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Combinatorics of the integral closure of edge ideals related to n-partite graphs

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Abstract. Combinatorial properties of some ideals related to n-partite graphs are examined. A description of the integral closure expressed through the log set of edge ideals of complete n-partite graphs is illustrated together with the fact that edge ideals of a strong quasi-n-partite graph are not integrally closed. Moreover, we are able to determine the structure and the invariants of the integral closure of the ideals of vertex covers for the edge ideals associated to a strong quasi-n-partite graph.

1. Introduction

In the present paper we consider classes of monomial ideals that can arise from graph theory [1, 4–6]. More precisely, we consider classes of n-partite graphs and study combinatorial properties of them.

Let G be a graph on the vertex set $V(G)$. A *quasi-n-partite* graph is an n-partite graph with vertex set $V(G)$ partitioned into $V_1 \cup V_2 \cup \cdots \cup V_n$ such that $V_i = \{x_{i1}, \ldots, x_{im_i}\}$ for $i = 1, \ldots, n$, and some vertices in $V(G)$ have loops. A *strong quasi-n-partite graph* is a complete n-partite graph having loops in all its vertices. When $n = 2$ these are the strong quasi-bipartite graphs. A great deal of knowledge on the strong quasi-bipartite graphs is accumulated in several papers [11–13].

Algebraic objects attached to G are the edge ideals $I(G)$. If G is a bipartite graph having bipartition $\{V_1, V_2\}$, edge ideals are monomial ideals of a polynomial ring in two sets of variables associated to such bipartition.

In detail, we are interested to handle the integral closure of edge ideals of strong quasi-n-partite graphs and to study algebraic aspects of it. Let K be a field and $S = K[x_1, \ldots, x_n]$ the polynomial ring in n variables over K with each x_i of degree 1. Next we consider the polynomial ring T over K in the variables

$$x_{11}, \ldots, x_{1m_1}, x_{21}, \ldots, x_{2m_2}, \ldots, x_{n1}, \ldots, x_{nm_n},$$

and let L be a monomial ideal of T.

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We use the notation $\mathbf{X}$ for the set $\{x_{11},\dots,x_{1m_1},x_{21},\dots,x_{2m_2},\dots,x_{n1},\dots,x_{nm_n}\}$. The integral closure $\overline{L}$ of L is the set of all elements of T which are integral over L. The integral closure of a monomial ideal is again a monomial ideal,

$$\overline{L} = (f \mid f \quad \text{is a monomial in T and} \quad f^k \in L^k, \text{for some} \quad k \geq 1).$$

Then L is integrally closed, if $L = \overline{L}$. Put

$$\mathbf{a}_j = (a_{j_{11}},\dots,a_{j_{1m_1}},\dots,a_{j_{n1}},\dots,a_{j_{nm_n}}) \in \mathbb{N}^{m_1} \oplus \cdots \oplus \mathbb{N}^{m_n}.$$

If L is generated by monomials $\mathbf{X}^{\mathbf{a}_1},\dots,\mathbf{X}^{\mathbf{a}_q}$, a combinatorial description of the integral closure of L is the following:

$$\overline{L} = (\mathbf{X}^{\lceil\alpha\rceil} \mid \alpha \in \operatorname{conv}(\mathbf{a}_1,\dots,\mathbf{a}_q)),$$

where $\alpha \in \mathbb{Q}_+^{m_1+\cdots+m_n}$, $\operatorname{conv}(\mathbf{a}_1,\dots,\mathbf{a}_q)$ is the set of all convex combinations of $\mathbf{a}_1,\dots,\mathbf{a}_q$, and $\lceil\ \rceil$ denotes the upper integer. When $\mathbf{X}^{\mathbf{a}} = x_{11}^{a_{11}}\cdots x_{1m_1}^{a_{1m_1}}\cdots x_{n1}^{a_{n1}}\cdots x_{nm_n}^{a_{nm_n}}$, we write $\log(\mathbf{X}^{\mathbf{a}})$ to indicate $\mathbf{a} = (a_{11},\dots,a_{1m_1},\dots,a_{n1},\dots,a_{nm_n}) \in \mathbb{Z}_+^{m_1+\cdots+m_n}$. Given a set $\mathcal{F}$ of monomials, the log set of $\mathcal{F}$, denoted by $\log(\mathcal{F})$, consists of all $\log(\mathbf{X}^{\mathbf{a}})$ such that $\mathbf{X}^{\mathbf{a}} \in \mathcal{F}$.

The present paper is organized as follows. In Section 2 we are able to give a description of the integral closure of an ideal L expressed by its log set for some classes of ideals associated to complete n-partite graphs. In Proposition 2.15 we prove that the integral closure of $I(G)$ of a strong quasi-n-partite graph G is generated by binomials of degree 2. Furthermore, we prove that the edge ideal of a strong quasi-n-partite graph G is not integrally closed and we give an expression for its integral closure, see Corollary 2.16 and Theorem 2.18.

In Section 3 algebraic and homological invariants of the integral closure of $I(G)$ related to a strong quasi-n-partite graph G are studied. There is a one to one correspondence between minimal vertex covers of any graph and minimal prime ideals of its edge ideal; we generalize to a complete n-partite graph with loops the notion of ideal of (minimal) vertex covers and determine the structure of the ideals of vertex covers $I_c(G)$ for the edge ideals associated to G.

In Corollary 3.2 we show that the integral closure of $I(G)$ associated to a strong quasi-n-partite graph G has a linear resolution. We also give formulae for invariants of $T/\overline{I(G)}$ and $T/\overline{I_c(G)}$ for the classes of edge ideals associated to a strong quasi-n-partite graph G such as dimension, projective dimension, depth, Castelnuovo-Mumford regularity.

2. Integral closure of edge ideals

Let K be a field, and let $S = K[x_1,\dots,x_n]$ be the polynomial ring in n variables over K with each x_i of degree 1.

We use the notation $\mathbf{X}$ for the set $\{x_{11},\dots,x_{1m_1},x_{21},\dots,x_{2m_2},\dots,x_{n1},\dots,x_{nm_n}\}$. Let $T = K[\mathbf{X}]$ be the polynomial ring over a field K in $m_1+\cdots+m_n$ variables, and let $L \subset T$ be a monomial ideal which is generated by monomials $\mathbf{X}^{\mathbf{a}_1},\dots,\mathbf{X}^{\mathbf{a}_q}$. Here

$$\mathbf{X}^{\mathbf{a}_j} = x_{11}^{a_{j_{11}}}\cdots x_{1m_1}^{a_{j_{1m_1}}} x_{21}^{a_{j_{21}}}\cdots x_{2m_2}^{a_{j_{2m_2}}}\cdots x_{n1}^{a_{j_{n1}}}\cdots x_{nm_n}^{a_{j_{nm_n}}}$$

for $\mathbf{a}_j = (a_{j_{11}},\dots,a_{j_{1m_1}},a_{j_{21}},\dots,a_{j_{2m_2}},\dots,a_{j_{n1}},\dots,a_{j_{nm_n}}) \in \mathbb{N}^{m_1} \oplus \cdots \oplus \mathbb{N}^{m_n}$.

The *integral closure* of L is the set of all elements of T which are integral over T. Since the integral closure of a monomial ideal is again a monomial ideal [7, Theorem 1.4.2], one has the following description of the integral closure of L:

$$\overline{L} = (f \mid f \quad \text{is a monomial in T and} \quad f^k \in L^k, \text{for some} \quad k \geq 1).$$

The ideal L is *integrally closed*, if $L = \overline{L}$.

Let $\beta \in \mathbb{Q}_+^n$, where $\mathbb{Q}_+$ is the set of nonnegative rational numbers. We define the *upper integer* or *ceiling* of β as the vector $\lceil \beta \rceil$ whose entries are given by $\lceil \beta \rceil_i$, where

$$\lceil \beta \rceil_i = \begin{cases} \beta_i & \text{if} \quad \beta_i \in \mathbb{N} \\ \lfloor \beta_i \rfloor + 1 & \text{if} \quad \beta_i \notin \mathbb{N} \end{cases}$$

and where $\lfloor \beta_i \rfloor$ stands for the integer part of β_i. In addition, we denote the set $\{x_{i1}, \ldots, x_{im_i}\}$ by $\mathbf{X}_i$ for $i = 1, \ldots, n$. Then the integral closure of L is the monomial ideal:

$$\overline{L} = (\mathbf{X}^{\lceil \alpha \rceil} \mid \alpha \in \operatorname{conv}(\mathbf{a}_1, \ldots, \mathbf{a}_q)),$$

where

$$\operatorname{conv}(\mathbf{a}_1, \ldots, \mathbf{a}_q) = \left\{ \sum_{j=1}^{q} \lambda_j \mathbf{a}_j \,\middle|\, \sum_{j=1}^{q} \lambda_j = 1, \lambda_j \in \mathbb{Q}_+ \right\}$$

is the set of all *convex combinations* of $\mathbf{a}_1, \ldots, \mathbf{a}_q$.

