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Linearly independent curvature tensors of half-symmetric affine connection

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Abstract. In this paper, a half-symmetric affine connection is considered. Based on this connection, the corresponding linearly independent curvature tensors are determined. Subsequently, a special half-symmetric affine connection (K. Yano, 1965) is examined. First, the dual and symmetric connections of this special half-symmetric connection are defined. Then, a necessary and sufficient condition for the torsion vanishing of the special half-symmetric affine connection is determined. Linearly independent curvature tensors of the special half-symmetric affine connection are then identified. Finally, necessary and sufficient conditions under which the dual and symmetric connections of the special half-symmetric affine connection are F-connections are established.

1. Introduction

In this research, we will continue the research about half-symmetric affine connection started by K. Yano [18]. Yano's research is a special case of the study about non-symmetric affine connection space started by L. P. Eisenhart [1], and continued by S. Minčić [4–9], M. Stanković [16], Lj. S. Velimirović [9, 11, 13, 16], and many others.

In this article, we will recall basic definitions about symmetric and non-symmetric affine connection spaces. After that, the curvature tensors of these spaces will be expressed. In the Section 2, we will present definition of half-symmetric connection [18] and correlate it with the corresponding non-symmetric affine connection. In the Section 3, we will obtain a family of curvature tensors with respect to half-symmetric affine connection. The last result in this study are linearly independent curvature tensors obtained with respect to the half-symmetric affine connection.

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1.1. Symmetric and non-symmetric affine connection space

A K-dimensional manifold $\mathcal{M}_K$ equipped with a symmetric metric affine connection $\overset{0}{\nabla}$, whose coefficients are $L^i_{\underline{jk}}$, $L^i_{\underline{jk}} = L^i_{\underline{kj}}$, is the symmetric affine connection space $\mathbb{A}_K$ $\big($see [2, 3, 15]$\big)$.

The covariant derivative of a tensor $\hat{a}$ of the type $(1,1)$, with respect to the symmetric affine connection $\overset{0}{\nabla}$ is [2, 3, 15]

$$a^i_{j|k} = a^i_{j,k} + L^i_{\underline{pk}} a^p_j - L^p_{\underline{jk}} a^i_p, \tag{1}$$

where partial derivation is marked by comma.

The corresponding Ricci identity is

$$a^i_{j|m|n} - a^i_{j|n|m} = a^p_j R^i_{pmn} - a^i_p R^p_{jmn}, \tag{2}$$

for the curvature tensor of the space $\mathbb{R}_K$ expressed as [15]

$$R^i_{jmn} = L^i_{\underline{jm},n} - L^i_{\underline{jn},m} + L^p_{\underline{jm}} L^i_{\underline{pn}} - L^p_{\underline{jn}} L^p_{\underline{jm}}. \tag{3}$$

A K-dimensional manifold $\mathcal{M}_K$ equiped with a non-symmetric affine connection ∇, whose coefficients are L^i_{jk}, $L^i_{jk} \neq L^i_{kj}$ for at least one pair of indices (j,k), is the non-symmetric affine connection space $\mathbb{GA}_K$ $\big($see [1, 4–9, 11–13, 16, 19]$\big)$.

The symmetric and anti-symmetric part of affine coefficients L^i_{jk} are

$$L^i_{\underline{jk}} = \frac{1}{2}\Big(L^i_{jk} + L^i_{kj}\Big), \quad T^i_{jk} = L^i_{\underset{\vee}{jk}} = \frac{1}{2}\Big(L^i_{jk} - L^i_{kj}\Big). \tag{4}$$

The components $L^i_{\underline{jk}}$ are components of coefficients of a symmetric affine connection. This symmetric affine connection is the affine connection of associated space of the space $\mathbb{GA}_K$. The components $L^i_{\underset{\vee}{jk}}$ are components of a tensor of the type $(1,2)$. The tensor $S^i_{jk} = 2L^i_{\underset{\vee}{jk}}$ is the torsion tensor of the space $\mathbb{GA}_K$. It holds the equality $L^i_{jk} = L^i_{\underline{jk}} + L^i_{\underset{\vee}{jk}}$.

