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Matching, odd $[1, b]$-factor and distance spectral radius of graphs with given some parameters

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Abstract. For a connected graph G, let $\mu(G)$ denote the distance spectral radius of G. A matching in a graph G is a set of disjoint edges of G. The maximum size of a matching in G is called the matching number of G, denoted by $\alpha(G)$. An odd $[1, b]$-factor of a graph G is a spanning subgraph G_0 such that the degree $d_{G_0}(v)$ of v in G_0 is odd and $1 \le d_{G_0}(v) \le b$ for every vertex $v \in V(G)$. In this paper, we give a sharp upper bound in terms of the distance spectral radius to guarantee $\alpha(G) > \frac{n-k}{2}$ in an n-vertex t-connected graph G, where $2 \le k \le n-2$ is an integer. We also present a sharp upper bound in terms of distance spectral radius for the existence of an odd $[1, b]$-factor in a graph with given minimum degree δ.

1. Introduction

All graphs considered are finite, simple, and connected throughout this paper. For a graph G, we use $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G)$ to denote the vertex set and the edge set of G, respectively. The set of neighbors of the vertex v_i is denoted by $N_G(v_i)$, which is defined as the set of vertices adjacent to v_i. The degree of the vertex v_i in G is the number of its neighbours, denoted by $d_G(v_i)$ (or simply $d(v_i)$), i.e., $d_G(v_i) = |N_G(v_i)|$. The minimum degree of G is denoted by $\delta(G)$ (or simply δ). For any two graphs G_1 and G_2, we use $G_1 + G_2$ to denote the disjoint union of G_1 and G_2. The join of G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph obtained from $G_1 + G_2$ by adding all possible edges between $V(G_1)$ and $V(G_2)$. For $S \subseteq V(G)$, we use $G - S$ to denote the subgraph obtained from G by deleting the vertices in S together with their incident edges. For $E' \subseteq E(G)$, we use $G - E'$ to denote the subgraph obtained from G by deleting the edges in E'. Let K_n denote an n-vertex complete graph. The connectivity of G is the minimum number of vertices whose deletion induces a non-connected graph or a single vertex. For $t \ge 0$, a graph G is called t-connected if the connectivity of G is at least t.

For a connected graph G of order n, the distance between vertices v_i and v_j denoted by d_{ij}, is the length of the shortest path between v_i and v_j. The distance matrix of G is defined as $D(G) = (d_{ij})_{n\times n}$, where (i, j)-entry is d_{ij}. The distance spectral radius of G is the largest eigenvalues of $D(G)$, denoted by $\mu(G)$.

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For a graph G, a matching of G is a set of pairwise nonadjacent edges of G. The maximum size of a matching in G is called matching number of G, denoted by $\alpha(G)$. A vertex v is said to be M-saturated if v is incident to some edge of M. A matching M is called perfect matching if every vertex of G is M-saturated. Therefore, if a graph G contains a perfect matching, it must have an even number of vertices and $\alpha(G) = \frac{|V(G)|}{2}$.

One of our results is to characterize the matching number of a graph using the distance spectral radius. Studying the matching of graphs using the spectral radius has received a lot of attention of researchers in recent years. For example, Feng et al. [3] gave a spectral radius condition for a graph with given matching number. O [14] proved a lower bound for the spectral radius in an n-vertex graph to guarantee the existence of a perfect matching. Zhang [18] characterized the extremal graphs with maximum spectral radius among all t-connected graphs on n vertices with matching number at most $\frac{n-k}{2}$, where $2 \le k \le n-2$ is an integer. Liu et al. [11] extended some results of [14] and [18], they proved sharp upper bounds for spectral radius of $A_\alpha(G)$ in an n-vertex t-connected graph with the matching number at most $\frac{n-k}{2}$. Zhang and van Dam [19] gave a sufficient condition in terms of distance spectral radius for the k-extendability of a graph and completely characterized the corresponding extremal graphs. Guo et al. [5] gave a spectral condition for a graph to have a rainbow matching. For more literature on studying the matching of graphs using the spectral radius, please refer to [7–9, 16, 20].

Inspired by [11] and [18], in this paper we firstly investigate the relation between the distance spectral radius of an n-vertex t-connected graph and its matching number.

$[a, b]$-factor plays important roles in solving the graph decomposability problem. An $[a, b]$-factor of a graph G is defined as a spanning subgraph G_0 such that $a \le d_{G_0}(v) \le b$ for each $v \in V(G)$. An odd $[1, b]$-factor of a graph G is defined as a spanning subgraph G_0 such that $d_{G_0}(v)$ is odd and $1 \le d_{G_0}(v) \le b$ for each $v \in V(G)$. Obviously, a perfect matching is a special odd $[1, b]$-factor when $b = 1$.

Recently, the existence of an $[a, b]$-factor in a graph has been investigated by many researchers. In [15], O provided some conditions for the existence of an $[a, b]$-factor in an h-edge-connected r-regular graph. In [2], Fan et al. provided spectral conditions for the existence of an odd $[1, b]$-factor in a connected graph with minimum degree δ and the existence of an $[a, b]$-factor in a graph, respectively. In [10], Li and Miao considered the edge condition for a connected graph to contain an odd $[1, b]$-factor. For more literature on $[a, b]$-factor, please refer to [6, 22, 23]

Motivated by [2], in this paper we provide a condition in terms of distance spectral radius for the existence of an odd $[1, b]$-factor in a graph with given minimum degree.

The rest of the paper is structured as follows. In Section 2, we recall some important known concepts and lemmas to prove the theorems in the following sections. In Section 3, we give a sharp upper bound in terms of the distance spectral radius to guarantee $\alpha(G) > \frac{n-k}{2}$ in an n-vertex t-connected graph G, where $2 \le k \le n-2$ is an integer. In Section 4, we provide a sharp upper bound in terms of distance spectral radius for the existence of an odd $[1, b]$-factor in a graph with given minimum degree δ.

2. Preliminaries

In this section, we give some concepts and useful lemmas which will be used in the follows. First of all, we give some known lemmas about matching number and $[1, b]$-factor in a graph G. Moreover, for any $S \subseteq V(G)$ of a graph G, $o(G-S)$ denotes the number of odd components in graph $G-S$.

Lemma 2.1. *([13]) Let G be a graph of order n. Then*

$$\alpha(G) = \frac{1}{2}(n - \max\{o(G-S) - |S| : \textit{for all } S \subseteq V(G)\}).$$

Lemma 2.2. *([1]) Let G be a graph and b be a positive odd integer. Then G contains an odd $[1, b]$-factor if and only if for every $S \subseteq V(G)$,*

$$o(G-S) \le b|S|.$$

Equitable quotient matrix plays an important role in the study of spectral graph theory. Thus, we will give the definition of the equitable quotient matrix and its some useful properties.

Definition 2.3. *([17]) Let M be a complex matrix of order n described in the following block form*

$$M = \begin{bmatrix} M_{11} & \cdots & M_{1t} \\ \vdots & \ddots & \vdots \\ M_{t1} & \cdots & M_{tt} \end{bmatrix},$$

where the blocks M_{ij} are the $n_i \times n_j$ matrices for any $1 \le i, j \le t$ and $n = n_1 + n_2 + \cdots + n_t$. For $1 \le i, j \le t$, let b_{ij} denote the average row sum of M_{ij}, i.e. b_{ij} is the sum of all entries in M_{ij} divided by the number of rows. Then $B(M) = (b_{ij})$(or simply B) is called the quotient matrix of M. If, in addition, for each pair i, j, M_{ij} has a constant row sum, then B is called the equitable quotient matrix of M.