Example 2.1. Let $T = K[x_{11}, x_{12}, x_{13}, x_{21}, x_{22}]$ be the polynomial ring over a field K, and let $L = (x_{11}^2 x_{12} x_{21}, x_{12} x_{13} x_{22}^2, x_{13} x_{21}^2)$ be the monomial ideal of T. By using Normalize [14], we obtain that

$$\begin{aligned} \overline{L} &= (\mathbf{X}_1^{\alpha} \mathbf{X}_2^{\alpha'} \mid (\alpha, \alpha') \in \operatorname{conv}((2,1,0,1,0), (0,1,1,0,2), (0,0,1,2,0))) \\ &= (\mathbf{X}_1^{\alpha} \mathbf{X}_2^{\alpha'} \mid (\alpha, \alpha') \in \{\lambda_1(2,1,0,1,0) + \lambda_2(0,1,1,0,2) + \lambda_3(0,0,1,2,0), \\ &\quad \lambda_1, \lambda_2, \lambda_3 \in \mathbb{Q}_+, \lambda_1 + \lambda_2 + \lambda_3 = 1\}) \\ &= (x_{11}^2 x_{12} x_{21}, x_{12} x_{13} x_{22}^2, x_{13} x_{21}^2, x_{12} x_{13} x_{21} x_{22}). \end{aligned}$$

The purpose of this section is to study the integral closure of monomial ideals in the polynomial ring $T = K[\mathbf{X}]$.

We recall the notions that come from the general theory for monomial ideals (see for instance [15]).

Definition 2.2. Let $T = K[\mathbf{X}]$ be a polynomial ring over a field K in the variables $x_{11}, \ldots, x_{1m_1}, \ldots, x_{n1}, \ldots, x_{nm_n}$. If $\mathbf{X}^{\mathbf{a}} = x_{11}^{a_{11}} \cdots x_{1m_1}^{a_{1m_1}} \cdots x_{n1}^{a_{n1}} \cdots x_{nm_n}^{a_{nm_n}}$, we set

$$\log(\mathbf{X}^{\mathbf{a}}) = \mathbf{a} = (a_{11}, \ldots, a_{1m_1}, \ldots, a_{n1}, \ldots, a_{nm_n}) \in \mathbb{Z}_+^{m_1 + \cdots + m_n}.$$

Given a set $\mathcal{F}$ of monomials, the *log set* of $\mathcal{F}$, denoted by $\log(\mathcal{F})$, consists of all $\log(\mathbf{X}^{\mathbf{a}})$, with $\mathbf{X}^{\mathbf{a}} \in \mathcal{F}$,

$$\log(\mathcal{F}) = \{\log(\mathbf{X}^{\mathbf{a}}) = \mathbf{a} \in \mathbb{Z}_+^{m_1 + \cdots + m_n} \mid \mathbf{X}^{\mathbf{a}} \in \mathcal{F}\}.$$

Example 2.3. Let $\mathcal{F} = \{x_{11}^2 x_{21}, x_{11}^3 x_{21}^2, x_{12} x_{22}\}$ be a set of monomials in the polynomial ring $T = K[x_{11}, x_{12}, x_{21}, x_{22}]$. The log set of $\mathcal{F}$ is $\log(\mathcal{F}) = \{(2,0,1,0), (3,0,2,0), (0,1,0,1)\}$.

Definition 2.4. Let L be an ideal of $T = K[\mathbf{X}]$ generated by the set of monomials of $\mathcal{F}$. We define $\log(L) = \{\log(\mathbf{X}^{\mathbf{a}}) = \mathbf{a} \in \mathbb{Z}_+^{m_1 + \cdots + m_n} \mid \mathbf{X}^{\mathbf{a}} \in L\}$.

Proposition 2.5. Let L be a monomial ideal of $T = K[\mathbf{X}]$. Then

$$\overline{L} = (\mathbf{X}^{\alpha} \mid \alpha \in \operatorname{conv}(\log(L)) \cap \mathbb{Z}^{m_1 + \cdots + m_n}).$$

Proof. See ([15]). $\square$

Now we introduce the following

Definition 2.6. Let $T = K[\mathbf{X}]$, and let $L = (\mathbf{X}_1^{a_{11}} \cdots \mathbf{X}_n^{a_{1n}}, \mathbf{X}_1^{a_{21}} \cdots \mathbf{X}_n^{a_{2n}}, \ldots, \mathbf{X}_1^{a_{q1}} \cdots \mathbf{X}_n^{a_{qn}})$ be a monomial ideal of T, where $\mathbf{X}_1^{a_{j1}} \cdots \mathbf{X}_n^{a_{jn}}$ stands for

$$x_{11}^{a_{j11}} \cdots x_{1m_1}^{a_{j1m_1}} x_{21}^{a_{j21}} \cdots x_{2m_2}^{a_{j2m_2}} \cdots x_{n1}^{a_{jn1}} \cdots x_{nm_n}^{a_{jnm_n}}$$

for $j = 1, \ldots, q$. Then for all $i = 1, \ldots, n$ we set $\mathcal{F}_i = \left\{ \mathbf{X}_i^{a_{ji}} \;\middle|\; j = 1, \ldots, q \right\}$. We define a monomial ideal L^* of T as

$$L^* = \left(\mathbf{X}_i^{\alpha_i} \mathbf{X}_{i'}^{\alpha_{i'}} \;\middle|\; \alpha_i \in \operatorname{conv}(\log(\mathcal{F}_i)) \cap \mathbb{Z}^{m_i}, \alpha_{i'} \in \operatorname{conv}(\log(\mathcal{F}_{i'})) \cap \mathbb{Z}^{m_{i'}} \right)$$

for all $1 \leq i \neq i' \leq n$.

Some good results about the inclusion relation between the integral closure and the log set of a monomial ideal are given for the edge ideals of n-partite graphs. Let G be a graph on the vertex set $V(G) = \{v_1, \ldots, v_n\}$. We put

$$E(G) = \{\{v_i, v_j\} \mid v_i \neq v_j, \quad v_i, v_j \in V(G)\}$$

the set of edges of G and $\mathcal{L}(G) = \{\{v_i, v_i\} \mid v_i \in V(G)\}$ the set of loops of G. Furthermore we set $\mathcal{W}(G) = \mathcal{L}(G) \cup E(G)$.

An algebraic object attached to G is the edge ideal $I(G) = (x_i x_j \mid \{v_i, v_j\} \in \mathcal{W}(G))$, a monomial ideal of S. If $\mathcal{L}(G) = \emptyset$, the graph G is said *simple* or *loopless*, otherwise, if $\mathcal{L}(G) \neq \emptyset$, G is a graph with loops.

Definition 2.7. A simple graph G is said to be *n-partite* if its vertex set $V(G)$ can be partitioned into n pairwise disjoint subsets such that no two vertices in the same subset are adjacent in G. When $n = 2$ these are the bipartite graphs.

Definition 2.8. A simple graph G is called *complete n-partite* if its vertex set can be partitioned into disjoint independent subsets $V_1, \ldots, V_n$ such that for all u and u' in different sets, $uu' \in E(G)$.

The following result gives a description of the integral closure expressed by its log set for the edge ideals of n-partite graphs.