S. M. Minčić founded four kinds of covariant derivatives of the tensor $\hat{a}$ of the type $(1,1)$ with respect to non-symmetric affine connection [4–9]

$$a^i_{j\underset{1}{|}k} = a^i_{j,k} + L^i_{pk} a^p_j - L^p_{jk} a^i_p, \tag{5}$$

$$a^i_{j\underset{2}{|}k} = a^i_{j,k} + L^i_{kp} a^p_j - L^p_{kj} a^i_p, \tag{6}$$

$$a^i_{j\underset{3}{|}k} = a^i_{j,k} + L^i_{pk} a^p_j - L^p_{kj} a^i_p, \tag{7}$$

$$a^i_{j\underset{4}{|}k} = a^i_{j,k} + L^i_{kp} a^p_j - L^p_{jk} a^i_p. \tag{8}$$

N. O. Vesić proved that three of these four kinds of covariant derivatives are linearly independent [17].

With respect to the four kinds of covariant derivatives (5, 6, 7, 8), S. M. Minčić obtained four curvature tensors, eight derived curvature tensors and fifteen curvature pseudotensors of the space $\mathbb{GA}_K$.

Minčić's work have continued many scientiests. Some of them are M. Stanković [16], M. Zlatanović [16, 19], Lj. S. Velimirović [9, 11, 13, 16], and many others.

N. O. Vesić [14, 17] and D. J. Simjanović [14] completed the research realized in [17] where it is proved that just curvature tensors may be obtained from the differences $a^i_{j\underset{p}{|}m\underset{q}{|}n} - a^i_{j\underset{r}{|}n\underset{s}{|}m}$, $p, q, r, s \in \{1, 2, 3, 4\}$.

The curvature tensors of the space $\mathbb{GA}_K$ are elements of family

$$K^i_{jmn} = R^i_{jmn} + uT^i_{jm|n} + u'T^i_{jn|m} + vT^p_{jm}T^i_{pn} + v'T^p_{jn}T^i_{pm} + wT^p_{mn}T^i_{pj}. \tag{9}$$

1.1.1. Linearly independent curvature tensors

In attempt to generalize initial research about curvature tensors of symmetric affine connection [2, 15], S. M. Minčić [4–8] concluded that from the difference $a^i_{j\underset{p}{|}m\underset{q}{|}n} - a^i_{j\underset{r}{|}n\underset{s}{|}m}$ five linearly independent curvature tensors may be obtained:

$$\underset{1}{R}{}^i_{jmn} = L^i_{jm,n} - L^i_{jn,m} + L^P_{jm}l^I_{pn} - L^p_{jn}L^i_{pm}, \tag{10}$$

$$\underset{2}{R}{}^i_{jmn} = L^i_{mj,n} - L^i_{nj,m} + L^p_{mj}L^i_{np} - L^p_{nj}L^i_{mp}, \tag{11}$$

$$\underset{3}{R}{}^i_{jmn} = L^i_{jm,n} - L^i_{nj,m} + L^p_{jm}L^i_{np} - L^p_{nj}L^i_{pm} + 2L^p_{nm}T^i_{pj}, \tag{12}$$

$$\underset{4}{R}{}^i_{jmn} = L^i_{jm,n} - L^i_{nj,m} + L^p_{jm}L^i_{np} - L^p_{nj}L^i_{pm} + 2L^p_{mn}T^i_{pj}, \tag{13}$$

$$\underset{5}{R}{}^i_{jmn} = L^i_{\underline{jm},n} - L^i_{\underline{jn},m} + \frac{1}{2}\Big(L^p_{jm}L^i_{pn} + L^p_{mj}L^i_{np} - L^p_{jn}L^i_{mp} - L^p_{nj}L^i_{pm}\Big). \tag{14}$$

These five linearly independent curvature tensors are expressed as functions of the curvature tensor R^i_{jmn} of the associated space $\mathbb{A}_N$

$$\underset{1}{R}{}^i_{jmn} = R^i_{jmn} + T^i_{jm|n} - T^i_{jn|m} + T^p_{jm}T^i_{pn} - T^p_{jn}T^i_{pm}, \tag{10A}$$

$$\underset{2}{R}{}^i_{jmn} = R^i_{jmn} - T^i_{jm|n} + T^i_{jn|m} + T^p_{jm}T^i_{pn} - T^p_{jn}T^i_{pm}, \tag{11A}$$

$$\underset{3}{R}{}^i_{jmn} = R^i_{jmn} + T^i_{jm|n} + T^i_{jn|m} - T^p_{jm}T^i_{pn} + T^p_{jn}T^i_{pm} - 2T^p_{mn}T^i_{pj}, \tag{12A}$$

$$\underset{4}{R}{}^i_{jmn} = R^i_{jmn} + T^i_{jm|n} + T^i_{jn|m} - T^p_{jm}T^i_{pn} + T^p_{jn}T^i_{pm} + 2T^p_{mn}T^i_{pj}, \tag{13A}$$

$$\underset{5}{R}{}^i_{jmn} = R^i_{jmn} + T^p_{jm}T^i_{pn} + T^p_{jn}T^i_{pm}. \tag{14A}$$

