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An approach to the study of infinite Markov chains from the Drazin inverse of a core-nilpotent operator

Pablo Rodríguez García[a]

[a]*Departamento de Matemáticas (Student), Universidad de Salamanca, 37008 Salamanca, España*

To my professor Fernando Pablos Romo, for his guidance, his support, and the trust he placed in me.

Abstract. The Drazin inverse is the most important pseudo-inverse of linear operators of finite-dimensional vector spaces. One of its applications is in finite Markov chains theory, where they are used to calculate certain properties such as the average number of times the chain passes through a particular state or the expected time it takes for the chain to enter it. Since it has been possible to generalize Drazin's inverse to linear operators in arbitrary dimension (Core-Nilpotent Operators), the aim of this paper is to use it to obtain the same results for infinite Markov chains. Furthermore, the results will be applied to a concrete example.

1. Introduction

It is well known that a linear operator may not be invertible, meaning its inverse does not exist. Therefore, it can be very useful to look for alternatives: given a linear operator, find another one that, even if not invertible, acts as the inverse operator or at least shares similar characteristics. For this reason, generalized inverses (or pseudo-inverses) were introduced: these have properties similar to inverse operators. One of the most studied generalized inverses is the Drazin inverse, named in honor of Michael Peter Drazin for his work in 1958 (see [3]).

The Drazin inverse is particularly useful in the analysis of dynamic systems, where operators are often defined by transition rules. In particular, Markov chains model dynamic systems that are stochastic rather than deterministic. In this context, there are linear operators that provide information about the probability distribution at each stage.

It is reasonable to think that the Drazin inverse can be useful in obtaining certain information about the chain. Specifically, it allows us to solve or analyze systems of linear equations associated with non-invertible linear operators. For example, in the calculation of stationary distributions, long-term behaviors of the chain, recurrence, among other aspects.

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Email address: pablo291101@usal.es (Pablo Rodríguez García)
ORCID iD: https://orcid.org/0009-0003-0191-5739 (Pablo Rodríguez García)

Another important aspect of Markov chains is the classification of states in terms of recurrence and transience. The Drazin inverse facilitates the study of these properties in finite Markov chains by providing another perspective on the transition matrix (see [2]).

In addition to the Drazin inverse, other generalized inverses are relevant. For instance, E. H. Moore and R. Penrose defined a generalized inverse called the Moore-Penrose inverse (see [7] and [9]). Nevertheless, this definition requires a vestor space endowed with an inner product. In contrast, the Drazin inverse just uses the vector space structure.

In general, generalized inverses have been shown to be useful tools for the study of Markov chains, especially in the finite case. Nevertheless, there is less theoretical development in the context of Markov chains with an infinite number of states. The Drazin inverse, which can be defined for linear operators over arbitrary vector spaces, namely infinite dimensional ones, using the Core-Nilpotent operators (see [8]), gives an appropriate framework to approach this type of systems.

For this reason, throughout this work, we will explore in detail the formal definition of the Drazin inverse and its generalization to the infinite-dimensional case, its fundamental properties, and its application in the theory of Markov chains with infinite number of states.

We will start with a review of linear operators in spaces with arbitrary dimension and the definition of the Drazin inverse, followed by an introduction to Markov chains and the application of the pseudo-inverse to chains with a finite number of states. We will continue with the generalization of some results to infinite Markov chains and, finally, we will end with the application of these results to an illustrative example.

The main objective of this work is to see that, through a deeper understanding of the Drazin inverse, the results that Campbell and Meyer proved in [2] are also correct on an arbitrary Markov chain. It is hoped that this exploration will serve as a motivation for future research about Drazin inverse and its applications in various disciplines that use linear operators and matrices.

2. Preliminaries

2.1. Drazin inverses and Core-Nilpotent operators

Let k be a field, and let V be a k-vector space. Like it was said before, a linear operator may not be invertible. For this reason, the theory of generalized inverses tried to deal with this problem. In this section, we will define a generalized inverse, called the Drazin inverse, and study when it can be defined.

Definition 2.1. *A linear operator $f : V \longrightarrow V$ is core-nilpotent (CN operator) when there exist two f-invariant subspaces, U_1 and U_2, such that $f_{|U_1}$ is an automorphism, $f_{|U_2}$ is nilpotent and*

$$V = U_1 \oplus U_2.$$

The decomposition $V = U_1 \oplus U_2$ is called AST-decomposition of f.

Definition 2.2. *A CN operator $f : V \longrightarrow V$ has index n, and it will be denoted as $Ind(f) = n$, when $Ker f^n = Ker f^{n+1}$ and $Im f^n = Im f^{n+1}$, but $Ker f^n \neq Ker f^{n-1}$ or $Im f^n \neq Im f^{n-1}$.*

If the convention where $f^0 = Id$ is used, the previous definition says that $\mathrm{Ind}(f) = 0$ if and only if f is an automorphism.

