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Existence and uniqueness of pseudo S-asymptotically (w,c)-periodic solutions with measure for some systems of neutral integral equations

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Abstract. In this article, we study a new extension of the (w,c)-periodic functions that we call measure pseudo S-asymptotically (w,c)-periodic functions using the measure theory. By employing an adequate fixed point theorem and differential inequality, we investigate the existence and uniqueness of μ-pseudo S-asymptotically (w,c)-periodic solutions for systems of neutral integral equations. For explanation, numerical example is given to clarify our main results.

1. Introduction

The qualitative and quantitative study of different methods is an important field, in particular on the existence and uniqueness of the almost periodic, pseudo almost periodic, pseudo almost periodic with measures and asymptotically omega periodic solutions for some models of evolution equations, for example, see [5–7, 17–35].
The first results of the category of (w,c)-periodic functions was given by Pinto in [37], which comprise the classical w-periodicity (for $c = 1$), w-anti-periodicity (for $c = -1$) and the Bloch w-periodicity (for $c = e^{ikw}$) as particular instances. Many researchers worked on this concept, in [4] Alvarez interested in the completeness, convolution and composition theorems for (w,c)-periodic functions in abstract spaces and he studied in [3] the existence of (w,c)-periodic solutions for abstract fractional difference equations also Mophou made an existence result of (w,c)-periodic mild solutions to some fractional differential equation (see [36]) and there are many results related to (w,c)-periodic functions and some applications are developed in [12–14].

Alvarez et al introduced an extension of the (w,c)-periodic functions, the so-called the (w,c)-asymptotically periodic functions (which contains asymptotically periodic, asymptotically antiperiodic, asymptotically Bloch-periodic and unbounded functions) and (w,c)-pseudo periodic functions in abstract spaces with applications to the first order abstract Cauchy problem and Lasota-Wazewska model with ergodic and unbounded oscillating production of red cells respectively (see [1, 2]). Recently, Chang and Zhao introduced

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a new expansion of (w,c)-periodic function called pseudo S-asymptotically (w,c)-periodic functions and established the completeness, convolution and superposition theorems for it in abstract spaces in [8].

Existence and uniqueness of (w,c)-periodic, (w,c)-asymptotically periodic, (w,c)-pseudo periodic and pseudo S-asymptotically (w,c)-periodic solutions are considered one of the most important and vital topics in the qualitative theory of ordinary or functional differential equations to applications in the physical sciences, mathematical biology...(see [1–4, 8, 13, 14, 36]).

The purpose of this work is to present an extension of pseudo S-asymptotically (w,c)-periodic function is the measure pseudo S-asymptotically (w,c)-periodic function and establish some fundamental properties. Obviously, this new notion generalize all the different notions recently studied in the literature. That's why we make an extensive use of theoretical measure theory to introduce and study the concept of measure pseudo S-asymptotically (w,c)-periodic function. In fact many authors interested in their works in using the measure theory, we can refer [10, 15, 16].
We are thus interested in this paper to study the existence and uniqueness of the pseudo S-asymptotically (w,c)-periodic solutions for the following equation:

$$\begin{cases} x(t) = \sigma_1(t)x(t-\rho_1) + \displaystyle\int_{t-\tau_1(t)}^{t} f_1(s, x(s-\lambda_1), y(s-\lambda_2))ds, \\ y(t) = \sigma_2(t)y(t-\rho_2) + \displaystyle\int_{t-\tau_2(t)}^{t} f_2(s, x(s-\lambda_1), y(s-\lambda_2))ds. \end{cases} \tag{1}$$

Let $\rho_1, \rho_2, \lambda_1, \lambda_2 \geq 0$ be fixed numbers and let $f_1, f_2 : \mathbb{R}\times\mathbb{C}\times\mathbb{C} \to \mathbb{C}$, $\sigma_1, \sigma_2 : \mathbb{R} \to \mathbb{R}_+$ and $\tau_1, \tau_2 : \mathbb{R} \to \mathbb{R}_+$ be appropriate functions verifying some hypothesis mentioned hereafter.