Lemma 2.4. *([17]) Let B be the equitable quotient matrix of M, where M is as shown in Definition 2.3. In addition, let M be a nonnegative matrix. Then the spectral radius relation satisfies $\rho(B) = \rho(M)$, where $\rho(B)$ and $\rho(M)$ denote the spectral radii of B and M respectively.*

Finally, we give some known results of the change of distance spectral radius caused by graph transformation.

Lemma 2.5. *([4]) Let e be an edge of G such that $G - e$ is connected. Then $\mu(G) < \mu(G - e)$.*

Lemma 2.6. *([21]) Let n, c, s and $n_i (1 \le i \le c)$ be positive integers with $n_1 \ge n_2 \ge \cdots \ge n_c \ge 1$ and $n_1 + n_2 + \cdots + n_c = n - s$. Then*

$$\mu(K_s \vee (K_{n_1} + K_{n_2} + \cdots + K_{n_c})) \ge \mu(K_s \vee (K_{n-s-(c-1)} + (c-1)K_1)),$$

with equality if and only if $(n_1, n_2, \cdots, n_c) = (n - s - (c-1), 1, \cdots, 1)$.

Lemma 2.7. *([12]) Let n, c, s, p and $n_i (1 \le i \le c)$ be positive integers with $n_1 \ge 2p$, $n_1 \ge n_2 \ge \cdots \ge n_c \ge p$ and $n_1 + n_2 + \cdots + n_c = n - s$. Then*

$$\mu(K_s \vee (K_{n_1} + K_{n_2} + \cdots + K_{n_c})) \ge \mu(K_s \vee (K_{n-s-p(c-1)} + (c-1)K_p)),$$

with equality if and only if $(n_1, n_2, \cdots, n_c) = (n - s - p(c-1), p, \cdots, p)$.

3. Matching number and distance spectral radius of t-connected graphs

First of all, we give a lemma which was proposed by Zhang [18].

Lemma 3.1. *([18]) Let G be a connected graph on n vertices with connectivity $t(G)$ and matching number $\alpha(G) < \lfloor \frac{n}{2} \rfloor$ (implying that $n - 2\alpha(G) \ge 2$). Then $t(G) \le \alpha(G)$.*

Let G be a t-connected graph on n vertices with $\alpha(G) \le \frac{n-k}{2}$, where $2 \le k \le n-2$ is an integer, and let $S \subseteq V(G)$ be a vertex subset such that $\alpha(G) = \frac{1}{2}(n - (o(G-S) - |S|))$. Based on Lemma 3.1, we have $t \le |S| \le \alpha(G) \le \frac{n-k}{2}$. It is natural to consider the following question.

Question 3.2: Can we find a condition in terms of distance spectral radius that makes the matching number of an n-vertex t-connected graph G more than $\frac{n-k}{2}$, where $2 \le k \le n-2$ is an integer? In addition, can we characterize the corresponding spectral extremal graphs?

Based on the question, we give the following theorem.

Theorem 3.2. *Let n, t and k be three positive integers, where $2 \le k \le n-2$, $1 \le t \le \frac{n-k}{2}$ and $n \equiv k (mod\ 2)$. Let G be a t-connected graph of order $n \ge 9k + 10t - 11$, and let $\alpha(G)$ be the matching number of G. If $\mu(G) \le \mu(K_t \vee (K_{n+1-2t-k} + (t+k-1)K_1))$, then $\alpha(G) > \frac{n-k}{2}$ unless $G \cong K_t \vee (K_{n+1-2t-k} + (t+k-1)K_1)$. (see Figure 1)*

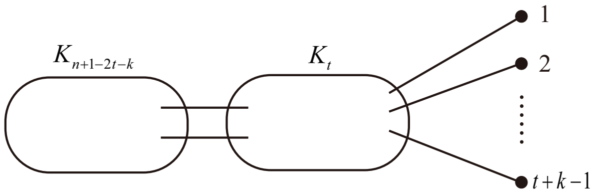

Figure 1: The extremal graph of Theorem 3.2.

Before we prove Theorem 3.2, we will prove the following lemmas.

Lemma 3.3. *Let n, t and k be three positive integers, where $2 \le k \le n-2$, $1 \le t \le \frac{n-k}{2}$ and $n \equiv k(mod\ 2)$. Let G be a t-connected graph of order n with matching number $\alpha(G)$. If $\alpha(G) \le \frac{n-k}{2}$, then $\mu(G) \ge \mu(K_s \vee (K_{n+1-2s-k} + (s+k-1)K_1))$, where $t \le s \le \frac{n-k}{2}$. Equality holds if and only if $G \cong K_s \vee (K_{n+1-2s-k} + (s+k-1)K_1)$.*

Proof. Suppose that the distance spectral radius of G is as small as possible among all t-connected graph on n vertices with matching number $\alpha \le \frac{n-k}{2}$. By Lemma 2.1, there exists a vertex subset $S \subseteq V(G)$ such that $\alpha(G) = \frac{1}{2}(n - (o(G-S) - |S|))$. Then, by Lemma 2.5, we can claim that all components of $G - S$ are odd components. Otherwise, we can randomly remove one vertex from each even component of $G - S$ to the set S until all components of $G - S$ are odd components. In this process, it can be checked that the number of vertices in set S and the number of odd components $G - S$ have the same increase. Therefore, the equality $o(G-S) - |S| = n - 2\alpha(G)$ always holds.

Let $s = |S|$ and $q = o(G-S)$. Since $o(G-S) - |S| = n - 2\alpha(G) \ge k$, we have $q \ge s + k$. Then we will prove the following claims.

Claim 1. Let $G_1 = K_s \vee (K_{n_1} + K_{n_2} + \cdots + K_{n_q})$, where $n_1 \ge n_2 \ge \cdots \ge n_q$ are positive odd integers. Then $\alpha(G_1) \le \frac{n-k}{2}$ and $\mu(G) \ge \mu(G_1)$ with the equality holds if and only if $G \cong G_1$.

Proof. Obviously, G is a spanning subgraph of G_1. By Lemma 2.5, $\mu(G) \ge \mu(G_1)$, where equality holds if and only if $G \cong G_1$. Note that $o(G_1 - S) = o(G-S) \ge s + k$ and $n - 2\alpha(G_1) = \max\{o(G_1 - K) - |K| : \text{for all } K \subseteq V(G_1)\} \ge o(G_1 - S) - |S| \ge k$, we get $\alpha(G_1) \le \frac{n-k}{2}$.

Claim 2. Let $G_2 = K_s \vee (K_{n'_1} + K_{n'_2} + \cdots + K_{n'_{s+k}})$, where $n'_1 = n_1 + \sum_{i=s+k+1}^{q} n_i$ and $n'_i = n_i$ for $i = 2, \cdots, s+k$. Then $\alpha(G_2) \le \frac{n-k}{2}$ and $\mu(G_1) \ge \mu(G_2)$ with the equality holds if and only if $G_1 \cong G_2$.

Proof. For $i = 1, 2, \cdots, q$, Since n_i is odd, we can take $n_i = 2k_i + 1$, where $k_i \ge 0$ and k_i is integer. Since $s + \sum_{i=1}^{q} n_i = s + q + \sum_{i=1}^{q}(2k_i) = n$, we have $q + s \equiv n \equiv k(mod\ 2)$. Thus $q - s - k = q + s - k - 2s$ is even and $n'_1 = n_1 + \sum_{i=s+k+1}^{q} n_i$ is odd. Obviously, $o(G_2 - S) = o(G_1 - S) - (q - s - k) = s + k$ and $n - 2\alpha(G_2) \ge o(G_2 - S) - |S|$. Hence $\alpha(G_2) \le \frac{n-k}{2}$. Since G_1 is a spanning subgraph of G_2, by Lemma 2.5, $\mu(G_1) \ge \mu(G_2)$, where the equality holds if and only if $G_1 \cong G_2$.