Proposition 2.9. *Let $T = K[\mathbf{X}]$ be the polynomial ring over a field K, and $I(G)$ be the edge ideal associated to an n-partite graph G. Then $I(G) = \overline{I(G)}$.*

Proof. Let G be be an n-partite graph on the vertex set $V(G) = V_1 \cup V_2 \cup \cdots \cup V_n$, where $V_i = \{x_{i1}, \ldots, x_{im_i}\}$ for $i = 1, \ldots, n$. Furthermore, let $I(G)$ be the edge ideal of G generated by the monomials $\mathbf{X}^{\mathbf{a}_1}, \ldots, \mathbf{X}^{\mathbf{a}_p}$, where

$$\mathbf{X}^{\mathbf{a}_j} = x_{11}^{a_{j11}} \cdots x_{1m_1}^{a_{j1m_1}} x_{21}^{a_{j21}} \cdots x_{2m_2}^{a_{j2m_2}} \cdots x_{n1}^{a_{jn1}} \cdots x_{nm_n}^{a_{jnm_n}}$$

for $\mathbf{a}_j = (a_{j_{11}}, \ldots, a_{j_{1m_1}}, a_{j_{21}}, \ldots, a_{j_{2m_2}}, \ldots, a_{j_{n1}}, \ldots, a_{j_{nm_n}}) \in \mathbb{N}^{m_1} \oplus \cdots \oplus \mathbb{N}^{m_n}$. Then by definition,

$$\overline{I(G)} = (\{\mathbf{X}^{\lceil \alpha \rceil} \mid \alpha \in \operatorname{conv}(\mathbf{a}_1, \ldots, \mathbf{a}_p)\})$$

and $\alpha = \sum_{j=1}^{p} \lambda_j \mathbf{a}_j \in \mathbb{Q}^{m_1 + \cdots + m_n}$ with $\sum_{j=1}^{p} \lambda_j = 1$.

Since G is a loopless graph on $m_1 + \cdots + m_n$ vertices, $I(G) = (x_{ih} x_{i'h'} \mid \{x_{ih}, x_{i'h'}\} \in E(G))$ for $1 \leq i \neq i' \leq n$ and $1 \leq h, h' \leq m_i$. Hence we set $\mathbf{X}^{\mathbf{a}_j} = x_{ih}^{a_{j_{ih}}} x_{i'h'}^{a_{j_{i'h'}}}$, where $a_{j_{ih}} = a_{j_{i'h'}} = 1$. By using the definition of integral closure we have the following situations:

if $\lambda_j = 1$ and $\lambda_r = 0$ for any $1 \leq j \neq r \leq p$, then

$$\alpha = (a_{j_{11}}, \ldots, a_{j_{1m_1}}, \ldots, a_{j_{n1}}, \ldots, a_{j_{nm_n}}) \in \mathbb{N}^{m_1} \oplus \cdots \oplus \mathbb{N}^{m_n}, \quad \mathbf{X}^{\lceil \alpha \rceil} = \mathbf{X}^{\mathbf{a}_j}, \quad 1 \leq j \leq p;$$

if $\lambda_j \in \mathbb{Q}_+$, then $\mathbf{X}^{\lceil \alpha \rceil} = \mathbf{X}^{\lfloor \alpha \rfloor + 1}$. Therefore the generators of $\overline{I(G)}$ as $\mathbf{X}^{\mathbf{a}_j}$ and $\mathbf{X}^{\lfloor \alpha \rfloor + 1}$, and hence $I(G) = \overline{I(G)}$. □

Corollary 2.10. *Let T and $I(G)$ be as in Proposition 2.9. Then $\overline{I(G)} \subseteq I^*(G)$.*

Proof. Let $x_{ih}x_{i'h'} \in \overline{I(G)}$; since $\mathcal{F}_i = \{x_{i1}, \ldots, x_{im_i}\}$ and $\mathcal{F}_{i'} = \{x_{i'1}, \ldots, x_{i'm_{i'}}\}$, it follows from Definition 2.6 that $x_{ih}x_{i'h'} \in I^*(G)$. □

For complete n-partite graphs the equality holds.

Proposition 2.11. *Let $T = K[\mathbf{X}]$ be the polynomial ring over a field K, and $I(G)$ be the edge ideal associated to a complete n-partite graph G. Then $\overline{I(G)} = I^*(G)$.*

Proof. We consider $I(G) = (x_{ih}x_{i'h'}, \quad \text{for all} \quad 1 \le i \ne i' \le n, 1 \le h, h' \le m_i)$ and

$$I^*(G) = \left(\mathbf{X}_i^{\alpha_i}\mathbf{X}_{i'}^{\alpha_{i'}} \;\middle|\; \alpha_i \in \operatorname{conv}(\log(\mathcal{F}_i)) \cap \mathbb{Z}^{m_i}, \alpha_{i'} \in \operatorname{conv}(\log(\mathcal{F}_{i'})) \cap \mathbb{Z}^{m_{i'}}\right),$$

where $\mathcal{F}_i = \{x_{i1}, \ldots, x_{im_i}\}$, $\mathcal{F}_{i'} = \{x_{i'1}, \ldots, x_{i'm_{i'}}\}$ for $1 \le i \ne i' \le n$.

From Corollary 2.10, it is enough to prove that $\overline{I(G)} \supseteq I^*(G)$. Let $\mathbf{X}_i^{\alpha_i}\mathbf{X}_{i'}^{\alpha_{i'}} \in I^*(G)$ with

$$\alpha_i = \sum_{j=1}^{q} \lambda_j a_{j_i} \in \mathbb{Z}^{m_i}, \quad \sum_{j=1}^{q} \lambda_j = 1, \quad \lambda_j \in \mathbb{Q}_+, a_{j_i} \in \log(\mathcal{F}_i),$$

and

$$\alpha_{i'} = \sum_{j=1}^{q} \vartheta_j a_{j_{i'}} \in \mathbb{Z}^{m_{i'}}, \quad \sum_{j=1}^{q} \vartheta_j = 1, \quad \vartheta_j \in \mathbb{Q}_+, a_{j_{i'}} \in \log(\mathcal{F}_{i'}).$$

This implies that $\mathbf{X}_i^{\alpha_i} = x_{ih}$ for all $1 \le i \le n$ and $\mathbf{X}_{i'}^{\alpha_{i'}} = x_{i'h'}$ for all $1 \le i' \le n$. By choosing $k \in \mathbb{N}_+$ such that $k\lambda_l \in \mathbb{N}$ and $k\vartheta_l \in \mathbb{N}$ for all $1 \le l \le q$, we have $\mathbf{X}_i^{k\alpha_i}\mathbf{X}_{i'}^{k\alpha_{i'}} = x_{ih}^k x_{i'h'}^k \in I(G)^k$, for all $1 \le i \ne i' \le n$ and $1 \le h, h' \le m_i$. Therefore, $\mathbf{X}_i^{\alpha_i}\mathbf{X}_{i'}^{\alpha_{i'}} \in \overline{I(G)}$, and hence $\overline{I(G)} = I^*(G)$. □

Example 2.12. Let $T = K[x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}]$ be the polynomial ring over a field K, and let

$$\begin{aligned} I(G) = \;& (x_{11}x_{21}, x_{11}x_{22}, x_{12}x_{21}, x_{12}x_{22}, x_{11}x_{31}, x_{11}x_{32}, x_{12}x_{31}, x_{12}x_{32}, x_{21}x_{31}, \\ & x_{21}x_{32}, x_{22}x_{31}, x_{22}x_{32}) \end{aligned}$$

be the edge ideal associated to complete 3-partite graph G with vertices $x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}$. Then Proposition 2.11 implies that $\overline{I(G)} = I^*(G)$.

Next we consider graphs with loops which edge ideals are not integrally closed and we compute the integral closure.

Definition 2.13. A *quasi-n-partite* graph is an n-partite graph with vertex set $V(G)$ partitioned into $V_1 \cup V_2 \cup \cdots \cup V_n$ such that $V_i = \{x_{i1}, \ldots, x_{im_i}\}$ for $i = 1, \ldots, n$, and some vertices in $V(G)$ have loops.

Definition 2.14. A quasi-n-partite graph G is called *strong* if it is a complete n-partite graph and all its vertices have loops.

Proposition 2.15. *Let $T = K[\mathbf{X}]$ be the polynomial ring over a field K, and $I(G)$ be the edge ideal of a strong quasi-n-partite graph G. Then $\overline{I(G)}$ is generated by binomials of degree 2.*

Proof. Let G be a strong quasi-n-partite with the vertex set $\mathbf{X}$, let $I(G)$ be its edge ideal. Let $\mathbf{X}^{\mathbf{a}_1}, \ldots, \mathbf{X}^{\mathbf{a}_p}$ be the generators of $I(G)$, where

$$\mathbf{X}^{\mathbf{a}_j} = x_{11}^{a_{j_{11}}} \cdots x_{1m_1}^{a_{j_{1m_1}}} x_{21}^{a_{j_{21}}} \cdots x_{2m_2}^{a_{j_{2m_2}}} \cdots x_{n1}^{a_{j_{n1}}} \cdots x_{nm_n}^{a_{j_{nm_n}}}$$

for $\mathbf{a}_j = (a_{j_{11}}, \ldots, a_{j_{1m_1}}, a_{j_{21}}, \ldots, a_{j_{2m_2}}, \ldots, a_{j_{n1}}, \ldots, a_{j_{nm_n}}) \in \mathbb{N}^{m_1} \oplus \cdots \oplus \mathbb{N}^{m_n}$.