In fact this type of equation is studied by several authors, in [9] Ding and N'Guérékata studied the existence of positive pseudo almost periodic solutions to a class of neutral integral equation:

$$x(t) = \alpha(t)x(t-\beta) + \int_{-\infty}^{t} a(t, t-s)f(s, x(s))ds, t \in \mathbb{R}.$$

In [38] Zhao and N'Guérékata investigated the existence of S-asymptotically periodic solutions for the following delay integral equation with super-linear perturbations:

$$x(t) = \alpha(t)x^n(t-\beta) + \int_{t-\tau(t)}^{t} f(s, x(s))ds, t \in \mathbb{R}.$$

Hamza and Abdellatif investigated in [11] the existence of positive weighted pseudo S-asymptotically periodic solution in Stepanov-like sense for some systems of nonlinear delay integral equations with super-linear perturbations of the following type:

$$\begin{cases} x(t) = \alpha_1(t)x^{\eta}(t-l) + \displaystyle\int_{0}^{\tau_1(t)} f(s, \sigma, x(s-\sigma-l), y(s-\sigma-l))ds, \\ y(t) = \alpha_2(t)y^{\nu}(t-l) + \displaystyle\int_{0}^{\tau_2(t)} g(s, \sigma, x(s-\sigma-l), y(s-\sigma-l))ds. \end{cases}$$

The rest of this article is structured as follows: in section 2 we list definitions, examples and basic results which will be used throughout this paper. In section 3 we discuss the existence and uniqueness of μ-pseudo S-asymptotically (w,c)-periodic solutions to the system (1). Finally, in Section 4 we give an example to check our results found.

2. Preliminaries:

In this section, we reminder some notations, definitions and auxiliary results which will be used all along this paper.
Let X be a Banach space, $\mathbb{R}$ be the set of all real numbers, $\mathbb{C}$ be the collection of all complex numbers, $\mathbb{N}$ be the set of all positive integers and $\mathcal{BC}(\mathbb{R}, X)$ denotes the Banach space of bounded continuous functions from $\mathbb{R}$ to X, equipped with the supremum norm

$$\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(t)\|.$$

Definition 2.1. *[37] For given $c \in \mathbb{C}^*$, $w \in \mathbb{R}^+$, a function $f \in \mathcal{BC}(\mathbb{R}, X)$ is said to be (w, c)-periodic if*

$$f(t + w) = cf(t), \forall t \in \mathbb{R}.$$

The collection of such functions will be denoted by $\mathcal{P}_{w,c}(\mathbb{R}, X)$.

For given $c \in \mathbb{C}^*$, $w \in \mathbb{R}^+$, we define the following set

$$\mathcal{BC}_{w,c}(\mathbb{R}, X) = \{f \in C(\mathbb{R}, X) : \sup_{t\in\mathbb{R}} \|c^\wedge(-t)f(t)\| < +\infty\},$$

with $c^\wedge(t) = c^{\frac{t}{w}} = \exp(\frac{t}{w}\log(c))$ and $|c^\wedge(t)| = |c|^\wedge(t) = |c|^{\frac{t}{w}}$.
We have the following properties for the set $\mathcal{BC}_{w,c}(\mathbb{R}, X)$:

Proposition 2.2. *[8]*

- *Let $f \in C(\mathbb{R}, X)$, then $f \in \mathcal{BC}_{w,c}(\mathbb{R}, X)$ if and only if $f(t) = c^\wedge(-t)u(t)$, $u(t) \in \mathcal{BC}(\mathbb{R}, X)$.*
- *The set $\mathcal{BC}_{w,c}(\mathbb{R}, X)$ is a Banach space with the norm*

$$\|f\|_{bwc} = \sup_{t\in\mathbb{R}} \|c^\wedge(-t)f(t)\|.$$

Definition 2.3. *[8] A function $f \in \mathcal{BC}_{w,c}(\mathbb{R}, X)$ is said to be S-asymptotically (w, c) periodic if*

$$\lim_{|t|\to+\infty} \|c^\wedge(-t)[f(t + w) - cf(t)]\| = 0.$$

The collection of such functions will be denoted by $\mathcal{SAP}_{w,c}(\mathbb{R}, X)$.

