $\chi(R)$ where $\chi(R) = \frac{Soc(_RR)}{Soc_s(_RR)}$ given in [4, Corollary 4.3] is a left Δ_{ss}-ring.*

In [14] a ring R is said to be *ss*-perfect if and only if R is semilocal and $Rad(R) \leq Soc(R)$. Using this concept we obtain another examples of left Δ_{ss}-rings.

Example 2.43. *Every ss-perfect ring is a left Δ_{ss}-ring: For an ss-perfect ring R, it is enough to show that every cyclic strongly δ-local R-module is ss-supplemented by Proposition 2.41. Let W be cyclic strongly δ-local. Thus, W has finite hollow dimension by [5, 18.10, (a $\Leftrightarrow$ b)]. It follows W satisfies ascending and descending chain condition on coclosed submodules by [5, 5.3]. Thus, $W = W_1 \oplus W_2 \oplus \cdots \oplus W_n$ where each W_i is an indecomposable submodule of W. By [28, Lemma 1.5] we have $\delta(W) = \delta(W_1) \oplus \delta(W_2) \oplus \cdots \oplus \delta(W_n)$. As W is strongly δ-local there exists an index $i_0 \in \{1, 2, ..., n\}$ such that $\delta(W_{i_0}) \leq W_{i_0}$ is maximal and $\delta(W_i) = W_i$ for every $i \neq i_0$ as $\frac{W}{\delta(W)} = \frac{W_1\oplus\cdots\oplus W_{i_0}\oplus\cdots\oplus W_n}{M_1\oplus\cdots\oplus\delta(W_{i_0})\oplus\cdots\oplus W_n} \cong \frac{W_{i_0}}{\delta(W_{i_0})}$ from the maximality of $\delta(W) \leq W$. Clearly, every W_i ($i \neq i_0$) is projective semisimple by [21, Lemma 2.9] since $\delta(W) \ll_\delta W \Leftrightarrow \delta(W_i) = W_i$ ($i \neq i_0$) and $\delta(W_{i_0}) \ll_\delta W_{i_0}$. In view of brevity let us assume that $K = W_1 \oplus W_2 \oplus \cdots \oplus W_n$ ($W_{i_0} \neq W_i$) and so $W = W_{i_0} \oplus K$. Note that K is semisimple by [10, Corollary 8.1.5] and obviously it is ss-supplemented. In the remaining part of the solution we aim to show that W_{i_0} is also ss-supplemented. If we show that W is strongly local, then the problem will be completed by [11, Proposition 15 and Corollary 24]. For any proper $X \leq W_{i_0}$ we have $\delta(W_{i_0}) + X = W_{i_0}$ as $\delta(W_{i_0}) \leq W_{i_0}$ is maximal. Since $\delta(W_{i_0}) \ll_\delta W_{i_0}$, a projective semisimple submodule $Y \leq \delta(W_{i_0})$ exists such that $Y \oplus X = W_{i_0}$. As W_{i_0} is indecomposable we get the contradiction $X = W_{i_0}$ or $Y = W_{i_0}$. So it must be true that $X = W_{i_0}$, that is, $\delta(W_{i_0}) \ll W_{i_0}$ and so $W_{i_0} \neq Rad(W_{i_0}) = \delta(W_{i_0}) \leq \delta(W)$. Clearly, $Rad(W_{i_0})$ is semisimple as W is strongly δ-local. Moreover, since W_{i_0} is finitely generated, then each proper submodule of it is small and so W_{i_0} is hollow. It follows that W_{i_0} is local by [5, 2.15(4)]. So we obtain that W_{i_0} is strongly local. Hence, $W = W_{i_0} \oplus K$ is ss-supplemented as a direct sum of two ss-supplemented modules.*

Example 2.44. *Let $R = \mathbb{Z}_4$. Then R is a semilocal ring as it is local. Also, since $Soc(R) = Rad(R) = 2\mathbb{Z}_4$, R is ss-perfect by [14, Theorem 2.15]. Hence, R is a left Δ_{ss}-ring by Example 2.43.*

Proposition 2.45. *Let W be a projective, semilocal and cofinitely δ_{ss}-supplemented module with $Rad(W) \ll W$. In that case, W is cofinitely ss-supplemented.*