Claim 3. Let $G_3 = K_s \vee (K_{n+1-2s-k} + (s+k-1)K_1)$. Then $\alpha(G_3) \le \frac{n-k}{2}$ and $\mu(G_2) \ge \mu(G_3)$ with the equality holds if and only if $G_2 \cong G_3$.

Proof. Obviously, $o(G_3 - S) = o(G_2 - S) = s + k$ and $n - 2\alpha(G_3) \ge o(G_3 - S) - |S|$. Therefore $\alpha(G_3) \le \frac{n-k}{2}$. Moreover, by Lemma 2.6, $\mu(G_2) \ge \mu(G_3)$, where the equality holds if and only if $G_2 \cong G_3$.

Based on the above results, we can conclude that if G is a t-connected graph of order n with $\alpha(G) \le \frac{n-k}{2}$, then $\mu(G) \ge \mu(G_3) = \mu(K_s \vee (K_{n+1-2s-k} + (s+k-1)K_1))$ with the equality holds if and only if $G \cong K_s \vee (K_{n+1-2s-k} + (s+k-1)K_1)$. This completes the proof. □

Lemma 3.4. *Let $n \ge 9k + 10t - 11$, t and k be three positive integers, where $2 \le k \le n-2$, $1 \le t \le \frac{n-k}{2}$ and $n \equiv k(mod\ 2)$. Then $\mu(K_s \vee (K_{n+1-2s-k} + (s+k-1)K_1)) \ge \mu(K_t \vee (K_{n+1-2t-k} + (t+k-1)K_1))$, where $t \le s \le \frac{n-k}{2}$. Equality holds if and only if $K_s \vee (K_{n+1-2s-k} + (s+k-1)K_1) \cong K_t \vee (K_{n+1-2t-k} + (t+k-1)K_1)$.*

Proof. For convenience, let $G_s = K_s \vee (K_{n+1-2s-k} + (s+k-1)K_1)$ and $G_t = K_t \vee (K_{n+1-2t-k} + (t+k-1)K_1)$. Since $t \le s \le \frac{n-k}{2}$, then we will discuss the proof in two ways according to the value of s.

Case 1. $s = t$.

Then $G_s \cong G_t$. Clearly, the result holds.

Case 2. $t + 1 \le s \le \frac{n-k}{2}$.

We divide $V(G_s)$ into three parts: $V(K_s)$, $V(K_{n+1-2s-k})$ and $V((s + k - 1)K_1)$. Then the distance matrix of G_s, denoted by $D(G_s)$, is

$$\begin{bmatrix} (J-I)_{s\times s} & J_{s\times(n+1-2s-k)} & J_{s\times(s+k-1)} \\ J_{(n+1-2s-k)\times s} & (J-I)_{(n+1-2s-k)\times(n+1-2s-k)} & 2J_{(n+1-2s-k)\times(s+k-1)} \\ J_{(s+k-1)\times s} & 2J_{(s+k-1)\times(n+1-2s-k)} & 2(J-I)_{(s+k-1)\times(s+k-1)} \end{bmatrix},$$

where $J_{i\times j}$ denotes the $i \times j$ all-one matrix and $I_{i\times i}$ denotes the $i \times i$ identity square matrix. Then the equitable quotient matrix of the distance matrix $D(G_s)$, denoted by M_s, with respect to the partition $V(K_s) \cup V(K_{n+1-2s-k}) \cup V((s + k - 1)K_1)$ is

$$M_s = \begin{bmatrix} s-1 & n+1-2s-k & s+k-1 \\ s & n-2s-k & 2(s+k-1) \\ s & 2(n+1-2s-k) & 2(s+k-2) \end{bmatrix}.$$

Through a simple calculation, the characteristic polynomial of M_s is

$$\begin{aligned} f_s(x) = {} & x^3 + (-s-n-k+5)x^2 + (5s^2 + (-2n+7k-8)s - 2kn - n + 2k^2 - 5k + 8)x \\ & - 2s^3 + (n-3k+8)s^2 + (kn - 3n - k^2 + 9k - 8)s - 2kn + 2k^2 - 4k + 4. \end{aligned}$$

We use $y_1(M_s)$ to denote the largest root of the equation $f_s(x) = 0$. By Lemma 2.4, $\mu(G_s) = y_1(M_s)$. What's more, we can get the equitable quotient matrix M_t of $G_t = K_t \vee (K_{n+1-2t-k} + (t + k - 1)K_1)$ by replacing s with t. Similarly, we can get the characteristic polynomial $f_t(x)$ of M_t and $\mu(G_t) = y_1(M_t)$ is the largest root of the equation $f_t(x) = 0$. By direct calculation, we have

$$\begin{aligned} f_s(x) - f_t(x) = {} & (t-s)[x^2 + (2n + 8 - 5(t+s) - 7k)x + 2s^2 + (2t - n + 3k - 8)s \\ & + 2t^2 + (-n + 3k - 8)t + 3n - kn + k^2 - 9k + 8]. \end{aligned}$$

Obviously, G_s and G_t are both spanning subgraphs of K_n, by Lemma 2.5, $\mu(G_s) > \mu(K_n) = n - 1$ and $\mu(G_t) > \mu(K_n) = n - 1$. Then we will give the proof that $f_s(x) - f_t(x) < 0$ for $x \in [n - 1, +\infty)$. Sine, $t < s$, thus we only need to prove that $p(x) > 0$ for $x \in [n - 1, +\infty)$, where

$$p(x) = x^2 + (2n + 8 - 5(t+s) - 7k)x + 2s^2(2t - n + 3k - 8)s + 2t^2 + (-n + 3k - 8)t + 3n - kn + k^2 - 9k + 8.$$

Since the symmetry axis of $p(x)$ is

$$\begin{aligned} \hat{x} &= \frac{5(t+s) + 7k - 2n - 8}{2} \\ &= \frac{5}{2}s + \frac{5}{2}t + \frac{7}{2}k - n - 4 \\ &\le \frac{5}{4}(n-k) + \frac{5}{2}t + \frac{7}{2}k - n - 4 \\ &= \frac{1}{4}n + \frac{5}{2}t + \frac{9}{4}k - 4, \end{aligned}$$

note that $n \ge 9k + 10t - 11 > 3k + \frac{10}{3}t - 4$ and $n > 3k + \frac{10}{3}t - 4 \iff \frac{1}{4}n + \frac{5}{2}t + \frac{9}{4}k - 4 < n - 1$, we get $\frac{5(t+s)+7k-2n-8}{2} < n - 1$. Thus, $p(x)$ is increasing with respect to $x \in [n - 1, +\infty)$, and

$$p(x) \ge p(n-1) = 2s^2 + (3k + 2t - 6n - 3)s + 3n^2 + 2t^2 + (3k - 6n - 3)t + 7n - 8kn + k^2 - 2k + 1.$$

Let

$$v(s) \triangleq p(n-1) = 2s^2 + (3k + 2t - 6n - 3)s + 3n^2 + 2t^2 + (3k - 6n - 3)t + 7n - 8kn + k^2 - 2k + 1.$$

Recall that $t+1 \le s \le \frac{n-k}{2}$ and $n \ge 9k+10t-11$, then

$$\begin{aligned}\frac{dv}{ds} &= 4s+3k+2t-6n-3\\ &\le 2n-2k+3k+2t-6n-3\\ &= -4n+k+2t-3<0.\end{aligned}$$

Thus, $v(s)$ is decreasing with respect to $s \in [t+1, \frac{n-k}{2}]$. Furthermore,

$$\begin{aligned}v(s) \ge v(\frac{n-k}{2}) &= \frac{1}{2}[n^2-(10t+9k-11)n+4t^2+(4k-6)t-k+2]\\ &> \frac{1}{2}[n^2-(10t+9k-11)n].\end{aligned}$$

Note that $n \ge 10t+9k-11$, we get $v(s)>0$. Therefore, $p(x) \ge p(n-1) = v(s) > 0$, which implies $f_s(x) < f_t(x)$ for $x \in [n-1, +\infty)$. In addition, by $\min\{\mu(G_s), \mu(G_t)\} > n-1$, we get that $\mu(G_s) > \mu(G_t)$. This completes the proof. $\square$

By Lemma 3.3 and Lemma 3.4, Theorem 3.2 clearly holds.

