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The transformation in the $\mathbb{P}$-Module by hyperbolic numbers

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Abstract. Hyperbolic numbers are one of the well-known number systems, like complex numbers. In this paper, we will talk about some properties of hyperbolic numbers, define the $\mathbb{P}-$modulus, and give some properties of the $\mathbb{P}-$modulus. We will also map the points on the unit hyperbolic sphere of the $\mathbb{P}-$module to some special directional lines in $\mathbb{E}^3$. Then, we will show the relation between the angle between these directional lines and the hyperbolic angle between the unit hyperbolic vectors.

1. Introduction

Hyperbolic numbers were introduced as “double numbers” by Clifford (1873, 1878). In the following periods, they were called by different names such as hyperbolic numbers, double numbers, split complex numbers, perplex numbers, and duplex numbers. After the hyperbolic numbers were introduced by Clifford, they received intense attention from physicists and were used in many fields. In addition, the mathematical analysis and physical applications of hyperbolic numbers is one of the areas that have developed in recent years [3, 7, 10, 13, 14, 21]. In a similar vein, the study of transformations, curves and surfaces in Euclidean spaces has also seen notable advancements. For example, recent works by Li et al. explore sweeping surfaces and directional developable surfaces [16, 29, 30], while other studies have focused on differential geometry of curves in Euclidean spaces [9, 17]. F. Catoni and P. Zampetti have made important contributions to the study of hyperbolic numbers by demonstrating that functions of a hyperbolic variable are best studied in a pseudo-Euclidean plane. Their work shows that, similar to complex analysis, a Cauchy integral formula holds for hyperbolic functions in this framework [7]. In analogy to complex numbers, the system of perplex numbers, introduced as $\mathbf{z} = x + jy$, where j satisfies $\left\|j\right\| = -1$, has been proposed. This system, developed by four freshmen at St. Olaf College, has potential relevance in physics, particularly in extending special relativity to superluminal velocities using a velocity parameter ϕ, where $v = c \tanh \phi$ [10]. Khrennikov developed the quantization formalism in a hyperbolic Hilbert space, generalizing Born’s probability interpretation and showing that unitary transformations in

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such a space represent a new class of probability transformations describing hyperbolic interference. His work also explores the potential of hyperbolic quantum formalism as a new theory of probability waves, which could be developed in parallel with standard quantum theory [13]. Also, hyperbolic numbers were used to extend quantum mechanics, and hyperbolic quantum mechanics was studied. Similarly, Hucks (1993) used hyperbolic numbers for Dirac spinors, and Kunstatter et al. (1983) used hyperbolic numbers for applications in the theory of gravity, [12, 15]. In addition, rotational movements in the Lorentz plane are also made with hyperbolic numbers. Properties and some applications of hyperbolic numbers are examined in detail in articles [1–6, 8, 22, 23]. For example, Çakır and Özdemir examined the use of split-complex (hyperbolic) numbers to describe the geometry of the Lorentzian plane and investigated the exponential of split-complex matrices in three different cases. Their work discusses the computation of the exponential for split-complex matrices, considering the presence of null split-complex numbers and their impact on the calculations [4].

The subject of this article concerns the question "Can we find a transformation in hyperbolic numbers as the E. Study transformation in dual numbers?". Some properties and applications related to the E. Study transform can be found in articles [11, 18–20, 24–28].

In this paper, we will examine the $\mathbb{P}-$module. First, we present an introduction to hyperbolic numbers to provide the necessary background. Then we pass to the $\mathbb{P}-$module by giving some properties of hyperbolic numbers. However, in the $\mathbb{P}-$module we define the basic operations and give some of their properties. In the next section, we map points on the unit hyperbolic sphere in the $\mathbb{P}-$module to directional lines at $\mathbb{E}^3$ whose vector moment norm is less than or equal to $\frac{1}{2}$. We then present a relationship between the angle between the directional lines corresponding to the unit hyperbolic vector and the angle between the unit hyperbolic vectors.

2. Preliminaries

The set of hyperbolic numbers $\mathbb{P}$ is expressed as follows:

$$\mathbb{P} = \{\mathbf{z} = z_1 + jz_2 : z_1, z_2 \in \mathbb{R}\}$$

where the hyperbolic unit j satisfies $j^2 = 1$ and $j \neq 1$. The sum and product of two hyperbolic numbers are defined as similar to complex numbers, but keep in mind that $j^2 = 1$. For any $\mathbf{z} = z_1 + jz_2 \in \mathbb{P}$, we define the real part of $\mathbf{z}$ as $\mathrm{Re}(\mathbf{z}) = z_1$ and the hyperbolic number part of $\mathbf{z}$ as $\mathrm{Hp}(\mathbf{z}) = z_2$. The conjugate of $\mathbf{z}$ is denoted by $\overline{\mathbf{z}}$ and it is $\overline{\mathbf{z}} = z_1 - jz_2$. The inner product of $\mathbf{z} = z_1 + jz_2$ and $\mathbf{w} = w_1 + jw_2$ is defined as

$$\begin{aligned}
&\langle \, , \, \rangle : \mathbb{P} \times \mathbb{P} \to \mathbb{R} \\
&\langle \mathbf{z}, \mathbf{w} \rangle = \mathrm{Re}\,(\overline{\mathbf{z}}\mathbf{w}) = w_1 z_1 - w_2 z_2.
\end{aligned}$$

The fact that the scalar product defined in the set of hyperbolic numbers is not positively defined will require us to classify the hyperbolic numbers. Accordingly, we will classify the hyperbolic number $\mathbf{z} = z_1 + jz_2$ as spacelike, timelike or null according to the following conditions.

If $\langle \mathbf{z}, \mathbf{z} \rangle = \overline{\mathbf{z}}\mathbf{z} > 0$,	spacelike hyperbolic number.
If $\langle \mathbf{z}, \mathbf{z} \rangle = \overline{\mathbf{z}}\mathbf{z} < 0$,	timelike hyperbolic number.
If $\langle \mathbf{z}, \mathbf{z} \rangle = \overline{\mathbf{z}}\mathbf{z} = 0$,	null or lightlike hyperbolic number.

Since $\langle \mathbf{z}, \mathbf{z} \rangle = \overline{\mathbf{z}}\mathbf{z} = z_1^2 - z_2^2$, this expression can be shortened as

$$\begin{cases} |z_1| > |z_2| & \Rightarrow \quad \text{spacelike hyperbolic number,} \\ |z_1| < |z_2| & \Rightarrow \quad \text{timelike hyperbolic number,} \\ z_1 = \pm z_2 & \Rightarrow \quad \text{null or lightlike hyperbolic number.} \end{cases}$$

The number $\mathbf{z}$ is said to be positive or negative, depending on the sign $\varepsilon(\mathbf{z}) = sgn(z_1 + z_2)$, where $\mathbf{z} = z_1 + jz_2$ is a hyperbolic number. If $z_1 + z_2 > 0$ $\mathbf{z}$ is positive, if $z_1 + z_2 < 0$ then it is negative. That is, it is defined as

$$\varepsilon(\mathbf{z}) = \begin{cases} +1 & z_1 + z_2 > 0 \\ -1 & z_1 + z_2 < 0 \\ 0 & z_1 + z_2 = 0 \end{cases}.$$

According to this definition, in the hyperbolic plane, all hyperbolic numbers on the line $y = -x$ have a zero sign. Numbers in the form $\mathbf{z} = z_1 - jz_1$ are considered equivalent to zero in the set of hyperbolic numbers. In the literature, a positive number of $\mathbf{z}$ positive is defined as future pointing in the Lorentz space, and past pointing when it is negative. The vector product in $\mathbb{P}$ is defined by