Using the geometric description of the integral closure of a monomial ideal ([15, Proposition 12.1.4]), we have

$$\overline{I(G)} = (\{\mathbf{X}^{\lceil \alpha \rceil} \mid \alpha \in \operatorname{conv}(\mathbf{a}_1, \ldots, \mathbf{a}_p)\}),$$

with

$$\mathrm{conv}(\mathbf{a}_1,\ldots,\mathbf{a}_p)=\left\{\sum_{j=1}^{p}\lambda_j\mathbf{a}_j \;\middle|\; \sum_{j=1}^{p}\lambda_j=1,\lambda_j\in\mathbb{Q}_+\right\}.$$

Let f be a generator of $\overline{I(G)}$. Then $f=\mathbf{X}^{\lceil\alpha\rceil}$ with $\alpha=\sum_{j=1}^{p}\lambda_j\mathbf{a}_j\in\mathrm{conv}(\mathbf{a}_1,\ldots,\mathbf{a}_p)$, $\sum_{j=1}^{p}\lambda_j=1$, $\lambda_j\in\mathbb{Q}_+$. Therefore,

$$\alpha=\left(\sum_{j=1}^{p}\lambda_j a_{j_{11}},\ldots,\sum_{j=1}^{p}\lambda_j a_{j_{1m_1}},\ldots,\sum_{j=1}^{p}\lambda_j a_{j_{n1}},\ldots,\sum_{j=1}^{p}\lambda_j a_{j_{nm_n}}\right)\in\mathbb{Q}_+^{m_1+\cdots+m_n}$$

with

$$\mathbf{a}_j=(a_{j_{11}},\ldots,a_{j_{1m_1}},a_{j_{21}},\ldots,a_{j_{2m_2}},\ldots,a_{j_{n1}},\ldots,a_{j_{nm_n}})\in\mathbb{N}^{m_1}\oplus\cdots\oplus\mathbb{N}^{m_n}$$

and $a_{j_{ih}}\in\{0,1,2\}$. The generic element of α, α_{ih}, $1\le i\le n$, $1\le h\le m_i$, is

$$\begin{aligned}\sum_{j=1}^{p}\lambda_j a_{j_{ih}} &= \lambda_1 a_{1_{ih}}+\lambda_2 a_{2_{ih}}+\cdots+\lambda_r.2+\cdots+\lambda_p a_{p_{ih}}\\ &= 2\lambda_r+(1-\lambda_r)\\ &= \lambda_r+1\le 2,\end{aligned}$$

as desired. □

Corollary 2.16. *Let G be a strong quasi-n-partite and $I(G)\subset T$ the edge ideal associated to G. Then $I(G)$ is not integrally closed.*

Proof. Let G be a graph on the vertex set $V(G)$, and let $I(G)$ be the edge ideal of a strong quasi-n-partite graph G. Let $\mathbf{X}^{\mathbf{a}_1},\ldots,\mathbf{X}^{\mathbf{a}_p}$ be the generators of $I(G)$, where

$$\mathbf{X}^{\mathbf{a}_j}=x_{11}^{a_{j_{11}}}\cdots x_{1m_1}^{a_{j_{1m_1}}}x_{21}^{a_{j_{21}}}\cdots x_{2m_2}^{a_{j_{2m_2}}}\cdots x_{n1}^{a_{j_{n1}}}\cdots x_{nm_n}^{a_{j_{nm_n}}}$$

for $\mathbf{a}_j=(a_{j_{11}},\ldots,a_{j_{1m_1}},a_{j_{21}},\ldots,a_{j_{2m_2}},\ldots,a_{j_{n1}},\ldots,a_{j_{nm_n}})\in\mathbb{N}^{m_1}\oplus\cdots\oplus\mathbb{N}^{m_n}$.

Now we may assume that $a_{j_{ir}}=a_{l_{ir'}}=2$ for some $1\le j,l\le p$, where $r\ne r'$. Then Proposition 2.15 implies that $x_{ir}^2,x_{ir'}^2$ are generators of $I(G)$. We consider the obtained convex hull placing $\lambda_j=\frac{1}{2}$ and $\lambda_l=\frac{1}{2}$ and $\lambda_t=0$ $\forall t\ne j,l$. Therefore,

$$\alpha=(0,\ldots,\underbrace{1}_{\alpha_{ir}},\ldots,\underbrace{1}_{\alpha_{ir'}},\ldots,0)\in\mathrm{conv}(\mathbf{a}_1,\ldots,\mathbf{a}_p),$$

hence $x_{ir}x_{ir'}$ is a generator of the integral closure of $I(G)$. Therefore $I(G)$ is not integrally closed, and $I(G)\ne\overline{I(G)}$. □

Example 2.17. Let G be a strong quasi-3-partite graph on the vertex set

$$V(G)=\{x_{11},x_{12},x_{21},x_{22},x_{31},x_{32}\}.$$

Then the edge ideal of G is the ideal:

$$\begin{aligned}I(G)=\ &(x_{11}^2,x_{12}^2,x_{21}^2,x_{22}^2,x_{31}^2,x_{32}^2,x_{11}x_{21},x_{11}x_{22},x_{12}x_{21},x_{12}x_{22},x_{11}x_{31},x_{11}x_{32},\\ &x_{12}x_{31},x_{12}x_{32},x_{21}x_{31},x_{21}x_{32},x_{22}x_{31},x_{22}x_{32})\subset K[x_{11},x_{12},x_{21},x_{22},x_{31},x_{32}].\end{aligned}$$

A computation with Normalize ([14]) gives

$$\begin{aligned}\overline{I(G)}=\ &(x_{11}^2,x_{12}^2,x_{21}^2,x_{22}^2,x_{31}^2,x_{32}^2,x_{11}x_{21},x_{11}x_{22},x_{12}x_{21},x_{12}x_{22},x_{11}x_{12},x_{11}x_{31},\\ &x_{11}x_{32},x_{12}x_{31},x_{12}x_{32},x_{21}x_{22},x_{21}x_{31},x_{21}x_{32},x_{22}x_{31},x_{22}x_{32},x_{31}x_{32}).\end{aligned}$$

Therefore, $I(G)\ne\overline{I(G)}$, and hence $I(G)$ is not integrally closed.

The structure of the integral closure of $I(G)$ associated to a strong quasi-n-partite graph G is given in the following result.

Theorem 2.18. *Let $T = K[\mathbf{X}]$ be the polynomial ring over a field K and G be a strong quasi-n-partite graph. Then*

$$\overline{I(G)} = \sum_{l_i \geq 0, \sum_{i=1}^{n} l_i = 2} I_{1l_1} I_{2l_2} \dots I_{nl_n},$$

where the ideals $I_{il_i} = (x_{i1}, \dots, x_{im_i})^{l_i}$ are the monomial ideals generated by all monomials of degree l_i in the variables $\mathbf{X}_i = \{x_{i1}, \dots, x_{im_i}\}$.

Proof. Let G be a strong quasi-n-partite graph on the vertex set $V(G) = V_1 \cup V_2 \cup \dots \cup V_n$ and $V_i = \{x_{i1}, \dots, x_{im_i}\}$ for $i = 1, \dots, n$. Let $\mathbf{X}^{\mathbf{a}_1}, \dots, \mathbf{X}^{\mathbf{a}_q}$ be the generators of $I(G)$, where $\mathbf{X}^{\mathbf{a}_j}$ is a monomial ideal of degree 2, namely x_{ir}^2 or $x_{ir}x_{i'r'}$ for all $1 \leq i \neq i' \leq n$.