In the research of N. O. Vesić [14, 17] and D. J. Simjanović [14], the sixth linearly independent curvature tensor of space $\mathbb{GA}_N$ was obtained. Six linearly independent curvature tensor of this space are

$$\underset{1}{\tilde{R}}{}^i_{jmn} = R^i_{jmn} + T^i_{jm|n} - T^i_{jn|m} + T^p_{jm}T^i_{pn} - T^p_{jn}T^i_{pm} + 2T^p_{mn}T^i_{pj}, \tag{15}$$

$$\underset{2}{\tilde{R}}{}^i_{jmn} = R^i_{jmn} + T^i_{jm|n} - T^i_{jn|m} + T^p_{jm}T^i_{pn} + T^p_{jn}T^i_{pm}, \tag{16}$$

$$\underset{3}{\tilde{R}}{}^i_{jmn} = R^i_{jmn} + T^i_{jm|n} - T^i_{jn|m} - T^p_{jm}T^i_{pn} - T^p_{jn}T^i_{pm}, \tag{17}$$

$$\underset{4}{\tilde{R}}{}^i_{jmn} = R^i_{jmn} + T^i_{jm|n} - T^i_{jn|m} - T^p_{jm}T^i_{pn} - 3T^p_{jn}T^i_{pm}, \tag{18}$$

$$\underset{5}{\tilde{R}}{}^i_{jmn} = R^i_{jmn} + T^i_{jm|n} + T^i_{jn|m} + T^p_{jm}T^i_{pn} + T^p_{jn}T^i_{pm}, \tag{19}$$

$$\underset{6}{\tilde{R}}{}^i_{jmn} = R^i_{jmn} - T^p_{jm|n} - T^p_{jn|m} - T^p_{jm}T^i_{pn} - T^p_{jn}T^i_{pm}. \tag{20}$$

The linearly independent curvature tensors $\underset{1}{R}{}^i_{jmn}, \dots, \underset{5}{R}{}^i_{jmn}$ given by (10A–14A) and $\underset{3}{\tilde{R}}{}^i_{jmn}$ given by (17) are linearly independent. Hence, our study in this research will be based on these six linearly independent curvature tensors. The above-mentioned Riemann curvature tensors can be subjected to decomposition theorems, which were studied in [10].

1.2. Research purposes

At the start of this research, in the Section 2, we will review the definitions of almost complex space and terms necessary for further investigation.

In the Section 3, we will obtain curvature tensors with respect to the half-symmetric affine connection such as the linearly independent curvature tensors with respect to this connection.

In the Subsection 3.1, we will define dual and symmetric connections of a half-symmetric affine connection. After that, the necessary and sufficient conditions that the special half-symmetric affine connection be torsion-free are presented. After that, we obtained the corresponding six linearly independent curvature tensors with respect to the special half-symmetric affine connection (curvature tensor with respect to the torsion-free affine connection and five other linearly independent curvature tensors). At the end of this research, necessary and sufficient conditions for the dual and symmetric connections to be the F-connections.

2. Almost complex manifolds

The $K = 2N$-dimensional affine connection spaces were studied in Yano's work [18]. A $2N$-dimensional manifold $\mathcal{M}_{2N} = \mathcal{M}_{2N}(x^1, \dots, x^{2N})$ equipped with a structural affinor F^h_i which satisfies the equality

$$F^s_i F^i_t = -\delta^s_t, \tag{21}$$

is almost complex manifold [18].

The operators O^{hs}_{ri} and ${}^*O^{hs}_{ri}$ are defined as

$$O^{hs}_{ri} = \frac{1}{2}\left(\delta^h_r \delta^s_i - F^h_r F^s_i\right), \tag{22}$$

$${}^*O^{hs}_{ri} = \frac{1}{2}\left(\delta^h_r \delta^s_i + F^h_r F^s_i\right) \tag{23}$$