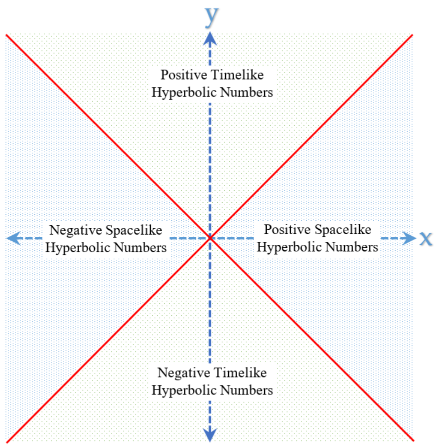

Figure 1: Classification of hyperbolic numbers.

$$\begin{aligned} \times \quad &: \quad \mathbb{P} \times \mathbb{P} \to \mathbb{P} \\ \mathbf{z} \times \mathbf{w} \quad &= \quad \mathrm{Hp}(\overline{\mathbf{z}}\mathbf{w})\, j = (z_1 w_2 - z_2 w_1)\, j, \end{aligned}$$

where $\mathbf{z} = z_1 + jz_2$ and $\mathbf{w} = w_1 + jw_2$. The norm of the hyperbolic number $\mathbf{z} = z_1 + jz_2$ is defined by

$$\begin{aligned} |\cdot| \quad &: \quad \mathbb{P} \to \mathbb{R}^+ \cup \{0\} \\ |\mathbf{z}| \quad &= \quad \sqrt{|\langle \mathbf{z}, \mathbf{z} \rangle|} = \sqrt{\left| z_1^2 - z_2^2 \right|}. \end{aligned}$$

$\mathbb{P}$ is a commutative ring with a unit element.

The square root of the hyperbolic number $\mathbf{z} = z_1 + jz_2$ can be found by

$$\sqrt{\mathbf{z}} = \frac{\sqrt{z_1 + z_2} + \sqrt{z_1 - z_2}}{2} + j\frac{\sqrt{z_1 + z_2} - \sqrt{z_1 - z_2}}{2}. \tag{1}$$

Corollary 2.1. *If the number $\mathbf{z} = z_1 + jz_2$ is a null hyperbolic number, then $z_1 = \pm z_2$. Therefore, according to Eq. (1), the square root of null hyperbolic numbers can be written as*

$$\begin{aligned} z_1 = z_2 \quad &\Rightarrow \quad \sqrt{z_1 + jz_2} = \tfrac{\sqrt{2z_1}}{2}(1 + j), \\ z_1 = -z_2 \quad &\Rightarrow \quad \sqrt{z_1 + jz_2} = \tfrac{\sqrt{2z_1}}{2}(1 - j). \end{aligned}$$

Corollary 2.2. *The necessary and sufficient condition for a non-lightlike hyperbolic number to have a square root is that the number is positive spacelike.*

3. $\mathbb{P}-$Module

The set

$$\mathbb{P}^3 = \{\mathbf{A} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) : \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbb{P}\}$$

is a module over the ring $\mathbb{P}$ which called a $\mathbb{P}$-module. The elements of $\mathbb{P}^3$ are called hyperbolic vectors. Thus a hyperbolic vector $\mathbf{A}$ can be written

$$\mathbf{A} = \overrightarrow{a} + j\overrightarrow{a^*}$$

where $\overrightarrow{a}$ and $\overrightarrow{a^*}$ are real vector at $\mathbb{R}^3$. For $\mathbf{A}, \mathbf{B} \in \mathbb{P}^3$ and $\alpha, \beta \in \mathbb{P}$, the followings are satisfied:

$$\begin{aligned} \alpha(\mathbf{A} + \mathbf{B}) &= \alpha\mathbf{A} + \alpha\mathbf{B}, \\ (\alpha + \beta)\mathbf{A} &= \alpha\mathbf{A} + \beta\mathbf{A}, \\ (\alpha\beta)\mathbf{A} &= \alpha(\beta\mathbf{A}), \\ 1\mathbf{A} &= \mathbf{A}. \end{aligned}$$

$\mathbb{P}^3$ is a module on $\mathbb{P}$, which is a ring of unitary and commutative hyperbolic numbers. The inner product of a vector on $\mathbb{P}^3$ is defined as

$$\begin{aligned} \langle\, ,\rangle &: \mathbb{P}^3 \times \mathbb{P}^3 \to \mathbb{P} \\ \langle \mathbf{A}, \mathbf{B}\rangle &= \left\langle \overrightarrow{a} + j\overrightarrow{a^*}, \overrightarrow{b} + j\overrightarrow{b^*} \right\rangle \end{aligned}$$

where $\mathbf{A} = \overrightarrow{a} + j\overrightarrow{a^*}$ and $\mathbf{B} = \overrightarrow{b} + j\overrightarrow{b^*}$. The norm of any vector in $\mathbb{P}^3$ is defined as

$$\begin{aligned} \|\cdot\| &: \mathbb{P}^3 \to \mathbb{P} \\ \|\mathbf{A}\| &= \sqrt{|\langle \mathbf{A}, \mathbf{A}\rangle|}. \end{aligned}$$

So for $\mathbf{A} = \overrightarrow{a} + j\overrightarrow{a^*}$, the norm of $\mathbf{A}$ is found by

$$\|\mathbf{A}\| = \frac{\left\|\overrightarrow{a} + \overrightarrow{a^*}\right\| + \left\|\overrightarrow{a} - \overrightarrow{a^*}\right\|}{2} + j\frac{\left\|\overrightarrow{a} + \overrightarrow{a^*}\right\| - \left\|\overrightarrow{a} - \overrightarrow{a^*}\right\|}{2}.$$

Hence, for a unit hyperbolic vector, that has a norm of 1, we define the hyperbolic vector unit condition as

$$\begin{aligned} \left\langle \overrightarrow{a}, \overrightarrow{a} \right\rangle + \left\langle \overrightarrow{a^*}, \overrightarrow{a^*} \right\rangle &= 1 \\ \left\langle \overrightarrow{a}, \overrightarrow{a^*} \right\rangle &= 0. \end{aligned}$$

The cross product of $\mathbf{A}$ and $\mathbf{B}$ for $\mathbf{A} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) \in \mathbb{P}^3$ and $\mathbf{B} = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3) \in \mathbb{P}^3$ is defined as

$$\mathbf{A} \times_{\mathbb{P}} \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{vmatrix}.$$

Proposition 3.1. *Let* $\mathbf{A} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) \in \mathbb{P}^3$ *and* $\mathbf{B} = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3) \in \mathbb{P}^3$

1. $\mathbf{A} \times_{\mathbb{P}} \mathbf{B} = -\mathbf{B} \times_{\mathbb{P}} \mathbf{A}$,

2. $\mathbf{A} \times_{\mathbb{P}} \mathbf{A} = \mathbf{0}$,

3. $(\lambda\mathbf{A}) \times_{\mathbb{P}} \mathbf{B} = \lambda(\mathbf{A} \times_{\mathbb{P}} \mathbf{B})$, *for* $\lambda \in \mathbb{R}$,

4. $\mathbf{0}\times_{\mathbb{P}}\mathbf{A} = \mathbf{A}\times_{\mathbb{P}}\mathbf{0} = \mathbf{0}$

5. $\mathbf{A}\times_{\mathbb{P}}\mathbf{B} = \mathbf{0} \Leftrightarrow \mathbf{A} = \lambda\mathbf{B}$, *for* $\lambda \in \mathbb{R}$.

