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Generalized Delannoy paths with cyclically shifting boundaries

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Abstract. In this paper, we consider the generalized Delannoy paths with steps $N_i = (0, i), i \geq 1$ and $H_i = (1, j), j \geq 0$, where all steps are weighted by u_i for N_i and v_j ($v_0 = 1$) for H_j. By the Riordan array theory, we provide a counting formula for the number of all generalized Delannoy paths dominated by a cyclically shifting piecewise linear boundary of varying slopes. Our main result can be viewed as a unified generalization of the well-known enumerative formulas for the generalized Dyck and Schröder paths from $(0, 0)$ to (kn, n) staying above the line $x = ky$. We also study the number of generalized Delannoy path boundary pairs $(P, \mathbf{a}_{n,k})$ with m-flaws, where P is a generalized Delannoy path (with steps N_i and H_0) from $(0, 0)$ to (n, k), $\mathbf{a}_{n,k}$ is a k-part composition of n, and a flaw is a horizontal step $(1, 0)$ of P below the boundary $\partial\mathbf{a}_{n,k}$.

1. Introduction

A *lattice path* in the xy-palne with steps in a given set $\mathcal{S} \subset \mathbb{N}^2$ is a sequence of points (called *nodes*) $\alpha = v_0 v_1 \cdots v_n$ with $v_i \in \mathbb{N}^2$ such that each vector $\overrightarrow{v_{i-1}v_i}$ is a member of the step set $\mathcal{S}$. Lattice paths have been studied for a very long time. They have close links with many combinatorial objects, such as pattern-avoiding permutations, integer partitions, graph theory, RNA structures, etc. [2,3,6,7,12,15-25]. A historical review of research related to lattice paths and their enumeration was presented in Humphreys [13]. In this paper, we consider the *generalized Delannoy paths* whose definition is given below.

Definition 1.1. The generalized Delannoy paths are nonempty lattice paths in the first quadrant of $\mathbb{N}^2$ starting at the origin, and consisting of step set

$$\mathcal{S} = \{N_i = (0, i) | i \geq 1\} \cup \{H_j = (1, j) | j \geq 0\},$$

where each step is labeled with weights u_i for N_i's and v_j ($v_0 = 1$) for H_j's, respectively.

Let $\mathbf{a}_{n,k} = (a_0, a_1 \cdots, a_{k-1})$ be a k-part composition of n (i.e., $a_0 + a_1 + \cdots + a_{k-1} = n$, and $a_0, \cdots, a_{k-1} \geq 1$). The piecewise linear boundary curve $\partial\mathbf{a}_{n,k}$ is defined by

$$x = a_i(y - i) + \sum_{j=0}^{i-1} a_j, \quad \text{for } y \in [i, i+1].$$

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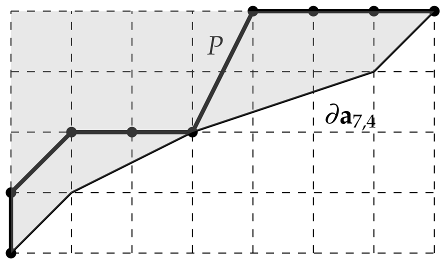

Fig. 1. A generalized Delannoy path dominated by $\mathbf{a}_{7,4} = (1, 2, 3, 1)$

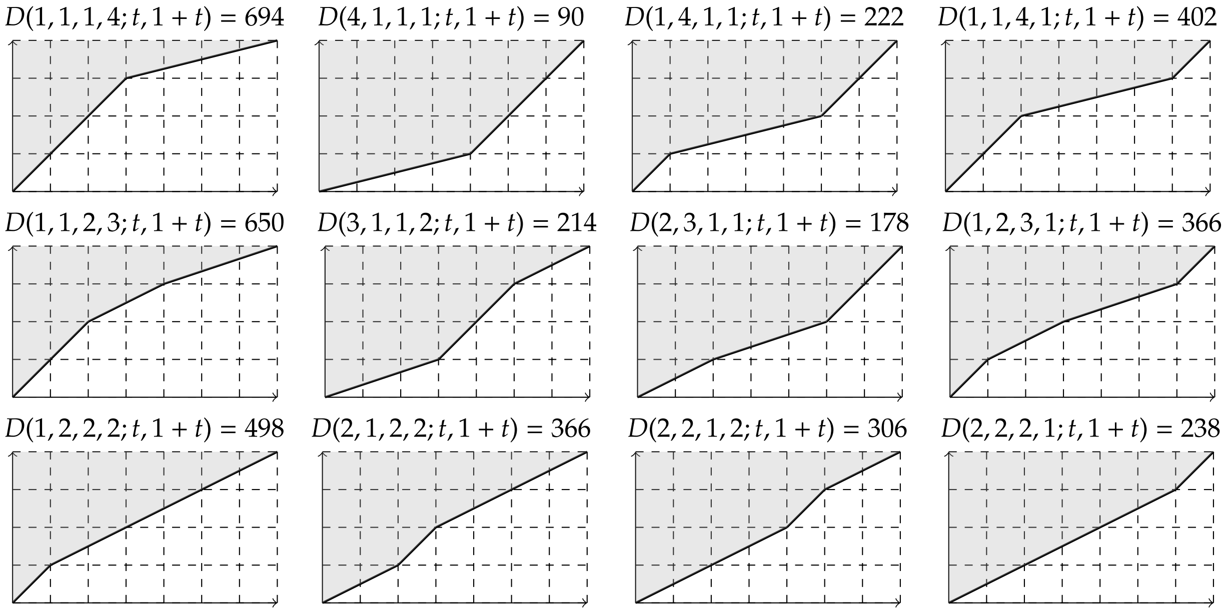

Fig. 2. The number of paths dominated by cyclically shifting boundaries

A generalized Delannoy path is said to be *dominated* by $\mathbf{a}_{n,k}$ if it lies weakly above the boundary $\partial\mathbf{a}_{n,k}$. For example, the boundary $\partial\mathbf{a}_{7,4}$ for $\mathbf{a}_{7,4} = (1, 2, 3, 1)$ is shown in Figure 1, together with a path it dominates.

The weight of a path is the product of the weights of all its steps. The weight of a set of paths is the sum of the weights of all paths. Let $D(\mathbf{a}_{n,k}; u, v)$ denote the sum of the weights of all generalized Delannoy paths from $(0, 0)$ to (n, k) dominated by $\mathbf{a}_{n,k}$, where weight functions $u := u(t) = \sum_{i\geq 1} u_i t^i$ and $v := v(t) = \sum_{i\geq 0} v_i t^i$. For example, Figure 2 provides numerical values of $D(\mathbf{a}_{n,k}; t, 1 + t)$ for various 4-part compositions of 7, along with their corresponding boundary curves. When $u(t) = t, v(t) = 1$ and all parts of $\mathbf{a}_{n,k}$ are the same, it is well known [8, Exercise 5.3.5] that $D(\mathbf{a}_{n,k}; t, 1)$ is a generalized Catalan number. Namely, we get

$$D(\underbrace{a, a, \cdots, a}_{k}; t, 1) = \frac{1}{(a+1)k+1}\binom{(a+1)k+1}{k}, \tag{1.1}$$

where the case $a = 1$ corresponds to the Dyck paths counted by the Catalan numbers OEIS[21, A000108]. When $u(t) = t, v(t) = 1 + t$ and all parts of $\mathbf{a}_{n,k}$ are the same, Song [24] showed that

$$D(\underbrace{a, a, \cdots, a}_{k}; t, 1 + t) = \frac{1}{k}\sum_{i=1}^{k}\binom{ak}{i-1}\binom{k}{i}2^i, \tag{1.2}$$

where the case $a = 1$ corresponds to the Schröder paths enumerate by the Schröder numbers OEIS[21, A006318]. It is easy to understand that for general $\mathbf{a}_{n,k}$ no enumerating formula for $D(\mathbf{a}_{n,k};u,v)$ is known, although Goulden and Jackson [8, Section 5.4.6] gave a determinant expression for $D(\mathbf{a}_{n,k};t,1)$.