The affine connection of almost complex space, whose coefficients are L^i_{jk}, is the F-connection if the affinor F^h_i is covariantly constant with respect to this connection, i.e. $F^j_{i\underset{1}{|}k} = 0$. Because

$$F^j_{i|k} = F^j_{i\underset{1}{|}k} - T^j_{lk} F^l_i + T^l_{ik} F^j_l, \tag{24}$$

the F-connection satisfies the equality

$$F^j_{i|k} = -T^j_{lk} F^l_i + T^l_{ik} F^j_l. \tag{25}$$

The F-connection L^i_{jk} is the half-symmetric connection if its torsion tensor satisfies the equality

$$O^{hs}_{rk} O^{kt}_{ij} S^r_{st} = 0. \tag{26}$$

For the half-symmetric affine connection, the next equalences are satisfied:

$$S^h_{ij} = F^s_j F^h_r S^r_{is} + F^t_i F^s_j S^h_{ts} + F^h_r F^t_i S^r_{tj}, \tag{27}$$

$$T^h_{ij} = F^s_j F^h_r T^r_{is} + F^t_i F^s_j T^h_{ts} + F^h_r F^t_i T^r_{tj}, \tag{28}$$

$$F^s_i F^i_t = -\delta^s_t, \tag{29}$$

$$F^j_{i|k} = -T^j_{lk} F_i{}^l + T^l_{ik} F_l{}^j. \tag{30}$$

3. Curvature tensors

With respect to the equations (27, 28, 29, 30), we will obtain the family of curvature tensors of a non-symmetric affine connection space analogous to the family (9).

The following equalities are satisfied

$$T^p_{jm}T^i_{pn} = \left(F^s_mF^p_rT^r_{js} + F^t_jF^s_mT^p_{ts} + F^p_rF^t_jT^r_{tm}\right)\left(F^a_nF^i_bT^b_{pa} + F^a_pF^b_nT^i_{ab} + F^i_aF^b_pT^a_{bn}\right) = \mathcal{Q}^{sbait}_{mrncj}T^r_{ts}T^c_{ba}, \tag{31}$$

for

$$\begin{aligned}\mathcal{Q}^{sbait}_{mrncj} &= F^s_mF^a_nF^b_rF^i_c\delta^t_j - F^s_mF^a_n\delta^b_r\delta^i_c\delta^t_j - F^s_mF^i_c\delta^b_r\delta^a_n\delta^t_j + F^s_mF^a_nF^t_jF^i_c\delta^b_r + F^s_mF^a_nF^t_jF^b_r\delta^i_c + F^s_mF^t_jF^i_cF^b_r\delta^a_n\\ &+ F^b_rF^t_jF^a_nF^i_c\delta^s_m - F^t_jF^a_n\delta^b_r\delta^s_m\delta^i_c - F^t_jF^i_c\delta^b_r\delta^s_m\delta^a_n.\end{aligned} \tag{32}$$

We need to determine $T^i_{jm|n}$ in terms of F^i_j.

$$\begin{aligned}T^i_{jm|n} &= \left(F^s_mF^i_rT^r_{js} + F^t_jF^s_mT^i_{ts} + F^i_rF^t_jT^r_{tm}\right)_{|n}\\ &= \left(-T^s_{ln}F^l_m + T^l_{mn}F^s_l\right)F^i_rT^r_{js} + F^s_m\left(-T^i_{ln}F^l_r + T^l_{rn}F^i_l\right)T^r_{js} + F^s_mF^i_rT^r_{js|n}\\ &+ \left(-T^t_{ln}F^l_j + T^l_{jn}F^t_l\right)F^s_mT^i_{ts} + F^t_j\left(-T^s_{ln}F^l_m + T^l_{mn}F^s_l\right)T^i_{ts} + F^t_jF^s_mT^i_{ts|n}\\ &+ \left(-T^i_{ln}F^l_r + T^l_{rn}F^i_l\right)F^t_jT^r_{tm} + F^i_r\left(-T^t_{ln}F^l_j + T^l_{jn}F^t_l\right)T^r_{tm} + F^i_rF^t_jT^r_{tm|n}\end{aligned}$$

After some computing, we get

$$T^i_{jm|n} = \widetilde{\mathcal{Q}}^{sbait}_{mrncj}T^c_{ba}T^r_{ts} + \widetilde{\mathcal{P}}^{sit}_{mrj}T^r_{ts|n}, \tag{33}$$

for

$$\widetilde{\mathcal{P}}^{sit}_{mrj} = F^s_mF^i_r\delta^t_j + F^t_jF^s_m\delta^i_r + F^i_rF^t_j\delta^s_m, \tag{34}$$

$$\begin{aligned}\widetilde{\mathcal{Q}}^{sbait}_{mrncj} &= -F^b_mF^i_r\delta^s_c\delta^a_n\delta^t_j + F^s_cF^i_r\delta^b_m\delta^a_n\delta^t_j - F^s_mF^b_r\delta^i_c\delta^a_n\delta^t_j + F^s_mF^i_c\delta^b_r\delta^a_n\delta^t_j - F^b_jF^s_m\delta^i_r\delta^t_c\delta^a_n + F^s_mF^t_c\delta^i_r\delta^b_j\delta^a_n\\ &- F^t_jF^b_m\delta^i_r\delta^s_c\delta^a_n + F^t_jF^s_c\delta^i_r\delta^b_m\delta^a_n - F^b_rF^t_j\delta^s_m\delta^i_c\delta^a_n + F^i_cF^t_j\delta^s_m\delta^b_r\delta^a_n - F^i_rF^b_j\delta^s_m\delta^c_t\delta^a_n + F^i_rF^t_c\delta^s_m\delta^b_j\delta^a_n.\end{aligned} \tag{35}$$