6. $\mathbf{A}\times_{\mathbb{P}}(\mathbf{B}+\mathbf{C}) = (\mathbf{A}\times_{\mathbb{P}}\mathbf{B}) + (\mathbf{A}\times_{\mathbb{P}}\mathbf{C})$

Definition 3.2. *The set* $\left\{\mathbf{A} = \overrightarrow{a} + j\overrightarrow{a^*} : \|\mathbf{A}\| = (1,0), \overrightarrow{a}, \overrightarrow{a^*} \in \mathbb{R}^3\right\}$ *is called unit hyperbolic sphere in* $\mathbb{P}-$*module.*

Theorem 3.3. *A point on the unit hyperbolic sphere of the* $\mathbb{P}-$*Module corresponds to directional lines in* $\mathbb{E}^3$ *that is less than or equal to the vector moment norm* $\frac{1}{2}$. *Conversely, directional lines in* $\mathbb{E}^3$ *whose vector moment norm is less than or equal to* $\frac{1}{2}$ *correspond to two different unit hyperbolic vectors in the* $\mathbb{P}$*-module.*

Proof. A line in $\mathbb{E}^3$ is completely determined by a point P on it with respect to an origin O and a vector $\overrightarrow{u} \in \mathbb{R}^3$ that specifies the direction of the line. In the $\mathbb{P}-$module, we know that for a vector $\mathbf{A} = \overrightarrow{a} + j\overrightarrow{a^*}$ on the unit hyperbolic sphere,

$$\left\|\overrightarrow{a}\right\|^2 + \left\|\overrightarrow{a^*}\right\|^2 = 1 \text{ and } \left\langle \overrightarrow{a}, \overrightarrow{a^*} \right\rangle = 0.$$

Let $\mathbf{A} = \overrightarrow{a} + j\overrightarrow{a^*}$ be a point on the unit hyperbolic sphere. So, if we draw the plane $\mathrm{Sp}\left\{\overrightarrow{a}, \overrightarrow{a^*}\right\}$ in $\mathbb{E}^3$, we can take $\overrightarrow{a^*} \times \overrightarrow{a}$ as the normal vector of this plane. As seen in Figure 2, if a d-direction line is drawn through the endpoints of vectors $\overrightarrow{a}$ and $\overrightarrow{a^*}$, the direction vector of this d line becomes $\overrightarrow{a} - \overrightarrow{a^*}$. Moreover, $\left\|\overrightarrow{a} - \overrightarrow{a^*}\right\| = 1$ since $\mathbf{A} = \overrightarrow{a} + j\overrightarrow{a^*}$ is the unit hyperbolic vector. If any P and X points are taken on the d line, we can write

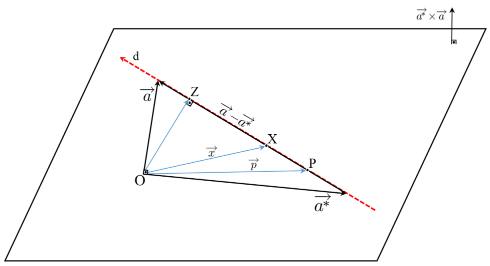

Figure 2: Directional line corresponding to the unit hyperbolic vector **A**.

it as

$$\left(\overrightarrow{x} - \overrightarrow{p}\right) \times \left(\overrightarrow{a} - \overrightarrow{a^*}\right) = 0.$$

Also, since the normal vector $\overrightarrow{a^*} \times \overrightarrow{a}$ of the plane is perpendicular to the vectors $\overrightarrow{x}, \overrightarrow{p}$ and $\overrightarrow{a} - \overrightarrow{a^*}$, if the equation $\overrightarrow{x} \times \left(\overrightarrow{a} - \overrightarrow{a^*}\right) = \overrightarrow{p} \times \left(\overrightarrow{a} - \overrightarrow{a^*}\right)$ is denoted by $\overrightarrow{a^*} \times \overrightarrow{a}$, the vector $\overrightarrow{a^*} \times \overrightarrow{a}$ can be chosen as the vector moment of the unit vector $\overrightarrow{a} - \overrightarrow{a^*}$ with respect to the origin. The vector $\overrightarrow{a} - \overrightarrow{a^*}$ is independent of the choice of the point on the line. If a point Y other than X is taken on the line, $\left(\overrightarrow{y} - \overrightarrow{p}\right) \times \left(\overrightarrow{a} - \overrightarrow{a^*}\right) = 0$ is obtained. Hence it is seen that

$$\overrightarrow{y} \times \left(\overrightarrow{a} - \overrightarrow{a^*}\right) = \overrightarrow{x} \times \left(\overrightarrow{a} - \overrightarrow{a^*}\right) = \overrightarrow{p} \times \left(\overrightarrow{a} - \overrightarrow{a^*}\right) = \overrightarrow{a^*} \times \overrightarrow{a}.$$

The length of the vector $\overrightarrow{a^*} \times \overrightarrow{a}$ is equal to the perpendicular distance of origin to the line. To illustrate this, let Z be the point where the perpendicular is drawn from the origin to the line crosses with the line. Since the vector $\overrightarrow{a^*} \times \overrightarrow{a}$ is independent of the choice of the point on the line, it can be written as

$$\overrightarrow{z} \times \left(\overrightarrow{a} - \overrightarrow{a^*}\right) = \overrightarrow{a^*} \times \overrightarrow{a}.$$

Thus, the length of the vector $\overrightarrow{a^*} \times \overrightarrow{a}$ can be found as

$$\begin{aligned} \left\|\overrightarrow{a^*} \times \overrightarrow{a}\right\| &= \left\|\overrightarrow{z}\right\| \left\|\overrightarrow{a} - \overrightarrow{a^*}\right\| \sin\frac{\pi}{2} \\ &= \left\|\overrightarrow{z}\right\|. \end{aligned}$$

It can be seen from here that the vector $\overrightarrow{a^*} \times \overrightarrow{a}$ depends on the choice of point O. Also, since $\left\|\overrightarrow{a}\right\|^2 + \left\|\overrightarrow{a^*}\right\|^2 = 1$, the length of the vector moment $\overrightarrow{a^*} \times \overrightarrow{a}$ is less than or equal to $\frac{1}{2}$. That is, $\left\|\overrightarrow{a^*} \times \overrightarrow{a}\right\| \leqslant \frac{1}{2}$.

Therefore, given the unit hyperbolic vector $\mathbf{A} = \overrightarrow{a} + j\overrightarrow{a^*}$, given the vector pair $\left(\overrightarrow{a}, \overrightarrow{a^*}\right)$, a line with a vector moment less than or equal to $\frac{1}{2}$ in $\mathbb{E}^3$ is determined in a completely uniquely way. That is, the unit hyperbolic vector $\mathbf{A} = \overrightarrow{a} + j\overrightarrow{a^*}$ corresponds to a directional line in $\mathbb{E}^3$ whose vector moment is less than or equal to $\frac{1}{2}$.

