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A generalization of fixed point result of nonlinear Ćirić type contraction on suprametric spaces

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Abstract. In this study, the nonlinear technique: (ψ,ϕ)-weak contraction, created by Dutta and Choudhury [6], is used to make the Ćirić type contraction nonlinear. Moreover, it is demonstrated that there is unique fixed point in suprametric space for this nonlinear Ćirić type contraction.

1. Introduction

The mathematical concept of a metric explains the majority of distances between two points. There are three well-known properties that a distance function must satisfy in order to represent a metric. Working with triangle inequality, which is one of the mentioned criteria of metric spaces, is one of the challenges we face when working in them. As a result, academics have recently begun working with a variety of metric spaces in which this property is weakened. In [9], Mathews defined partial metric spaces and extended the Banach contraction principle by changing the triangle inequality. In [5], Czerwick introduced b-metric spaces, which also have a weaker triangle inequality. In 2022, Berzig used a weaker triangle inequality to establish suprametrics based on this concept [2] defined as follows:

Definition 1.1. *Let X be a nonempty set. A function $d : X \times X \to \mathbf{R}^+$ is called suprametric if for all $x, y, z \in X$ the following properties hold:*

$(m1)$ $d(x, y) = 0$ *if and only if* $x = y$,
$(m2)$ $d(x, y) = d(y, x)$,
$(m3)$ $d(x, y) \leq d(x, z) + d(z, y) + \rho d(x, z)d(z, y)$, *for some constant* $\rho \in \mathbf{R}^+$.

A suprametric space is a pair (X, d), where X is a nonempty set and d is a suprametric.

One of the significant advantages of working in such spaces is that structures exist that are suprametric spaces while not metric spaces. For some specific examples, we recommend consulting the article [2]. Furthermore, the main definitions and in-depth examples are found in [2]; therefore, in order to avoid repetition, we have not included them herein (See Also: [10], [3]).

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The famous Banach contraction principle is the key concept of fixed point theory. Some generalizations of this core principle have been studied for years in different areas.

A nonlinear generalization of the Banach contraction principle was established for Hilbert spaces by Alber and Guerre-Delabriere in [1]. Later on, this new generalization is called the ψ-weak contraction in literature where ψ is an altering distance function satisfying the following properties:

- $\psi : [0,\infty) \to [0,\infty)$,
- ψ is a continuous and non-decreasing function,
- $\psi(t) = 0$ if and only if $t = 0$.

Then, Rhoades extended the results in [1] to arbitrary Banach spaces in [11].

In 2009, Dutta and Choudhury took the idea of ψ-weak contraction a step further. They introduced (ψ,ϕ)-weak contraction concept in [6]. Recently, in light of the benefits of working with suprametrics, attempts have been made to get fixed points with contractions defined on suprametric spaces. The following research question was whether a contraction defined in the same way as in [6] would have a unique fixed point in the complete suprametric space. Yeşilkaya answered this question by obtaining the existence and uniqueness of the fixed point of (ψ,ϕ)-weak contraction in suprametric spaces in [12].

The existence and uniqueness of the fixed point of a linear extension of Ćirić type contraction in interpolative metric spaces were demonstrated in our earlier work [4]. In this paper, we explore a nonlinear extension of Ćirić type contractions in suprametric spaces. This is achieved by integrating the concepts from the linear extension of Ćirić type contractions and (ψ,ϕ)-weak contractions.

2. Main Results

In this section, we first recall the contractions on which our study is based. In [6], Dutta and Choudhury introduced a nonlinear extension of contractions, which is (ψ,ϕ)-weak contraction with the help of alternating distance functions $\varphi, \phi : [0,\infty) \to [0,\infty)$ so that for each $x, y \in X$,

$$\phi\left(d\left(Tx, Ty\right)\right) \leq \phi\left(d\left(x, y\right)\right) - \varphi\left(d\left(x, y\right)\right) \tag{1}$$

on any usual metric space X. On the other side, one of the known linear contractions is Ćirić type contraction which is defined as follows: there exists a number $0 < k < 1$, so that, for each $x, y \in X$

$$d(Tx, Ty) \leq k \max\left(d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Tx) + d(y, Tx)}{2}\right). \tag{2}$$