By the geometric description of the integral closure of a monomial ideal in [15, Proposition 12.1.4], we have

$$\overline{I(G)} = (\{\mathbf{X}^{\lceil \alpha \rceil} \mid \alpha \in \operatorname{conv}(\mathbf{a}_1, \dots, \mathbf{a}_q)\}),$$

where

$$\operatorname{conv}(\mathbf{a}_1, \dots, \mathbf{a}_q) = \left\{ \sum_{j=1}^{q} \lambda_j \mathbf{a}_j \,\middle|\, \sum_{j=1}^{q} \lambda_j = 1, \lambda_j \in \mathbb{Q}_+ \right\}.$$

Now let $f = \mathbf{X}^{\lceil \alpha \rceil}$ be a generator of $\overline{I(G)}$ with $\alpha = \sum_{j=1}^{q} \lambda_j \mathbf{a}_j \in \operatorname{conv}(\mathbf{a}_1, \dots, \mathbf{a}_q)$, $\sum_{j=1}^{q} \lambda_j = 1$, $\lambda_j \in \mathbb{Q}_+$. It then follows that

$$\alpha = \left(\sum_{j=1}^{q} \lambda_j a_{j_{11}}, \dots, \sum_{j=1}^{q} \lambda_j a_{j_{1m_1}}, \dots, \sum_{j=1}^{q} \lambda_j a_{j_{n1}}, \dots, \sum_{j=1}^{q} \lambda_j a_{j_{nm_n}} \right) \in \mathbb{Q}_+^{m_1 + \dots + m_n},$$

with $\mathbf{a}_j = (a_{j_{11}}, \dots, a_{j_{1m_1}}, a_{j_{21}}, \dots, a_{j_{2m_2}}, \dots, a_{j_{n1}}, \dots, a_{j_{nm_n}}) \in \mathbb{N}^{m_1} \oplus \dots \oplus \mathbb{N}^{m_n}$. By definition of $I(G)$, in each generator $\mathbf{X}^{\mathbf{a}_j}$ we have $a_{j_{ip}} = 0, 1, 2$. We set

$$\mathcal{M}[i] = m_1 + \dots + m_{i-1} + m_{i+1} + \dots + m_n$$

for every $i = 1, \dots, n$. Thus

$$\sum_{j=1}^{q} \lambda_j a_{j_{ih}} = \lambda_{j_{it_1}} + \dots + \lambda_{j_{it_{\mathcal{M}[i]}}} + 2\lambda_{j_{ih}},$$

such that $1 \leq \{it_1\} < \{it_2\} < \dots < \{it_{\mathcal{M}[i]}\} \leq q$ and $\{ih\} \neq \{it_1\}, \{it_2\}, \dots, \{it_{\mathcal{M}[i]}\}$.

If $\lambda_j \in \mathbb{N}$ with $\sum_{j=1}^{q} \lambda_j = 1$ we obtain that $\mathbf{X}^{\lceil \alpha \rceil} = \mathbf{X}^{\mathbf{a}_j}$, $\forall 1 \leq j \leq q$, that is $\mathbf{X}^{\lceil \alpha \rceil} = x_{ir}^2$ or $\mathbf{X}^{\lceil \alpha \rceil} = x_{ir}x_{i'r''}$ for $1 \leq i \neq i' \leq n$.

On the other hand, if $\lambda_{j_{ih}} = \frac{1}{2}$ with $\sum_j \lambda_j = 1$, it follows that the monomials $\mathbf{X}^{\lceil \alpha \rceil}$ match $x_{ir}x_{ir'}$, where $r < r'$ for all $1 \leq r \leq m_r$ and $1 \leq r' \leq m_{r'}$.

Otherwise, if $\lambda_j \in \mathbb{Q}_+ \setminus \mathbb{N}$ one obtains a monomial $\mathbf{X}^{\lceil \alpha \rceil}$ with $\lceil \alpha \rceil \geq \mathbf{a}_j$, that is $\alpha_{ir} \geq a_{j_{ir}}$. Hence the minimal system of generators of $\overline{I(G)}$ is $\{x_{ir}^2, x_{ir}x_{ir'}, x_{ir}x_{i'r''}\}$ for all $1 \leq i \neq i' \leq n$, where $r \neq r'$. Then the assertion follows. □

Definition 2.19. Let $T = K[\mathbf{X}]$, and let L be a monomial ideal of T generated by the monomials

$$\mathbf{X}_1^{a_{1_1}} \cdots \mathbf{X}_n^{a_{1_n}}, \mathbf{X}_1^{a_{2_1}} \cdots \mathbf{X}_n^{a_{2_n}}, \dots, \mathbf{X}_1^{a_{q_1}} \cdots \mathbf{X}_n^{a_{q_n}},$$

where $\mathbf{X}_1^{a_{j_1}} \cdots \mathbf{X}_n^{a_{j_n}}$ stands for $x_{11}^{a_{j_{11}}} \cdots x_{1m_1}^{a_{j_{1m_1}}} x_{21}^{a_{j_{21}}} \cdots x_{2m_2}^{a_{j_{2m_2}}} \cdots x_{n1}^{a_{j_{n1}}} \cdots x_{nm_n}^{a_{j_{nm_n}}}$ for $j = 1, \dots, q$. We define the *integral bi-closure* of L as the following monomial ideal of T:

$$\overline{\overline{L}} = \left(\left\{ \mathbf{X}_1^{\lceil \alpha_1 \rceil} \cdots \mathbf{X}_n^{\lceil \alpha_n \rceil} \,\middle|\, \alpha_i \in \operatorname{conv}(a_{1_i}, \dots, a_{q_i}) \quad \text{for} \quad i = 1, \dots, n \right\} \right)$$

with $\alpha_i \in \mathbb{Q}_+^{m_i}$.

Proposition 2.20. *Let $T = K[\mathbf{X}]$ be the polynomial ring over a field K and G be a strong quasi-n-partite graph. Then $\overline{I(G)} \subseteq \overline{\overline{I(G)}}$.*

Proof. Let G be a strong quasi-n-partite graph on the vertex set $V(G) = V_1 \cup V_2 \cup \cdots \cup V_n$ and $V_i = \{x_{i1}, \ldots, x_{im_i}\}$ for $i = 1, \ldots, n$. Let $I(G)$ be the edge ideal of G generated by the monomials $\mathbf{X}_1^{a_{1_1}} \cdots \mathbf{X}_n^{a_{1_n}}, \mathbf{X}_1^{a_{2_1}} \cdots \mathbf{X}_n^{a_{2_n}}, \ldots, \mathbf{X}_1^{a_{q_1}} \cdots \mathbf{X}_n^{a_{q_n}}$, where $\mathbf{X}_1^{a_{j_1}} \cdots \mathbf{X}_n^{a_{j_n}}$ stands for

$$x_{11}^{a_{j_{11}}} \cdots x_{1m_1}^{a_{j_{1m_1}}} x_{21}^{a_{j_{21}}} \cdots x_{2m_2}^{a_{j_{2m_2}}} \cdots x_{n1}^{a_{j_{n1}}} \cdots x_{nm_n}^{a_{j_{nm_n}}}$$

for $j = 1, \ldots, q$. Then the integral closure of $I(G)$ is the ideal:

$$\overline{I(G)} = \left(\prod_{i=1}^{n} \mathbf{X}_i^{\lceil \alpha_i \rceil} \;\middle|\; (\alpha_1, \ldots, \alpha_n) \in \operatorname{conv}((a_{1_1}, \ldots, a_{1_n}), \ldots, (a_{q_1}, \ldots, a_{q_n})) \right).$$

By definition

$$\overline{\overline{I(G)}} = \left(\left\{ \mathbf{X}_1^{\lceil \alpha_1 \rceil} \cdots \mathbf{X}_n^{\lceil \alpha_n \rceil} \;\middle|\; \alpha_i \in \operatorname{conv}(a_{1_i}, \ldots, a_{q_i}) \quad \text{for} \quad i = 1, \ldots, n \right\} \right)$$

and let $f = \mathbf{X}_1^{\lceil \alpha_1 \rceil} \cdots \mathbf{X}_n^{\lceil \alpha_n \rceil}$ be a generator of $\overline{\overline{I(G)}}$.

By hypothesis $\alpha_i = \sum_{j=1}^{q} \lambda_j a_{j_i} \in \operatorname{conv}(a_{1_i}, \ldots, a_{q_i})$ with $\sum_{j=1}^{q} \lambda_j = 1, 1 \leq i \leq n$. Then $\alpha_i = (\sum_{j=1}^{q} \lambda_j a_{j_{i1}}, \ldots, \sum_{j=1}^{q} \lambda_j a_{j_{im_i}})$. Furthermore, we put $\mathcal{M}[i] = m_1 + \cdots + m_{i-1} + m_{i+1} + \cdots + m_n$ for every $i = 1, \ldots, n$. Hence

$$\sum_{j=1}^{q} \lambda_j a_{j_{ir}} = \lambda_{j_{ip_1}} + \cdots + \lambda_{j_{ip_{\mathcal{M}[i]}}} + 2\lambda_{j_{ir}},$$

$1 \leq \{ip_1\} < \{ip_2\} < \cdots < \{ip_{\mathcal{M}[i]}\} \leq q$ and $\{ir\} \neq \{ip_1\}, \{ip_2\}, \ldots, \{ip_{\mathcal{M}[i]}\}$.