In this way, we obtained that the next equation holds.

$$\mathcal{K}^i_{jmn} = R^i_{jmn} + u\widetilde{\mathcal{P}}^{sit}_{mrj}T^r_{ts|n} + u'\widetilde{\mathcal{P}}^{sit}_{nrj}T^r_{ts|m} + \left(u\widetilde{\mathcal{Q}}^{sbait}_{mrncj} + u'\widetilde{\mathcal{Q}}^{sbait}_{nrmcj} + v\mathcal{Q}^{sbait}_{mrncj} + v'\mathcal{Q}^{sbait}_{nrmcj} + w\mathcal{Q}^{sbait}_{nrjcm}\right)T^c_{ba}T^r_{ts}. \tag{36}$$

The next theorem holds.

Theorem 3.1. *Let $\mathbb{GA}_{2N}$ be a generalized Riemannian space equipped with a half-symmetric affine connection. The family of curvature tensors of this space is given as* (36). □

For different u, u', v, v', w, S. M. Minčić obtained five curvature tensors [4–9]. N. O. Vesić [17] and D. J. Simjanović [14, 17] obtained that there six linearly independent curvature tensors of a non-symmetric affine connection space. The sixth linearly independent curvature tensor, which was not obtained in Minčić's works and rest, is the curvature tensor of the associated space $\mathbb{A}_{2N}$. We will list five linearly independent curvature tensors of the space with a half-symmetric affine connection analogues to the five ones which

Minčić and his colleagues obtained. These curvature tensors are:

$$\begin{aligned}\underset{1}{R}{}^{i}_{jmn} &= R^{i}_{jmn} + \widetilde{\mathcal{P}}^{sit}_{mrj} T^{r}_{ts|n} - \widetilde{\mathcal{P}}^{sit}_{nrj} T^{r}_{ts|m} \\ &\quad + \left(\widetilde{\mathcal{Q}}^{sbait}_{mrncj} - \widetilde{\mathcal{Q}}^{sbait}_{nrmcj} + \mathcal{Q}^{sbait}_{mrncj} - \mathcal{Q}^{sbait}_{nrmcj}\right) T^{c}_{ba} T^{r}_{ts},\end{aligned} \tag{37}$$

$$\begin{aligned}\underset{2}{R}{}^{i}_{jmn} &= R^{i}_{jmn} - \widetilde{\mathcal{P}}^{sit}_{mrj} T^{r}_{ts|n} + \widetilde{\mathcal{P}}^{sit}_{nrj} T^{r}_{ts|m} \\ &\quad - \left(\widetilde{\mathcal{Q}}^{sbait}_{mrncj} - \widetilde{\mathcal{Q}}^{sbait}_{nrmcj} - \mathcal{Q}^{sbait}_{mrncj} + \mathcal{Q}^{sbait}_{nrmcj}\right) T^{c}_{ba} T^{r}_{ts},\end{aligned} \tag{38}$$

$$\begin{aligned}\underset{3}{R}{}^{i}_{jmn} &= R^{i}_{jmn} + \widetilde{\mathcal{P}}^{sit}_{mrj} T^{r}_{ts|n} + \widetilde{\mathcal{P}}^{sit}_{nrj} T^{r}_{ts|m} \\ &\quad + \left(\widetilde{\mathcal{Q}}^{sbait}_{mrncj} + \widetilde{\mathcal{Q}}^{sbait}_{nrmcj} - \mathcal{Q}^{sbait}_{mrncj} + \mathcal{Q}^{sbait}_{nrmcj} - 2\mathcal{Q}^{sbait}_{nrjcm}\right) T^{c}_{ba} T^{r}_{ts},\end{aligned} \tag{39}$$