Let $k=2$ in Theorem 3.2, we can get a condition in terms of distance spectral radius in a graph G with order n and connectivity t such that $\alpha(G) > \frac{n}{2}-1$. In addition, the matching number of G is no more then $\frac{n}{2}$. Hence, we have $\alpha(G) = \frac{n}{2}$. Then we can obtain the following corollary about the perfect matching based on the distance spectral radius.

Corollary 3.5. *Let n be an even integer. Suppose that t is a positive integer, where $1 \le t \le \frac{n-2}{2}$. Let G be a graph of order $n \ge 10t+7$ with connectivity t. If $\mu(G) \le \mu(K_t \vee (K_{n-2t-1}+(t+1)K_1))$, then G contains a perfect matching unless $G \cong K_t \vee (K_{n-2t-1}+(t+1)K_1)$.*

4. Odd $[1,b]$-factor and distance spectral radius of graph with given minimum degree

Amahashi [1] gave a sufficient and necessary condition for a graph contains an odd $[1,b]$-factor. Therefore, it is natural to consider the following question.

Question 4.1: Can we obtain a distance spectral radius condition that makes a graph G with given minimum degree δ having an odd $[1,b]$-factor? In addition, can we characterize the corresponding spectral extremal graphs?

Based on the question, we give the following theorem.

Theorem 4.1. *Let G be a connected graph of even order $n \ge \max\{2b\delta^2, (\frac{3}{b}+5+2b)\delta+1+\frac{3(b+1)}{b^2}\}$ with minimum degree $\delta \ge 3$, where b is a positive odd integer. If $\mu(G) \le \mu(K_\delta \vee (K_{n-(b+1)\delta-1}+(b\delta+1)K_1))$, then G has an odd $[1,b]$-factor unless $G \cong K_\delta \vee (K_{n-(b+1)\delta-1}+(b\delta+1)K_1)$.(see Figure 2)*

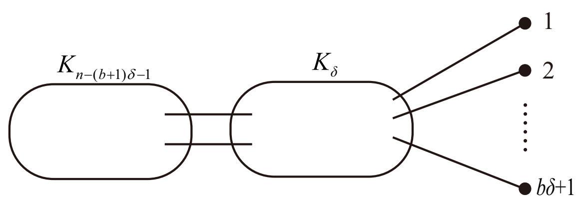

Figure 2: The extremal graph of Theorem 4.1.

Proof. Let G be a connected graph of even order $n \geq \max\{2b\delta^2, (\frac{3}{b}+5+2b)\delta+1+\frac{3(b+1)}{b^2}\}$, where δ is the minimum degree of G and $\delta \geq 3$, b is a positive odd integer. Suppose to the contrary that G has no odd $[1,b]$-factor. Then by Lemma 2.2, there exists a vertex subset $S \subseteq V(G)$ such that $o(G-S) > b|S|$. Let $|S| = s$ and $o(G-S) = q$. Since n is even, it is easy to see that s and q have the same parity. Since b is odd, we have that bs and q have the same parity. Thus $q = o(G-S) \geq bs+2$. Let $t = bs+2$. Since $s+t \leq s+q \leq n$, we have $s \leq \frac{n-2}{b+1}$. Obviously, G is a spanning subgraph of $K_s \vee (K_{n_1}+K_{n_2}+\cdots+K_{n_t})$ for some positive odd integer $n_1 \geq n_2 \geq \cdots \geq n_t$ with $\sum_{i=1}^{t} n_i + s = n$. Let $G_1 = K_s \vee (K_{n_1}+K_{n_2}+\cdots+K_{n_t})$. By Lemma 2.5,

$$\mu(G) \geq \mu(G_1), \tag{1}$$

where equality holds if and only if $G \cong G_1$. Let $G_\delta = K_\delta \vee (K_{n-(b+1)\delta-1}+(b\delta+1)K_1)$. Let S_1 be the vertex set of K_s. Since there exists a vertex subset $S_1 \subseteq V(G_1)$ such that $o(G_1-S_1) = t = bs+2 > bs = b|S_1|$, we have that G_1 contains no odd $[1,b]$-factor. Recall that $s \leq \frac{n-2}{b+1}$, then we will discuss the proof in three ways according to the value of s.

Case 1. $s = \delta$.

In this case, we have $G_1 = K_s \vee (K_{n_1}+K_{n_2}+\cdots+K_{n_t}) = K_\delta \vee (K_{n_1}+K_{n_2}+\cdots+K_{n_{b\delta+2}})$.

By Lemma 2.6,

$$\mu(G_\delta) = \mu(K_\delta \vee (K_{n-(b+1)\delta-1}+(b\delta+1)K_1)) \leq \mu(G_1), \tag{2}$$

with equality if and only if $G_\delta \cong G_1$.

Furthermore, combining with (1) and (2),

$$\mu(G) \geq \mu(G_\delta) = \mu(K_\delta \vee (K_{n-(b+1)\delta-1}+(b\delta+1)K_1)),$$

with equality if and only if $G_\delta \cong G$.

Moreover, according to the assumed condition $\mu(G) \leq \mu(K_\delta \vee (K_{n-(b+1)\delta-1}+(b\delta+1)K_1))$, we have $\mu(G) = \mu(K_\delta \vee (K_{n-(b+1)\delta-1}+(b\delta+1)K_1))$.

Based on the above results, we conclude that $G \cong K_\delta \vee (K_{n-(b+1)\delta-1}+(b\delta+1)K_1))$. In addition, take $S = V(K_\delta)$, then $o(G-S) = b\delta+2 = bs+2 > bs = b|S|$, which implies G does not have an odd $[1,b]$-factor.

Case 2. $\delta+1 \leq s \leq \frac{n-2}{b+1}$.

Let $G_s = K_s \vee (K_{n-(b+1)s-1}+(bs+1)K_1)$. By Lemma 2.6,

$$\mu(G_s) = \mu(K_s \vee (K_{n-(b+1)s-1}+(bs+1)K_1)) \leq \mu(G_1), \tag{3}$$

with equality if and only if $G_s \cong G_1$. For the graph G_s, let $D(G_s)$ denote the distance matrix of G_s. Then the equitable quotient matrix of $D(G_s)$, denoted by M_s, with respect to the partition $V(K_s) \cup V(K_{n-(b+1)s-1}) \cup V((bs+1)K_1)$ is

$$M_s = \begin{bmatrix} s-1 & n-(b+1)s-1 & bs+1 \\ s & n-(b+1)s-2 & 2(bs+1) \\ s & 2(n-(b+1)s-1) & 2bs \end{bmatrix},$$

and the characteristic polynomial of M_s is

$$\begin{aligned} f_s(x) = {} & x^3 + (-bs-n+3)x^2 + (2b^2s^2+3bs^2-2bns+3bs+3s-5n+6)x - (b^2+b)s^3 \\ & + (bn+2b^2+b-1)s^2 + (n-2bn+4b+2)s + 4 - 4n. \end{aligned}$$