If $\lambda_j \in \mathbb{N}$ with $\sum_{j=1}^{m} \lambda_j = 1$ we obtain that $\mathbf{X}_i^{\lceil \alpha_i \rceil} = x_{ip}$ or $\mathbf{X}_i^{\lceil \alpha_i \rceil} = x_{ip}^2$ $\forall 1 \leq i \leq n$ or $\mathbf{X}_i^{\lceil \alpha_i \rceil} = 1$.

On the other hand, if $\lambda_j \in \mathbb{Q}_+ \setminus \mathbb{N}$ it follows that $\mathbf{X}_i^{\lceil \alpha_i \rceil} = x_{i1} \cdots x_{im_i}$. Therefore, $\overline{\overline{I(G)}}$ is generated by all the products of monomials $\mathbf{X}_i^{\lceil \alpha_i \rceil}$ before defined. Hence the assertion follows. □

3. Regularity and projective dimension

Let as before, K be a field and $T = K[\mathbf{X}]$ be the polynomial ring over K in the variables

$$x_{11}, \ldots, x_{1m_1}, x_{21}, \ldots, x_{2m_2}, \ldots, x_{n1}, \ldots, x_{nm_n},$$

and let $L \subset T$ be a monomial ideal. We denote by $G(L)$ its unique minimal set of monomial generators.

In this section we study the regularity, depth, dim and projective dimension of monomial ideals corresponding to quasi-n-partite graphs with loops.

A monomial ideal L is said to have *linear quotients* if there is an ordering $f_1, \ldots, f_q$ of monomials belonging to $G(L)$ with $\deg(f_1) \leq \cdots \leq \deg(f_q)$ such that the colon ideal $(f_1, \ldots, f_{j-1}) : (f_j)$ is generated by a subset of $\mathbf{X}$ for each $2 \leq j \leq q$. Let r_j denote the number of variables which is required to generate $(f_1, \ldots, f_{j-1}) : (f_j)$, set $\mathrm{r}(L) = \max_{2 \leq j \leq q} r_j$. For this topic we refer the reader to [15, Definition 6.3.45].

Proposition 3.1. *Let $T = K[\mathbf{X}]$ be the polynomial ring over a field K and G be a strong quasi-n-partite graph. Then $\overline{I(G)}$ has linear quotients.*

Proof. Let G be a strong quasi-n-partite on the vertex set $V(G) = V_1 \cup V_2 \cup \cdots \cup V_n$, where $V_i = \{x_{i1}, \ldots, x_{im_i}\}$ for $i = 1, \ldots, n$. Then G is a complete n-partite graph and all its vertices have loops.

Let $\overline{I(G)}$ be the integral closure of $I(G)$ with a set of minimal monomial generators $G(\overline{I(G)}) = \{f_{11}, \ldots, f_{1m_1}, \ldots, f_{q1}, \ldots, f_{qm_q}\}$. We claim that $\overline{I(G)}$ has linear quotients with respect to the ordering

$$f_{11}, \ldots, f_{1m_1}, \ldots, f_{q1}, \ldots, f_{qm_q} \tag{1}$$

of $G(\overline{I(G)})$, where $f_{i1} <_{Lex} \cdots <_{Lex} f_{im_i}$ by the ordering

$$x_{11} > \cdots > x_{1i_1} > \cdots > x_{n1} > \cdots > x_{ni_n}$$

for all i.

Now let $u, v \in G(\overline{I(G)})$ be two monomials such that in (1) the monomial u appears before v. In order to show that $\overline{I(G)}$ has linear quotients with respect to the above mentioned order, we must show that there exists a variable x_{ij} and a monomial $w \in G(\overline{I(G)})$ such that $x_{ij}|(u/\gcd(u, v))$, in (1) the monomial w comes before v, and $x_{ij} = w/\gcd(w, v)$.

We define a K-algebra homomorphism $\varphi : T \to S$ by $\varphi(x_{ij}) = x_i$ for all i, j. We suppose that $\varphi(u) = \varphi(v)$, and x_{ij} is the greatest variable with respect to the given order on the variables such that $x_{ij}|(u/\gcd(u, v))$. Let $w \in G(\overline{I(G)})$ be the monomial ideal with $\varphi(w) = \varphi(v)$ and $w = x_{ij}\gcd(v, w)$. Since the order of monomials in (1) are given by the lexicographical order, it then follows that w comes before v in (1).

Next assume that $\varphi(u) \neq \varphi(v)$. Let x_{ij} be the variable which divides $u/\gcd(u, v)$. Hence there exists a monomial $w \in G(\overline{I(G)})$ such that $\varphi(w)$ coming before $\varphi(v)$. Then there exists a variable x_i such that $x_i|(\varphi(u)/\gcd(\varphi(u), \varphi(v))$ and $x_i = \varphi(w)/\gcd(\varphi(w), \varphi(v))$. Therefore, the monomial w before v in (1) and

$$w = x_{ij}\gcd(w, v),$$

as desired. □

As an immediate consequence we obtain the following important result

Corollary 3.2. *Let $T = K[\mathbf{X}]$ be the polynomial ring over a field K and G be a strong quasi-n-partite graph. Then $\overline{I(G)}$ has a linear resolution.*

Proof. Follows from the general fact that the ideals generated in the same degree with linear quotients have a linear resolution (see [7, Proposition 8.2.1]). □

A *vertex cover* of a monomial ideal $L \subset T$ is a subset C of $\mathbf{X}$ such that each $u \in G(L)$ is divided by some x_{ij} on C. The vertex cover C is called *minimal* if no proper subset of C is a vertex cover of L.

Now we investigate algebraic and homological invariants of $T/\overline{I(G)}$.

Lemma 3.3. *Let $I(G) \subset T$ be the edge ideal of a strong quasi-n-partite graph G. Then* $\text{height}(\overline{I(G)}) = m_1 + \cdots + m_n$.

Proof. Let G be a strong quasi-n-partite on the vertex set $V(G)$ and $I(G)$ be its edge ideal. In fact, P is a minimal prime ideal of $I(G)$ if and only if $P = (C)$, for some minimal vertex cover C of G ([15, Proposition 6.1.16]). The minimal cardinality of the vertex covers of $\overline{I(G)}$ is $\text{height}(\overline{I(G)}) = m_1 + \cdots + m_n$ being $C = \{x_{11}, \ldots, x_{1m_1}, x_{21}, \ldots, x_{2m_2}, \ldots, x_{n1}, \ldots, x_{nm_n}\}$ a minimal vertex cover of $\overline{I(G)}$ by construction. □

Consider the minimal graded free resolution of $M = T/L$ as an T-module:

$$\mathbb{F}: \qquad 0 \to \bigoplus_j T(-j)^{b_{gj}} \to \cdots \bigoplus_j T(-j)^{b_{1j}} \to T \to T/L \to 0.$$

The *Castelnuovo-Mumford regularity* of M is defined as:

$$\text{reg}(M) = \max\{j - i \mid b_{ij} \neq 0\}.$$

The numbers $b_{ij} = \dim Tor_i(K, M)_j$ are called the *graded Betti numbers* of M, and $b_i = \sum_j b_{ij}$ is called the *ith Betti number* of M.

Theorem 3.4. *Let $T = K[\mathbf{X}]$ be the polynomial ring over a field K and G be a strong quasi-n-partite graph. Then* *(i)* $\dim(T/\overline{I(G)}) = 0$;

(ii) $\operatorname{proj}\dim(T/\overline{I(G)}) = m_1 + \cdots + m_n$;

(iii) $\operatorname{depth}(T/\overline{I(G)}) = 0$;

(iv) $\operatorname{reg}(T/\overline{I(G)}) = 1$.

Proof. (i) Let G be a strong quasi-n-partite graph on the vertex set $V(G)$ and let $I(G)$ be its edge ideal. By [15, Corollary 7.2.5] we have $\dim(T/\overline{I(G)}) = \dim T - \operatorname{height}(\overline{I(G)})$. Hence Lemma 3.3 implies that $\dim(T/\overline{I(G)}) = 0$.