Theorem 2.3. ***(Characterization of CN operators).*** *If $f : V \longrightarrow V$ is a linear operator, then the following conditions are equivalent:*

1. *f is CN operator.*
2. *$Ker f^n = Ker f^{n+1}$ and $Im f^n = Im f^{n+1}$ for a certain $n \in \mathbb{N}$.*
3. *$V = Ker f^n \oplus Im f^n$ for a certain $n \in \mathbb{N}$.*
4. *There exists two linear operators $f_1, f_2 : V \longrightarrow V$ such that $f = f_1 + f_2$, $Ind(f_1) \leq 1$, f_2 is nilpotent and $f_1 \circ f_2 = f_2 \circ f_1 = 0$.*

Proof. See Theorem 3.6 in [8]. □

As an immediate consequence of the previous theorem and with the previous notations

Corollary 2.4. *Given $f : V \longrightarrow V$ a linear operator*

$$Ind(f) \leq 1 \text{ if and only if } V = Ker f \oplus Im f.$$

Proposition 2.5. *If V is finite-dimensional, then every linear operator is CN.*

Proof. One can find a explanation in [8]. □

Now, we will introduce the definition of the Drazin inverse.

Definition 2.6. *A linear operator $f : V \longrightarrow V$ has a Drazin inverse when there exists $f^D : V \longrightarrow V$ such that*

- $f^{m+1} \circ f^D = f^m$ *for $m \geq n$ where n is a certain non-negative integer,*
- $f^D \circ f \circ f^D = f^D$,
- $f^D \circ f = f \circ f^D$.

Notice that if f is an isomorphism, then his Drazin inverse exists and coincides with his inverse.

Proposition 2.7. *A linear operator $f : V \longrightarrow V$ is a CN operator if and only if there exists an unique Drazin inverse f^D where $n = Ind(f)$.*

Proof. We know that $V = U_1 \oplus U_2$ where $f_{|U_1}$ is an isomorphism. We define

$$f^D(u_1 + u_2) = \left(f_{|U_1}\right)^{-1}(u_1). \tag{1}$$

It is easy to see that it verifies the previous definition. Conversely, one can take $f_1 = f \circ f^D \circ f$ and $f_2 = f - f \circ f^D \circ f$. It is easy to prove that they verify condition 4 of the Theorem 2.3. For a detailed proof, check [8]. □

Proposition 2.8. *Let $f : V \longrightarrow V$ a CN operator and f^D his Drazin inverse. The following properties are satisfied:*

- $\left(f^D\right)^D = f$ *if and only if $Ind(f) \leq 1$.*
- $f \circ f^D = Id_{|U_1}$ *and* $Id - f \circ f^D = Id_{|U_2}$ *where $V = U_1 \oplus U_2$ is the AST-decomposition.*
- $(f^m)^D = \left(f^D\right)^m$ *for all $m \in \mathbb{N}$.*

Proof. One can easily see all of them from (1). □

2.2. Markov Chains

Let $\{X_i\}_{i \in \mathbb{N}}$ be a Markov chain and let E be the states set (finite or countable) of the chain. We denote by $p_{i,j}^{(n)}$ the probability to change to j-state from the i-state in the n-th step; and by $d^{(n)} = \left(d_i^{(n)}\right)_{i \in E}$ the probability distribution of the chain. That is,

$$p_{i,j}^{(n)} = \mathcal{P}(X_{n+1} = j \,|X_n = i), \qquad d_i^{(n)} = \mathcal{P}(X_n = i).$$

By the law of total probability, the following holds:

$$d_j^{(n+1)} = \sum_{i \in E} d_i^{(n)} p_{i,j}^{(n)}, \tag{2}$$

or equivalently

$$d^{(n+1)} = d^{(n)} \cdot \mathscr{P}^{(n)},$$

where $\mathscr{P}^{(n)} = \left(p_{i,j}^{(n)}\right)_{i,j \in E}$ is the transition matrix.

Definition 2.9. *A Markov chain is homogeneous when the probability of change states does not depend on the step, that is*

$$p_{i,j}^{(n)} = p_{i,j}^{(0)} \qquad \forall n \in \mathbb{N}.$$

Unless otherwise indicated, we will always consider homogeneous Markov chains from this point on.

Definition 2.10. *If E is a finite set, the Markov chain is said to be finite. On the contrary, it is said to be infinite.*

In the next section, we will focus on a specific type of Markov chain. Therefore, it is important to classify the states and the Markov chains.

Before proceeding, a definition will be introduced to aid in the states classification.

Definition 2.11. *Let $e_1, e_2 \in E$ be two states of a Markov chain, it is said that e_1 communicates with e_2, and it will be denoted by $e_1 \to e_2$, if it is possible to go from e_1 to e_2 in a finite number of steps. That is*

$$\mathcal{P}(X_n = e_2 | X_0 = e_1) > 0 \qquad \textit{for some } n > 0,$$

and it is said that e_1 and e_2 intercommunicate, denoted by $e_1 \leftrightarrow e_2$, if $e_1 \to e_2$ and $e_2 \to e_1$.