We use $y_1(M_s)$ to denote the largest root of the equation $f_s(x) = 0$. By Lemma 2.4, we can get $\mu(G_s) = y_1(M_s)$. Moreover, we obtain the equitable quotient matrix M_δ of $G_\delta = K_\delta \vee (K_{n-(b+1)\delta-1}+(b\delta+1)K_1)$ by replacing s with δ. Similarly, we can get the characteristic polynomial $f_\delta(x)$ of M_δ and $\mu(G_\delta) = y_1(M_\delta)$ is the largest root of the equation $f_\delta(x) = 0$. By a calculation, we have

$$\begin{aligned} f_s(x) - f_\delta(x) = {} & (\delta-s)[bx^2 + (-2b^2s-3bs+2bn-2b^2\delta-3b\delta-3b-3)x \\ & + (b^2s+bs-bn+b^2\delta+b\delta-2b^2-b+1)s \\ & + (-bn+b^2\delta+b\delta-2b^2-b+1)\delta - n + 2bn - 4b - 2]. \end{aligned}$$

Since G_s and G_δ are spanning subgraphs of K_n, by lemma 2.5, $\mu(G_s) > \mu(K_n) = n-1$ and $\mu(G_\delta) > \mu(K_n) = n-1$. Then we will prove that $f_s(x) - f_\delta(x) < 0$ for $x \in [n-1, +\infty)$. Since $\delta < s$, we only need to prove $c(x) > 0$, where

$$\begin{aligned} c(x) &= bx^2 + (-2b^2 s - 3bs + 2bn - 2b^2\delta - 3b\delta - 3b - 3)x \\ &\quad + (b^2 s + bs - bn + b^2\delta + b\delta - 2b^2 - b + 1)s \\ &\quad + (-bn + b^2\delta + b\delta - 2b^2 - b + 1)\delta - n + 2bn - 4b - 2. \end{aligned}$$

By direct calculation, the symmetry axis of $c(x)$ is

$$\begin{aligned} \hat{x} &= -\frac{(-2b^2 s - 3bs + 2bn - 2b^2\delta - 3b\delta - 3b - 3)}{2b} \\ &= \frac{3}{2b} + \frac{3}{2} + (\frac{3}{2} + b)\delta + \frac{3}{2}s + bs - n \\ &< 3 + (\frac{3}{2} + b)\delta + (\frac{3}{2} + b)s - n \\ &\leq 3 + (\frac{3}{2} + b)\delta + (\frac{3}{2} + b)\frac{n-2}{b+1} - n \\ &= 1 + (\frac{3}{2} + b)\delta + \frac{1}{2(b+1)}n - \frac{1}{b+1}. \end{aligned}$$

Since $\delta \geq 3$ and $n \geq 2b\delta^2$, we have $n \geq 2b\delta^2 > 2 + \frac{2}{3}(3+2b)\delta > 2 + \frac{b+1}{2b+1}(3+2b)\delta$. Note that $n > 2 + \frac{b+1}{2b+1}(3+2b)\delta \iff 1 + (\frac{3}{2}+b)\delta + \frac{1}{2(b+1)}n - \frac{1}{b+1} < n-1$, we get $-\frac{(-2b^2s-3bs+2bn-2b^2\delta-3b\delta-3b-3)}{2b} < n-1$, which implies $c(x)$ is increasing with respect to $x \in [n-1, +\infty)$. Hence,

$$\begin{aligned} c(x) \geq c(n-1) &= (b^2+b)s^2 + (1 + 2b + b\delta + b^2\delta - 4bn - 2b^2 n)s + 3bn^2 \\ &\quad + b(b+1)\delta^2 + (1 + 2b - 4bn - 2b^2 n)\delta - 4n - 5bn + 1. \end{aligned}$$

Let

$$\begin{aligned} h(s) \triangleq c(n-1) &= (b^2+b)s^2 + (1 + 2b + b\delta + b^2\delta - 4bn - 2b^2 n)s + 3bn^2 \\ &\quad + b(b+1)\delta^2 + (1 + 2b - 4bn - 2b^2 n)\delta - 4n - 5bn + 1. \end{aligned}$$

Recall that $\delta < s \leq \frac{n-2}{b+1}$ and $n \geq 2b\delta^2$, we obtain

$$\begin{aligned} \frac{dh}{ds} &= 2b(b+1)s + 1 + 2b + b\delta + b^2\delta - 4bn - 2b^2 n \\ &\leq 2b(n-2) + 1 + 2b + b\delta + b^2\delta - 4bn - 2b^2 n \\ &= -2bn + 1 - 2b + b\delta + b^2\delta - 2b^2 n < 0. \end{aligned}$$

Thus, $h(s)$ is decreasing with respect to $s \in [\delta+1, \frac{n-2}{b+1}]$. By direct calculation,

$$\begin{aligned} h(s) \geq h(\frac{n-2}{b+1}) &= \frac{1}{b+1}[b^2 n^2 - (3 + 3b + b^2 + (3b + 5b^2 + 2b^3)\delta)n \\ &\quad + b^3\delta^2 + 2\delta^2 b^2 + b\delta^2 + b^2\delta + (b+1)\delta + b - 1] \\ &> \frac{1}{b+1}[b^2 n^2 - (3 + 3b + b^2 + (3b + 5b^2 + 2b^3)\delta)n] \\ &= \frac{b^2}{b+1}[n^2 - ((\frac{3}{b} + 5 + 2b)\delta + 1 + \frac{3(b+1)}{b^2})n]. \end{aligned}$$

Note that $n \geq \max\{2b\delta^2, (\frac{3}{b}+5+2b)\delta+1+\frac{3(b+1)}{b^2}\} \geq (\frac{3}{b}+5+2b)\delta+1+\frac{3(b+1)}{b^2}$. Hence $c(x) \geq c(n-1) = h(s) > 0$, which implies $f_s(x) < f_\delta(x)$ for $x \in [n-1, +\infty)$. Recall that $\min\{\mu(G_s), \mu(G_\delta)\} > n-1$, we have $\mu(G_s) > \mu(G_\delta)$. Furthermore, combining with (1) and (3), we get

$$\mu(G) \geq \mu(G_1) \geq \mu(G_s) > \mu(G_\delta) = \mu(K_\delta \vee (K_{n-(b+1)\delta-1} + (b\delta+1)K_1),$$

a contradiction.

Case 3. $1 \leq s < \delta$.

Since G is a spanning subgraph of $G_1 = K_s \vee (K_{n_1} + K_{n_2} + \cdots + K_{n_t})$, where $n_1 \geq n_2 \geq \cdots \geq n_t$ is odd integer, $t = bs+2$ and $n_1 + n_2 + \cdots + n_t = n - s$. It is easy to see that $\delta(G_1) \geq \delta(G) = \delta$, we have $n_t - 1 + s \geq \delta$. Thus, $n_1 \geq n_2 \geq \cdots \geq n_t \geq \delta - s + 1$. Then we will proof that $n_1 \geq 2(\delta - s + 1)$. If $n_1 < 2(\delta - s + 1)$, then $n_1 \leq 2\delta - 2s + 1$. Since $n_1 \geq n_2 \geq \cdots \geq n_t$ and $1 \leq s < \delta$ and $\delta \geq 3$, we obtain

$$\begin{aligned}
n &= s + n_1 + n_2 + \cdots + n_t \\
&\leq s + (bs+2)(2\delta - 2s + 1) \\
&= -2bs^2 + (-3 + b + 2b\delta)s + 4\delta + 2 \\
&\leq -2b(\frac{\delta}{2} + \frac{1}{4} - \frac{3}{4b})^2 + (-3 + b + 2b\delta)(\frac{\delta}{2} + \frac{1}{4} - \frac{3}{4b}) + 4\delta + 2 \\
&= \frac{b\delta^2}{2} + \frac{b+5}{2}\delta + \frac{b}{8} + \frac{9}{8b} + \frac{5}{4} \\
&< \frac{b\delta^2}{2} + \frac{b+5}{2}\delta + \frac{b}{8} + 3.
\end{aligned}$$

Let

$$l(b) = 2b\delta^2 - (\frac{b\delta^2}{2} + \frac{b+5}{2}\delta + \frac{b}{8} + 3) = \frac{3}{2}b\delta^2 - (\frac{b+5}{2}\delta + \frac{b}{8} + 3).$$

Note that $l'(b) = \frac{3}{2}\delta^2 - \frac{1}{2}\delta - \frac{1}{8} > 0$, we have $l(b) \geq l(1) = \frac{3}{2}\delta^2 - 3\delta - \frac{25}{8} > 0$. Thus $n < 2b\delta^2$. This is a contradiction with $n \geq 2b\delta^2$. Hence, $n_1 \geq 2(\delta - s + 1)$. Let $G_s = K_s \vee (K_{n-s-(\delta+1-s)(bs+1)} + (bs+1)K_{\delta+1-s})$. By Lemma 2.7,

$$\mu(G_1) \geq \mu(G_s), \tag{4}$$

where equality holds if and only if $G_1 \cong G_s$. In what follows, we will discuss three subcases by classifying the value of s.