Irving and Rattan [14] gave an explicit formula for classical lattice paths (with steps $E=(0,1), N=(1,0)$) dominated by *all cyclic shifts* of an arbitrary composition. Precisely, for any $j\in\mathbb{N}^+$, let $\mathbf{a}_{n,k}^{\langle j\rangle}$ denote the j shift of $\mathbf{a}_{n,k}$, namely,

$$\mathbf{a}_{n,k}^{\langle j\rangle}=(a_{-j},a_{-j+1},\cdots,a_{-j+k-1}),$$

where the indices are understood modulo k. Irving and Rottan [14] found that, for any k-part composition of n,

$$D\left(\mathbf{a}_{n,k};t,1\right)+D\left(\mathbf{a}_{n,k}^{\langle 1\rangle};t,1\right)+\cdots+D\left(\mathbf{a}_{n,k}^{\langle k-1\rangle};t,1\right)=\binom{n+k}{k-1}. \tag{1.3}$$

The rows of Figure 2 illustrate the boundaries $\mathbf{a}_{7,4}$, $\mathbf{a}_{7,4}^{\langle 1\rangle}$, $\mathbf{a}_{7,4}^{\langle 2\rangle}$, and $\mathbf{a}_{7,4}^{\langle 3\rangle}$ for the compositions $\mathbf{a}_{7,4}=(1,1,1,4)$, $(1,1,2,3)$, and $(1,2,2,2)$. Notice that in each row of Figure 2, there are a total of 1408 dominated Delannoy paths (with steps $E=(1,0), D=(1,1), N=(0,1)$) from $(0,0)$ to $(7,4)$. Motivated by this phenomenon, we get the following result.

Theorem 1.1. *Let $\mathbf{a}_{n,k}$ and $\mathbf{b}_{n,k}$ be distinct k-part compositions of n. Let $D(\mathbf{a}_{n,k};u,v)$ and $D(\mathbf{b}_{n,k};u,v)$ be the sum of the weights of all generalized Delannoy paths dominated by $\mathbf{a}_{n,k}$ and $\mathbf{b}_{n,k}$, respectively. Then, we have*

$$\sum_{j=0}^{k-1}D\left(\mathbf{a}_{n,k}^{\langle j\rangle};u,v\right)=\sum_{j=0}^{k-1}D\left(\mathbf{b}_{n,k}^{\langle j\rangle};u,v\right).$$

Let $H\left(\mathbf{a}_{n,k};u,v\right)$ be the sum of the weights of all the generalized Delannoy paths dominated by all cyclic shifts of a composition $\mathbf{a}_{n,k}$, i.e.,

$$H\left(\mathbf{a}_{n,k};u,v\right)=\sum_{j=0}^{k-1}D\left(\mathbf{a}_{n,k}^{\langle j\rangle};u,v\right).$$

Define the generalized Delannoy matrix $\mathcal{H}(u,v)=\left[H_{n,k}(u,v)\right]_{n,k\in\mathbb{N}}$ through the relation

$$H_{n,k}(u,v)=H\left(\mathbf{a}_{n+1,n-k+1};u,v\right),$$

where $\mathbf{a}_{n+1,n-k+1}$ is a $(n-k+1)$-part composition of $n+1$. We show that this matrix is a Riordan array.

Theorem 1.2. *The generalized Delannoy matrix $\mathcal{H}\left(u,v\right)=\left[H_{n,k}\left(u,v\right)\right]_{n,k\in\mathbb{N}}$ is a Riordan array given by*

$$\mathcal{H}\left(u,v\right)=\left(\left(\frac{h(t)}{t}\right)',h(t)\right).$$

Then

$$H\left(\mathbf{a}_{n,k};u,v\right)=H_{n-1,n-k}(u,v)=[t^{n-1}]\left(\frac{h(t)}{t}\right)'\cdot h(t)^{n-k},$$

where $h(t)$ satisfies the equation $h(t)=\frac{tv(h(t))}{1-u(h(t))}$.

Apparently, Theorem 1.2 is a generalization of Song's formula (1.2) and Irving-Rattan's formula (1.3). The following two special cases deserve mention.

Corollary 1.1. [14, Corollary 2] *When $u(t) = t,\ v(t) = 1$, we get $h(t) = \frac{1-\sqrt{1-4t}}{2}$, thus*

$$\mathcal{H}(t,1) = \left(\frac{1-2t-\sqrt{1-4t}}{2t^2\sqrt{1-4t}}, \frac{1-\sqrt{1-4t}}{2}\right),$$

$$H(\mathbf{a}_{n,k};t,1) = [t^{n-1}]\frac{1-2t-\sqrt{1-4t}}{2t^2\sqrt{1-4t}} \cdot \left(\frac{1-\sqrt{1-4t}}{2}\right)^{n-k} = \binom{n+k}{k-1}.$$

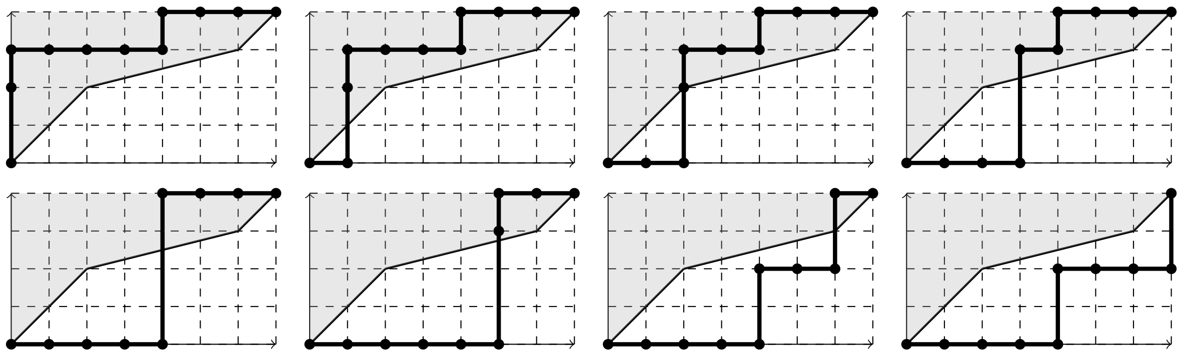

Fig. 3. 8 GDPBPs with $0, 1, \cdots, 7$ flaws, respectively

Corollary 1.2. *When $u(t) = t,\ v(t) = 1 + t$, we have $h(t) = \frac{1-t-\sqrt{1-6t+t^2}}{2}$, then*

$$\mathcal{H}(t,1+t) = \left(\frac{1-3t-\sqrt{1-6t+t^2}}{2t^2\sqrt{1-6t+t^2}}, \frac{1-t-\sqrt{1-6t+t^2}}{2}\right),$$

and

$$H(\mathbf{a}_{n,k};t,1+t) = [t^{n-1}]\frac{1-3t-\sqrt{1-6t+t^2}}{2t^2\sqrt{1-6t+t^2}} \cdot \left(\frac{1-t-\sqrt{1-6t+t^2}}{2}\right)^{n-k} = \sum_{i=1}^{k}\binom{n}{i-1}\binom{k}{i}2^i, \tag{1.4}$$

which is a generalization of the formula (1.2).