Now let's take a line d in $\mathbb{E}^3$ with the unit direction vector $\overrightarrow{u} = (u_1, u_2, u_3)$ and the vector moment norm less than or equal to $\frac{1}{2}$. So we can write

$$\begin{cases} d : \frac{x-x_0}{u_1} = \frac{y-y_0}{u_2} = \frac{z-z_0}{u_3} = \lambda, \\ u_1^2 + u_2^2 + u_3^2 = 1, \\ P(x_0, y_0, z_o) \in d. \end{cases}$$

From here, the vector moment of the line d is found as

$$\overrightarrow{u^*} = \overrightarrow{OP} \times \overrightarrow{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_0 & y_0 & z_0 \\ u_1 & u_2 & u_3 \end{vmatrix}.$$

Since we have chosen the norm of the vector moment of the line d less than or equal to $\frac{1}{2}$, we can choose the points $A(x, y, z)$ and $B(x - u_1, y - u_2, z - u_3)$ on the line so that

$$\left\|\overrightarrow{OA}\right\|^2 + \left\|\overrightarrow{OB}\right\|^2 = 1,$$

where $x = x_0 + \lambda u_1$, $y = y_0 + \lambda u_2$, $z = z_0 + \lambda u_3$. From here, we get

$$\begin{aligned} \lambda_1 &= \frac{1}{2} - u_1x_0 - u_2y_0 - u_3z_0 + \frac{1}{2}\sqrt{1 - 4\left\|\overrightarrow{u^*}\right\|^2}, \\ \lambda_2 &= \frac{1}{2} - u_1x_0 - u_2y_0 - u_3z_0 - \frac{1}{2}\sqrt{1 - 4\left\|\overrightarrow{u^*}\right\|^2}. \end{aligned}$$

Therefore, given a line in $\mathbb{E}^3$ with a vector moment norm less than or equal to $\frac{1}{2}$, there are two unit hyperbolic vectors

$$\begin{aligned} \mathbf{A} = {} & (x_0+\lambda_1u_1, y_0 + \lambda_1u_2, z_0 + \lambda_1u_3) \\ & +j\left(x_0+(\lambda_1-1)u_1, y_0+(\lambda_1-1)u_2, z_0+(\lambda_1-1)u_3\right) \end{aligned}$$

and

$$\begin{aligned} \mathbf{B} = {} & (x_0+\lambda_2u_1, y_0 + \lambda_2u_2, z_0 + \lambda_2u_3) \\ & +j\left(x_0+(\lambda_2-1)u_1, y_0+(\lambda_2-1)u_2, z_0+(\lambda_2-1)u_3\right) \end{aligned}$$

where

$$\lambda_1 = \frac{1}{2} - u_1x_0 - u_2y_0 - u_3z_0 + \frac{1}{2}\sqrt{1 - 4\left\|\overrightarrow{u^*}\right\|^2},$$
$$\lambda_2 = \frac{1}{2} - u_1x_0 - u_2y_0 - u_3z_0 - \frac{1}{2}\sqrt{1 - 4\left\|\overrightarrow{u^*}\right\|^2}.$$

□

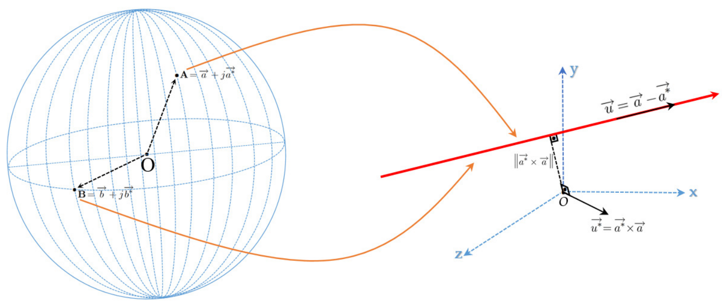

Figure 3: Representation of two points on the unit hyperbolic sphere corresponding to the directional line whose vector moment norm is less than or equal to $\frac{1}{2}$ in $\mathbb{E}^3$.

Remark 3.4. *With the mapping used in the proof of the above theorem, we will call the line corresponding to the unit hyperbolic vector as type 1 directional line. Similarly, the unit hyperbolic vector* $\mathbf{A} = \overrightarrow{a} + j\overrightarrow{a^*}$ *can be corresponded to a directional line, with the directing vector* $\overrightarrow{a} + \overrightarrow{a^*}$ *and the vector moment* $\overrightarrow{a^*} \times \overrightarrow{a}$. *We will call such lines as type 2 directional lines.*

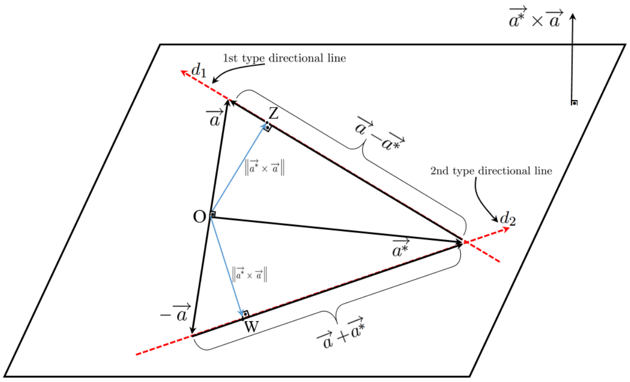

Figure 4: Representation of type 1 and type 2 directional line corresponding to the unit hyperbolic vector **A**.

Example 3.5. *Find the type 1 and type 2 directional lines in* $\mathbb{E}^3$ *corresponding to the unit hyperbolic vectors* $\mathbf{A} = \left(\frac{4}{5}, 0, 0\right) + j\left(0, 0, \frac{3}{5}\right)$ *and* $\mathbf{B} = \left(0, -\frac{3}{13}, \frac{4}{13}\right) + j\left(\frac{12}{13}, 0, 0\right)$.

First, let's find the vectors $\overrightarrow{a}-\overrightarrow{a^*}$, $\overrightarrow{a}+\overrightarrow{a^*}$ and $\overrightarrow{a^*}\times\overrightarrow{a}$ to find the type 1 and type 2 directional line corresponding to the unit hyperbolic vector **A** and **B**. The vectors $\overrightarrow{a}-\overrightarrow{a^*}$, $\overrightarrow{a}+\overrightarrow{a^*}$ and $\overrightarrow{a^*}\times\overrightarrow{a}$ are calculated as

$$\begin{aligned}\overrightarrow{a}-\overrightarrow{a^*} &= \left(\frac{4}{5},0,0\right)-\left(0,0,\frac{3}{5}\right)=\left(\frac{4}{5},0,-\frac{3}{5}\right),\\ \overrightarrow{a}+\overrightarrow{a^*} &= \left(\frac{4}{5},0,0\right)+\left(0,0,\frac{3}{5}\right)=\left(\frac{4}{5},0,\frac{3}{5}\right),\end{aligned}$$

and

$$\overrightarrow{a^*}\times\overrightarrow{a}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 0 & \frac{3}{5}\\ \frac{4}{5} & 0 & 0\end{vmatrix}=\left(0,\frac{12}{25},0\right).$$

Since $\left\|\overrightarrow{a^*}\times\overrightarrow{a}\right\|=\left\|\overrightarrow{z}\right\|$ for $\overrightarrow{z}=(z_1,z_2,z_3)$,

$$z_1^2+z_2^2+z_3^2=\left(\frac{12}{25}\right)^2. \tag{2}$$

Also, using $\overrightarrow{z}\times\left(\overrightarrow{a}-\overrightarrow{a^*}\right)=\overrightarrow{a^*}\times\overrightarrow{a}$ for the type 1 directional line,

$$\begin{aligned}\left(0,\frac{12}{25},0\right) &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ z_1 & z_2 & z_3\\ \frac{4}{5} & 0 & -\frac{3}{5}\end{vmatrix}=\left(-\frac{3z_2}{5},\frac{3z_1+4z_3}{5},-\frac{4z_2}{5}\right)\\ &\Rightarrow\quad z_2=0 \text{ and } 3z_1+4z_3=\frac{12}{5}.\end{aligned} \tag{3}$$