Case 3.1. $s = 1$.

In this case, $G_s = K_1 \vee (K_{n-1-\delta(b+1)} + (b+1)K_\delta)$, and the equitable quotient matrix of its distance matrix is

$$M_1 = \begin{bmatrix} 0 & n-(b+1)\delta-1 & (b+1)\delta \\ 1 & n-(b+1)\delta-2 & 2(b+1)\delta \\ 1 & 2(n-(b+1)\delta-1) & 2b\delta+\delta-1 \end{bmatrix}.$$

By a simple calculation, the characteristic polynomial of M_1 is

$$\begin{aligned}
f_1(x) = {} & x^3 + (3 - b\delta - n)x^2 + (3 + 3\delta + b\delta + 3\delta^2 + 5b\delta^2 + 2b^2\delta^2 - 2n - 3\delta n - 2b\delta n)x \\
& + (b^2 + 3b + 2)\delta^2 + (-bn - 2n + b + 2)\delta - n + 1.
\end{aligned}$$

Recall that in Case 2, by replacing s with δ, we can get the equitable quotient matrix M_δ of $G_\delta = K_\delta \vee (K_{n-(b+1)\delta-1} + (b\delta+1)K_1)$. Thus, the characteristic polynomial of M_δ is

$$\begin{aligned}
f_\delta(x) = {} & x^3 + (-b\delta - n + 3)x^2 + (2b^2\delta^2 + 3b\delta^2 - 2bn\delta + 3b\delta + 3\delta - 5n + 6)x - (b^2 + b)\delta^3 \\
& + (bn + 2b^2 + b - 1)\delta^2 + (n - 2bn + 4b + 2)\delta + 4 - 4n.
\end{aligned}$$

Since $\delta \geq 3$ and $n \geq 2b\delta^2$, for $x \in [n-1, +\infty)$, we have

$$\begin{aligned} f_\delta(x) - f_1(x) &= [3n(\delta-1) - 2b\delta^2 - 3\delta^2 + 2b\delta + 3]x \\ &\quad - (b^2+b)\delta^3 + (bn + b^2 - 2b - 3)\delta^2 + (-bn + 3n + 3b)\delta - 3n + 3 \\ &\geq [3n(\delta-1) - 2b\delta^2 - 3\delta^2 + 2b\delta + 3](n-1) \\ &\quad - (b^2+b)\delta^3 + (bn + b^2 - 2b - 3)\delta^2 + (-bn + 3n + 3b)\delta - 3n + 3 \\ &= (\delta-1)[3n^2 - (b\delta + 3\delta + 3)n - b^2\delta^2 - b\delta^2 - b\delta] \\ &\triangleq (\delta-1)m(n). \end{aligned}$$

Observe that G_δ and G_s are spanning subgraphs of K_n, by lemma 2.5, $\mu(G_\delta) > \mu(K_n) = n-1$ and $\mu(G_s) > \mu(K_n) = n-1$. Then we will prove that $f_\delta(x) - f_1(x) > 0$ for $x \in [n-1, +\infty)$. Since $\delta \geq 3$, we only need to prove $m(n) > 0$.

The symmetry axis of $m(n)$ is

$$\hat{n} = \frac{b\delta + 3\delta + 3}{6} < 2b\delta^2.$$

Thus, $m(n)$ is increasing with respect to $n \in [2b\delta^2, +\infty)$. By a simple calculation, we have

$$m(n) \geq m(2b\delta^2) = b\delta[12b\delta^3 - (2b+6)\delta^2 - (7+b)\delta - 1] \triangleq b\delta h(b).$$

It is easy to see that $h(b)$ is increasing with respect to $b \in [1, +\infty)$. Thus $h(b) \geq h(1) = \delta(12\delta^2 - 8\delta - 8) - 1 > 0$ and $m(n) \geq m(2b\delta^2) = b\delta h(b) > 0$, which implies $f_\delta(x) > f_1(x)$ for $x \in [n-1, +\infty)$. Note that $\min\{\mu(G_\delta), \mu(G_s)\} > n-1$, we have $\mu(G_s) > \mu(G_\delta)$. Furthermore, by (1) and (4),

$$\mu(G) \geq \mu(G_1) \geq \mu(G_s) > \mu(G_\delta) = \mu(K_\delta \vee (K_{n-(b+1)\delta-1} + (b\delta+1)K_1),$$

a contradiction.

Case 3.2. $2 \leq s \leq \delta - 1$.

Note that $G_s = K_s \vee (K_{n-s-(\delta+1-s)(bs+1)} + (bs+1)K_{\delta+1-s})$. The distance matrix $D(G_s)$ of G_s is

$$\begin{array}{c} \\ K_s \\ K_{n-s-(\delta+1-s)(bs+1)} \\ K_{\delta+1-s} \\ \vdots \\ K_{\delta+1-s} \end{array} \begin{array}{c} \begin{array}{ccccc} K_s & K_{n-s-(\delta+1-s)(bs+1)} & K_{\delta+1-s} & \cdots & K_{\delta+1-s} \end{array} \\ \begin{pmatrix} J-I & J & J & \cdots & J \\ J & J-I & 2J & \cdots & 2J \\ J & 2J & J-I & \cdots & 2J \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ J & 2J & 2J & \cdots & J-I \end{pmatrix} \end{array},$$

where J denotes the all-one matrix and I denotes the identity square matrix. Then we use P_s to denote the equitable quotient matrix of the distance matrix $D(G_s)$ for the partition $V(K_s) \cup V(K_{n-s-(\delta+1-s)(bs+1)}) \cup V((bs+1)K_{\delta+1-s})$. Thus

$$P_s = \begin{bmatrix} s-1 & n-s-(\delta+1-s)(bs+1) & (\delta+1-s)(bs+1) \\ s & n-s-(\delta+1-s)(bs+1)-1 & 2(\delta+1-s)(bs+1) \\ s & 2(n-s-(\delta+1-s)(bs+1)) & 2bs(\delta+1-s)+(\delta-s) \end{bmatrix}.$$

and the characteristic polynomial of P_s is

$$\begin{aligned}
f_s(x) = {} & x^3 + (3 - n - bs - b\delta s + bs^2)x^2 \\
& + [2b^2s^4 + (2b - 4b^2 - 4b^2\delta)s^3 + (-5b + 2b^2 - 7b\delta + 4b^2\delta + 2b^2\delta^2 + 2bn)s^2 \\
& + (-3 + 3b - 3\delta + 8b\delta + 5b\delta^2 + 3n - 2bn - 2b\delta n)s - 3\delta n - 5n + 3\delta^2 + 6\delta + 6]x \\
& - b^2s^5 + (-b + 4b^2 + 2b^2\delta)s^4 + (5b - 5b^2 + 3b\delta - 6b^2\delta - b^2\delta^2 - bn)s^3 \\
& + (1 - 8b + 2b^2 + \delta - 11b\delta + 4b^2\delta - 2b\delta^2 + 2b^2\delta^2 - n + 3bn + b\delta n)s^2 \\
& + (-4 + 4b - 5\delta + 9b\delta - \delta^2 + 5b\delta^2 + 4n - 2bn + \delta n - 2b\delta n)s \\
& - 3\delta n - 4n + 3\delta^2 + 6\delta + 4.
\end{aligned}$$