(ii) The length of the minimal free resolution of $T/\overline{I(G)}$ over T is equal to $\mathfrak{r}(\overline{I(G)}) + 1$ ([10, Corollary 1.6]). Then Proposition 3.1 yields $\operatorname{proj}\dim(T/\overline{I(G)}) = m_1 + \cdots + m_n$.

(iii) By Lemma 3.3 we conclude that $\operatorname{height}(\overline{I(G)}) = m_1 + \cdots + m_n$. Therefore $\overline{I(G)}$ is Cohen-Macaulay, and hence $\dim(T/\overline{I(G)}) = \operatorname{depth}(T/\overline{I(G)}) = 0$.

(iv) It follows from Theorem 3.2 that $\overline{I(G)}$ has a linear resolution. Then $\operatorname{reg}(\overline{I(G)}) = 2$, the assertion follows. □

Example 3.5. Let $T = K[x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}]$ be a polynomial ring over a field K. Let G be a strong quasi-bipartite graph on the vertex set $V(G) = \{x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}\}$. Then

$$\begin{aligned} I(G) = \ & (x_{11}^2, x_{12}^2, x_{13}^2, x_{21}^2, x_{22}^2, x_{23}^2, x_{11}x_{21}, x_{11}x_{22}, x_{11}x_{23}, x_{12}x_{21}, x_{12}x_{22}, x_{12}x_{23},\\ & x_{13}x_{21}, x_{13}x_{22}, x_{13}x_{23}). \end{aligned}$$

A computation with Normalize ([14]) gives

$$\begin{aligned} \overline{I(G)} = \ & (x_{11}^2, x_{12}^2, x_{13}^2, x_{21}^2, x_{22}^2, x_{23}^2, x_{11}x_{12}, x_{11}x_{13}, x_{11}x_{21}, x_{11}x_{22}, x_{11}x_{23}, x_{12}x_{21}, x_{12}x_{22},\\ & x_{12}x_{23}, x_{12}x_{13}, x_{21}x_{22}, x_{21}x_{23}, x_{22}x_{23}, x_{13}x_{21}, x_{13}x_{22}, x_{13}x_{23}). \end{aligned}$$

There exists a one to one correspondence between the minimal vertex covers of G and the minimal prime ideals of $I(G)$. Then the irredundant primary decomposition of $\overline{I(G)}$ is

$$\begin{aligned} \overline{I(G)} &= (x_{11}, x_{12}, x_{13}, x_{21}^2, x_{22}^2, x_{23}^2, x_{21}x_{22}, x_{21}x_{23}, x_{22}x_{23})\\ &\quad \cap (x_{11}^2, x_{12}^2, x_{13}^2, x_{21}, x_{22}, x_{23}, x_{11}x_{12}, x_{11}x_{13}, x_{12}x_{13}). \end{aligned}$$

Hence Lemma 3.3 implies that $\operatorname{height}(\overline{I(G)}) = 6$.

By [10, Corollary 1.6] it follows that the length of the minimal free resolution of $T/\overline{I(G)}$ over T is equal to $\mathfrak{r}(\overline{I(G)}) + 1 = 6$ ([10, Corollary 1.6]). Therefore, Theorem 3.4 yields

(i) $\dim(T/\overline{I(G)}) = 0$;

(ii) $\operatorname{proj}\dim(T/\overline{I(G)}) = 6$;

(iii) $\operatorname{depth}(T/\overline{I(G)}) = 0$;

(iv) $\operatorname{reg}(T/\overline{I(G)}) = 1$.

In the following, we compute the Betti numbers of the integral closure of $I(G)$.

Theorem 3.6. *Let $T = K[\mathbf{X}]$ be the polynomial ring over a field K and G be a strong quasi-n-partite graph. Then for $i \geq 0$, we have*

$$b_i(\overline{I(G)}) = \binom{m_1 + \cdots + m_n + 1}{m_1 + \cdots + m_n - i - 1}\binom{i+1}{i}.$$

Proof. Let G be a strong quasi-n-partite on the vertex set $V(G)$, and let $I(G)$ be its edge ideal. Corollary 3.2 implies that $\overline{I(G)}$ has a linear resolution. Then Eagon-Northcatt complex resolving $\overline{I(G)}$ gives the Betti numbers; see for example [2]. Alternatively one can use the Herzog-Kühl formula to obtain the Betti numbers [8, Theorem 1]. Next by Auslander-Buchsbaum formula, one has $\operatorname{proj}\dim(\overline{I(G)}) = m_1+\cdots+m_n-1$. Therefore, by Herzog-Kühl formula one obtains that

$$\begin{aligned} b_i(\overline{I(G)}) &= \frac{(m_1+\cdots+m_n+1)!}{(m_1+\cdots+m_n-i-1)!(2+i)!} \times \frac{(i+1)!}{i!} \\ &= \binom{m_1+\cdots+m_n+1}{m_1+\cdots+m_n-i-1}\binom{i+1}{i}, \end{aligned}$$

as desired. □

Let $I \subset S = K[x_1,\ldots,x_n]$ be a graded ideal. We consider S/I as a standard graded K-algebra. We have the following (see [15]):

Proposition 3.7. *Let S/I be a Cohen-Macaulay ring then the type of S/I is equal to the last Betti number in the minimal free resolution of S/I as an S-module.*

Proposition 3.8. *Let G be a strong quasi-n-partite graph and $I(G) \subset T$ its edge ideal. Then*

$$\operatorname{type}(T/\overline{I(G)}) = m_1+\cdots+m_n.$$

Proof. By Theorem 3.4, we have $T/\overline{I(G)}$ is Cohen-Macaulay, then $\dim(T/\overline{I(G)}) = 0$. By Auslander-Buchsbaum formula we obtain that $\operatorname{proj}\dim(T/\overline{I(G)}) = m_1+\cdots+m_n$. Hence Theorem 3.6 together with Proposition 3.7 now yields

$$\begin{aligned} b_{m_1+\cdots+m_n-1}(\overline{I(G)}) &= \binom{m_1+\cdots+m_n+1}{0}\binom{m_1+\cdots+m_n-1+1}{m_1+\cdots+m_n-1} \\ &= m_1+\cdots+m_n. \end{aligned}$$

Then the assertion follows. □

Next we want to study the ideals of vertex covers for the class of edge ideals associated to quasi-n-partite graphs.

Let $I \subset S = K[x_1,\ldots,x_n]$ be a monomial ideal. The ideal of *(minimal) covers* of I, denoted by I_c, is the ideal of S generated by all monomials $x_{i_1}\cdots x_{i_r}$ such that $(x_{i_1},\ldots,x_{i_r})$ is an associated (minimal) prime ideal of I. Let G be a graph and let $I(G)$ be its edge ideal. We define $I_c(G)$ the ideal of *vertex covers* of $I(G)$. Then

$$I_c(G) = \Big(\bigcap_{\{v_i,v_j\}\in E(G), i\neq j}(x_i,x_j)\Big) \cap \Big(x_p \mid \{v_p,v_p\} \in \mathcal{L}(G), \quad p \neq i,j\Big).$$

Proposition 3.9. *Let G be a graph with loops, and let $I_c(G)$ be the ideal of vertex covers of $I(G)$. Then for all $k \geq 1$ we have*

$$\overline{I_c(G)^k} = \left(\left\{\mathbf{x}^{\lceil\alpha\rceil} \;\middle|\; \alpha \in \operatorname{conv}(k\log(I_c(G))\right\}\right).$$

Proof. Let G be a graph on the vertex set $V(G) = \{x_1,\ldots,x_n\}$, and let $I_c(G)$ be the ideal of vertex covers of $I(G)$ generated by the monomials $\mathbf{x}^{\mathbf{v}_1},\ldots,\mathbf{x}^{\mathbf{v}_q}$. We put $\log(I_c(G)) = \{\mathbf{v}_1,\ldots,\mathbf{v}_q\} \subset \mathbb{N}^n$ and we define the set

$$k\log(I_c(G)) = \{\mathbf{v}_{p_1}+\mathbf{v}_{p_2}+\cdots+\mathbf{v}_{p_k} \mid 1 \leq p_1 \leq \cdots \leq p_k \leq q\}.$$

We assume that $k\log(I_c(G)) = \{\mathfrak{v}_1, \dots, \mathfrak{v}_r\}$, where $\mathfrak{v}_p = \mathbf{v}_{p_1} + \mathbf{v}_{p_2} + \cdots + \mathbf{v}_{p_k}$, with $1 \le p_1 \le \cdots \le p_k \le q$. Then there are $r = \binom{k+q-1}{k}$ elements in $k\log(I_c(G))$, and hence

$$\operatorname{conv}(k\log(I_c(G))) = \left\{ \sum_{i=1}^{r} \lambda_i \mathfrak{v}_i \;\middle|\; \sum_{i=1}^{r} \lambda_i = 1, \lambda_i \in \mathbb{Q}_+ \right\}$$

is the convex hull of $k\log(I_c(G))$.