Example 2.4. *$\forall c \in \mathbb{C}^*$ we consider $f(t) = \frac{c^\wedge(t)}{1+t^2}$, $t \in \mathbb{R}$. f is S-asymptotically (w, c)-periodic. Indeed*

$$c^\wedge(-t)[f(t + w) - cf(t)] = c^\wedge(-t)[\frac{c^\wedge(t + w)}{1 + (t + w)^2} - c\frac{c^\wedge(t)}{1 + t^2}] = \frac{c^\wedge(w)}{1 + (t + w)^2} - c\frac{1}{1 + t^2}$$

we have $\lim_{|t|\to+\infty} [\frac{c^\wedge(w)}{1+(t+w)^2} - c\frac{1}{1+t^2}] = 0$ then we obtain $\lim_{|t|\to+\infty} |c^\wedge(-t)[f(t + w) - cf(t)]| = 0$.

Remark 2.5. *For $c = 1$ we obtain*

$$\lim_{|t|\to+\infty} \|f(t + w) - f(t)\| = 0.$$

f is said to be S-asymptotically w-periodic.

Proposition 2.6. *[8] Let $f, f_2, f_2 \in \mathcal{SAP}_{w,c}(\mathbb{R}, X)$. Then the following results hold:*

- *$f_1 + f_2 \in \mathcal{SAP}_{w,c}(\mathbb{R}, X)$ and $kf \in \mathcal{SAP}_{w,c}(\mathbb{R}, X)$ for any $k \in \mathbb{C}$.*

- *The space $\mathcal{SAP}_{w,c}(\mathbb{R}, X)$ is translation invariant.*
- *The space $(\mathcal{SAP}_{w,c}(\mathbb{R}, X), ||.||_{bwc})$ is a Banach space.*

Definition 2.7. *[8] A function $f \in \mathcal{BC}_{w,c}(\mathbb{R}, X)$ is said to be pseudo S-asymptotically (w, c) periodic if*

$$\lim_{r\to+\infty} \frac{1}{2r} \int_{-r}^{r} \left\| c^\wedge(-t)[f(t+w) - cf(t)] \right\| dt = 0, \forall t \in \mathbb{R}.$$

The set of all such functions will be denoted by $\mathcal{PSAP}_{w,c}(\mathbb{R}, X)$.

Example 2.8. *$\forall c \in \mathbb{C}^*$ we consider $f(t) = \frac{c^\wedge(t)}{1+t^2}$, $t \in \mathbb{R}$. f is pseudo S-asymptotically (w, c)-periodic. Indeed:*

$$\begin{aligned} \frac{1}{2r} \int_{-r}^{r} |c^\wedge(-t)[f(t+w) - cf(t)]| dt &= \frac{1}{2r} \int_{-r}^{r} |\frac{c^\wedge(w)}{1+(t+w)^2} - c\frac{1}{1+t^2}| dt \\ &\leq \frac{|c|^\wedge(w)}{2r} \int_{-r}^{r} \frac{1}{1+(t+w)^2} dt + \frac{|c|}{2r} \int_{-r}^{r} \frac{1}{1+t^2} dt \\ &\leq \frac{|c|^\wedge(w)}{2r} [\arctan(w+r) - \arctan(w-r)] + \frac{|c|}{r} \arctan(r) \\ &\to 0, (r \to \infty). \end{aligned}$$

Remark 2.9. *For $c = 1$ we obtain*

$$\lim_{r\to+\infty} \frac{1}{2r} \int_{-r}^{r} \left\| f(t+w) - f(t) \right\| dt = 0, \forall t \in \mathbb{R}.$$

f is said to be pseudo S-asymptotically w-periodic.

Proposition 2.10. *[8] Let $f, f_1, f_2 \in \mathcal{PSAP}_{w,c}(\mathbb{R}, X)$. Then the following results hold:*

- *$f_1 + f_2 \in \mathcal{PSAP}_{w,c}(\mathbb{R}, X)$ and $kf \in \mathcal{PSAP}_{w,c}(\mathbb{R}, X)$ for any $k \in \mathbb{C}$.*
- *The space $\mathcal{PSAP}_{w,c}(\mathbb{R}, X)$ is translation invariant.*
- *The space $(\mathcal{PSAP}_{w,c}(\mathbb{R}, X), ||.||_{bwc})$ is a Banach space.*