Now we are ready to introduce a nonlinear version of Ćirić type contraction and the main result of our study in the following theorem:

Theorem 2.1. *Let (X, d) be a complete suprametric space and $T : X \to X$ be a continuous mapping. Moreover, let Ω be the set of continuous and non-decreasing mappings $\omega : \mathbf{R}_0^+ \to \mathbf{R}_0^+$ such that $\omega(0) = 0$ if and only if $t = 0$.*

Suppose that there exist $\psi, \phi \in \Omega$ such that

$$\psi(d(Tx, Ty)) \leq \psi(M_d(x, y)) - \phi(M_d(x, y)) \tag{3}$$

for all $x, y \in X$ where

$$M_d(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{q(d(x, Ty) + d(y, Tx))}{2(1 + \rho d(y, Ty))}\right\},\ q \in (0, 1).$$

Then, T has a unique fixed point in X. Moreover, for every $x_0 \in X$ the iterative sequence defined by $x_n = Tx_{n-1}$, $n \in \mathbb{N}$ converges to this fixed point.

Proof. We can define a sequence $\{x_n\}$ as follows:

$$x_{n+1} = Tx_n \quad \text{for all} \quad n \in \mathbb{N}$$

where x_0 is an arbitrary point in X.

For any $n_0 \in \mathbf{N}$, if two successive terms are equal, that is, $x_{n_0} = x_{n_0+1}$, then we have $x_{n_0} = Tx_{n_0}$ since $x_{n_0+1} = Tx_{n_0}$. This proves that x_{n_0} is a fixed point and ends the proof.

Now, let's suppose that any successive terms in the sequence x_n are distinct, that is, $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. This debate thus automatically leads to the conclusion that, for all $n \in \mathbb{N}$, $d(x_n, x_{n+1}) > 0$.

Based on the assumption stated by equation (3) in Theorem 2.1, we have

$$\begin{aligned} \psi(d(x_n, x_{n+1})) &= \psi(d(Tx_{n-1}, Tx_n)) \\ &\leq \psi(M_d(x_{n-1}, x_n)) - \phi(M_d(x_{n-1}, x_n)) \end{aligned} \tag{4}$$

On the other side,

$$\begin{aligned} M_d(x_{n-1}, x_n) &= \max \left\{ \begin{array}{l} d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \\ \dfrac{q(d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1}))}{2(1 + \rho d(x_n, Tx_n))} \end{array} \right\} \\ &= \max \left\{ \begin{array}{l} d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \\ \dfrac{q(d(x_{n-1}, x_{n+1}) + d(x_n, x_n))}{2(1 + \rho d(x_n, x_{n+1}))} \end{array} \right\} \\ &= \max \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{qd(x_{n-1}, x_{n+1})}{2(1 + \rho d(x_n, x_{n+1}))} \right\} \end{aligned} \tag{5}$$

As can be seen from the equality (5), there are three possible values for $M_d(x_{n-1}, x_n)$. If $M_d(x_{n-1}, x_n) = d(x_n, x_{n+1})$ then, the inequality (4) results the following inequality:

$$\psi(d(x_n, x_{n+1})) \leq \psi(d(x_n, x_{n+1})) - \phi(d(x_n, x_{n+1})).$$

That is not possible because $\psi, \phi : \mathbf{R}_0^+ \to \mathbf{R}_0^+$, such that $\psi(0) = 0$ and $\phi(0) = 0$ if and only if $t = 0$. On the other side, we know that $d(x_n, x_{n+1}) > 0$ since $x_n \neq x_{n+1}$. As a result, $\phi(d(x_n, x_{n+1})) > 0$. This means that $M_d(x_{n-1}, x_n) \neq d(x_n, x_{n+1})$. Hence, either

$$M_d(x_{n-1}, x_n) = d(x_{n-1}, x_n),$$

or

$$M_d(x_{n-1}, x_n) = \frac{qd(x_{n-1}, x_{n+1})}{2(1 + \rho d(x_n, x_{n+1}))}.$$