The first and most important distinction between states is based on the system's ability to return to a specific state.

Definition 2.12. *Let $e \in E$. We say that e is transient if it is not certain that the chain will return to it after leaving. Otherwise, e is called recurrent.*

Definition 2.13. *Let $e \in E$ be a recurrent state:*

- *it is positive recurrent if the mean time it takes to return to e is finite. Otherwise, it is null recurrent.*
- *it is said to be periodic with period t if it is only possible to return to that state after a number of steps that is a multiple of t. In the case where $t = 1$, the state is said to be aperiodic.*
- *it is ergodic if it is both positive recurrent and aperiodic.*

Proposition 2.14. *Given a Markov chain, and $e_1, e_2 \in E$ such that $e_1 \to e_2$. If e_1 is recurrent, e_2 is also recurrent and if e_2 is transient, e_1 must also be transient.*

Proof. If $e_1 \to e_2$ and e_1 is recurrent, then the Markov chain is in state e_1 an infinite number of times and, due to $\mathcal{P}(X_n = e_2 | X_0 = e_1) > 0$, the chain returns to e_2 with absolute certainty.

Now, suppose that e_2 is transient. If e_1 was recurrent, we have just proved that e_2 is recurrent too, but this leads to a contradiction. □

Corollary 2.15. *If two states intercommunicate, they are of the same type (positive recurrent, null recurrent, aperiodic or transient).*

During this work, we will only consider ergodic Markov chains. Let us formally define this concept.

Definition 2.16. *A Markov chain is said to be ergodic if all states are ergodic and they intercommunicate with each other.*

Theorem 2.17. *Consider an ergodic Markov chain. Then there is a unique stationary distribution, that is, there exists a unique $\pi = (\pi_i)_{i \in E}$ where $0 \le \pi_i \le 1$ and $\sum \pi_i = 1$, such that*

$$\pi = \pi \cdot \mathscr{P}, \tag{3}$$

and, in that case,

$$\pi = \lim_{n \to \infty} d^{(0)} \cdot \mathscr{P}^n \tag{4}$$

for all initial probability distribution $d^{(0)}$.

Proof. A detailed proof can be found in Section 1.5 of [6]. □

There is a similar theorem, but more useful in some later proofs.

Theorem 2.18. *Given a Markov chain whose states all intercommunicate with each other, a stationary distribution π exists if and only if there is a positive recurrent state. In that case, all states are positive recurrent and π is unique.*

Proof. See Section 6 of [4]. □

2.3. *Aplication of the Drazin inverse to finite Markov chains*

Let $\mathscr{P}$ be a transition matrix of a Markov chain with m states. Then, for $A = \mathrm{Id} - \mathscr{P}$, it holds that $\mathrm{Ind}(A) = 1$, as demonstrated by Campbell in Section 8 of [2].

Theorem 2.19. *Let us consider an ergodic Markov chain and A as defined above. One has*

$$Id - A\,A^D = \lim_{n\to\infty} \mathscr{P}^n, \tag{5}$$

where A^D is the Drazin inverse of A.

Proof. See Theorem 8.2.2 in [1]. □

Henceforth, we will write $W = \mathrm{Id} - A\,A^D$. It is easier to calculate W than A^D, because each row of W corresponds to the stationary distribution. For this reason, the next result is useful.

Theorem 2.20. *With the notation of above, one has that*

$$A^D = \sum_{n=0}^{\infty} (\mathscr{P}^n - W). \tag{6}$$

Proof. See Theorem 8.3.1 in [1]. □

Another characterization for A^D using W is given in the following theorem.

Theorem 2.21. *For $A = Id - \mathscr{P}$, A^D is the unique solution of the equations $W\,X = 0$ and $A\,X = I - W$.*

Proof. See Theorem 8.5.5 in [1]. □

Theorem 2.22. *Let $N^{(n)}$ be a matrix whose (i, j)-th entry is the expected number of times the chain is in j-th state in n steps when the chain was initially in i-th state. Then,*

$$A^D = \lim_{n\to\infty} \left(N^{(n)} - (n+1)W\right). \tag{7}$$

Proof. See Theorem 8.3.2 in [1]. □

Theorem 2.23. *Let M be the matrix whose (i, j)-th entry is the expected number of steps before entering the j-th state for the first time from the i-th state. Then,*

$$M = (Id - A^D + J\,A^D_d)\,D, \tag{8}$$

where J is a matrix of all 1's, X_d is the diagonal matrix of X, obtained setting all off-diagonal entries of X equal to zero and $D = (W_d)^{-1}$

Proof. See Theorem 8.4.1 in [1]. □

3. Application of the Drazin Inverse to Infinite Markov Chains

Throughout this section, we will consider a homogeneous Markov chain $\{X_n\}_{n\geq 0}$. We will assume that the state space is $E = \mathbb{N}$ (whether $0 \in \mathbb{N}$ or $0 \notin \mathbb{N}$ depends on the context in which the Markov chain is defined).