$$\begin{aligned}\underset{4}{R}{}^{i}_{jmn} &= R^{i}_{jmn} + \widetilde{\mathcal{P}}^{sit}_{mrj} T^{r}_{ts|n} + \widetilde{\mathcal{P}}^{sit}_{nrj} T^{r}_{ts|m} \\ &\quad + \left(\widetilde{\mathcal{Q}}^{sbait}_{mrncj} + \widetilde{\mathcal{Q}}^{sbait}_{nrmcj} - \mathcal{Q}^{sbait}_{mrncj} + \mathcal{Q}^{sbait}_{nrmcj} + 2\mathcal{Q}^{sbait}_{nrjcm}\right) T^{c}_{ba} T^{r}_{ts},\end{aligned} \tag{40}$$

$$\underset{5}{R}{}^{i}_{jmn} = R^{i}_{jmn} + \left(\mathcal{Q}^{sbait}_{mrncj} + \mathcal{Q}^{sbait}_{nrmcj}\right) T^{c}_{ba} T^{r}_{ts}. \tag{41}$$

The sixth linearly independent curvature tensor of semi-symmetric affine connection space $\mathbb{GA}_{2N}$ is

$$\underset{3}{\tilde{R}}{}^{i}_{jmn} = R^{i}_{jmn} + \widetilde{\mathcal{P}}^{sit}_{mrj} T^{r}_{ts|n} - \widetilde{\mathcal{P}}^{sit}_{nrj} T^{r}_{ts|m} - \left(\widetilde{\mathcal{Q}}^{sbait}_{nrmcj} + \widetilde{\mathcal{Q}}^{sbait}_{nrjcm}\right) T^{c}_{ba} T^{r}_{ts}. \tag{42}$$

Following the previous research, we proved that the next theorem holds.

Theorem 3.2. *The set of curvature tensors given by the equation* (36)*, is completely generated by six linearly independent curvature tensors. These six tensors are the curvature tensor of the associated space and the curvature tensors given by* (37–42). □

3.1. Special half-symmetric connection

Let $L^{i}_{\underline{jk}}$ be coefficients of a symmetric affine connection. Based on the Theorem 1.7 ([18], Chapter 12), we conclude that

$$\overset{1}{L}{}^{i}_{jk} = L^{i}_{\underline{jk}} - \frac{1}{2} F^{i}_{p} F^{p}_{j|k} \tag{43}$$

are coefficients of a half-symmetric F connection. Hence, for the affine connection $\overset{1}{L}{}^{i}_{jk}$ we obtain that is:

$$\overset{1}{S}{}^{i}_{jk} = 2\overset{1}{L}{}^{i}_{\underset{\vee}{jk}} = \frac{1}{2} F^{i}_{p} \left(F^{p}_{k|j} - F^{p}_{j|k}\right), \tag{44}$$

$$\overset{1}{T}{}^{i}_{jk} = \overset{1}{L}{}^{i}_{\underset{\vee}{jk}} = \frac{1}{4} F^{i}_{p} \left(F^{p}_{k|j} - F^{p}_{j|k}\right). \tag{45}$$

Dual and symmetric connection of the connection (43) are respectively given below:

$$\overset{2}{L}{}^{i}_{jk} = \overset{1}{L}{}^{i}_{kj} = \overset{1}{L}{}^{i}_{jk} - \overset{1}{S}{}^{i}_{jk} = L^{i}_{\underline{jk}} - \frac{1}{2} F^{i}_{p} F^{p}_{k|j}. \tag{46}$$

$$\overset{1}{L}{}^{i}_{\underline{jk}} = \frac{1}{2}\left(\overset{1}{L}{}^{i}_{jk} + \overset{1}{L}{}^{i}_{kj}\right) = L^{i}_{\underline{jk}} - \frac{1}{4} F^{i}_{p} \left(F^{p}_{j|k} + F^{p}_{k|j}\right), \tag{47}$$

The connection (43) may be expressed as

$$\overset{1}{L}{}^{i}_{jk} = \overset{1}{L}{}^{i}_{\underline{jk}} + \overset{1}{L}{}^{i}_{\underset{\vee}{jk}} = \overset{1}{L}{}^{i}_{\underline{jk}} + \overset{1}{T}{}^{i}_{jk} = \overset{1}{L}{}^{i}_{\underline{jk}} + \frac{1}{4}F^{i}_{p}\left(F^{p}_{k|j} - F^{p}_{j|k}\right). \tag{48}$$

Theorem 3.3. *Let in an almost complex space be defined a symmetric affine connection whose coeffcients are* $L^{i}_{\underline{jk}}$. *For this connection, the next equalites are satisfied*

$$\overset{1}{S}{}^{i}_{jk} = 0 \iff F^{i}_{j|k} = F^{i}_{k|j}, \tag{49}$$

$$\overset{1}{T}{}^{i}_{jk} = 0 \iff F^{i}_{j|k} = F^{i}_{k|j}, \tag{50}$$

i.e. $\overset{1}{L}{}^{i}_{jk} = \overset{1}{L}{}^{i}_{\underline{jk}}$, *if and only if the tensor* $F^{i}_{j|k}$ *is symmetric by* j *and* k. □