Using the equations (2) and (3) , z_1,z_2 and z_3 are calculated as

$$z_1=\frac{36}{125},\ z_2=0,\ z_3=\frac{48}{125}.$$

Thus, the type 1 directional line corresponding to the unit hyperbolic vector $\mathbf{A}=\left(\frac{4}{5},0,0\right)+j\left(0,0,\frac{3}{5}\right)$ is found as

$$\begin{aligned} d_1 &: \quad \frac{x-\frac{36}{125}}{\frac{4}{5}}=\frac{z-\frac{48}{125}}{-\frac{3}{5}}, y=0,\\ d_1 &: \quad \frac{125x-36}{4}=\frac{125z-48}{-3}, y=0.\end{aligned}$$

Since $\left\|\overrightarrow{a^*}\times\overrightarrow{a}\right\|=\left\|\overrightarrow{w}\right\|$ for $\overrightarrow{w}=(w_1,w_2,w_3)$,

$$w_1^2+w_2^2+w_3^2=\left(\frac{12}{25}\right)^2. \tag{4}$$

Also, using $\overrightarrow{w}\times\left(\overrightarrow{a}+\overrightarrow{a^*}\right)=\overrightarrow{a^*}\times\overrightarrow{a}$ for the type 2 directional line,

$$\begin{aligned}\left(0,\frac{12}{25},0\right) &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ w_1 & w_2 & w_3\\ \frac{4}{5} & 0 & \frac{3}{5}\end{vmatrix}=\left(\frac{3w_2}{5},\frac{4w_3-3w_1}{5},-\frac{4w_2}{5}\right)\\ &\Rightarrow\quad w_2=0 \text{ and } 4w_3-3w_1=\frac{12}{5}.\end{aligned} \tag{5}$$

Using the equations (4) and (5), w_1, w_2 and w_3 are calculated as

$$w_1 = -\frac{36}{125},\ w_2 = 0,\ w_3 = \frac{48}{125}.$$

Thus, the type 2 directional line corresponding to the unit hyperbolic vector $\mathbf{A} = \left(\frac{4}{5}, 0, 0\right) + j\left(0, 0, \frac{3}{5}\right)$ is found as

$$\begin{aligned} d_2 &: \frac{x + \frac{36}{125}}{\frac{4}{5}} = \frac{z - \frac{48}{125}}{\frac{3}{5}}, y = 0, \\ d_2 &: \frac{125x + 36}{4} = \frac{125z - 48}{3}, y = 0. \end{aligned}$$

Now let's find the type 1 and type 2 directional line corresponding to the unit hyperbolic vector $\mathbf{B} = \left(0, -\frac{3}{13}, \frac{4}{13}\right) + j\left(\frac{12}{13}, 0, 0\right)$. The vectors $\overrightarrow{b} - \overrightarrow{b^*}$, $\overrightarrow{b} + \overrightarrow{b^*}$ and $\overrightarrow{b^*} \times \overrightarrow{b}$ are calculated as

$$\begin{aligned} \overrightarrow{b} - \overrightarrow{b^*} &= \left(0, -\frac{3}{13}, \frac{4}{13}\right) - \left(\frac{12}{13}, 0, 0\right) = \left(-\frac{12}{13}, -\frac{3}{13}, \frac{4}{13}\right), \\ \overrightarrow{b} + \overrightarrow{b^*} &= \left(0, -\frac{3}{13}, \frac{4}{13}\right) + \left(\frac{12}{13}, 0, 0\right) = \left(\frac{12}{13}, -\frac{3}{13}, \frac{4}{13}\right), \end{aligned}$$

and

$$\overrightarrow{b^*} \times \overrightarrow{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{12}{13} & 0 & 0 \\ 0 & -\frac{3}{13} & \frac{4}{13} \end{vmatrix} = \left(0, -\frac{48}{169}, -\frac{36}{169}\right).$$

Since $\left\| \overrightarrow{b^*} \times \overrightarrow{b} \right\| = \left\| \overrightarrow{p} \right\|$ for $\overrightarrow{p} = (p_1, p_2, p_3)$,

$$p_1^2 + p_2^2 + p_3^2 = \left(\frac{60}{169}\right)^2. \tag{6}$$

Also, using $\overrightarrow{p} \times \left(\overrightarrow{b} - \overrightarrow{b^*}\right) = \overrightarrow{b^*} \times \overrightarrow{b}$ for the type 1 directional line,

$$\begin{aligned} \left(0, -\frac{48}{169}, -\frac{36}{169}\right) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ p_1 & p_2 & p_3 \\ -\frac{12}{13} & -\frac{3}{13} & \frac{4}{13} \end{vmatrix} && (7) \\ &= \left(\frac{4p_2 + 3p_3}{13}, \frac{-4p_1 - 12p_3}{13}, \frac{-3p_1 + 12p_2}{13}\right) && (8) \\ &\Rightarrow\ 4p_2 + 3p_3 = 0,\ p_1 + 3p_3 = \frac{12}{13} \text{ and } p_1 - 4p_2 = \frac{12}{13}. \end{aligned}$$

Using the equations (6) and (7), p_1, p_2 and p_3 are calculated as

$$p_1 = \frac{300}{2197},\ p_2 = -\frac{432}{2197},\ p_3 = \frac{576}{2197}.$$

Thus, the type 1 directional line corresponding to the unit hyperbolic vector $\mathbf{B} = \left(0, -\frac{3}{13}, \frac{4}{13}\right) + j\left(\frac{12}{13}, 0, 0\right)$ is found as

$$\begin{aligned} k_1 &: \frac{x - \frac{300}{2197}}{-\frac{12}{13}} = \frac{y + \frac{432}{2197}}{-\frac{3}{13}} = \frac{z - \frac{576}{2197}}{\frac{4}{13}}, \\ k_1 &: \frac{300 - 2197x}{2028} = \frac{-2197y - 432}{507} = \frac{2197z - 576}{676} \end{aligned}$$

Since $\left\|\overrightarrow{b^*} \times \overrightarrow{b}\right\| = \left\|\overrightarrow{q}\right\|$ for $\overrightarrow{q} = (q_1, q_2, q_3)$,

$$q_1^2 + q_2^2 + q_3^2 = \left(\frac{60}{169}\right)^2. \tag{9}$$

Also, using $\overrightarrow{q} \times \left(\overrightarrow{b} + \overrightarrow{b^*}\right) = \overrightarrow{b^*} \times \overrightarrow{b}$ for the type 2 directional line,

$$\begin{aligned}
\left(0, -\frac{48}{169}, -\frac{36}{169}\right) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ q_1 & q_2 & q_3 \\ \frac{12}{13} & -\frac{3}{13} & \frac{4}{13} \end{vmatrix} && (10) \\
&= \left(\frac{4q_2 + 3q_3}{13}, \frac{12q_3 - 4q_1}{13}, \frac{-3q_1 - 12q_2}{13}\right) && (11) \\
&\Rightarrow \quad 4q_2 + 3q_3 = 0,\ q_1 - 3q_3 = \frac{12}{13} \text{ and } q_1 + 4q_2 = \frac{12}{13}.
\end{aligned}$$