We propose in this article to generalize the pseudo S-asymptotically (w, c) periodic functions studied recently by using Lebesgue measure. For that let $\mathcal{B}$ denote the lebesque $\sigma - fieled$ of $\mathbb{R}$ and let $\mathcal{M}$ be the set of all positive measures μ on $\mathcal{B}$ satisfying

$$\mu(\mathbb{R}) = +\infty \text{ and } \mu([a, b]) < \infty, \text{ for all } a, b \in \mathbb{R} (a \leq b).$$

We formulate the following hypothesis:

(H0) For $\mu \in \mathcal{M}$, $\tau \in \mathbb{R}$, there exist $\beta > 0$ and a bounded interval I such that:

$$\mu(a + \tau : a \in A) \leq \beta, \text{ when } A \in \mathcal{B} \text{ satisfies } A \cap I = \emptyset.$$

Remark 2.11. *(**H0**) implies*

$$\text{for all } \sigma > 0, \lim_{r\to+\infty} \sup \tfrac{\mu[-r-\sigma, r+\sigma]}{\mu[-r,r]} < \infty.$$

Definition 2.12. *Let $\mu \in \mathcal{M}$. A function $f \in \mathcal{BC}_{w,c}(\mathbb{R}, X)$ is said to be μ-pseudo S-asymptotically (w, c) periodic if*

$$\lim_{r\to+\infty} \frac{1}{\mu([-r, r])} \int_{-r}^{r} \left\| c^\wedge(-t)[f(t+w) - cf(t)] \right\| d\mu(t) = 0, \forall t \in \mathbb{R}.$$

The collection of such functions will be denoted by $\mathcal{PSAP}_{w,c}(\mathbb{R}, X, \mu)$.

Example 2.13. *$\forall c \in \mathbb{C}^*$ we consider $f(t) = \frac{c^\wedge(t)}{1+t^2}$, $t \in \mathbb{R}$ and we pose the measure μ defined by $d\mu(t) = \rho(t)dt$ where*

$$\rho(t) = \begin{cases} t & if\, t > 0 \\ 1 & if\, t \leq 0 \end{cases}$$

we obtain $\mu([-r,r]) = r + \frac{r^2}{2}$, then

$$\begin{aligned}
& \frac{1}{\mu([-r,r])}\int_{-r}^{r} |c^\wedge(-t)[f(t+w) - cf(t)]| d\mu(t) \\
= \; & \frac{1}{\mu([-r,r])}\int_{-r}^{r} |\frac{c^\wedge(w)}{1+(t+w)^2} - c\frac{1}{1+t^2}|\rho(t)dt \\
\leq \; & \frac{|c|^\wedge(w)}{\mu([-r,r])}\int_{-r}^{0} \frac{1}{1+(t+w)^2}dt + \frac{|c|^\wedge(w)}{\mu([-r,r])}\int_{0}^{r} \frac{t}{1+(t+w)^2}dt \\
+ \; & \frac{|c|}{\mu([-r,r])}\int_{-r}^{0} \frac{1}{1+t^2}dt + \frac{|c|}{\mu([-r,r])}\int_{0}^{r} \frac{t}{1+t^2}dt \\
\leq \; & \frac{1}{r+\frac{r^2}{2}}\Big[|c|^\wedge(w)[\arctan(w) - \arctan(w-r)] + \frac{|c|^\wedge(w)}{2}[\ln(r+w) - \ln(w)] \\
- \; & w|c|^\wedge(w)[\arctan(r+w) - \arctan(w)] + |c|\arctan(r) + \frac{|c|}{2}\ln(1+r^2)\Big] \\
& \to 0, (r \to +\infty)
\end{aligned}$$

as a result $f \in \mathcal{PSAP}_{w,c}(\mathbb{R}, X, \mu)$.

Remark 2.14. *For $c = 1$ we obtain*

$$\lim_{r\to+\infty} \frac{1}{\mu([-r,r])}\int_{-r}^{r} \left\| f(t+w) - f(t) \right\| d\mu(t) = 0, \forall t \in \mathbb{R}.$$

f is said to be μ-pseudo S-asymptotically w-periodic.