Proof. Note that $Soc_s(W)$ is a direct summand of $Soc(W)$ as $Soc(W)$ is semisimple. Then $Soc(W) = Soc_s(W) \oplus T$ for some $T \leq Soc(W)$. Since W is semilocal, $W = T + N$ and $T \cap N \ll W$ for some $N \leq W$. So it is clear that $T \cap N \leq Rad(W)$. From here $T \cap N \leq T \cap Rad(W) = [T \cap Soc(W)] \cap Rad(W) = T \cap [Soc(W) \cap Rad(W)] = T \cap Soc_s(W) = 0$. Also it is obtained that $Rad(W) = Rad(T) \oplus Rad(N) = Rad(N)$ as T is semisimple. Note that N is projective as a direct summand of the projective module W. Now it will be shown that $\delta(N) = Rad(N)$. With this aim it is necessary to verify that N has no simple projective direct summand by [20, Proposition 2.4]. Let us suppose that S is a simple projective direct summand of N. Thus $N = S \oplus K$ for some $K \leq N$. From here, $S \ll_\delta S \leq N$ and so $S \leq Soc_\delta(N) \leq Soc(N)$ as S is projective semisimple. We get, $Soc(N) = Soc(W) \cap N = [Soc_s(W) \oplus T] \cap N = [\{Soc(W) \cap Rad(W)\} \oplus T] \cap N = [\{Soc(W) \cap Rad(N)\}] \oplus (T \cap N) = Soc(W) \cap Rad(N) \leq Rad(N)$ from modularity and so $S \leq Soc(N) \leq Rad(N) = Rad(W) \ll W$ is got. Since Y is a direct summand of W, $S \ll N$ then the contradiction $K = N$ is got. Thus, it requires that $\delta(N) = Rad(N)$. Since W is cofinitely δ_{ss}-supplemented, then N is cofinitely δ_{ss}-supplemented from Corollary 2.4 . Thus, for every cofinite submodule U of N, a submodule V of N exists such that $U + V = N$ and $U \cap V \leq Soc_\delta(N)$. It follows that $U \cap V \leq \delta(N) = Rad(N) \ll W$ and so $U \cap V \ll N$ as $N \leq_\oplus W$. Therefore, $U \cap V \leq Soc_s(N)$, that is, N is cofinitely *ss*-supplemented. So, $W = T \oplus N$ is cofinitely *ss*-supplemented by [23, Proposition 2]. □

Corollary 2.46. *The following statements are equivalent for a ring R :*

(1) R is semiperfect and $Rad(R) \leq Soc(_RR)$.

(2) R is left δ_{ss}-perfect and semilocal.

(3) R is left δ_{ss}-perfect and $\frac{Soc(_RR)}{Soc_s(R)}$ is finitely generated.

Proof. (1) ⇒ (2) : From hypothesis $_RR$ is *ss*-supplemented from [11, Theorem 41] and so it is δ_{ss}-supplemented. Thus R is left δ_{ss}-perfect from [22, Theorem 5.3]. And, R is semilocal as it is semiperfect by [27, 42.6].

(2) ⇔ (3) : It follows from [4, Lemma 4.1].

(3) ⇒ (1) : Since R is left δ_{ss}-perfect, then R is δ-semiperfect and $\delta(R) = Soc(_RR)$ by [22, Theorem 5.3]. So, $Rad(R) \leq Soc(_RR)$. Moreover, R is semiperfect by [4, Corollary 4.3]. □

3. Conclusion

Owing to this article a middle module form (cofinitely δ_{ss}-supplemented module) takes place between cofinitely *ss*-supplemented modules and cofinitely δ-supplemented modules as it can be seen from the definitions. Many new generalizations of ss-supplemented modules have emerged in recent years ([8], [12], [16], [24], [25]). And each of these with the interpretation of singularity can be handled again with respect to cofinite submodules of a module. In this way, sustainability of our study is possible by establishing new definitions.