Yeşilkaya analyzed the case $M_d(x_{n-1}, x_n) = d(x_{n-1}, x_n)$ in [12] and concluded that T has a unique fixed point in X and, additionally, the iterative sequence described by $x_n = Tx_{n-1}$, $n \in \mathbb{N}$, converges to this fixed point for every $x_0 \in X$. Now, let's assume that the last possibility is valid:

$$M_d(x_{n-1}, x_n) = \frac{qd(x_{n-1}, x_{n+1})}{2(1 + \rho d(x_n, x_{n+1}))}.$$

With this assumption, the inequality (4) gives the following:

$$\psi(d(x_n, x_{n+1})) \leq \psi\left(\frac{qd(x_{n-1}, x_{n+1})}{2(1 + \rho d(x_n, x_{n+1}))}\right) - \phi\left(\frac{qd(x_{n-1}, x_{n+1})}{2(1 + \rho d(x_n, x_{n+1}))}\right)$$

which implies that

$$\psi(d(x_n, x_{n+1})) \leq \psi\left(\frac{qd(x_{n-1}, x_{n+1})}{2(1+\rho d(x_n, x_{n+1}))}\right).$$

because $\phi : \mathbf{R}_0^+ \to \mathbf{R}_0^+$, such that $\phi(0) = 0$ if and only if $t = 0$. On the other side, as ψ is a non-decreasing function, we arrive at

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \frac{qd(x_{n-1}, x_{n+1})}{2(1+\rho d(x_n, x_{n+1}))} \\ &= \frac{q}{2(1+\rho d(x_n, x_{n+1}))} d(x_{n-1}, x_{n+1}). \end{aligned} \tag{6}$$

Moreover, since (X, d) is a suprametric space, we know that

$$d(x_{n-1}, x_{n+1}) \leq d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + \rho d(x_{n-1}, x_n) d(x_n, x_{n+1}).$$

Thus, using (6) and the previous inequality, we obtain the following outcome:

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \frac{q}{2(1+\rho d(x_n, x_{n+1}))} d(x_{n-1}, x_{n+1}) \\ &\leq \frac{q}{2(1+\rho d(x_n, x_{n+1}))} \left\{ \begin{array}{l} d(x_{n-1}, x_n) + d(x_n, x_{n+1}) \\ +\rho d(x_{n-1}, x_n) d(x_n, x_{n+1}) \end{array} \right\} \\ &= \frac{q}{2(1+\rho d(x_n, x_{n+1}))} \left\{ \begin{array}{l} (1+\rho d(x_n, x_{n+1})) d(x_{n-1}, x_n) \\ +d(x_n, x_{n+1}) \end{array} \right\} \\ &= \frac{q}{2} \left(\begin{array}{l} \dfrac{1+\rho d(x_n, x_{n+1})}{1+\rho d(x_n, x_{n+1})} d(x_{n-1}, x_n) \\ + \dfrac{1}{1+\rho d(x_n, x_{n+1})} d(x_n, x_{n+1}) \end{array} \right) \\ &= \frac{q}{2} d(x_{n-1}, x_n) + \left(\frac{q}{2}\right) \frac{1}{1+\rho d(x_n, x_{n+1})} d(x_n, x_{n+1}) \end{aligned}$$

Being $\rho \in \mathbf{R}_0^+$ and $d(x_n, x_{n+1}) > 0$ gives $1 + \rho d(x_n, x_{n+1}) > 1$. This fact yields the following result:

$$d(x_n, x_{n+1}) \leq \frac{q}{2} d(x_{n-1}, x_n) + \frac{q}{2} d(x_n, x_{n+1})$$

which provides us

$$\left(1 - \frac{q}{2}\right) d(x_n, x_{n+1}) \leq \frac{q}{2} d(x_{n-1}, x_n)$$

$$\Rightarrow d(x_n, x_{n+1}) \leq \frac{q}{2-q} d(x_{n-1}, x_n)$$

Because $q \in (0, 1)$, we have

$$\frac{q}{2-q} d(x_{n-1}, x_n) \leq qd(x_{n-1}, x_n) \text{ for all } n \in \mathbf{N}$$