Now let $\alpha = \sum_{i=1}^{r} \lambda_i \mathfrak{v}_i$ with $\mathfrak{v}_i \in k\log(I_c(G))$, $\sum_{i=1}^{r} \lambda_i = 1$, $\lambda_i \in \mathbb{Q}_+$. We know that $\lceil\alpha\rceil \ge \alpha$, there is $\beta \in \mathbb{Q}_+^n$ such that $\lceil\alpha\rceil = \alpha + \beta$. Then there is $0 \ne h \in \mathbb{N}$ so that $h\beta \in \mathbb{N}^n$ and $h\lambda_i \in \mathbb{N}$ for all i. Therefore

$$\mathbf{x}^{h\lceil\alpha\rceil} = \mathbf{x}^{h\beta}\mathbf{x}^{h\alpha} = \mathbf{x}^{h\beta}(\mathbf{x}^{\mathfrak{v}_1})^{h\lambda_1} \cdots (\mathbf{x}^{\mathfrak{v}_r})^{h\lambda_r} \in (I_c(G)^k)^h \Rightarrow \mathbf{x}^{\lceil\alpha\rceil} \in \overline{I_c(G)^k}.$$

Conversely, let $\mathbf{x}^\gamma \in \overline{I_c(G)^k}$, that is $\mathbf{x}^{h\gamma} \in (I_c(G)^k)^h$ for some $0 \ne h \in \mathbb{N}$. There are nonnegative integers $a_1, \dots, a_r$ such that

$$\mathbf{x}^{h\gamma} = \mathbf{x}^{\vartheta}(\mathbf{x}^{\mathfrak{v}_1})^{a_1} \cdots (\mathbf{x}^{\mathfrak{v}_r})^{a_r} \quad \text{and} \quad a_1 + \cdots + a_r = h.$$

It then follows that $\gamma = (\vartheta/h) + \sum_{i=1}^{r}(a_i/h)\mathfrak{v}_i$. In addition, we set $\alpha = \sum_{i=1}^{r}(a_i/h)\mathfrak{v}_i$. By dividing the entries of ϑ by h we can write $\gamma = \beta + \xi + \alpha$, where $0 \le \xi_i < 1$ for all i and $\beta \in \mathbb{N}^n$. Note that $\xi + \alpha \in \mathbb{N}^n$. This implies that $\lceil\alpha\rceil = \xi + \alpha$. Therefore $\mathbf{x}^\gamma = \mathbf{x}^\beta \mathbf{x}^{\lceil\alpha\rceil}$, where $\alpha \in \operatorname{conv}(\mathfrak{v}_1, \dots, \mathfrak{v}_r)$ as desired. □

Let L be a monomial ideal of $T = K[\mathbf{X}]$. The *big height of* L, denoted by $\operatorname{bight}(L)$, is the maximum among the heights of the associated primes of L, namely

$$\operatorname{bight}(L) = \max\{\operatorname{height}(P) \mid P \in \operatorname{Ass}(T/L)\}.$$

Lemma 3.10. *Let G be a strong quasi-n-partite graph on the vertex set $V(G)$. Then*

$$\operatorname{bight}(\overline{I_c(G)}) = 1.$$

Proof. Let G be a strong quasi-n-partite graph on the vertex set $V(G) = V_1 \cup V_2 \cup \cdots \cup V_n$ and $V_i = \{x_{i1}, \dots, x_{im_i}\}$ for $i = 1, \dots, n$. Let $I(G)$ be the edge ideal of G generated by the monomials $\mathbf{X}_1^{a_{1_1}} \cdots \mathbf{X}_n^{a_{1_n}}, \mathbf{X}_1^{a_{2_1}} \cdots \mathbf{X}_n^{a_{2_n}}, \dots, \mathbf{X}_1^{a_{q_1}} \cdots \mathbf{X}_n^{a_{q_n}}$, where $\mathbf{X}_1^{a_{j_1}} \cdots \mathbf{X}_n^{a_{j_n}}$ stands for

$$x_{11}^{a_{j_{11}}} \cdots x_{1m_1}^{a_{j_{1m_1}}} x_{21}^{a_{j_{21}}} \cdots x_{2m_2}^{a_{j_{2m_2}}} \cdots x_{n1}^{a_{j_{n1}}} \cdots x_{nm_n}^{a_{j_{nm_n}}}$$

for $j = 1, \dots, q$. We assume that $\mathbf{v} = (\underbrace{1, \dots, 1}_{m_1-\text{times}}, \dots, \underbrace{1, \dots, 1}_{m_n-\text{times}}) \in \mathbb{N}^{m_1} \oplus \cdots \oplus \mathbb{N}^{m_n}$ be a vector. Later by using [15, Proposition 12.1.4]) it turns out that $\overline{I_c(G)} = (\{\mathbf{X}^{\lceil\alpha\rceil} \mid \alpha \in \operatorname{conv}(\mathbf{v})\})$, with

$$\operatorname{conv}(\mathbf{v}) = \lambda_j \mathbf{v} \quad \text{with} \quad \lambda_j \in \mathbb{Q}_+.$$

Let f be a generator of $\overline{I_c(G)}$. Then $f = \mathbf{X}^{\lceil\alpha\rceil}$ with $\alpha = (\lambda_j, \dots, \lambda_j)$, $\lambda_j = 1$. Then $\mathbf{X}^{\lceil\alpha\rceil} = \mathbf{X}^{\mathbf{v}}$, that is $\mathbf{X}^{\lceil\alpha\rceil} = x_{11} \cdots x_{1m_1} \cdots x_{n1} \cdots x_{nm_n}$. Therefore $I_c(G)$ is integrally closed, and $\overline{I_c(G)} = I_c(G)$. The maximal cardinality of the vertex covers of $I(G)$ is $\operatorname{bight}(I_c(G)) = 1$ being $\{x_{ij}\}$ a maximal vertex cover of $I(G)$ by construction. □

Theorem 3.11. *Let $T = K[\mathbf{X}]$ be the polynomial ring over a field K and G be a strong quasi-n-partite graph. Then*

(i) $\dim(T/\overline{I_c(G)}) = m_1 + \cdots + m_n - 1$;

(ii) $\operatorname{proj}\dim(T/\overline{I_c(G)}) = 1$;

(iii) $\operatorname{depth}(T/\overline{I_c(G)}) = m_1 + \cdots + m_n - 1$;

(iv) $\operatorname{reg}(T/\overline{I_c(G)}) = m_1 + \cdots + m_n - 1$.

Proof. (i) Let G be a strong quasi-n-partite graph on the vertex set $V(G)$, and let $I(G)$ be its edge ideal. The minimal cardinality of the vertex covers of $I_c(G)$ is height$(I_c(G)) = 1$ being $C = \{x_{il}\}$ a minimal vertex cover of $I_c(G)$ by construction. Therefore, [15, Corollary 7.2.5] implies that

$$\dim(T/\overline{I_c(G)}) = m_1 + \cdots + m_n - 1.$$

(ii) Using Lemma 3.10 and [7, Theorem 12.6.7], together with [15, Corollary 6.4.20] now yield

$$\operatorname{proj}\dim(T/\overline{I_c(G)}) = \operatorname{bight}(\overline{I_c(G)}) = 1.$$

(iii) By the Auslander-Buchsbaum formula (see [15, Theorem 3.5.13]), one has the equality

$$\operatorname{depth}(T/\overline{I_c(G)}) = \dim T - \operatorname{proj}\dim(T/\overline{I_c(G)}) = m_1 + \cdots + m_n - 1.$$

(iv) The ideal $\overline{I_c(G)}$ generated in degree $m_1 + \cdots + m_n$. Then [7, Proposition 8.2.1] and [7, Theorem 12.6.2] says that $\overline{I_c(G)}$ has a $m_1 + \cdots + m_n$-linear resolution. Therefore, $\operatorname{reg}(\overline{I_c(G)}) = m_1 + \cdots + m_n$, as desired. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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