Declarations

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References

[1] K. Al-Takhman, Cofinitely *δ-supplemented and cofinitely δ-semiperfect modules*, Int. J. Algebra, **12** (2007), 601-613.
[2] R. Alizade, G. Bilhan and P.F. Smith, *Modules whose maximal submodules have supplements*, Commun. Algebra, **29(6)** (1999), 2389-2405.
[3] F.W. Anderson and K.R. Fuller, *Rings and categories of modules*, Commun. Algebra, **31** (2003), 5377-5390.
[4] E. Büyükaşık and C. Lomp, *When δ-semiperfect rings are semiperfect*, Turk. J. of Math., **34** (2010), 317-324.
[5] J. Clark, C. Lomp, N. Vanaja and R. Wisbauer, *Lifting modules, supplements and projectivity in module theory series*: Frontiers in Mathematics, 2006.
[6] F. Eryılmaz, *Cofinitely semisimple ss-lifting modules*, Yüzüncü Yıl Üniv. Journal of the Inst. of Nat. and App. Sci., **28(2)** (2023), 424-431.
[7] F. Eryılmaz, *ss-Lifting modules and rings*, Miskolc Math. Notes, **22(2)** (2021), 655-662.
[8] F. Gömleksiz and B.N. Türkmen, Goldie *ss-supplemented modules*, Montes Taurus J. Pure Appl. Math, **5(1)** (2023), 65-70.
[9] A.R.M. Hamzekolaee, *H-supplemented modules and singularity*, Algebr. Struct. and Their Appl., **7(1)** (2020), 49-57.
[10] F. Kasch, *Modules and rings*, London: Academiz Press Inc., 1982.
[11] E. Kaynar, H. Çalışıcı and E. Türkmen, *ss-Supplemented modules*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., **69(1)** (2020), 473-485.
[12] E.Ö. Kır, *A note on ss-supplement submodules*, Turk. J. Math., **47(2)** (2023), 502-515.
[13] M.T. Koşan, *δ-Lifting and δ-supplemented modules*, Algebra Colloq., **14** (2007), 53-60.
[14] A. Olgun and E. Türkmen, *On a class of perfect rings*, Honam Math. J., **3** (2020), 591-600.
[15] E. Öztürk Sözen and E. Eryaşar, *On ss-lifting modules in view of singularity*, Sinop Univ. J. of Nat. Sci., **8(2)** (2023), 145-155.
[16] I. Soydan and E. Türkmen, *Generalizations of ss-supplemented modules*, Carpathian Math. Publ., **13(1)** (2021), 119-126.
[17] Y. Talebi, R. Tribak and A.R.M. Hamzekolaee, *On H-cofinitely supplemented modules*, Bull. of the Iran. Math. Soc., **39(2)** (2013), 325-346.
[18] Y. Talebi and A.R.M Hamzekolaee, *On a generalization of lifting modules via SSP-modules*, Bull. of the Iran. Math. Soc., **48** (2022), 343-349.
[19] J. Tian, E. Ozturk Sozen, and A.R.M Hamzekolaee, *Some variations of δ-supplemented modules with regard to a hereditary torsion theory*. J. Math., **2023(1)** (2023), 9968793.
[20] R. Tribak, *When finitely generated δ-supplemented modules are amply δ-supplemented*, Algebra Colloq., **22(1)** (2015), 119-130.
[21] R. Tribak, *On δ-Local modules and amply δ-supplemented modules*, J. Algebra Appl., **12(2)** (2013), 1250144(14 p.).
[22] B.N. Türkmen and E. Türkmen, *δ_{ss}-Supplemented modules and rings*, An. St. Univ. Ovidius Constanta, **28(3)** (2020), 193-216.
[23] B.N. Türkmen and B. Kılıç, *On cofinitely ss-supplemented modules*, Algebra Discrete Math., **34(1)** (2023).
[24] B.N. Türkmen and Y.M. Demirci, *On a class of Harada rings*, Open Math., **20(1)** (2022), 1833-1837.
[25] B.N. Türkmen and F. Eryılmaz, *ss-Discrete modules*, Filomat, **35(10)** (2021), 3423-3431.
[26] Y. Wang and D. Wu, *A note on cofinitely δ-lifting (supplemented) modules*, Int. J. Alg., **4(9)** (2010), 439-446.
[27] R. Wisbauer, *Foundations of module and ring theory*, Gordon and Breach Science Publishers, Reading, 1991.
[28] Y. Zhou, *Generalizations of perfect, semiperfect and semiregular rings*, Algebra Colloq., **3** (2000), 305-318.