and consequently

$$d(x_n, x_{n+1}) \leq qd(x_{n-1}, x_n) \text{ for all } n \in \mathbf{N}.$$

We can use this result repeatedly to obtain the following inequality:

$$d(x_n, x_{n+1}) \leq q^{n-k} d(x_k, x_{k+1}) \text{ for all } n > k \in \mathbf{N},$$

which generates

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{7}$$

As per the result mentioned in equation (8), there exists a natural number k which satisfies the condition

$$d(x_n, x_{n+1}) \leq 1 \text{ for all } n \geq k. \tag{8}$$

Consider an arbitrary $n \in \mathbf{N}$ such that $n > k$ and an arbitrary $r \in \mathbf{N}$. Either $x_n = x_{n+r}$ or $x_n \neq x_{n+r}$ are the two possible outcomes. Assuming $x_n = x_{n+r}$, we obtain

$$T^n(x_0) = T^{n+r}(x_0) \Rightarrow T^n(x_0) = T^r(T^n(x_0)).$$

This indicates that a fixed point of T^r is $T^n(x_0)$. Besides, we possess

$$T(T^r(T^n(x_0))) = T^r(T(T^n(x_0))) \quad \text{and} \quad T(T^r(T^n(x_0))) = T(T^n(x_0)),$$

and hence

$$T^r(T(T^n(x_0))) = T(T^n(x_0)),$$

which denotes that $T(T^n(x_0))$ is the fixed point of T^r. Therefore,

$$T(T^n(x_0)) = T^n(x_0)$$

Thus, the fixed point of T is $T^n(x_0)$.

We shall employ a Cauchy sequence in a complete metric space to ensure the existence of a fixed point if $x_n \neq x_{n+r}$. Let us demonstrate by induction that the created iterative sequence $\{x_n\}$ is a Cauchy sequence.

The distance between x_n and x_{n+r+1} when $n \to \infty$ is what we wish to examine. For sufficiently large n, r such that $n > k$, the suprametric d given in the Definition 1.1 and the inequality (8) yields us:

$$\begin{aligned}
d(x_n, x_{n+r}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+r}) \\
&\quad + \rho d(x_n, x_{n+1}) d(x_{n+1}, x_{n+r}) \\
&\leq q^{n-k} d(x_k, x_{k+1}) + d(x_{n+1}, x_{n+r}) \\
&\quad + \rho q^{n-k} d(x_k, x_{k+1}) d(x_{n+1}, x_{n+r}) \\
&\leq q^{n-k} + \left(1 + \rho q^{n-k}\right) d(x_{n+1}, x_{n+r})
\end{aligned}$$

where

$$\begin{aligned}
d(x_{n+1}, x_{n+r}) &\leq d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+r}) \\
&\quad + \rho d(x_{n+1}, x_{n+2}) d(x_{n+2}, x_{n+r}) \\
&\leq q^{n+1-k} d(x_k, x_{k+1}) + d(x_{n+2}, x_{n+r}) \\
&\quad + \rho q^{n+1-k} d(x_k, x_{k+1}) d(x_{n+2}, x_{n+r}) \\
&\leq q^{n+1-k} + \left(1 + \rho q^{n+1-k}\right) d(x_{n+2}, x_{n+r})
\end{aligned}$$

Then these two inequalities together give that

$$\begin{aligned} d(x_n, x_{n+r}) &\leq q^{n-k} + \left(1 + \rho q^{n-k}\right)\left(q^{n+1-k} + \left(1 + \rho q^{n+1-k}\right) d(x_{n+2}, x_{n+r})\right) \\ &= q^{n-k} + \left(1 + \rho q^{n-k}\right)\left(q^{n+1-k}\right) \\ &\quad + \left(1 + \rho q^{n-k}\right)\left(1 + \rho q^{n+1-k}\right) d(x_{n+2}, x_{n+r}) \end{aligned}$$

We can continue this process and use (8) in every term of the sum, until we get

$$d(x_n, x_{n+r}) \leq q^{n-k} \sum_{i=0}^{r-1} q^i \prod_{j=0}^{i-1} \left(1 + \rho q^{n+j-k}\right)$$

Since $q \in (0, 1)$, it follows that

$$d(x_n, x_{n+r}) \leq q^{n-k} \sum_{i=0}^{r-1} q^i \prod_{j=0}^{i-1} \left(1 + \rho q^j\right)$$

Here it is easy to see that the series

$$\sum_{i=0}^{r-1} q^i \prod_{j=0}^{i-1} \left(1 + \rho q^j\right)$$

converges. Thus, the sequence $d(x_n, x_{n+r})$ aproaches zero as n, r goes to infinity. This implies that the constructed iterative sequence, $\{x_n\}$, is a Cauchy sequence. Due to the fact that (X, d) is a complete suprametric space, the sequence $\{x_n\}$ converges to $y \in X$.