Formula (1.4) explains that there are $\sum_{i=1}^{4}\binom{7}{i-1}\binom{4}{i}2^i = 1408$ dominated Delannoy paths for each row of Figure 2. The initial parts of matrices $\mathcal{H}(t,1)$ and $\mathcal{H}(t,1+t)$ are

$$\mathcal{H}(t,1) = \begin{bmatrix} 1 & & & & & \\ 4 & 1 & & & & \\ 15 & 5 & 1 & & & \\ 56 & 21 & 6 & 1 & & \\ 210 & 84 & 28 & 7 & 1 & \\ 792 & 330 & 120 & 36 & 8 & 1 \end{bmatrix}, \quad \mathcal{H}(t,1+t) = \begin{bmatrix} 2 & & & & & \\ 12 & 2 & & & & \\ 66 & 16 & 2 & & & \\ 360 & 102 & 20 & 2 & & \\ 1970 & 608 & 146 & 24 & 2 & \\ 10386 & 3530 & 952 & 198 & 28 & 2 \end{bmatrix}.$$

A *generalized Delannoy path boundary pair (GDPBP)* is an ordered pair $(P, \mathbf{a}_{n,k})$, where P is a generalized Delannoy path from $(0,0)$ to (n,k) and $\mathbf{a}_{n,k}$ is a k-part composition of n. We say that a GDPBP $(P, \mathbf{a}_{n,k})$ has m-flaws if there are exactly m horizontal steps of P lying below the boundary $\partial\mathbf{a}_{n,k}$. Thus, a path P is dominated by $\mathbf{a}_{n,k}$ if and only if $(P, \mathbf{a}_{n,k})$ has no flaws. For $\mathbf{a}_{7,4} = (1,1,4,1)$, eight GDPBPs with different numbers of flaws are shown in Figure 3. The sum of the weights of all GDPBPs $(P, \mathbf{a}_{n,k})$ with m-flaws is denoted by $D_m(\mathbf{a}_{n,k};u,v)$, and let $H_m(\mathbf{a}_{n,k};u,v) = \sum_{i=0}^{k-1} D_m(\mathbf{a}_{n,k}^{\langle i\rangle};u,v)$. It seems difficult to obtain an explicit formula for $H_m(\mathbf{a}_{n,k};u,v)$. However, when $v(t) = 1$, we have the following result, which is a refinement of Theorem 2.1 in [9].

Theorem 1.3. *Let $\mathbf{a}_{n,k}$ be the k-part composition of n, and let $0 \le m \le n$. Then, we have*

$$H_m(\mathbf{a}_{n,k}; u, 1) = \sum_{j=0}^{k-1} D_m(\mathbf{a}_{n,k}^{\langle j \rangle}; u, 1) = [t^{n-1}] \left(\frac{h(t)}{t} \right)' \cdot h(t)^{n-k},$$

where $h(t)$ satisfies the equation $h(t) = \frac{t}{1-u(h(t))}$.

The remainder of this paper is organized as follows. In the next section, we review the concept of Riordan arrays and then use the Riordan arrays to study enumerative problems about generalized Schröder paths and primitive generalized Schröder paths. In Section 3, we obtain a new class of generalized Delannoy matrices and give the bijective proofs of Theorem 1.1 and Theorem 1.2. In Section 4, we will show a bijective proof of Theorem 1.3.

2. Riordan arrays and enumeration of generalized Schröder paths

Since Shapiro et al. [22] introduced the concept of Riordan arrays, many authors have applied them to several counting problems [10,15,18,27-31]. An infinite lower triangular matrix $\mathcal{D} = [d_{n,k}]_{n,k\in\mathbb{N}}$ is called a Riordan array if its column k has generating function $g(t)f(t)^k$, where $g(t)$ and $f(t)$ are formal power series with $g(0) \neq 0, f(0) = 0$ and $f'(0) \neq 0$. The matrix corresponding to the pair $g(t)$, $f(t)$ is denoted by $\mathcal{D} = (g(t), f(t))$. Recently, many generalizations of the Riordan arrays have been studied. One of them is the almost-Riordan arrays [1,4]. An almost-Riordan array $(l(t)|g(t), f(t))$ is a matrix that consists of an initial column vector $(l_0, l_1, \cdots)^T$ with generating function $l(t) = \sum_{n\ge 0} l_n t^n$, followed by a vertically shifted Riordan array $(g(t), f(t))$ as illustrated below

$$\left[\begin{array}{c|ccccc} l_0 & 0 & \cdots & 0 & 0 & \cdots \\ \hline l_1 & & & & & \\ l_2 & & & & & \\ l_3 & & & & & \\ l_4 & & & (g(t), f(t)) & & \\ l_5 & & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}\right].$$

The multiplication of the two Riordan arrays is defined as follows

$$(g(t), f(t))(g^*(t), f^*(t)) = (g(t)g^*(f(t)), f^*(f(t))). \tag{2.1}$$

The set of Riordan arrays is a group under multiplication with (2.1). The identity element of the Riordan arrays is $(1, t)$, and the inverse of the Riordan array $(g(t), f(t))$ is given below

$$(g(t), f(t))^{-1} = \left(\frac{1}{g(\bar{f}(t))}, \bar{f}(t) \right), \tag{2.2}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$, i.e., $\bar{f}(f(t)) = f(\bar{f}(t)) = t$.

A Riordan array $(g(t), f(t))$ can also be characterized by two sequences, the A-sequence $(a_n)_{n\in\mathbb{N}}$ and the Z-sequence $(z_n)_{n\in\mathbb{N}}$ (see [5,11]), such that

$$\begin{aligned} d_{n+1,0} &= z_0 d_{n,0} + z_1 d_{n,1} + z_2 d_{n,2} + \cdots, \\ d_{n+1,k+1} &= a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \cdots, (k \ge 1). \end{aligned} \tag{2.3}$$

If $A(t)$ and $Z(t)$ are the generating functions for the corresponding A- and Z-sequence, respectively, then it follows that

$$A(t) = \frac{t}{\bar{f}(t)}, \quad Z(t) = \frac{1 - d_{0,0}g(t)}{\bar{f}(t)}.$$

A Riordan array $\mathcal{D} = [d_{n,k}]_{n,k\in\mathbb{N}}$ can also be characterized by another matrix as follows (see [18]).