Let $V = \left\{(d_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} \mathbb{R} : \sum_{i\in\mathbb{N}} |d_i| < \infty\right\}$. There exists an operator $\mathscr{P} : V \longrightarrow V$ called the transition operator, defined as

$$\mathscr{P}((d_i)_{i\in\mathbb{N}}) = \left(\sum_{i\in\mathbb{N}} d_i\, p_{i,j}\right)_{j\in\mathbb{N}}. \tag{9}$$

Lemma 3.1. *The transition operator defined as (9) is well defined.*

Proof. Since $p_{i,j} \leq 1$, we have
$$\sum_{i\in\mathbb{N}} d_i\, p_{i,j} \leq \sum_{i\in\mathbb{N}} d_i < \infty.$$
Furthermore, it should be

$$\sum_{j\in\mathbb{N}} \left|\sum_{i\in\mathbb{N}} d_i p_{i,j}\right| \leq \sum_{j\in\mathbb{N}} \sum_{i\in\mathbb{N}} \left|d_i p_{i,j}\right| = \sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} |d_i| \left|p_{i,j}\right| \leq \sum_{i\in\mathbb{N}} |d_i| < \infty.$$

□

We define $A = \mathrm{Id} - \mathscr{P}$. We will restrict ourselves to the case where the chain is ergodic and where A is a CN operator of index 1. Then $V = \mathrm{Ker}A \oplus \mathrm{Im}A$ such that A over $\mathrm{Im}A$ is an isomorphism.

Furthermore, since the chain is ergodic, there exists a unique stationary distribution $\pi = (\pi_i)_{i\in\mathbb{N}}$.

In this section, we will attempt to generalize the existing results for finite Markov chains to arbitrary Markov chains using the theory of CN operators.

The first result deals with the relationship between the stationary distribution and A^D.

Theorem 3.2. *Let $\{X_n\}_{n\geq 0}$ be a homogeneous ergodic Markov chain and let $\mathscr{P}$ its transition operator. For $A = Id - \mathscr{P}$*

$$Id - A\,A^D = \lim_{n\to\infty} \mathscr{P}^n. \tag{10}$$

Proof. First, we know that if $\sum_{i\in\mathbb{N}} d_i = 1$, $\lim_{n\to\infty} \mathscr{P}^n(d)$ exists and coincides with π. By proposition 2.8, $\mathrm{Id} - A\,A^D$ vanishes over $\mathrm{Im}A$ and it acts as the identity on $\mathrm{Ker}A$. Let us see that the limit in the right side of the equation (10) verifies the same. Let $v \in \mathrm{Ker}A$. Then $\mathscr{P}(v) = v$, so

$$\lim_{n\to\infty} \mathscr{P}^n(v) = \lim_{n\to\infty} v = v.$$

Now, let $A(d) \in \mathrm{Im}A$, then

$$\lim_{n\to\infty} \mathscr{P}^n(\mathrm{Id} - \mathscr{P})(d) = \lim_{n\to\infty} (\mathscr{P}^n - \mathscr{P}^{n+1})(d) = s\,\pi - s\,\pi = 0,$$

where $s = \sum_{i\in\mathbb{N}} d_i$. □

This result is significant because it connects the long-term behavior of the Markov chain, as described by the limit of $\mathscr{P}^n$, with the Drazin inverse.

For the rest of the section, we will denote by W the projection onto $\mathrm{Ker}A$, $W = I - A\,A^D$.

Theorem 3.3. *Given an ergodic arbitrary Markov chain, it holds that*

$$A^D = \sum_{k=0}^{\infty}(\mathscr{P}^k - W). \tag{11}$$

Proof. Let $v \in \mathrm{Ker}A$, then

$$\sum_{k=0}^{\infty}(\mathscr{P}^k - W)(v) = \sum_{k=0}^{\infty}(v - v) = 0.$$

On the other hand, if $A(d) \in \mathrm{Im}A$,

$$\begin{aligned}\sum_{k=0}^{\infty}(\mathscr{P}^k - W)(A(d)) &= \sum_{k=0}^{\infty}(\mathscr{P}^k(d) - \mathscr{P}^{k+1}(d))\\ &= \lim_{k\to\infty}(d - \mathscr{P}^{k+1}(d)) = d - W(d)\end{aligned}$$

Note that $A(d - W(d)) = A(d)$ and that $d - W(d) \in \mathrm{Im}A$, we conclude therefore, $A^D(d) = d - W(d)$. □

This series can be very useful in practical computations, especially in iterative algorithms for approximating the Drazin inverse, since calculating Drazin inverse explicitly may be expensive in terms of time computing or even be impossible.