Using the equations (9) and (10) , q_1, q_2 and q_3 are calculated as

$$q_1 = \frac{300}{2197},\ q_2 = \frac{432}{2197},\ q_3 = -\frac{576}{2197}.$$

Thus, the type 2 directional line corresponding to the unit hyperbolic vector $\mathbf{B} = \left(0, -\frac{3}{13}, \frac{4}{13}\right) + j\left(\frac{12}{13}, 0, 0\right)$ is found as

$$\begin{aligned}
k_2 \quad &: \quad \frac{x - \frac{300}{2197}}{\frac{12}{13}} = \frac{y - \frac{432}{2197}}{-\frac{3}{13}} = \frac{z + \frac{576}{2197}}{\frac{4}{13}}, \\
k_2 \quad &: \quad \frac{2197x - 300}{2028} = \frac{432 - 2197y}{507} = \frac{2197z + 576}{676}.
\end{aligned}$$

3.1. Hyperbolic Angle in $\mathbb{P}$−Module

Let $\mathbf{A} = \overrightarrow{a} + j\overrightarrow{a^*}$ and $\mathbf{B} = \overrightarrow{b} + j\overrightarrow{b^*}$ be two unit hyperbolic vectors. From here, the dot product of two unit hyperbolic vectors is

$$\begin{aligned}
\langle \mathbf{A}, \mathbf{B} \rangle &= \left\langle \overrightarrow{a} + j\overrightarrow{a^*}, \overrightarrow{b} + j\overrightarrow{b^*} \right\rangle \\
&= \left\langle \overrightarrow{a}, \overrightarrow{b} \right\rangle + \left\langle \overrightarrow{a^*}, \overrightarrow{b^*} \right\rangle + j\left(\left\langle \overrightarrow{a}, \overrightarrow{b^*} \right\rangle + \left\langle \overrightarrow{a^*}, \overrightarrow{b} \right\rangle\right).
\end{aligned}$$

On the other hand, type 1 and type 2 direction lines corresponding to the unit hyperbolic vector $\mathbf{A} = \overrightarrow{a} + j\overrightarrow{a^*}$ are d_1 and d_2, respectively, and type 1 and type 2 direction lines corresponding to the unit hyperbolic vector $\mathbf{B} = \overrightarrow{b} + j\overrightarrow{b^*}$ are k_1 and k_2, respectively. Also, if the angle between the lines d_1 and k_1 is called α, and the angle between the lines d_2 and k_2 is called β, we can get

$$\begin{aligned}
&\left\langle \overrightarrow{a} - \overrightarrow{a^*}, \overrightarrow{b} - \overrightarrow{b^*} \right\rangle = \left\|\overrightarrow{a} - \overrightarrow{a^*}\right\| \left\|\overrightarrow{a} - \overrightarrow{a^*}\right\| \cos\alpha \\
&\Rightarrow \cos\alpha = \left\langle \overrightarrow{a}, \overrightarrow{b} \right\rangle + \left\langle \overrightarrow{a^*}, \overrightarrow{b^*} \right\rangle - \left\langle \overrightarrow{a}, \overrightarrow{b^*} \right\rangle - \left\langle \overrightarrow{a^*}, \overrightarrow{b} \right\rangle \qquad (12)
\end{aligned}$$

and

$$\begin{aligned}
&\left\langle \overrightarrow{a} + \overrightarrow{a^*}, \overrightarrow{b} + \overrightarrow{b^*} \right\rangle = \left\|\overrightarrow{a} + \overrightarrow{a^*}\right\| \left\|\overrightarrow{a} + \overrightarrow{a^*}\right\| \cos\beta \\
&\Rightarrow \cos\beta = \left\langle \overrightarrow{a}, \overrightarrow{b} \right\rangle + \left\langle \overrightarrow{a^*}, \overrightarrow{b^*} \right\rangle + \left\langle \overrightarrow{a}, \overrightarrow{b^*} \right\rangle + \left\langle \overrightarrow{a^*}, \overrightarrow{b} \right\rangle. \qquad (13)
\end{aligned}$$

Thus, using equations (12) and (13), we get

$$\begin{aligned}\langle \mathbf{A}, \mathbf{B}\rangle &= \left\langle \overrightarrow{a}, \overrightarrow{b} \right\rangle + \left\langle \overrightarrow{a^*}, \overrightarrow{b^*} \right\rangle + j\left[\left\langle \overrightarrow{a}, \overrightarrow{b^*} \right\rangle + \left\langle \overrightarrow{a^*}, \overrightarrow{b} \right\rangle\right] \\ &= \frac{\cos\beta + \cos\alpha}{2} + j\frac{\cos\beta - \cos\alpha}{2} \\ &= \cos\left(\frac{\beta+\alpha}{2}\right)\cos\left(\frac{\beta-\alpha}{2}\right) - j\sin\left(\frac{\beta+\alpha}{2}\right)\sin\left(\frac{\beta-\alpha}{2}\right) \\ &= \cos\left(\frac{\beta+\alpha}{2} + j\frac{\beta-\alpha}{2}\right).\end{aligned}$$

Also, if we use that $\langle \mathbf{A}, \mathbf{B}\rangle = \|\mathbf{A}\|\,\|\mathbf{B}\|\cos\theta$ for the hyperbolic angle $\theta = \theta + j\theta^*$, then

$$\theta = \frac{\beta+\alpha}{2} \text{ and } \theta^* = \frac{\beta-\alpha}{2}.$$

Example 3.6. *Find the unit hyperbolic vectors corresponding to the type 1 directional line $d_1 : x = \lambda, y = -\frac{1}{4}, z = 0$.*

First, to find the unit hyperbolic vector corresponding to the type 1 directional line d_1, let's find the vector moment of the line using the unit direction vector and a point P on it. Let $P\left(1, -\frac{1}{4}, 0\right)$ and $\overrightarrow{u} = (1, 0, 0)$. Then we get

$$\overrightarrow{u^*} = \overrightarrow{OP} \times \overrightarrow{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -\frac{1}{4} & 0 \\ 1 & 0 & 0 \end{vmatrix} = \left(0, 0, \frac{1}{4}\right),$$

$$\Rightarrow \left\|\overrightarrow{u^*}\right\| = \frac{1}{4} \leqslant \frac{1}{2}.$$

Since the vector moment norm of the line d_1 is less than or equal to $\frac{1}{2}$, there are points $A\left(\lambda, -\frac{1}{4}, 0\right)$ and $B\left(\lambda - 1, -\frac{1}{4}, 0\right)$ such that

$$\left\|\overrightarrow{OA}\right\|^2 + \left\|\overrightarrow{OB}\right\|^2 = 1.$$

From here, we get

$$\lambda^2 + \left(-\frac{1}{4}\right)^2 + (\lambda-1)^2 + \left(-\frac{1}{4}\right)^2 = 1$$

$$\Rightarrow \begin{cases} \lambda_1 = \frac{\sqrt{3}+2}{4}, \\ \lambda_2 = \frac{2-\sqrt{3}}{4}, \end{cases}$$

$$\mathbf{A} = \left(\tfrac{\sqrt{3}+2}{4}, -\frac{1}{4}, 0\right) + j\left(\tfrac{\sqrt{3}-2}{4}, -\frac{1}{4}, 0\right)$$

and

$$\mathbf{B} = \left(\tfrac{2-\sqrt{3}}{4}, -\frac{1}{4}, 0\right) + j\left(\tfrac{-2-\sqrt{3}}{4}, -\frac{1}{4}, 0\right).$$

Therefore, the unit hyperbolic vectors $\mathbf{A}$ and $\mathbf{B}$ correspond to the type 1 directional line d_1, whose vector moment norm is less than $\frac{1}{2}$.