Lemma 2.1. *A lower triangular array $\mathcal{D} = [d_{n,k}]_{n,k\in\mathbb{N}}$ is a Riordan array if and only if there exists an array $A = [\alpha_{n,k}]_{n,k\in\mathbb{N}}$, with $\alpha_{0,0} \neq 0$, and a sequence $\rho = [\rho_n]_{n\in\mathbb{N}}$ such that*

$$d_{n+1,k+1} = \sum_{i\geq 0}\sum_{j\geq 0} \alpha_{i,j} d_{n-i,k+j} + \sum_{j\geq 0} \rho_j d_{n+1,k+j+2}. \tag{2.4}$$

The array $[\alpha_{n,k}]_{n,k\in\mathbb{N}}$ in the above lemma is called the A-matrix of the Riordan array $(g(t), f(t))$. Let $\mathcal{L}^{[i]}(t) = \sum_{j\geq 0} \alpha_{i,j} t^j, i \geq 0$, and $\mathcal{L}^*(t) = \sum_{j\geq 0} \rho_j t^j$. Then $f(t)$ for $\mathcal{D}$ is given by

$$f(t) = \sum_{i\geq 0} t^{1+i} \mathcal{L}^{[i]}(f(t)) + f(t)^2 \mathcal{L}^*(f(t)). \tag{2.5}$$

Let $B = [\beta_{n,k}]_{n,k\in\mathbb{N}}$ and $\eta = [\eta_n]_{n\geq 0}$. If column 0 of the Riordan array $\mathcal{D}$ is defined by

$$d_{n+1,0} = \sum_{i\geq 0}\sum_{j\geq 0} \beta_{i,j} d_{n-i,j} + \sum_{j\geq 0} \eta_j d_{n+1,j+1}, \tag{2.6}$$

then $g(t)$ for $\mathcal{D}$ is

$$g(t) = \frac{d_{0,0}}{1 - \sum_{i\geq 0} t^{1+i} \mathcal{M}^{[i]}(f(t)) - \mathcal{M}^*(f(t))}, \tag{2.7}$$

where $\mathcal{M}^{[i]}(t) = \sum_{j\geq 0} \beta_{i,j} t^j, i \geq 0$, and $\mathcal{M}^*(t) = \sum_{j\geq 0} \eta_j t^j$.

Now we can use the Riordan arrays to study enumerative problems about generalized Schröder paths and primitive generalized Schröder paths, whose definitions are given below.

Definition 2.1. [16] A generalized Schröder path of order n is a generalized Delannoy path from $(0,0)$ to (n,n), stay weakly below the mian diagonal $y = x$. A primitive generalized Schröder path is a generalized Schröder path that its nodes (excluding the initial and ending nodes) never on the $y = x$.

Let $R(n,k)$ be the set of all partial generalized Schröder paths from $(0,0)$ to $(n, n-k)$, and $R_{n,k} = |R(n,k)|$. Let $P(n,k)$ be the set of all partial primitive generalized Schröder paths from $(0,0)$ to $(n, n-k)$, and $P_{n,k} = |P(n,k)|$. We call the matrices $\mathcal{R} = [R_{n,k}]_{n,k\in\mathbb{N}}$ and $\mathcal{P} = [P_{n,k}]_{n,k\in\mathbb{N}}$ the generalized Schröder matrix and the primitive generalized Schröder matrix, respectively.

Theorem 2.2. [16] *The generalized Schröder matrix $\mathcal{R} = [R_{n,k}]_{n,k\in\mathbb{N}}$ is a Riordan array given by*

$$\mathcal{R} = \left(\frac{h(t)}{t}, h(t)\right) = \left(\frac{1-u(t)}{v(t)}, \frac{t(1-u(t))}{v(t)}\right)^{-1},$$

where $h(t)$ satisfies the equation $h(t) = \frac{tv(h(t))}{1-u(h(t))}$, and $u(t) = \sum_{i\geq 1} u_i t^i$, $v(t) = \sum_{i\geq 0} v_i t^i$.

Proof. See proof of [16, Theorem 3]. □

Theorem 2.3. *The primitive generalized Schröder matrix $\mathcal{P} = [P_{n,k}]_{n,k\in\mathbb{N}}$ is a almost-Riordan array given by*

$$\mathcal{P} = \left(\frac{2h(t)-t}{h(t)} \middle| \frac{h(t)}{t}, h(t)\right),$$

where $h(t)$ satisfies the equation $h(t) = \frac{tv(h(t))}{1-u(h(t))}$.

Proof. For any $P \in P(n,k)$, according to the last step of P, one can deduce the following recurrences for $P_{n,k}$, i.e.

$$P_{1,0} = v_1 P_{0,0} + u_1 P_{1,1},$$

$$P_{n+1,0} = \sum_{i=1}^{n} v_{i+1} P_{n,i} + \sum_{i=1}^{n+1} u_i P_{n+1,i},\ (n \geq 1), \tag{2.8}$$

$$P_{n+1,1} = \sum_{i=1}^{n} v_i P_{n,i} + \sum_{i=1}^{n} u_i P_{n+1,i+1}, \tag{2.9}$$

$$P_{n+1,k+1} = P_{n,k} + \sum_{i=1}^{n} v_i P_{n,k+i} + \sum_{i=1}^{n} u_i P_{n+1,k+i+1},\ (k \geq 1), \tag{2.10}$$

with initial conditions $P_{n,n} = 1$ for $n \geq 0$. Let $\overline{\mathcal{P}} = [P_{n,k}]_{n,k \in \mathbb{N}^+}$. By (2.9), (2.10), and Lemma 2.1, we get

$$\overline{\mathcal{P}} = \mathcal{R} = \left(\frac{h(t)}{t}, h(t) \right),$$

where $h(t)$ satisfies the equation $h(t) = \frac{tv(h(t))}{1-u(h(t))}$. Let $d(t)$ denote the generating function of the first column of $\mathcal{P}$. By (2.8), we have

$$d(t) = 1 + u(h(t)) + \frac{t(v(h(t)) - 1)}{h(t)} = \frac{2h(t) - t}{h(t)}.$$

Thus,

$$\mathcal{P} = \left(\frac{2h(t) - t}{h(t)} \middle| \frac{h(t)}{t}, h(t) \right).$$

This completes the proof. □

Remark 2.4. Let $Q(n)$ be the set of all generalized Delannoy paths from $(0,0)$ to (n,n) stay weakly above the diagonal $y = x$. and let $S(n)$ the subset of $Q(n)$ whose paths touch $y = x$ only at $(0,0)$ and (n,n). Let $q_n = |Q(n)|$ and $s_n = |S(n)|$. It is easy to know that $q_n = R_{n,0}$ and $s_n = P_{n,0}$. Thus, we have

$$\sum_{i\geq 1} i q_i t^{i-1} = \left(\frac{h(t)}{t} \right)', \qquad \sum_{i\geq 1} s_i t^i = \frac{h(t) - t}{h(t)}, \tag{2.11}$$

where $h(t)$ is determined by equation $h(t) = \frac{tv(h(t))}{1-u(h(t))}$.

We give the following examples of primitive generalized Schröder matrices.

Example 2.5. If $u(t) = t, v(t) = 1 + t$, we have

$$\mathcal{P} = \left(\frac{3 + t - \sqrt{1 - 6t + t^2}}{2} \middle| \frac{1 - t - \sqrt{1 - 6t + t^2}}{2t}, \frac{1 - t - \sqrt{1 - 6t + t^2}}{2} \right)$$

$$= \begin{bmatrix} 1 & & & & & & & \\ 2 & 1 & & & & & & \\ 2 & 2 & 1 & & & & & \\ 6 & 6 & 4 & 1 & & & & \\ 22 & 22 & 16 & 6 & 1 & & & \\ 90 & 90 & 68 & 30 & 8 & 1 & & \\ 394 & 394 & 304 & 146 & 48 & 10 & 1 & \\ \vdots & & \vdots & & \vdots & & & \ddots \end{bmatrix}.$$

The first column of the matrix $\mathcal{P}$ corresponds to the sequence A108524 in OEIS [21].