Lemma 2.15. *Let μ satisfies (**H0**) and $f, f_1, f_2 \in \mathcal{PSAP}_{w,c}(\mathbb{R}, X, \mu)$. Then the following results hold:*

- *$f_1 + f_2 \in \mathcal{PSAP}_{w,c}(\mathbb{R}, X, \mu)$ and $kf \in \mathcal{PSAP}_{w,c}(\mathbb{R}, X, \mu)$ for any $k \in \mathbb{C}$.*
- *The space $\mathcal{PSAP}_{w,c}(\mathbb{R}, X, \mu)$ is translation invariant.*
- *The space $(\mathcal{PSAP}_{w,c}(\mathbb{R}, X, \mu), \|.\|_{bwc})$ is a Banach space.*

Proof.
- We pose $F = f_1 + f_2$

$$\begin{aligned}
& \frac{1}{\mu([-r,r])}\int_{-r}^{r} \left\| c^\wedge(-t)[F(t+w) - cF(t)] \right\| d\mu(t) \\
= \; & \frac{1}{\mu([-r,r])}\int_{-r}^{r} \left\| c^\wedge(-t)[f_1(t+w) + f_2(t+w) - c(f_1(t) + f_2(t))] \right\| d\mu(t) \\
\leq \; & \frac{1}{\mu([-r,r])}\int_{-r}^{r} \left\| c^\wedge(-t)[f_1(t+w) - cf_1(t)] \right\| d\mu(t) + \frac{1}{\mu([-r,r])}\int_{-r}^{r} \left\| c^\wedge(-t)[f_2(t+w) - cf_2(t)] \right\| d\mu(t)
\end{aligned}$$

since $f_1, f_2 \in \mathcal{PSAP}_{w,c}(\mathbb{R}, X, \mu)$ so $\lim_{r\to+\infty} \frac{1}{\mu([-r,r])}\int_{-r}^{r} \left\| c^\wedge(-t)[F(t+w) - cF(t)] \right\| d\mu(t) = 0.$

Thus

$$\begin{aligned}
\lim_{r\to+\infty} \frac{1}{\mu([-r,r])}\int_{-r}^{r} \left\| c^\wedge(-t)[kf(t+w) - ckf(t)] \right\| d\mu(t) & \leq \lim_{r\to+\infty} |k| \frac{1}{\mu([-r,r])}\int_{-r}^{r} \left\| c^\wedge(-t)[f(t+w) - cf(t)] \right\| d\mu(t) \\
& = 0.
\end{aligned}$$

- For a given $a \in \mathbb{R}$

$$\begin{aligned}
& \frac{1}{\mu([-r,r])}\int_{-r}^{r}\left\|c^\wedge(-t)[f_a(t+w)-cf_a(t)]\right\|d\mu(t)\\
= & \frac{1}{\mu([-r,r])}\int_{-r}^{r}\left\|c^\wedge(-t)[f(t+a+w)-cf(t+a)]\right\|d\mu(t)\\
= & \frac{1}{\mu([-r,r])}\int_{-r+a}^{r+a}\left\|c^\wedge(-t+a)[f(t+w)-cf(t)]\right\|d\mu(t-a)\\
\leq & \frac{1}{\mu([-r,r])}\int_{-r-|a|}^{r+|a|}|c^\wedge(a)|\left\|c^\wedge(-t)[f(t+w)-cf(t)]\right\|\beta d\mu(t)\\
\leq & \frac{\mu([-r-|a|,r+|a|])}{\mu([-r,r])}\frac{\beta|c^\wedge(a)|}{\mu([-r-|a|,r+|a|])}\int_{-r-|a|}^{r+|a|}\left\|c^\wedge(-t)[f(t+w)-cf(t)]\right\|d\mu(t)\\
\leq & cst\frac{1}{\mu([-r-|a|,r+|a|])}\int_{-r+|a|}^{r+|a|}\left\|c^\wedge(-t)[f(t+w)-cf(t)]\right\|d\mu(t)\\
& \to 0,(r\to\infty).
\end{aligned}$$

- Let $(f_n)_n$ be a sequence of $\mathcal{PSAP}_{w,c}(\mathbb{R},X,\mu)$ such that $f_n \to f, n\to\infty$ then for any $\epsilon>0, \exists n_\epsilon>0$ and $\exists r_\epsilon>0$ such that:

  $*$ $\displaystyle\frac{1}{\mu([-r,r])}\int_{-r}^{r}\left\|c^\wedge(-t)[f_n(t+w)-cf_n(t)]\right\|d\mu(t)\leq\frac{\varepsilon}{3}$

  $*$ $\|f_n-f\|_{bwc} = \sup_{t\in\mathbb{R}}\|c^\wedge(-t)[f_n(t)-f(t)]\| \leq \frac{\varepsilon}{3|c|}$

  for $n>n_\varepsilon$ and $r>r_\varepsilon$. Thus

$$\begin{aligned}
& \frac{1}{\mu([-r,r])}\int_{-r}^{r}\left\|c^\wedge(-t)[f(t+w)-cf(t)]\right\|d\mu(t)\\
= & \frac{1}{\mu([-r,r])}\int_{-r}^{r}\left\|c^\wedge(-t)[f(t+w)-f_n(t+w)+f_n(t+w)-cf_n(t)+cf_n(t)-cf(t)]\right\|d\mu(t)\\
\leq & \frac{1}{\mu([-r,r])}\int_{-r}^{r}|c|\left\|c^\wedge(-t-w)[f(t+w)-f_n(t+w)]\right\|d\mu(t)+\frac{1}{\mu([-r,r])}\int_{-r}^{r}\left\|c^\wedge(-t)[f_n(t+w)-cf_n(t)]\right\|d\mu(t)\\
+ & \frac{1}{\mu([-r,r])}\int_{-r}^{r}|c|\left\|c^\wedge(-t)[f_n(t)-f(t)]\right\|d\mu(t)\\
\leq & 2|c|\|f_n-f\|_{bwc}+\frac{1}{\mu([-r,r])}\int_{-r}^{r}\left\|c^\wedge(-t)[f_n(t+w)-cf_n(t)]\right\|d\mu(t)\\
\leq & \varepsilon.
\end{aligned}$$

Which implies that the $\mathcal{PSAP}_{w,c}(\mathbb{R},X,\mu)$ is a closed sub-space of $\mathcal{BC}_{w,c}(\mathbb{R},X,\mu)$. So it is a Banach space equipped with $\|.\|_{bwc}$. $\square$

Lemma 2.16. *Let $f\in\mathcal{BC}_{w,c}(\mathbb{R},X)$. Then the following assertions are equivalent:*

(a) $\displaystyle\lim_{r\to+\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\left\|c^\wedge(-t)[f(t+w)-cf(t)]\right\|d\mu(t)=0.$

(b) *For each $\varepsilon>0$, $\displaystyle\lim_{r\to+\infty}\frac{1}{\mu([-r,r])}\int_{M_{r,\varepsilon}(f)}dt=0$, where*

$$\mathrm{M}_{r,\varepsilon}(f) = \{t \in [-r, r] : \left\| c^\wedge(-t)[f(t + w) - cf(t)] \right\| \geq \varepsilon\}.$$

Proof. Since

$$\begin{aligned}
&\frac{1}{\mu([-r, r])} \int_{-r}^{r} \left\| c^\wedge(-t)[f(t + w) - cf(t)] \right\| d\mu(t) \\
= \;& \frac{1}{\mu([-r, r])} \int_{[-r,r]\setminus \mathrm{M}_{r,\varepsilon}(f)} \left\| c^\wedge(-t)[f(t + w) - cf(t)] \right\| d\mu(t) + \frac{1}{\mu([-r, r])} \int_{\mathrm{M}_{r,\varepsilon}(f)} \left\| c^\wedge(-t)[f(t + w) - cf(t)] \right\| d\mu(t) \\
\geq \;& \frac{1}{\mu([-r, r])} \int_{\mathrm{M}_{r,\varepsilon}(f)} \left\| c^\wedge(-t)[f(t + w) - cf(t)] \right\| d\mu(t) \\
\geq \;& \frac{\varepsilon}{\mu([-r, r])} \int_{\mathrm{M}_{r,\varepsilon}(f)} d\mu(t) \geq 0.
\end{aligned}$$