Based on the equation (3), we obtain the curvature tensor with respect to the symmetric affine connection $\overset{1}{L}{}^{i}_{\underline{jk}}$:

$$\begin{aligned}
\overset{1}{R}{}^{i}_{jmn} &= \overset{1}{L}{}^{i}_{\underline{jm},n} - \overset{1}{L}{}^{i}_{\underline{jn},m} + \overset{1}{L}{}^{p}_{\underline{jm}}\overset{1}{L}{}^{i}_{\underline{pn}} - \overset{1}{L}{}^{p}_{\underline{jn}}\overset{1}{L}{}^{i}_{\underline{pm}} \\
&= \frac{3}{4}R^{i}_{jmn} - \frac{1}{4}F^{i}_{p}F^{s}_{j}R^{p}_{smn} - \frac{1}{4}F^{i}_{p}\left(F^{p}_{m|jn} - F^{p}_{n|jm}\right) \\
&- \frac{1}{16}\left(3F^{i}_{p|n} + F^{s}_{p}F^{q}_{n}F^{i}_{q|s}\right)\left(F^{p}_{j|m} + F^{p}_{m|j}\right) \\
&+ \frac{1}{16}\left(3F^{i}_{p|m} + F^{s}_{p}F^{q}_{m}F^{i}_{q|s}\right)\left(F^{p}_{j|n} + F^{p}_{n|j}\right),
\end{aligned} \tag{51}$$

where R^{i}_{jmn} and (|) are curvature tensor and covariant derivative obtained by symmetric affine connection, respectively. From the equation (37–42), (45) i (51) we obtain:

$$\underset{1}{\overset{1}{R}}{}^{i}_{jmn} = \frac{1}{2}R^{i}_{jmn} - \frac{1}{2}F^{i}_{p}F^{s}_{j}R^{p}_{smn} - \frac{1}{4}F^{i}_{p|n}F^{p}_{j|m} + \frac{1}{4}F^{i}_{p|m}F^{p}_{j|n}, \tag{52}$$

$$\begin{aligned}
\underset{2}{\overset{1}{R}}{}^{i}_{jmn} &= R^{i}_{jmn} - \frac{1}{2}F^{i}_{p}\left(F^{p}_{m|jn} - F^{p}_{n|jm}\right) - \frac{1}{2}F^{i}_{p|n}F^{p}_{m|j} + \frac{1}{2}F^{i}_{p|m}F^{p}_{n|j} \\
&- \frac{1}{4}F^{i}_{p|q}F^{q}_{s}\left(F^{s}_{m|j}F^{p}_{n} - F^{s}_{n|j}F^{p}_{m}\right),
\end{aligned} \tag{53}$$

$$\begin{aligned}
\underset{3}{\overset{1}{R}}{}^{i}_{jmn} &= R^{i}_{jmn} - \frac{1}{2}F^{i}_{p}\left(F^{p}_{j|mn} - F^{p}_{n|jm}\right) - \frac{1}{2}F^{i}_{p|n}F^{p}_{j|m} + \frac{1}{4}F^{i}_{p|m}F^{p}_{n|j} + \frac{1}{4}F^{p}_{n|m}F^{i}_{p|j} \\
&- \frac{1}{4}F^{i}_{p|q}F^{q}_{s}\left(F^{s}_{j|m}F^{p}_{n} - F^{s}_{n|m}F^{p}_{j}\right),
\end{aligned} \tag{54}$$

$$\begin{aligned}
\underset{4}{\overset{1}{R}}{}^{i}_{jmn} &= R^{i}_{jmn} - \frac{1}{2}F^{i}_{p}\left(F^{p}_{j|mn} - F^{p}_{n|jm}\right) - \frac{1}{2}F^{i}_{p|n}F^{p}_{j|m} + \frac{1}{4}F^{i}_{p|m}F^{p}_{n|j} + \frac{1}{4}F^{p}_{m|n}F^{i}_{p|j} \\
&- \frac{1}{4}F^{i}_{p|q}F^{q}_{s}\left(F^{s}_{j|m}F^{p}_{n} - F^{s}_{m|n}F^{p}_{j}\right),
\end{aligned} \tag{55}$$