Example 2.6. Let $u(t) = \frac{t}{1-t}$, $v(t) = 1$, we obtain

$$\mathcal{P} = \left(\frac{3 - t - \sqrt{1 - 6t + t^2}}{2} \middle| \frac{1 + t - \sqrt{1 - 6t + t^2}}{4t}, \frac{1 + t - \sqrt{1 - 6t + t^2}}{4} \right)$$

$$= \begin{bmatrix} 1 & & & & & & & \\ 1 & 1 & & & & & & \\ 2 & 1 & 1 & & & & & \\ 6 & 3 & 2 & 1 & & & & \\ 22 & 11 & 7 & 3 & 1 & & & \\ 90 & 45 & 28 & 12 & 4 & 1 & & \\ 394 & 197 & 121 & 52 & 18 & 5 & 1 & \\ \vdots & & \vdots & & \vdots & & & \ddots \end{bmatrix}.$$

Example 2.7. If $u(t) = \frac{t}{1-t}$, $v(t) = \frac{1}{1-t}$, we derive

$$\mathcal{P} = \left(\frac{3-\sqrt{1-8t}}{2} \middle| \frac{1-\sqrt{1-8t}}{2t}, \frac{1-\sqrt{1-8t}}{2} \right)$$

$$= \begin{bmatrix} 1 & & & & & & & \\ 2 & 1 & & & & & & \\ 8 & 4 & 1 & & & & & \\ 40 & 20 & 6 & 1 & & & & \\ 224 & 112 & 36 & 8 & 1 & & & \\ 1344 & 672 & 224 & 56 & 10 & 1 & & \\ 8448 & 4224 & 1440 & 384 & 80 & 12 & 1 & \\ \vdots & & \vdots & & \vdots & & & \ddots \end{bmatrix}.$$

The first column of the matrix $\mathcal{P}$ corresponds to the sequence A175962 in OEIS [21].

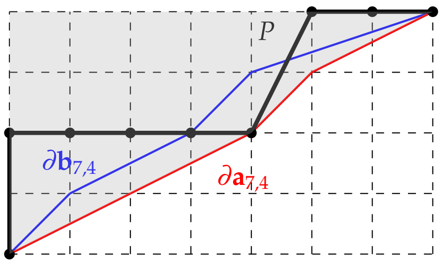

Fig. 4. A path dominated by $\mathbf{a}_{7,4} \setminus \mathbf{b}_{7,4}$

3. The proofs of Theorem 1.1 and Theorem 1.2

Let $\mathbf{a}_{n,k} = (a_1, a_2, \ldots, a_k)$ and $\mathbf{b}_{n,k} = (b_1, b_2, \ldots, b_k)$ denote two arbitrary k-part compositions of n, such that the boundary curve $\partial\mathbf{b}_{n,k}$ lies weakly above $\partial\mathbf{a}_{n,k}$. A generalized Delannoy path P is said to be dominated by $\mathbf{a}_{n,k} \setminus \mathbf{b}_{n,k}$ if it is dominated by $\partial\mathbf{a}_{n,k}$ but not dominated by $\partial\mathbf{b}_{n,k}$. For example, if $\mathbf{a}_{7,4} = (2,2,1,2)$ and $\mathbf{b}_{7,4} = (1,2,1,3)$, Figure 4 shows a generalized Delannoy path dominated by $\partial\mathbf{a}_{7,4}$ but not by $\partial\mathbf{b}_{7,4}$. Let $\mathcal{D}(\mathbf{a}_{n,k} \setminus \mathbf{b}_{n,k}; u, v)$ denote the set of all generalized Delannoy paths from $(0,0)$ to (n,k) that are dominated by $\mathbf{a}_{n,k} \setminus \mathbf{b}_{n,k}$, and let $D(\mathbf{a}_{n,k} \setminus \mathbf{b}_{n,k}; u, v) = |\mathcal{D}(\mathbf{a}_{n,k} \setminus \mathbf{b}_{n,k}; u, v)|$.

Lemma 3.1. *Let $i \in \{0, 1, \ldots, k-2\}$. Assume that $\mathbf{a}_{n,k,i} = (a_0, \cdots, a_i, a_{i+1}, \cdots, a_{k-1})$ and $\mathbf{b}_{n,k,i} = (a_0, \cdots, a_i - 1, a_{i+1} + 1, \cdots, a_{k-1})$ are k-part compositions of n. Then there exists a bijection between*

$$\bigcup_{\substack{j=0 \\ j \neq k-i-1}}^{k-1} \mathcal{D}\left(\mathbf{a}_{n,k,i}^{\langle j \rangle} \setminus \mathbf{b}_{n,k,i}^{\langle j \rangle}; u, v\right) \text{ and } \mathcal{D}\left(\mathbf{b}_{n,k,i}^{\langle k-i-1 \rangle} \setminus \mathbf{a}_{n,k,i}^{\langle k-i-1 \rangle}; u, v\right).$$

This bijection induces the following identity:

$$\sum_{j=0}^{k-1} D\left(\mathbf{a}_{n,k,i}^{\langle j \rangle}; u, v\right) = \sum_{j=0}^{k-1} D\left(\mathbf{b}_{n,k,i}^{\langle j \rangle}; u, v\right). \tag{3.1}$$

Proof. We construct the required bijection as follows:

$$\theta : \bigcup_{\substack{j=0 \\ j\neq k-i-1}}^{k-1} \mathcal{D}\left(\mathbf{a}_{n,k,i}^{\langle j\rangle} \setminus \mathbf{b}_{n,k,i}^{\langle j\rangle}; u, v\right) \to \mathcal{D}\left(\mathbf{b}_{n,k,i}^{\langle k-i-1\rangle} \setminus \mathbf{a}_{n,k,i}^{\langle k-i-1\rangle}; u, v\right).$$

If $j \in \{0, \ldots, k-i-2, k-i, \ldots, k-1\}$, for these indices, the boundary curve $\partial\mathbf{b}_{n,k,i}^{\langle j\rangle}$ is weakly above the boundary curve $\partial\mathbf{a}_{n,k,i}^{\langle j\rangle}$. A generalized Delannoy path P is dominated by $\mathbf{a}_{n,k,i}^{\langle j\rangle} \setminus \mathbf{b}_{n,k,i}^{\langle j\rangle}$ if and only if P is dominated by $\mathbf{a}_{n,k,i}^{\langle j\rangle}$ and a horizontal step of P touches the boundary $\partial\mathbf{a}_{n,k,i}^{\langle j\rangle}$ at $(x, (i+j+1) \bmod k)$. If $j = k-i-1$, here the boundary curve $\partial\mathbf{a}_{n,k,i}^{\langle k-i-1\rangle}$ is weakly above the boundary curve $\partial\mathbf{b}_{n,k,i}^{\langle k-i-1\rangle}$. A generalized Delannoy path P is dominated by $\mathbf{b}_{n,k,i}^{\langle k-i-1\rangle} \setminus \mathbf{a}_{n,k,i}^{\langle k-i-1\rangle}$ if and only if P is dominated by $\mathbf{b}_{n,k,i}^{\langle k-i-1\rangle}$ and there exists a horizontal step of P that intersects with $\partial\mathbf{b}_{n,k,i}^{\langle k-i-1\rangle}$ excluding the terminal point (n,k).