The assertion (a) implies the assertion (b). On the other hand,

$$\begin{aligned}
&\frac{1}{\mu([-r, r])} \int_{-r}^{r} \left\| c^\wedge(-t)[f(t + w) - cf(t)] \right\| d\mu(t) \\
= \;& \frac{1}{\mu([-r, r])} \int_{[-r,r]\setminus \mathrm{M}_{r,\varepsilon}(f)} \left\| c^\wedge(-t)[f(t + w) - cf(t)] \right\| d\mu(t) + \frac{1}{\mu([-r, r])} \int_{\mathrm{M}_{r,\varepsilon}(f)} \left\| c^\wedge(-t)[f(t + w) - cf(t)] \right\| d\mu(t) \\
\leq \;& \Big(1 - \frac{1}{\mu([-r, r])} \int_{\mathrm{M}_{r,\varepsilon}(f)} dt\Big)\varepsilon + \frac{1}{\mu([-r, r])} 2|c| \|f\|_{bwc} \int_{\mathrm{M}_{r,\varepsilon}(f)} dt.
\end{aligned}$$

So we can affirm statement (a) by the sooth of statement (b). □

Lemma 2.17. *Under the conditions $x \in \mathcal{PSAP}_{w,c}(\mathbb{R}, X, \mu)$ and σ is w-periodic, we have*

$$t \mapsto \sigma(t)x(t) \in \mathcal{PSAP}_{w,c}(\mathbb{R}, X, \mu).$$

Proof.

$$\begin{aligned}
\frac{1}{\mu([-r, r])} \int_{-r}^{r} \left\| c^\wedge(-t)[\sigma(t + w)x(t + w) - c\sigma(t)x(t)] \right\| d\mu(t) &= \frac{1}{\mu([-r, r])} \int_{-r}^{r} \left\| c^\wedge(-t)[\sigma(t)x(t + w) - c\sigma(t)x(t)] \right\| d\mu(t) \\
&\leq \|\sigma\|_\infty \frac{1}{\mu([-r, r])} \int_{-r}^{r} \left\| c^\wedge(-t)[x(t + w) - cx(t)] \right\| d\mu(t)
\end{aligned}$$

since $x \in \mathcal{PSAP}_{w,c}(\mathbb{R}, X, \mu)$ so $t \mapsto \sigma(t)x(t) \in \mathcal{PSAP}_{w,c}(\mathbb{R}, X, \mu)$. □

Lemma 2.18. *For $\phi \in \mathcal{PSAP}_{w,c}(\mathbb{R}, X, \mu)$ and τ is w-periodic, we have $t \mapsto \int_{t-\tau(t)}^{t} \phi(s)ds \in \mathcal{PSAP}_{w,c}(\mathbb{R}, X, \mu)$.*

Proof. We pose $\varphi(t) = \displaystyle\int_{t-\tau(t)}^{t} \phi(s)ds$

$$\begin{aligned}
& \frac{1}{\mu([-r,r])}\int_{-r}^{r}\left\|c^{\wedge}(-t)\Big[\varphi(t+w)-c\varphi(t)\Big]\right\|d\mu(t)\\
= \;& \frac{1}{\mu([-r,r])}\int_{-r}^{r}\left\|c^{\wedge}(-t)\Big[\int_{t+w-\tau(t+w)}^{t+w}\phi(s)ds - c\int_{t-\tau(t)}^{t}\phi(s)ds\Big]\right\|d\mu(t)\\
= \;& \frac{1}{\mu([-r,r])}\int_{-r}^{r}\left\|c^{\wedge}(-t)\Big[\int_{-\tau(t)}^{0}\phi(s+t+w)ds - c\int_{-\tau(t)}^{0}\phi(s+t)ds\Big]\right\|d\mu(t)\\
= \;& \frac{1}{\mu([-r,r])}\int_{-r}^{r}\int_{-\tau(t)}^{0}\left\|c^{\wedge}(-t)\Big[\phi(s+t+w)-c\phi(s+t)\Big]\right\|dsd\mu(t)\\
\leq \;& \int_{-\|\tau\|_{\infty}}^{0}\Big(\frac{1}{\mu([-r,r])}\int_{-r}^{r}\left\|c^{\wedge}(-t)\Big[\phi(s+t+w)-c\phi(s+t)\Big]\right\|d\mu(t)\Big)ds
\end{aligned}$$

according to lemma 1.6 we have $\displaystyle\lim_{r\to+\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\left\|c^{\wedge}(-t)\Big[\phi(s+t+w)-c\phi(s+t)\Big]\right\|d\mu(t)=0$ and from the dominated convergence theorem we get the result. □

Lemma 2.19. *Let $f \in C(\mathbb{R}\times X\times X, X)$ satisfy the following conditions:*

(A1) **(a)** $\sup_{t\in\mathbb{R}}\|c^{\wedge}(-t)f(t,x,y)\| < +\infty$ *uniformly for* $x, y \in X$.