We use $y_1(M_s)$ to denote the largest real root of the equation $f_s(x) = 0$. By Lemma 2.4, we have that $\mu(G_s) = y_1(M_s)$. Recall that in Case 3.1, we can get the equitable quotient matrix M_δ of $G_\delta = K_\delta \vee (K_{n-(b+1)\delta-1} + (b\delta + 1)K_1)$. Thus, the characteristic polynomial of M_δ is

$$\begin{aligned}
f_\delta(x) = {} & x^3 + (-b\delta - n + 3)x^2 + (2b^2\delta^2 + 3b\delta^2 - 2bn\delta + 3b\delta + 3\delta - 5n + 6)x - (b^2 + b)\delta^3 \\
& + (bn + 2b^2 + b - 1)\delta^2 + (n - 2bn + 4b + 2)\delta + 4 - 4n,
\end{aligned}$$

and

$$\begin{aligned}
f_\delta(x) - f_s(x) = {} & (\delta - s)[(-b + bs)x^2 + (-3 + 3b - 3\delta + 3b\delta + 2b^2\delta + 3n - 2bn \\
& + (-5b + 2b^2 - 5b\delta + 2bn)s + (2b - 4b^2 - 2b^2\delta)s^2 + 2b^2s^3)x \\
& - b^2s^4 + (-b + 4b^2 + b^2\delta)s^3 + (5b - 5b^2 + 2b\delta - 2b^2\delta - bn)s^2 \\
& + (1 - 8b + 2b^2 + \delta - 6b\delta - b^2\delta - n + 3bn)s \\
& + (b\delta - 2b + 4)n - (b^2 + b)\delta^2 + (2b^2 + b - 4)\delta + 4b - 4] \\
\triangleq {} & (\delta - s)H(x).
\end{aligned}$$

Since $s \leq \delta - 1$, then we will prove that $H(x) > 0$. In what follows, we will show $H(x) > 0$ in two steps.

Step 1. $H(n - 1) > 0$.

$$\begin{aligned}
H(n - 1) = {} & 3(1 - b + bs)n^2 \\
& + (2b^2s^3 + (b - 4b^2 - 2b^2\delta)s^2 + (-1 - 6b + 2b^2 - 5b\delta)s + (2b^2 + 4b - 3)\delta + 5b - 2)n \\
& - b^2s^4 + (-b + 2b^2 + b^2\delta)s^3 + (3b - b^2 + 2b\delta)s^2 + (1 - 2b + \delta - b\delta - b^2\delta)s \\
& - (b^2 + b)\delta^2 - (2b + 1)\delta - 1.
\end{aligned}$$

Let $q(n) \triangleq H(n - 1)$, then the symmetry axis of $q(n)$ is

$$\hat{n} = -\frac{2b^2s^3 + (b - 4b^2 - 2b^2\delta)s^2 + (-1 - 6b + 2b^2 - 5b\delta)s + (2b^2 + 4b - 3)\delta + 5b - 2}{6(1 - b + bs)}.$$

Note that

$$\begin{aligned}
& -(2b^2s^3 + (b - 4b^2 - 2b^2\delta)s^2 + (-1 - 6b + 2b^2 - 5b\delta)s + (2b^2 + 4b - 3)\delta + 5b - 2) \\
& = 2 + 3\delta - 4b\delta + s + 5b\delta s + b(s - 1)(5 - s) - 2b^2s(s - 1)^2 + 2b^2\delta(s^2 - 1) \\
& = 2 + 3\delta - 4b\delta + s + 5b\delta s + b(s - 1)[2bs(\delta - s) + 2b\delta + 2bs - s + 5] > 0.
\end{aligned}$$

Since $2 \leq s \leq \delta - 1$, we have

$$\begin{aligned}
\hat{n} &= -\frac{2b^2s^3 + (b - 4b^2 - 2b^2\delta)s^2 + (-1 - 6b + 2b^2 - 5b\delta)s + (2b^2 + 4b - 3)\delta + 5b - 2}{6(1 - b + bs)} \\
&< \frac{2 + 3\delta - 4b\delta + s + 5b\delta s + b(s-1)[2bs(\delta - s) + 2b\delta + 2bs - s + 5]}{6b(s-1)} \\
&= \frac{2 + 3\delta - 4b\delta + s + 5b\delta s}{6b(s-1)} + \frac{5 - s + 2bs(\delta - s) + 2b\delta + 2bs}{6} \\
&< \frac{2 + 5b\delta s}{6b} + \frac{3 + 2bs\delta + 2b\delta + 2bs}{6} \\
&< \frac{2 + 5b\delta^2}{6b} + \frac{3 + 2b\delta^2 + 2b\delta + 2b\delta}{6} \\
&< 1 + \frac{2}{3}b\delta + (\frac{5}{6} + \frac{b}{3})\delta^2.
\end{aligned}$$

Note that $\delta \geq 3$, it can be checked that $1 + \frac{2}{3}b\delta + (\frac{5}{6} + \frac{b}{3})\delta^2 < 2b\delta^2$, which implies that $q(n)$ is increasing with respect to $n \in [2b\delta^2, +\infty)$. By a simple calculation,

$$\begin{aligned}
q(n) &\geq q(2b\delta^2) \\
&= 12b^2(1 + b(s-1))\delta^4 + (-6b + 8b^2 + 4b^3 - 10b^2s - 4b^3s^2)\delta^3 \\
&\quad + (-5b + 9b^2 + (-2b - 12b^2 + 4b^3)s + (2b^2 - 8b^3)s^2 + 4b^3s^3)\delta^2 \\
&\quad + (-1 - 2b + (1 - b - b^2)s + 2bs^2 + b^2s^3)\delta \\
&\quad - 1 + (1 - 2b)s + (3b - b^2)s^2 + (-b + 2b^2)s^3 - b^2s^4.
\end{aligned}$$

Next, we will prove that $q(2b\delta^2) > 0$ progressively scaling. Since $\delta \geq s + 1$ and $s \geq 2$, we have

$$\begin{aligned}
&12b^2(1 + b(s-1))\delta^4 + (-6b + 8b^2 + 4b^3 - 10b^2s - 4b^3s^2)\delta^3 \\
&= \delta^3[12b^2(1 + b(s-1))\delta + (-6b + 8b^2 + 4b^3 - 10b^2s - 4b^3s^2)] \\
&\geq \delta^3[12b^2(1 + b(s-1))(s+1) + (-6b + 8b^2 + 4b^3 - 10b^2s - 4b^3s^2)] \\
&= \delta^3(-6b + 20b^2 - 8b^3 + 2b^2s + 8b^3s^2) > 0.
\end{aligned}$$