Corollary 3.7. *The closest point to the origin of the type 1 directional line corresponding to the unit hyperbolic vector* $\mathbf{A} = \overrightarrow{a} + j\overrightarrow{a^*}$ *is found with*

$$H_1\left(\left\|\overrightarrow{a^*}\right\|^2 \overrightarrow{a} + \left\|\overrightarrow{a}\right\|^2 \overrightarrow{a^*}\right).$$

Corollary 3.8. *The closest point to the origin of the type 2 directional line corresponding to the unit hyperbolic vector* $\mathbf{A} = \overrightarrow{a} + j\overrightarrow{a^*}$ *is found with*

$$H_2\left(\left\|\overrightarrow{a}\right\|^2 \overrightarrow{a^*} - \left\|\overrightarrow{a^*}\right\|^2 \overrightarrow{a}\right).$$

Corollary 3.9. *If one of the unit hyperbolic vectors corresponding to the type 1 directional line is* $\mathbf{A} = \overrightarrow{a} + j\overrightarrow{a^*}$*, the other unit hyperbolic vector is calculated with*

$$\mathbf{B} = \overrightarrow{a^*} + 2\left\|\overrightarrow{a^*}\right\|^2\left(\overrightarrow{a} - \overrightarrow{a^*}\right) + j\left(\overrightarrow{a} - 2\left\|\overrightarrow{a}\right\|^2\left(\overrightarrow{a} - \overrightarrow{a^*}\right)\right).$$

Corollary 3.10. *If one of the unit hyperbolic vectors corresponding to the type 2 directional line is* $\mathbf{A} = \overrightarrow{a} + j\overrightarrow{a^*}$*, the other unit hyperbolic vector is calculated with*

$$\mathbf{B} = -\overrightarrow{a^*} + 2\left\|\overrightarrow{a^*}\right\|^2\left(\overrightarrow{a} + \overrightarrow{a^*}\right) + j\left(-\overrightarrow{a} + 2\left\|\overrightarrow{a}\right\|^2\left(\overrightarrow{a} + \overrightarrow{a^*}\right)\right).$$

Theorem 3.11. *When another point P is chosen instead of the origin with a distance less than or equal to $\frac{1}{2}$ to the directional line, the unit hyperbolic vectors that denote the line with respect to the point P are*

$$\begin{aligned}\mathbf{A} &= (x_0+\lambda_1 u_1 - p_1, y_0 + \lambda_1 u_2 - p_2, z_0 + \lambda_1 u_3 - p_3) \\ &\quad + j\left(x_0 + (\lambda_1 - 1)u_1 - p_1, y_0 + (\lambda_1 - 1)u_2 - p_2, z_0 + (\lambda_1 - 1)u_3 - p_3\right)\end{aligned}$$

and

$$\begin{aligned}\mathbf{B} &= (x_0+\lambda_2 u_1 - p_1, y_0 + \lambda_2 u_2 - p_2, z_0 + \lambda_2 u_3 - p_3) \\ &\quad + j\left(x_0 + (\lambda_2 - 1)u_1 - p_1, y_0 + (\lambda_2 - 1)u_2 - p_2, z_0 + (\lambda_2 - 1)u_3 - p_3\right)\end{aligned}$$

where

$$\begin{aligned}\lambda_1 &= \frac{1}{2} - \left\langle \overrightarrow{PQ}, \overrightarrow{u}\right\rangle + \frac{1}{2}\sqrt{1 - 4\left\|\overrightarrow{PQ} \times \overrightarrow{u}\right\|^2}, \\ \lambda_2 &= \frac{1}{2} - \left\langle \overrightarrow{PQ}, \overrightarrow{u}\right\rangle - \frac{1}{2}\sqrt{1 - 4\left\|\overrightarrow{PQ} \times \overrightarrow{u}\right\|^2}.\end{aligned}$$

Proof. Let's take a line d in $\mathbb{E}^3$ with the unit direction vector $\overrightarrow{u} = (u_1, u_2, u_3)$ and the vector moment norm less than or equal to $\frac{1}{2}$. So we can write

$$\begin{cases} d : \frac{x-x_0}{u_1} = \frac{y-y_0}{u_2} = \frac{z-z_0}{u_3} = \lambda, \\ u_1^2 + u_2^2 + u_3^2 = 1, \\ Q(x_0, y_0, z_o) \in d. \end{cases}$$

Given any point $P(p_1, p_2, p_3)$ that is less than or equal to $\frac{1}{2}$ the distance from the line d, we can choose points $X(x, y, z)$ and $Y(x - u_1, y - u_2, z - u_3)$ on the line d such that

$$\begin{aligned}\overrightarrow{PX} &= (x_0 + \lambda u_1 - p_1, y_0 + \lambda u_2 - p_2, z_0 + \lambda u_3 - p_3) \\ \overrightarrow{PY} &= (x_0 + (\lambda - 1)u_1 - p_1, y_0 + (\lambda - 1)u_2 - p_2, z_0 + (\lambda - 1)u_3 - p_3)\end{aligned}$$

and

$$\left\|\overrightarrow{PX}\right\|^2+\left\|\overrightarrow{PY}\right\|^2=1,$$

where $x=x_0+\lambda u_1$, $y=y_0+\lambda u_2$, $z=z_0+\lambda u_3$. Also, we can write the vector moment of the line d with respect to the point P as

$$\begin{aligned}
\overrightarrow{u_p^*} &= \overrightarrow{PY}\times\overrightarrow{PX}=\left(\overrightarrow{PO}+\overrightarrow{y}\right)\times\left(\overrightarrow{PO}+\overrightarrow{x}\right)\\
&= \left(\overrightarrow{PO}\times\overrightarrow{x}\right)+\left(\overrightarrow{y}\times\overrightarrow{PO}\right)+\overrightarrow{y}\times\overrightarrow{x}\\
&= \left(\overrightarrow{PO}\times\overrightarrow{u}\right)+\left(\overrightarrow{OQ}\times\overrightarrow{u}\right).\\
&= \left(\overrightarrow{PO}+\overrightarrow{OQ}\right)\times\overrightarrow{u}\\
&= \overrightarrow{PQ}\times\overrightarrow{u}
\end{aligned}$$

From here, we get the values of λ as

$$\begin{aligned}
\lambda_1 &= \frac{1}{2}-\left\langle\overrightarrow{PQ},\overrightarrow{u}\right\rangle+\frac{1}{2}\sqrt{1-4\left\|\overrightarrow{PQ}\times\overrightarrow{u}\right\|^2},\\
\lambda_2 &= \frac{1}{2}-\left\langle\overrightarrow{PQ},\overrightarrow{u}\right\rangle-\frac{1}{2}\sqrt{1-4\left\|\overrightarrow{PQ}\times\overrightarrow{u}\right\|^2}.
\end{aligned}$$

Therefore, when choosing a point P other than the origin whose distance to the directional line in $\mathbb{E}^3$ is less than or equal to $\frac{1}{2}$, the unit hyperbolic vectors denoting the directional line are found with

$$\begin{aligned}
\mathbf{A} &= (x_0+\lambda_1u_1-p_1, y_0+\lambda_1u_2-p_2, z_0+\lambda_1u_3-p_3)\\
&\quad +j\left(x_0+(\lambda_1-1)\,u_1-p_1, y_0+(\lambda_1-1)\,u_2-p_2, z_0+(\lambda_1-1)\,u_3-p_3\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{B} &= (x_0+\lambda_2u_1-p_1, y_0+\lambda_2u_2-p_2, z_0+\lambda_2u_3-p_3)\\
&\quad +j\left(x_0+(\lambda_2-1)\,u_1-p_1, y_0+(\lambda_2-1)\,u_2-p_2, z_0+(\lambda_2-1)\,u_3-p_3\right)
\end{aligned}$$

where

$$\begin{aligned}
\lambda_1 &= \frac{1}{2}-\left\langle\overrightarrow{PQ},\overrightarrow{u}\right\rangle+\frac{1}{2}\sqrt{1-4\left\|\overrightarrow{PQ}\times\overrightarrow{u}\right\|^2},\\
\lambda_2 &= \frac{1}{2}-\left\langle\overrightarrow{PQ},\overrightarrow{u}\right\rangle-\frac{1}{2}\sqrt{1-4\left\|\overrightarrow{PQ}\times\overrightarrow{u}\right\|^2}.
\end{aligned}$$

□

4. Conclusion

In this study, we have explored the properties of hyperbolic numbers and introduced the concept of the $\mathbb{P}$-module. By examining the fundamental operations and defining the $\mathbb{P}$-modulus, we have provided a deeper understanding of hyperbolic numbers and their structure. We also mapped the points on the unit hyperbolic sphere of the $\mathbb{P}$-module to directional lines in $\mathbb{E}^3$, focusing on the relationship between the angles of these lines and the hyperbolic angles between the corresponding unit vectors.