$$\begin{aligned}
\underset{5}{\overset{1}{R}}{}^{i}_{jmn} &= \frac{3}{4}R^{i}_{jmn} - \frac{1}{4}F^{i}_{p}F^{s}_{j}R^{p}_{smn} - \frac{1}{4}F^{i}_{p}\left(F^{p}_{m|jn} - F^{p}_{n|jm}\right) - \frac{1}{8}F^{i}_{p|n}F^{p}_{j|m} - \frac{1}{4}F^{i}_{p|n}F^{p}_{m|j} \\
&+ \frac{1}{4}F^{i}_{p|m}F^{p}_{j|n} + \frac{1}{8}F^{i}_{p|m}F^{p}_{n|j} - \frac{1}{8}F^{i}_{p|q}F^{q}_{s}\left(F^{s}_{m|j}F^{p}_{n} - F^{s}_{j|n}F^{p}_{m}\right).
\end{aligned} \tag{56}$$

$$\begin{aligned}
\underset{3}{\overset{1}{\widetilde{R}}}{}^{i}_{jmn} &= \frac{1}{2}R^{i}_{jmn} - \frac{1}{2}F^{i}_{p}F^{s}_{j}R^{p}_{smn} - \frac{1}{8}F^{i}_{p|n}F^{p}_{j|m} + \frac{1}{4}F^{i}_{p|m}F^{p}_{j|n} - \frac{1}{8}F^{i}_{p|n}F^{p}_{m|j} \\
&+ \frac{1}{8}F^{i}_{p|q}F^{q}_{s}F^{p}_{n}\left(F^{s}_{j|m} - F^{s}_{m|j}\right).
\end{aligned} \tag{57}$$

Because $\overset{1}{L}{}^{i}_{jk}$ is the F-connection, we get $F^{i}_{j\underset{1}{|}k} = 0$, where $\underset{1}{|}$ is covariant derivative with respect to the connection $\overset{1}{L}{}^{i}_{jk}$. From the equality $F^{i}_{j\underset{1}{|}k} = 0$ and equation (46), we obtain

$$F^{i}_{j\underset{2}{|}k} = F^{i}_{j|k} - \frac{1}{2}F^{i}_{k|j} + \frac{1}{2}F^{i}_{p|q}F^{q}_{j}F^{p}_{k}, \tag{58}$$

where $\underset{2}{|}$ denotes covariant derivative with respect to the affine connection $\overset{2}{L}{}^{i}_{jk}$.

Theorem 3.4. *Dual connection $\overset{2}{L}{}^{i}_{jk}$ of the connection $\overset{1}{L}{}^{i}_{jk}$ is F-connection if and only if*

$$F^{i}_{j|k} = \frac{1}{2}F^{i}_{k|j} - \frac{1}{2}F^{i}_{p|q}F^{q}_{j}F^{p}_{k}.$$

□

From the equality $F^{i}_{j\underset{1}{|}k} = 0$ and the equation (47) we obtain the following relation

$$F^{i}_{j\underset{0}{|}k} = \frac{1}{2}\left(F^{i}_{j|k} - \frac{1}{2}F^{i}_{k|j} + \frac{1}{2}F^{i}_{p|q}F^{q}_{j}F^{p}_{k}\right), \tag{59}$$

where $\underset{0}{|}$ is covariant derivative with respect to the symmetric affine connection $\overset{1}{L}{}^{i}_{\underline{jk}}$.

Theorem 3.5. *Symmetric affine connection $\overset{1}{L}{}^{i}_{\underline{jk}}$ of the connection $\overset{1}{L}{}^{i}_{jk}$ is F-connection if and only if*

$$F^{i}_{j|k} = \frac{1}{2}F^{i}_{k|j} - \frac{1}{2}F^{i}_{p|q}F^{q}_{j}F^{p}_{k}.$$

□

4. Conclusion

In Chapter 2, we reviewed the definition of almost complex manifolds and the operators necessary for further considerations in this work, following the work of K. Yano [18].

In Chapter 3, we determined the curvature tensors of half-symmetric affine connection space. In Subection 3.1, we discussed a special half-symmetric affine connection. Firstly, we defined the dual connection of this special half-symmetric affine connection. Then, we presented the necessary and sufficient condition for the special half-symmetric connection to be symmetric. Subsequently, six linearly independent curvature tensors were determined based on the special half-symmetric affine connection (one curvature tensor with respect to the symmetric affine connection and five curvature tensors with respect to the non-symmetric affine connection of the special half-symmetric connection). At the end of this research, the necessary and sufficient conditions under which the dual and symmetric connections of the special half-symmetric connection are F-connections were determined.

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