Let $j \in \{0, \ldots, k-i-2, k-i, \ldots, k-1\}$ and let $\left(P, \mathbf{a}_{n,k,i}^{\langle j\rangle} \setminus \mathbf{b}_{n,k,i}^{\langle j\rangle}\right)$ be a GDPBP in $\mathcal{D}\left(\mathbf{a}_{n,k,i}^{\langle j\rangle} \setminus \mathbf{b}_{n,k,i}^{\langle j\rangle}; u, v\right)$. Then $P = AH_0^*B$, where H_0^* is the horizontal step that touches the boundary $\partial\mathbf{a}_{n,k,i}^{\langle j\rangle}$ at $(x, (i+j+1) \bmod k)$. We define $P' = BH_0^*A$, then

$$\theta\left(P, \mathbf{a}_{n,k,i}^{\langle j\rangle} \setminus \mathbf{b}_{n,k,i}^{\langle j\rangle}\right) = (P', \mathbf{b}_{n,k,i}^{\langle k-i-1\rangle} \setminus \mathbf{a}_{n,k,i}^{\langle k-i-1\rangle}) \in \mathcal{D}\left(\mathbf{b}_{n,k,i}^{\langle k-i-1\rangle} \setminus \mathbf{a}_{n,k,i}^{\langle k-i-1\rangle}; u, v\right).$$

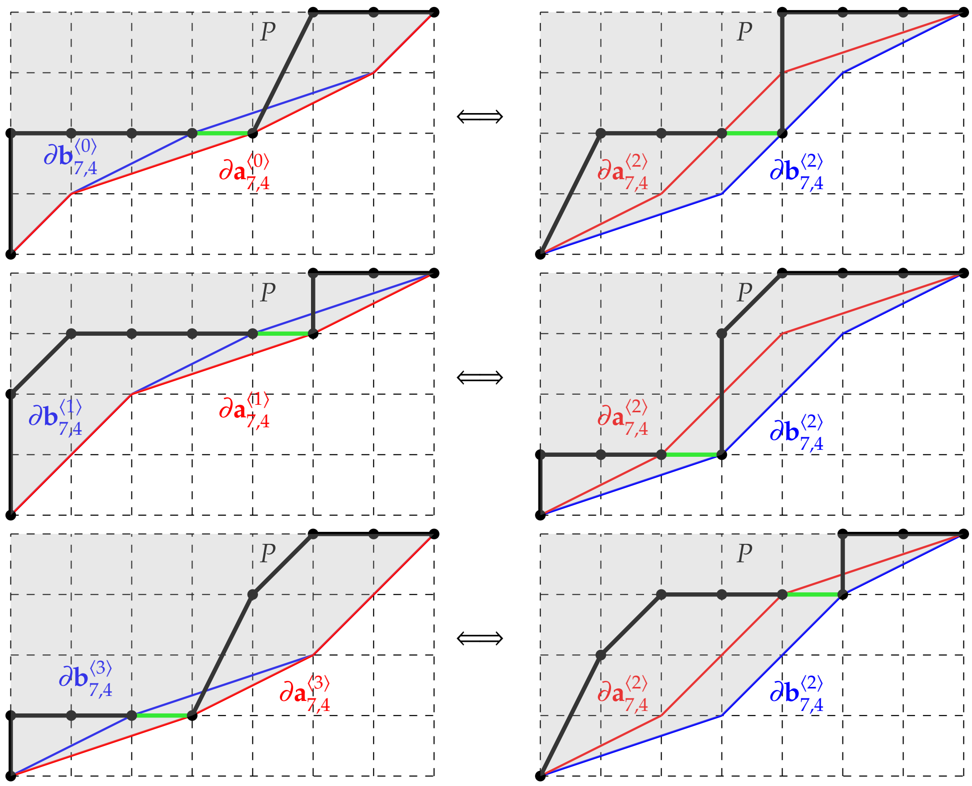

Fig. 5. The bijection θ in the proof of Lemma 3.1.

To establish bijectivity, we construct the inverse mapping θ^{-1} as follows. Let $(P', \mathbf{b}_{n,k,i}^{\langle k-i-1\rangle} \setminus \mathbf{a}_{n,k,i}^{\langle k-i-1\rangle})$ be a GDPBP in $\mathcal{D}\left(\mathbf{b}_{n,k,i}^{\langle k-i-1\rangle} \setminus \mathbf{a}_{n,k,i}^{\langle k-i-1\rangle}; u, v\right)$. Then $P' = AH_0^{(s)}B$, where $H_0^{(s)}$ marks the first horizontal step that touches the boundary $\partial\mathbf{b}_{n,k,i}^{<k-i-1>}$ at level s. We define $P = BH_0A$, then

$$\theta^{-1}(P', \mathbf{b}_{n,k,i}^{\langle k-i-1\rangle} \setminus \mathbf{a}_{n,k,i}^{\langle k-i-1\rangle}) = \left(P, \mathbf{a}_{n,k,i}^{\langle (k-i-1-s) \bmod k\rangle} \setminus \mathbf{b}_{n,k,i}^{\langle (k-i-1-s) \bmod k\rangle}\right) \in \bigcup_{\substack{j=0\\ j\neq k-i-1}}^{k-1} \mathcal{D}\left(\mathbf{a}_{n,k,i}^{\langle j\rangle} \setminus \mathbf{b}_{n,k,i}^{\langle j\rangle}; u, v\right).$$

For example, if $\mathbf{a}_{7,4,1} = (1, 3, 2, 1)$ and $\mathbf{b}_{7,4,1} = (1, 2, 3, 1)$, the bijection θ is given in Figure 5. □

Proof of Theorem 1.1. Let $\mathbf{a}_{n,k} = (a_0, \dots, a_{k-1})$ and $\mathbf{b}_{n,k} = (b_0, \dots, b_{k-1})$ be k-part compositions of n satisfying the condition:

$$\sum_{i=0}^{s} a_i \geq \sum_{i=0}^{s} b_i \quad \forall s \in \{0, \dots, k-1\}.$$

Define the cumulative difference $x_s := \sum_{i=0}^{s}(a_i - b_i)$. Then, by Lemma 3.1, we obtain

$$H(\mathbf{a}_{n,k}; u, v) = H(a_0 - x_0,\ a_1 + x_0 - x_1,\ \dots,\ a_{k-1} + x_{k-2} - x_{k-1}; u, v) = H(\mathbf{b}_{n,k}; u, v). \tag{3.2}$$

For arbitrary compositions, let $y_s := \max\left(\sum_{i=0}^{s} a_i,\ \sum_{i=0}^{s} b_i\right)$. The transformed composition

$$\mathbf{y}_{n,k} = \left(y_0,\ y_1 - y_0,\ \dots,\ y_{k-2} - y_{k-3},\ y_{k-1} - y_{k-2}\right).$$