Then

$$\begin{aligned}
&\delta^2[(-6b + 20b^2 - 8b^3 + 2b^2s + 8b^3s^2)\delta + (-5b + 9b^2 + (-2b - 12b^2 + 4b^3)s \\
&\quad + (2b^2 - 8b^3)s^2 + 4b^3s^3)] \\
&\geq \delta^2[(-6b + 20b^2 - 8b^3 + 2b^2s + 8b^3s^2)(s+1) \\
&\quad + (-5b + 9b^2 + (-2b - 12b^2 + 4b^3)s + (2b^2 - 8b^3)s^2 + 4b^3s^3)] \\
&= \delta^2(-11b + 29b^2 - 8b^3 + (-8b + 10b^2 - 4b^3)s + 4b^2s^2 + 12b^3s^3).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\delta[(-11b + 29b^2 - 8b^3 + (-8b + 10b^2 - 4b^3)s + 4b^2s^2 + 12b^3s^3)\delta \\
&\quad + (-1 - 2b + (1 - b - b^2)s + 2bs^2 + b^2s^3)] \\
&\geq \delta[(-11b + 29b^2 - 8b^3 + (-8b + 10b^2 - 4b^3)s + 4b^2s^2 + 12b^3s^3)(s+1) \\
&\quad + (-1 - 2b + (1 - b - b^2)s + 2bs^2 + b^2s^3)] \\
&= \delta[-1 - 13b + 29b^2 - 8b^3 + (1 - 20b + 38b^2 - 12b^3)s \\
&\quad + (-6b + 14b^2 - 4b^3)s^2 + (5b^2 + 12b^3)s^3 + 12b^3s^4].
\end{aligned}$$

Finally, we get

$$\begin{aligned}
&\delta[-1-13b+29b^2-8b^3+(1-20b+38b^2-12b^3)s+(-6b+14b^2-4b^3)s^2+(5b^2+12b^3)s^3+12b^3s^4]\\
&+(-1+(1-2b)s+(3b-b^2)s^2+(-b+2b^2)s^3-b^2s^4)\\
&>-1-13b+29b^2-8b^3+(1-20b+38b^2-12b^3)s+(-6b+14b^2-4b^3)s^2+(5b^2+12b^3)s^3+12b^3s^4\\
&\quad+(-1+(1-2b)s+(3b-b^2)s^2+(-b+2b^2)s^3-b^2s^4)\\
&=-2-13b+29b^2-8b^3+(2-22b+38b^2-12b^3)s+(-3b+13b^2-4b^3)s^2\\
&\quad+(-b+7b^2+12b^3)s^3+(-b^2+12b^3)s^4\\
&>11b^3s^4+12b^3s^3-4b^3s^2-12b^3s-8b^3>0.
\end{aligned}$$

According to the above calculation process, we obtain $q(n)\geq q(2b\delta^2)>0$, which implies that $H(n-1)>0$.

Step 2. $H'(x)>0$ for $x\in[n-1,+\infty)$.

Recall that

$$\begin{aligned}
H(x)=&(-b+bs)x^2+(-3+3b-3\delta+3b\delta+2b^2\delta+3n-2bn\\
&+(-5b+2b^2-5b\delta+2bn)s+(2b-4b^2-2b^2\delta)s^2+2b^2s^3)x\\
&-b^2s^4+(-b+4b^2+b^2\delta)s^3+(5b-5b^2+2b\delta-2b^2\delta-bn)s^2\\
&+(1-8b+2b^2+\delta-6b\delta-b^2\delta-n+3bn)s\\
&+(b\delta-2b+4)n-(b^2+b)\delta^2+(2b^2+b-4)\delta+4b-4.
\end{aligned}$$

Then

$$\begin{aligned}
H'(x)=&2(-b+bs)x+(-3+3b-3\delta+3b\delta+2b^2\delta+3n-2bn\\
&+(-5b+2b^2-5b\delta+2bn)s+(2b-4b^2-2b^2\delta)s^2+2b^2s^3)\\
\geq&2(-b+bs)(n-1)+(-3+3b-3\delta+3b\delta+2b^2\delta+3n-2bn\\
&+(-5b+2b^2-5b\delta+2bn)s+(2b-4b^2-2b^2\delta)s^2+2b^2s^3)\\
=&2b^2s^3+(2b-4b^2-2b^2\delta)s^2+(-7b+2b^2-5b\delta+4bn)s\\
&+2b^2\delta+(5+3\delta-4n)b+3(n-\delta-1)\\
\triangleq&g(s).
\end{aligned}$$

Next we prove that $g(s)>0$ for $2\leq s\leq\delta-1$. By direct calculation, we deduce that

$$g'(s)=6b^2s^2+b(4-4b(2+\delta))s+b(-7+2b-5\delta+4n),$$

and the symmetry axis of $g'(s)$ is $\hat{s}=\frac{\delta}{3}+\frac{2}{3}-\frac{1}{3b}$.

Since $n\geq 2b\delta^2$ and $\delta\geq 3$,

$$g'(s)\geq g'(\frac{\delta}{3}+\frac{2}{3}-\frac{1}{3b})=4bn-\frac{1}{3}(2+13b+11b\delta+2b^2(1+4\delta+\delta^2)).$$

Note that

$$\begin{aligned}
g'(s)\geq&4bn-\frac{1}{3}(2+13b+11b\delta+2b^2(1+4\delta+\delta^2))\\
\geq&\frac{1}{3}[24b^2\delta^2-(2+13b+11b\delta+2b^2(1+4\delta+\delta^2))]\\
=&\frac{1}{3}[22b^2\delta^2-2-13b-11b\delta-2b^2-8b^2\delta]\\
\triangleq&\frac{1}{3}v(b).
\end{aligned}$$

For $b\in[1,+\infty)$, we have

$$v'(b)=44b\delta^2-13-11\delta-4b-16b\delta>0,$$

Thus, $v'(b)>0$ and $v(b)\geq v(1)=22\delta^2-17-19\delta>0$.

Therefore, $g'(s) > 0$ and $g(s)$ is increasing with respect to $s \in [2, \delta - 1]$. Hence

$$g(s) \geq g(2) = (3 + 4b)n + 4(2b - 4b^2 - 2b^2\delta) + (2b^2 - 7b - 3)\delta + 20b^2 - 9b - 3.$$

Since $\delta \geq 3$, it can be checked that

$$\begin{aligned} g(2) &= (3 + 4b)n + 4(2b - 4b^2 - 2b^2\delta) + (2b^2 - 7b - 3)\delta + 20b^2 - 9b - 3 \\ &> (3 + 4b)2b\delta^2 - 4(4b^2 + 2b^2\delta) - (7b + 3)\delta \\ &> 6b\delta^2 + 8b^2\delta^2 - 24b^2\delta - 10b\delta \\ &> 18b\delta + 24b^2\delta - 24b^2\delta - 10b\delta \\ &= 8b\delta > 0. \end{aligned}$$

Thus, we have that $H'(x) > 0$ for $x \in [n - 1, +\infty)$.

Combining with Step 1 and Step 2, we get $H(x) > 0$ for $x \in [n-1, +\infty)$, which implies $f_\delta(x) > f_s(x)$ for $x \in [n-1, +\infty)$. Observe that $\min\{\mu(G_\delta), \mu(G_s)\} > n - 1$, we have $\mu(G_s) > \mu(G_\delta)$.

Furthermore, by (1) and (4),

$$\mu(G) \geq \mu(G_1) \geq \mu(G_s) > \mu(G_\delta) = \mu(K_\delta \vee (K_{n-(b+1)\delta-1} + (b\delta + 1)K_1),$$

a contradiction. This completes the proof. □

Note that a perfect matching is a special odd $[1, b]$-factor when $b = 1$. Let $b = 1$, then we can obtain a condition in terms of distance spectral radius about perfect matching with given minimum degree.

Corollary 4.2. *Let G be a connected graph of even order $n \geq \max\{2\delta^2, 10\delta + 7\}$ with minimum degree $\delta \geq 3$. If $\mu(G) \leq \mu(K_\delta \vee (K_{n-2\delta-1} + (\delta + 1)K_1))$, then G has a perfect matching unless $G \cong K_\delta \vee (K_{n-2\delta-1} + (\delta + 1)K_1)$.*

Statements and Declarations

The authors declare that they have no conflict of interest.

Data availability

No data was used for the research described in the paper.

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