The following theorem gives one of the most interesting uses of the Drazin inverse.

Theorem 3.4. *Let $\{X_n\}_{n\geq 0}$ be an ergodic Markov chain and let $N^{n)}_{i,j}$ be the expected number of times the chain enters state j in n steps, having started from state i. If $N^{n)} = (N^{n)}_{i,j})$, then it holds that*

$$A^D = \lim_{n\to\infty}(N^{n)} - (n+1)W). \tag{12}$$

Proof. It is known that $N^{n)} = \sum_{k=0}^{n}\mathscr{P}^k$. Applying the previous theorem, we get

$$\lim_{n\to\infty}(N^{n)} - (n+1)W) = \lim_{n\to\infty}\sum_{k=0}^{n}(\mathscr{P}^k - W) = A^D.$$

□

This result implies that if n is large enough, we can approximate $N^{n)}$ by $A^D + (n+1)W$.

$$N^{n)} \approx A^D + (n+1)W.$$

We have talked about the number of times the chain passes through a certain state, now the question is, how long will it take the chain to reach a certain state? The following theorem allows us to calculate it using the Drazin inverse.

Theorem 3.5. *For an arbitrary ergodic Markov chain, let $m_{i,j}$ be the expected number of steps it takes for the system to reach state j when it initially starts in state i. Then, it holds that*

$$m_{i,j} = \frac{1}{\pi_j}\left(\delta_{i,j} + A^D(\mu^j - \mu^i)_j\right), \tag{13}$$

where $\delta_{i,j}$ is the Kronecker delta, $\mu^i = (\delta_{i,k})_{k\in\mathbb{N}}$, $A^D(d)_j$ is the jth coordinate of $A^D(d)$, and $\pi = (\pi_j)_{j\in\mathbb{N}}$ is the stationary distribution of the Markov chain.

Proof. The expected number of steps to go from state i to state j satisfies the following system of equations

$$m_{i,j} = 1 + \sum_{k\in\mathbb{N}} p_{i,k}m_{k,j} - p_{i,j}m_{j,j}.$$

Let's see that the expression in the statement satisfies it. Noting that $\sum\limits_{k\in\mathbb{N}} p_{i,k}A^D(\mu^k)_j = A^D(\mathscr{P}(\mu^i))_j$ and that $A\,A^D(\mu^i) = (\mathrm{Id} - W)(\mu^i) = \mu^i - \pi$, we have

$$\begin{aligned}
&1 + \sum_{k\in\mathbb{N}} p_{i,k}\frac{1}{\pi_j}(\delta_{k,j} + A^D(\mu^j - \mu^k)_j) - p_{i,j}\frac{1}{\pi_j} \\
=&1 + \frac{1}{\pi_j}A^D(\mu^j)_j - \frac{1}{\pi_j}A^D(\mathscr{P}(\mu^i))_j = 1 + \frac{1}{\pi_j}A^D(\mu^j)_j - \frac{1}{\pi_j}A^D(\mathrm{Id} - A)(\mu^i)_j \\
=&1 + \frac{1}{\pi_j}A^D(\mu^j)_j - \frac{1}{\pi_j}A^D(\mu^i) + \frac{1}{\pi_j}(\delta_{i,j} - \pi_j) = \frac{1}{\pi_j}\left(\delta_{i,j} + A^D(\mu^j - \mu^i)_j\right)
\end{aligned}$$

□

This theorem provides a powerful method for computing mean passage times using the Drazin inverse in arbitrary Markov chains. It proves that the Drazin inverse not only helps in understanding the stationary behavior of Markov chains but also in calculating important dynamical quantities like the expected passage times between states.

4. Illustrative Example

The most typical example of an infinite Markov chain is the random walk with reflection. Consider a particle that moves randomly on the natural numbers, forward, with probability p, or backward with probability $1 - p$. In the special case of 0, since it cannot move further backward, we consider that it can stay in place with probability $1 - p$. Then

$$\mathcal{P}(X_{n+1} = j|X_n = i) = p_{i,j} = \begin{cases} \;\;p & \text{if} \quad j = i+1 \\ 1-p & \text{if} \quad j = i-1 \text{ or } j = i = 0 \\ \;\;0 & \quad\;\;\,\text{in other case} \end{cases} \tag{14}$$

We can represent this chain with the following diagram.

$$\begin{tikzcd}
0 \arrow[loop left, "1-p"] \arrow[r, bend left, "p"] & 1 \arrow[l, bend left, "1-p"] \arrow[r, bend left, "p"] & 2 \arrow[l, bend left, "1-p"] \arrow[r, bend left, "p"] & \cdots \arrow[l, bend left, "1-p"] \arrow[r, bend left, "p"] & n \arrow[l, bend left, "1-p"] \arrow[r, bend left, "p"] & \cdots \arrow[l, bend left, "1-p"]
\end{tikzcd}$$

In matrix form, the following matrix is obtained

$$\mathscr{P} = \begin{pmatrix} 1-p & p & 0 & \cdots \\ 1-p & 0 & p & \cdots \\ 0 & 1-p & 0 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

It is easily seen that any state can be accessed from any other, but it is not so immediate that all states are ergodic.