We now declare that y is the fixed point of T. Let us now verify this argument. To begin with, by (3) we know that

$$\begin{aligned} \psi(d(x_{n+1}, Ty)) &= \psi(d(Tx_n, Ty)) \\ &\leq \psi(M_d(x_n, y)) - \phi(M_d(x_n, y)) \end{aligned} \tag{9}$$

where

$$M_d(x_n, y) = \max\left\{d(x_n, y), d(x_n, Tx_n), d(y, Ty), \frac{q(d(x_n, Ty) + d(y, Tx_n))}{2(1 + \rho d(y, Ty))}\right\}. \tag{10}$$

Letting $n \to \infty$ in both sides of the inequality (10) yields

$$\begin{aligned} M_d(x_n, y) &= \max\left\{d(y, y), d(y, Ty), d(y, Ty), \frac{q(d(y, Ty) + d(y, Ty))}{2(1 + \rho d(y, Ty))}\right\} \\ &= d(y, Ty) \end{aligned}$$

Hence, (9) turns to

$$\psi(d(y, Ty)) \leq \psi(d(y, Ty)) - \phi(d(y, Ty))$$

which is possible if and only if $d(y, Ty) = 0$. So, $y = Ty$. This means that y is a fixed point of T.

The second assumption is that T contains two separate fixed points, y_1 and y_2. Now let's focus on $d(y_1, y_2) > 0$. Since,

$$\begin{aligned} M_d(y_1, y_2) &= \max\left\{d(y_1, y_2), d(y_1, Ty_1), d(y_2, Ty_2), \frac{q(d(y_1, Ty_2) + d(y_2, Ty_1))}{2(1 + \rho d(y_2, Ty_2))}\right\} \\ &= d(y_1, y_2) \end{aligned}$$

we have

$$\begin{aligned}\psi(d(y_1,y_2)) &= \psi(d(Ty_1,Ty_2))\\ &\leq \psi(M_d(y_1,y_2)) - \phi(M_d(y_1,y_2))\\ &= \psi(d(y_1,y_2)) - \phi(d(y_1,y_2)).\end{aligned}$$

This can occur if and only if $d(y_1,y_2) = 0$ which implies $y_1 = y_2$. We can therefore conclude that there is a unique fixed point for T. □

If we substitute $\psi(t) = t$ in Theorem 2.1, then we obtain the following result which is given by Dutta and Choudhury in [6]:

Corollary 2.2. *Let (X, d) be a complete suprametric space and $T : X \to X$ be a self mapping on X. Assume that there exist an $\omega \in \Omega$ such that*

$$d(Tx, Ty) \leq M_d(x, y) - \varphi(M_d(x, y)) \tag{11}$$

for all $x, y \in X$ where

$$M_d(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{q(d(x,Ty) + d(y,Tx))}{2(1+\rho d(y,Ty))}\right\},\ q \in (0,1).$$

Then, there is a unique fixed point for T in X. Additionally, the iterative sequence described by $x_n = Tx_{n-1}$, $n \in \mathbb{N}$, converges to this fixed point for every $x_0 \in X$.

3. Conclusion

Although suprametric spaces have just recently been defined in fixed point theory [2], they have already drawn the interest of many researchers. Undoubtedly, the argument for this interest is mainly because they provide advantages to work simple as well as prevent congestion in metric spaces. Our goal in this study is to obtain a nonlinear fixed point theorem in suprametric spaces. Future research may be shaped by the response to the question of whether many of the results currently obtained from metric fixed point theory can be transferred to suprametric spaces.

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