By (3.2), we have

$$H(\mathbf{y}_{n,k}; u, v) = H(\mathbf{a}_{n,k}; u, v) = H(\mathbf{b}_{n,k}; u, v). \quad \square$$

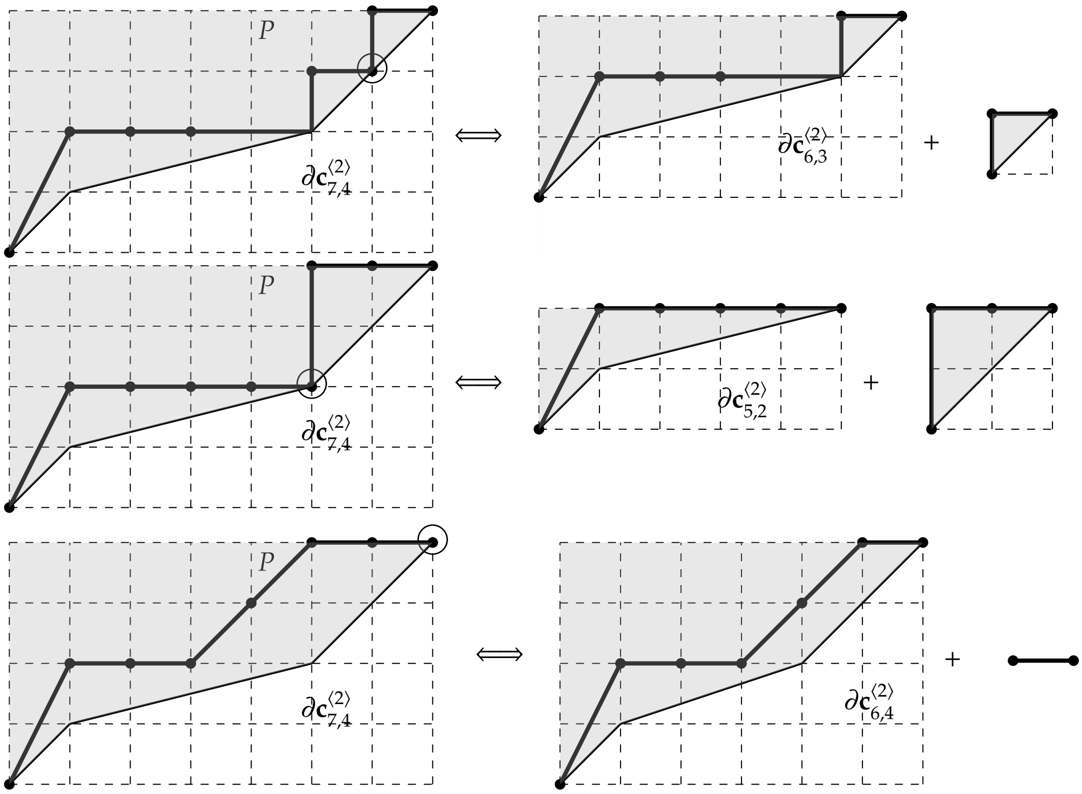

Fig. 6. The bijection ϕ in the proof of Lemma 3.2.

As mentioned in Section 2, let $S(n)$ be the set of all primitive generalized Schröder paths of order n, and let $s_n = |s(n)|$. We derive the following recurrence relation for $H(\mathbf{a}_{n,k}; u, v)$.

Lemma 3.2. *Let $\mathbf{a}_{n,k}$ be the k-part composition of n. The number $H(\mathbf{a}_{n,k}; u, v)$ satisfies the following recurrence relation:*

$$H(\mathbf{a}_{n,k}; u, v) = \sum_{i=1}^{k-1} s_i H(\mathbf{a}_{n-i,k-i}; u, v) + H(\mathbf{a}_{n-1,k}; u, v). \tag{3.3}$$

Proof. We assume that $\mathbf{c}_{n,k} = (\underbrace{1, \cdots, 1}_{k-1}, n-k+1)$. Then we construct a bijection ϕ between $\mathcal{D}\left(\mathbf{c}_{n,k}^{\langle j \rangle}, u, v\right)$ and the following disjoint subsets:

$$\bigcup_{i=1}^{k-j} \left[\mathcal{D}\left(\mathbf{c}_{n-i,k-i}^{\langle j \rangle}, u, v\right) \times S(i)\right] \cup \left[\mathcal{D}\left(\mathbf{c}_{n-1,k}^{\langle j \rangle}, u, v\right) \times \{(1,0)\}\right].$$

We consider generalized Delannoy paths dominated by $\mathbf{c}_{n,k}^{\langle j \rangle}$ for $j = 0, 1, \cdots, k-1$. We will decompose these paths into an initial subpath and a final subpath. Let $1 \le i \le k-j$, and let $(n-i, k-i)$ be the penultimate note on the boundary $\partial \mathbf{c}_{n,k}^{\langle j \rangle}$. The initial subpaths from $(0,0)$ to $(n-i, k-i)$ are dominated by $\mathbf{c}_{n-i,k-i}^{\langle j \rangle}$, and the final subpaths from $(n-i, k-i)$ to (n, k) belong to $S(i)$. If the penultimate note on the boundary $\partial \mathbf{c}_{n,k}^{\langle j \rangle}$ is not belong to $\{(n-i, k-i) | i = 1, 2, \cdots, k-j\}$, the initial subpaths from $(0,0)$ to $(n-1, k)$ are dominated by $\partial \mathbf{c}_{n-1,k}^{\langle j \rangle}$, and the final subpaths are horizontal steps. In both cases, the decomposition is reversible, and hence, bijective. By this decomposition, we have

$$\begin{aligned}
H(\mathbf{c}_{n,k}; u, v) &= \sum_{j=0}^{k-1} \left|\mathcal{D}\left(\mathbf{c}_{n,k}^{\langle j \rangle}, u, v\right)\right| \\
&= \sum_{j=0}^{k-1} \sum_{i=1}^{k-j} \left|\mathcal{D}\left(\mathbf{c}_{n-i,k-i}^{\langle j \rangle}, u, v\right)\right| \cdot |S(i)| + \sum_{j=0}^{k-1} \left|\mathcal{D}\left(\mathbf{c}_{n-1,k}^{\langle j \rangle}, u, v\right)\right| \\
&= \sum_{i=1}^{k-1} s_i H(\mathbf{c}_{n-i,k-i}; u, v) + H(\mathbf{c}_{n-1,k}; u, v).
\end{aligned}$$

For an arbitrary $\mathbf{a}_{n,k}$, Lemma 3.1 implies that $D(\mathbf{a}_{n,k}; u, v) = D(\mathbf{c}_{n,k}; u, v)$. Thus, $H(\mathbf{a}_{n,k}; u, v)$ satisfies (3.3). An example of the bijection ϕ is given in Figure 6. □

Proof of Theorem 1.2. From (3.3), we find that the generalized Delannoy matrix $\mathcal{H}(u, v) = [H_{n,k}(u, v)]_{n,k \in \mathbf{N}}$ satisfies the following recurrence relation:

$$H_{n+1,k+1}(u, v) = H_{n,k}(u, v) + \sum_{i=1}^{n-k} s_i H_{n-i+1,k+1}, \quad n, k \ge 0.$$