Theorem 4.1. *If $p < \frac{1}{2}$, the previous Markov chain is ergodic and, therefore, it has a unique stationary distribution.*

Proof. One can find a similar proof of this result in [5]. Let us see if there exists a stationary distribution, $\pi = (\pi_i)_{i\in E}$. This distribution should be a solution of the following system,

$$\pi \cdot \mathscr{P} = \pi$$

therefore, it has to verify

$$\begin{cases} \pi_0 = & \pi_0(1-p) + \pi_1(1-p) \\ \pi_i = & \pi_{i-1}\,p + \pi_{i+1}\,(1-p) \qquad \forall i > 0 \end{cases} \tag{15}$$

From the first equation, it follows that

$$\pi_1 = \frac{p}{1-p}\pi_0,$$

and from the second with $i = 1$ we have that

$$\pi_2 = \frac{1}{1-p}(\pi_1 - \pi_0\,p) = \left(\frac{p}{1-p}\right)^2 \pi_0.$$

Applying induction, we can conclude that

$$\pi_{i+1} = \frac{1}{1-p}(\pi_i - \pi_{i-1}(1-p)) = \frac{1}{1-p}\left(\frac{p^i}{(1-p)^i}\pi_0 - \frac{p^i}{(1-p)^{i-1}}\pi_0\right)$$
$$= \left(\frac{p}{1-p}\right)^{i+1}\pi_0,$$

and that $\left(\frac{p^i}{(1-p)^i}\pi_0\right)_{i\in E}$ is a solution of the system (15). For it to indeed be a solution, it must satisfy

$$\sum_{i=0}^{\infty}\left(\frac{p}{1-p}\right)^i \pi_0 = 1.$$

Since $p < \frac{1}{2}$, then $\frac{p}{1-p} < 1$ and

$$\sum_{i=0}^{\infty}\left(\frac{p}{1-p}\right)^i \pi_0 = \frac{\pi_0}{1-\frac{p}{1-p}}.$$

Taking $\pi_0 = 1 - \frac{p}{1-p} = \frac{1-2p}{1-p}$, we determine a stationary distribution. By Theorem 2.18 all states are positive recurrent and, therefore, the chain is ergodic due to the fact that the states are aperiodic. □

Theorem 4.2. *If $\mathscr{P}$ is the transition operator of the Markov chain of the random walk with reflection, defined in (14), the operator $A = Id - \mathscr{P}$ has index 1.*

Proof. Let us see $V = \{(d_i)_{i\geq 0} : \sum |d_i| < \infty\}$ decompose into $V = \mathrm{Ker}A \oplus \mathrm{Im}A$. Firstly, one has that

$$A((d_i)_{i\geq 0}) = (p\,d_0 + (p-1)d_1, ..., -p\,d_{i-1} + d_i + (p-1)d_{i+1}, ...)$$

Furthermore, in the previous proof we just demonstrate that $\mathrm{Ker}A = \left\langle \left(\left(\frac{p}{1-p}\right)^i\right)_{i\geq 0} \right\rangle$.

Suppose by contradiction that $\mathrm{Ker}A \cap \mathrm{Im}A \neq 0$, so there exists $(d_i)_{i\geq 0} \in V$ such that

$$\begin{cases} p\,d_0 + (p-1)d_1 = & 1 \\ -p\,d_{i-1} + d_i + (p-1)d_{i+1} = & \frac{p^i}{(1-p)^i} \qquad \forall i > 0 \end{cases}$$

Adding first equation with second one when $i = 1$, we get that

$$p\,d_1 + (p-1)d_2 = 1 + \frac{p}{1-p}.$$

Applying induction, we obtain that

$$p\,d_i + (p-1)d_{i+1} = \sum_{k=0}^{i} \left(\frac{p}{1-p}\right)^k.$$

Since $\sum_{i\geq 0} d_i < \infty$, $\lim_{i\to\infty} d_i = 0$. Then, the left member of the expression above goes to 0, but the right-hand side goes to $\frac{1}{1-\frac{p}{1-p}}$, which leads to a contradiction.