Our findings suggest that the study of hyperbolic numbers, particularly through the $\mathbb{P}$-module framework, opens new avenues for both mathematical and physical applications. The connections between hyperbolic angles and directional lines provide valuable insights into the geometric properties of hyperbolic numbers, which could be applied in areas such as quantum mechanics, gravity theory, and other

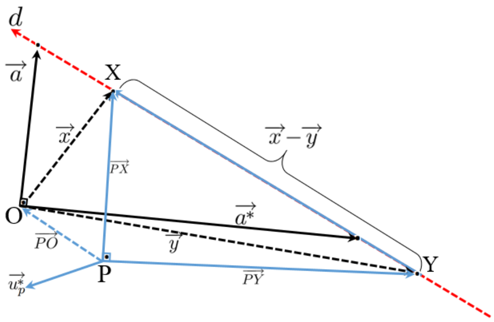

Figure 5: The unit hyperbolic vector corresponding to a point P other than the origin.

physical domains where hyperbolic structures are utilized. Future work may involve further exploration of transformations in hyperbolic numbers, akin to the E. Study transformation in dual numbers, and their potential applications in advanced mathematical and physical models.

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Conflict of interest

The authors declare no potential conflict of interests.

References

[1] N. A. Borota, E. Flores, and T. J. Osler, *Spacetime numbers the easy way*, Math. Comput. Educ. **34** (2000), 159-168.
[2] N. A. Borota and T. J. Osler, *Functions of a spacetime variable*, Math. Comput. Educ. **36** (2002), 231-239.
[3] J. J. Callahan, *The geometry of spacetime: an introduction to special and general relativity*, Springer, New York, 2013.
[4] H. Çakır and M. Özdemir, *Explicit formulas for exponential of 2×2 split-complex matrices*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **71** (2022), 518-532.
[5] H. Çakır, *Algebraic and geometric applications of hyperbolic number and hyperbolic number matrix*, Master Thesis, Akdeniz Univ., Antalya, 2017.
[6] F. Catoni, D. Boccaletti, R. Cannata, V. Catoni, E. Nichelatti, and P. Zampetti, *The mathematics of Minkowski space-time: with an introduction to commutative hypercomplex numbers*, Springer, Basel, 2013.
[7] F. Catoni, P. Zampetti, *Cauchy-like integral formula for functions of a hyperbolic variable*, Adv. Appl. Clifford Algebras **22** (2012), 23-37.
[8] F. Catoni, R. Cannata, V. Catoni, and P. Zampetti, *Hyperbolic trigonometry in two-dimensional space-time geometry*, Nuovo Cimento Soc. Ital. Fis. B **118** (2003), 475-492.
[9] Y. Cheng, Y. Li, P. Badyal, K. Singh, and S. Sharma, *Conformal interactions of osculating curves on regular surfaces in Euclidean 3-space*, Mathematics **13** (2025), 881.
[10] P. Fjelstad, *Extending special relativity via the perplex numbers*, Amer. J. Phys. **54** (1986), 416-422.
[11] H. H. Hacısalihoğlu, *Hareket geometrisi ve kuaterniyonlar teorisi*, Gazi Üniv., Ankara, 2013.
[12] J. Hucks, *Hyperbolic complex structures in physics*, J. Math. Phys. **34** (1993), 5986-6008.
[13] A. Khrennikov, *Hyperbolic quantum mechanics*, Adv. Appl. Clifford Algebras **13** (2003), 1-9.
[14] A. Khrennikov, *An introduction to hyperbolic analysis*, arXiv preprint math-ph/0507053 (2005).
[15] G. Kunstatter, J. W. Moffat, and J. Malzan, *Geometrical interpretation of a generalized theory of gravitation*, J. Math. Phys. **24** (1983), 886-889.

[16] Y. Li, K. Eren, S. Ersoy, and A. Savic, *Modified sweeping surfaces in Euclidean 3-space*, Axioms **13** (2024), 800.

[17] Y. Li, O. Kecilioglu, and K. Ilarslan, *Generalized Bertrand curve pairs in Euclidean four-dimensional space*, Axioms **14** (2025), 253.

[18] Y. Li, D. Pei, *Evolutes of dual spherical curves for ruled surfaces*, Math. Method Appl. Sci. **39** (2016), 3005-3015.

[19] Y. Li, Z. Wang, T. Zhao, *Geometric algebra of singular ruled surfaces*, Adv. Appl. Clifford Algebras **31** (2021), 1-19.

[20] Y. Li, F. Mofarreh, R. A. Abdel-Baky, *Kinematic-geometry of a line trajectory and the invariants of the axodes*, Demonstr. Math. **56** (2023), 20220252.

[21] A. E. Motter and M. A. F. Rosa, *Hyperbolic calculus*, Adv. Appl. Clifford Algebras **8** (1998), 109-128.

[22] R. D. Poodiack and K. J. LeClair, *Fundamental theorems of algebra for the perplexes*, College Math. J. **40** (2009), 322-336.

[23] G. Sobczyk, *The hyperbolic number plane*, College Math. J. **26** (1995), 268-280.

[24] E. Study, *Geometrie der Dynamen*, Teubner, Leipzig, 1903.

[25] H. H. Uğurlu and A. Çalışkan, *The Study mapping for directed space-like and time-like in Minkowski 3-space* $\mathbb{R}^3_1$, Math. Comput. Appl. **1** (1996), 142-148.

[26] Y. Yayli, A. Çalışkan, and H. H. Uğurlu, *The E. Study maps of circles on dual hyperbolic and Lorentzian unit spheres* $\mathbb{H}^2_0$ *and* $\mathbb{S}^2_1$, Proc. R. Ir. Acad. Sect. A Math. Phys. Sci. **102A** (2002), 37-47.

[27] Ö. G. Yıldız, *Study map of the orthotomic*, Ph.D. Thesis, Anadolu Univ., Eskişehir, 2015.

[28] S. Yüce, *On the E. Study maps for the dual quaternions*, Appl. Math. E-Notes **21** (2021), 365-374.

[29] Y. Zhu, Y. Li, K. Eren, and S. Ersoy, *Sweeping surfaces of polynomial curves in Euclidean 3-space*, Ann. Ştiinţ. Univ. Ovidius Constanţa Ser. Mat. **33**(1) (2025), 293-311.

[30] M. Zhu, H. Yang, Y. Li, R. A. Abdel-Baky, A. AL-Jedani, and M. Khalifa, *Directional developable surfaces and their singularities in Euclidean 3-space*, Filomat **38**(32) (2024), 11333-11347.