From this recurrence, the associated A-matrix and ρ-sequence are

$$A = \begin{bmatrix} 1 & s_1 & 0 & 0 & \cdots \\ 0 & s_2 & 0 & 0 & \cdots \\ 0 & s_3 & 0 & 0 & \cdots \\ 0 & s_4 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad \rho = (0, 0, 0, \cdots).$$

Hence, from Lemma 2.1, the matrix $\mathcal{H}(u, v)$ is a Riordan array $(g(t), f(t))$. The generating functions of rows of A-matrix are $\mathcal{L}^{[0]}(t) = 1 + s_1 t$, and $\mathcal{L}^{[i]}(t) = s_{i+1} t$ for all $i \ge 1$. The generating function of the ρ-sequence is $\mathcal{L}^*(t) = 0$. Using (2.5) and (2.11), we get $f(t) = t + s_1 t f(t) + s_2 t^2 f(t) + s_3 t^3 f(t) + \cdots$. Thus, $f(t) = h(t)$, where $h(t)$ is determined by equation $h(t) = \frac{tv(h(t))}{1-u(h(t))}$. By Remark 2.4, we have $g(t) = \sum_{i \ge 1} i q_i t^{i-1} = \left(\frac{h(t)}{t}\right)'$. □.

4. The proof of Theorem 1.3

Our proof of Theorem 1.3 is bijective and is motivated by Guo and Wang's bijective proof of [9, Theorem 2.1]. Let $\mathcal{D}_m(\mathbf{a}_{n,k}; u, v)$ denote the set of all GDPBPs $(P, \mathbf{a}_{n,k})$ with m flaws. We will construct a bijection

$$\psi : \bigcup_{j=0}^{k-1} \mathcal{D}_m\left(\mathbf{a}_{n,k}^{\langle j\rangle}; u, 1\right) \to \bigcup_{j=0}^{k-1} \mathcal{D}_{m-1}\left(\mathbf{a}_{n,k}^{\langle j\rangle}; u, 1\right), \tag{4.1}$$

which keeps the numbers of the vertical steps unchanged, to prove this result.

Let $1 \le m \le n$ and $\left(P, \mathbf{a}_{n,k}^{\langle j\rangle}\right)$ be a GDPBP in $\mathcal{D}_m\left(\mathbf{a}_{n,k}^{\langle j\rangle}; u, 1\right)$. Then $P = A'H_0B$, where H_0 is the rightmost horizontal step intersecting the boundary $\partial\mathbf{a}_{n,k}^{\langle j\rangle}$, and stays weakly below $\partial\mathbf{a}_{n,k}^{\langle j\rangle}$. Since $k \ge 1$, such H_0 must exist. We have two cases:

- If A' is empty or A' ends with a horizontal step, then we define $P' = A'BH_0$ and $\psi\left(P, \mathbf{a}_{n,k}^{\langle j\rangle}\right) = \left(P', \mathbf{a}_{n,k}^{\langle j\rangle}\right)$.
- If A' ends with a vertical step N_i, and the length of all vertical steps N_i of A' is equal to d, then we define $P' = BH_0A'$, and $\psi\left(P, \mathbf{a}_{n,k}^{\langle j\rangle}\right) = \left(P', \mathbf{a}_{n,k}^{\langle j-d\rangle}\right)$.

It is easy to see that $\psi\left(P, \mathbf{a}_{n,k}^{\langle j\rangle}\right) \in \bigcup_{j=0}^{k-1} \mathcal{D}_{m-1}\left(\mathbf{a}_{n,k}^{\langle j\rangle}; u, 1\right)$. To prove that the mapping ψ is a bijection, we construct its inverse ψ^{-1} as follows. Suppose that $\left(P', \mathbf{a}_{n,k}^{\langle j\rangle}\right)$ is a GDPBP in $\mathcal{D}_{m-1}\left(\mathbf{a}_{n,k}^{\langle j\rangle}; u, 1\right)$. We consider the following two cases:

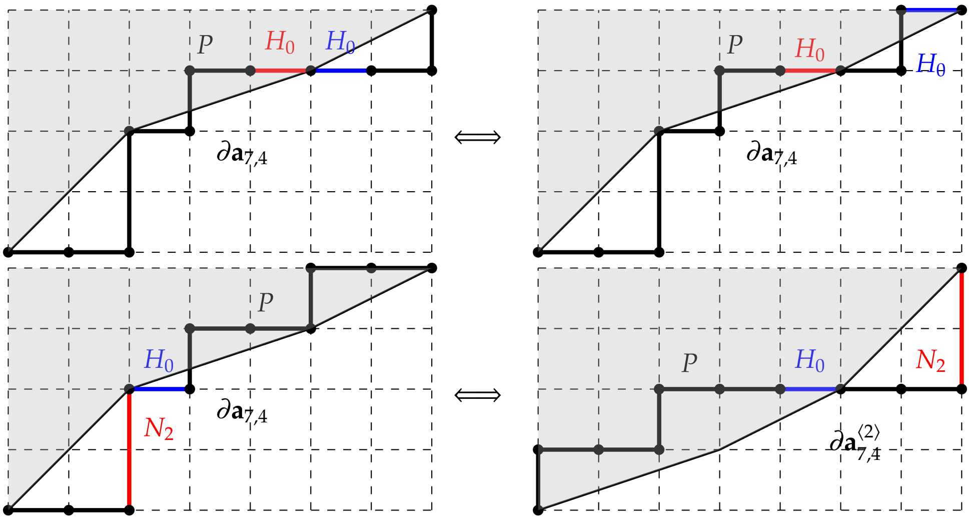

Fig. 7. The bijection ψ in the proof of Theorem 1.3.

- P' is end with H_0. We write $P' = A'BH_0$, where $A' = AH_0$ and this H_0 is the rightmost flaw in $A'B$ that intersects $\mathbf{a}_{n,k}^{\langle j\rangle}$ if such H_0 exists, and we write $P' = BH_0(A' = 0)$ otherwise. Then $P = A'H_0B$ and $\psi^{-1}\left(P', \mathbf{a}_{n,k}^{\langle j\rangle}\right) = \left(P, \mathbf{a}_{n,k}^{\langle j\rangle}\right)$.
- P' is end with N_i. We write $P' = BH_0A' = BH_0AN_i$, where H_0 is the leftmost flaw in $\left(P, \mathbf{a}_{n,k}^{\langle j\rangle}\right)$ that intersects $\mathbf{a}_{n,k}^{\langle j\rangle}$. Assume that the length of all vertical steps N_i of A' is equal to d. then we let $P = A'H_0B$ and $\psi^{-1}\left(P', \mathbf{a}_{n,k}^{\langle j\rangle}\right) = \left(P, \mathbf{a}_{n,k}^{\langle j+d\rangle}\right)$.

The example of the bijection ψ is given in Figure 7. From Theorem 1.2 and bijection (4.1), for $1 \le m \le n$, we have

$$H_m(\mathbf{a}_{n,k}; u, 1) = \sum_{j=0}^{k-1} D_m(\mathbf{a}_{n,k}^{\langle j \rangle}; u, 1) = H_0(\mathbf{a}_{n,k}; u, 1) = [t^{n-1}] \left(\frac{h(t)}{t} \right)' \cdot h(t)^{n-k},$$

where $h(t)$ satisfies the equation $h(t) = \frac{t}{1-u(h(t))}$. □

Declaration

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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