Now, let $(\mu_i)_{i\geq 0} \in V$. To see if it is in $\mathrm{Ker}A + \mathrm{Im}A$, let us see if the following system has a solution.

$$\begin{cases} p\,d_0 + (p-1)d_1 + a = & \mu_0 \\ -p\,d_{i-1} + d_i + (p-1)d_{i+1} + a\frac{p^i}{(1-p)^i} = & \mu_i \qquad \forall i > 0 \end{cases}$$

Reasoning analogously to before, we conclude that for any $i > 0$ it holds that

$$p\,d_i + (p-1)d_{i+1} + a\sum_{k=0}^{i} \left(\frac{p}{1-p}\right)^k = \sum_{k=0}^{i} \mu_k. \tag{16}$$

And as $i \to \infty$, we obtain that

$$a = \frac{1-2p}{1-p}\sum_{k=0}^{\infty} \mu_k.$$

Finally, solving from (16), one determines $(d_i)_{i\geq 0}$ through the recurrence formula

$$d_{i+1} = \frac{1}{p-1}\left(\sum_{k=0}^{i} \mu_k - a\sum_{k=0}^{i}\left(\frac{p}{1-p}\right)^k - p\,d_i\right).$$

We need to check if $(d_i)_{i\geq 0} \in V$. For this, we use the ratio criterion.

$$\lim_{i\to\infty} \frac{|d_{i+1}|}{|d_i|} = \lim_{i\to\infty} \frac{\left|\sum_{k=0}^{i} \mu_k - a\sum_{k=0}^{i}\left(\frac{p}{1-p}\right)^k - p\,d_i\right|}{|d_i|(p-1)} = \frac{p}{1-p} < 1.$$

Thus, the series $\sum |d_i|$ converges.

Based on the above, it has been proven that $V = \mathrm{Ker}A \oplus \mathrm{Im}A$ and by Theorem 2.3, A has index 1. □

Now that we have shown that we are under the hypotheses of the theory from the previous section, we can apply it. To proceed, we need to compute the Drazin inverse of A. Let $\mu = (\mu_i)_{i\geq 0}$. We know that there exist $d = (d_i)_{i\geq 0} \in V$ and $v \in \mathrm{Ker}A$ such that $\mu = A(d) + v$. In the same way, we can write $d = A(\overline{d}) + \overline{v}$, so $d - \overline{v} \in \mathrm{Im}A$ and $A(d) = A(d - \overline{v})$. Consequently, $A^D(\mu) = A^{-1}(A(d)) = d - \overline{v}$. It is easy to check that $\overline{v} = \left(\frac{1-2p}{1-p}\sum d_j\right)\left(\left(\frac{p}{1-p}\right)^i\right)_{i\geq 0}$ and

$$A^D(\mu) = \left(d_i - \left(\frac{1-2p}{1-p}\sum_{j=0}^{\infty} d_j\right)\left(\frac{p}{1-p}\right)^i\right)_{i\geq 0}$$

where $(d_i)_{i\geq 0}$ is recurrently calculated by

$$d_{i+1} = \frac{1}{p-1}\left(\sum_{k=0}^{i}\mu_k - a\sum_{k=0}^{i}\left(\frac{p}{1-p}\right)^k - p\,d_i\right).$$

By Theorem 3.4, the expected number of times the chain is in jth state in n stages when the chain was initially in ith state, $N_{i,j}^{n)}$, can be approximated by (i,j)-th entry of $A^D + (n+1)\,W$. For example, if we want to know how many times the chain is in 0 state, we need to calculate the 0th coordinate of $A^D(\mu^i) + (n+1)\,W(\mu^i)$ where $\mu^i_j = \delta_{i,j}$, and $\delta_{i,j}$ is the Kronecker delta.

If $A^D\left(\mu^i\right) = d^{i)} - \overline{v}^{i)}$ and $d^{i)} = \left(d^{i)}_j\right)$, with patience, one can find

$$d^{i)}_j = \begin{cases} \frac{1-2p}{(1-p)^2}\sum\limits_{k=0}^{j-1}(k+1)\left(\frac{p}{1-p}\right)^k + \left(\frac{p}{1-p}\right)^j d_0 & \text{if } j \leq i \\ \frac{1}{p-1}\sum\limits_{k=0}^{j-i-1}\left(\frac{p}{1-p}\right)^k + \frac{1-2p}{(1-p)^2}\sum\limits_{k=0}^{j-1}(k+1)\left(\frac{p}{1-p}\right)^k + \left(\frac{p}{1-p}\right)^j d_0 & \text{if } j > i \end{cases}$$

Consequently, the 0th coordinate of $A^D(\mu^i)$ matches with

$$\begin{aligned} & d_0 - \frac{1-2p}{1-p}\sum_{j=0}^{\infty}\left(\frac{1}{p-1}\sum_{k=0}^{j-i-1}\left(\frac{p}{1-p}\right)^k + \frac{1-2p}{(1-p)^2}\sum_{k=0}^{j-1}(k+1)\left(\frac{p}{1-p}\right)^k + \left(\frac{p}{1-p}\right)^j d_0\right) \\ =& d_0 - \frac{1-2p}{1-p}\sum_{j=0}^{\infty}\overline{d}_j^{i)} - \frac{1-2p}{1-p}\sum_{j=0}^{\infty}\left(\frac{p}{1-p}\right)^j d_0 = -\frac{1-2p}{1-p}\sum_{j=0}^{\infty}\overline{d}_j^{i)}, \end{aligned}$$

where $\overline{d}_j^{i)} = \frac{1}{p-1}\sum\limits_{k=0}^{j-i-1}\left(\frac{p}{1-p}\right)^k + \frac{1-2p}{(1-p)^2}\sum\limits_{k=0}^{j-1}(k+1)\left(\frac{p}{1-p}\right)^k$ and we used the convention $\sum\limits_{k=0}^{j} a_k = 0$ if $j<0$. Notice that it does not depend on the $(d_i)_{i\geq 0}$ we used to calculate it.

Given that $W = \lim\limits_{n\to\infty}\mathscr{P}^n$, it follows that $W(\mu^i) = \pi$, where π is the stationary distribution. Finally, we have that

$$N_{i,0}^{n)} = -\frac{1-2p}{1-p}\sum_{j=0}^{\infty}\overline{d}_j^{0)} + (n+1)\frac{1-2p}{1-p}.$$

Now, we can consider the expected number of steps the chain needs to reach the state j from the state i. By Theorem 3.5, this number matches with $\frac{1}{\pi_j}\left(\delta_{i,j} + A^D(\mu^j - \mu^i)_j\right)$. Since

$$A^D(\mu^i)_j = \overline{d}_j^{i)} - \frac{1-2p}{1-p}\left(\frac{p}{1-p}\right)^j\sum_{k\geq 0}\overline{d}_k^{i)},$$

one can get

$$m_{i,j} = \frac{1-p}{1-2p}\left(\frac{1-p}{p}\right)^j\left(\delta_{i,j} + \overline{d}_j^{j)} - \overline{d}_j^{i)}\right) - \sum_{k\geq 0}\left(\overline{d}_k^{j)} - \overline{d}_k^{i)}\right).$$

To conclude the section, we will use the software *Wolfram Mathematica* to perform some computation and verify experimentally that the theoretical calculations are correct.

First, a function to compute $\overline{d}_j^{i)}$ is created (Figure 1). Then, using $p = 0.1$, we compute the expected number of times the chain is in 0th state in 1000 stages when the chain was initially in 41th state ,$N_{41,0}^{1000)}$, and the expected time to reach the 0th state for the first time, $m_{41,0}$ (Figure 2). The results are 844.361 and 51.25 respectively.

```
In[1]:= d[p_, i_, j_] := 1/(p - 1) Sum[(p/(1 - p))^k, {k, 0, j - i - 1}] + (1 - 2 p)/(1 - p)^2 Sum[(k + 1) (p/(1 - p))^k, {k, 0, j - 1}]
```

$$d[p\_, i\_, j\_] := \frac{1}{p-1}\sum_{k=0}^{j-i-1}\left(\frac{p}{1-p}\right)^k + \frac{1-2p}{(1-p)^2}\sum_{k=0}^{j-1}(k+1)\left(\frac{p}{1-p}\right)^k$$

Figure 1: Function to compute $\overline{d}_j^{\,i}$.

```
In[12]:= p = 0.10;
i = 41;
n = 1000;
jmax = 2000;
```

$$-\frac{1-2p}{1-p}\sum_{j=0}^{jmax} d[p, i, j] + (n+1)\,\frac{1-2p}{1-p}$$

$$\sum_{j=0}^{jmax} (d[p, i, j] - d[p, 0, j])$$

```
Out[16]= 844.361

Out[17]= 51.25
```

Figure 2: Computation of $N_{41,0}^{1000)}$ and $m_{41,0}$. To reduce computing time, the series $\sum \overline{d}_j^{\,i}$ is approximated by a partial sum.

In Figure 3, one can see the code for approximating $N_{41,0}^{1000)}$. For this propose, 2000 simulations of the random walk have been done, and the number of times it passes thought 0th state has been measured. Finally, the average of this data has been calculated.

```
In[18]:=

meanRecurrenceTimes = {};
Do[
 in = i;
 count = 0;
 Do[
  in = If[in == 0, in + RandomChoice[{1 - p, p} → {0, 1}],
    in + RandomChoice[{1 - p, p} → {-1, 1}]];
  If[in == 0, count++];
  , n];
 AppendTo[meanRecurrenceTimes, count];
 , 2000]
Mean[meanRecurrenceTimes] // N

Out[20]= 844.426
```

Figure 3: Experimental calculation for $N_{41,0}^{1000)}$.

Similarly, the analogous process has been carried out to approximate $m_{41,0}$. The code can be seen in Figure 4.

```
meanPassageTimes = {};
Do[
 in = i;
 time = 0;
 While[in ≠ 0,
  in = in + RandomChoice[{1 - p, p} → {-1, 1}];
  time++;
 ];
 AppendTo[meanPassageTimes, time];
 , 2000]
Mean[meanPassageTimes] // N

Out[23]= 51.292
```

Figure 4: Experimental calculation for $m_{41,0}$.

The results obtained experimentally coincide with the theoretical ones.

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