(b) $\displaystyle\lim_{r\to+\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\left\|c^{\wedge}(-t)[f(t+w,cx,cy)-cf(t,x,y)]\right\|d\mu(t)=0$ *uniformly for* $x,y\in X$.

(A2) *There exists constants $L_1, L_2 > 0$ such that for all $x_1, x_2, y_1, y_2 \in X$ and $t \in \mathbb{R}$,*

$$\|f(t,x_1,y_1)-f(t,x_2,y_2)\| \leq L_1\|x_1-x_2\| + L_2\|y_1-y_2\|.$$

Then for each $\psi_1, \psi_2 \in \mathcal{PSAP}_{w,c}(\mathbb{R},X,\mu)$, $f(.,\psi_1(.),\psi_2(.)) \in \mathcal{PSAP}_{w,c}(\mathbb{R},X,\mu)$.

Proof. It is clear from (A1)(a) that $f(.,\psi_1(.),\psi_2(.)) \in \mathcal{BC}_{w,c}(\mathbb{R},X)$. For each $\psi_1, \psi_2 \in \mathcal{PSAP}_{w,c}(\mathbb{R},X,\mu)$, we have

$$\lim_{r\to+\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\left\|c^{\wedge}(-t)[\psi_i(t+w)-c\psi_i(t)]\right\|d\mu(t)=0, i=1,2.$$

On the other hand

$$\begin{aligned}
& \frac{1}{\mu(r,\rho)}\int_{-r}^{r}\left\|c^{\wedge}(-t)[f(t+w,\psi_1(t+w),\psi_2(t+w))-cf(t,\psi_1(t),\psi_2(t))]\right\|d\mu(t)\\
= \;& \frac{1}{\mu(r,\rho)}\int_{-r}^{r}\Big\|c^{\wedge}(-t)[f(t+w,\psi_1(t+w),\psi_2(t+w))-cf(t,\frac{1}{c}\psi_1(t+w),\frac{1}{c}\psi_2(t+w))\\
+ \;& cf(t,\frac{1}{c}\psi_1(t+w),\frac{1}{c}\psi_2(t+w))-cf(t,\psi_1(t),\psi_2(t))]\Big\|d\mu(t)\\
\leq \;& \frac{1}{\mu(r,\rho)}\int_{-r}^{r}\left\|c^{\wedge}(-t)[f(t+w,\psi_1(t+w),\psi_2(t+w))-cf(t,\frac{1}{c}\psi_1(t+w),\frac{1}{c}\psi_2(t+w))]\right\|d\mu(t)\\
+ \;& \frac{1}{\mu(r,\rho)}\int_{-r}^{r}\left\|c^{\wedge}(-t)[cf(t,\frac{1}{c}\psi_1(t+w),\frac{1}{c}\psi_2(t+w))-cf(t,\psi_1(t),\psi_2(t))]\right\|d\mu(t)\\
= \;& J_1 + J_2.
\end{aligned}$$

By (A1)(b), we have $J_1 \to 0$, as $r \to \infty$.
For J_2, we have

$$\begin{aligned}
& \frac{1}{\mu([-r,r])}\int_{-r}^{r}\Big\|c^\wedge(-t)[cf(t,\frac{1}{c}\psi_1(t+w),\frac{1}{c}\psi_2(t+w))-cf(t,\psi_1(t),\psi_2(t))]\Big\|d\mu(t)\\
\leq\ & \frac{1}{\mu([-r,r])}\int_{-r}^{r}|c^\wedge(-t)c|\Big\|f(t,\frac{1}{c}\psi_1(t+w),\frac{1}{c}\psi_2(t+w))-f(t,\psi_1(t),\psi_2(t))\Big\|d\mu(t)\\
\leq\ & \frac{1}{\mu([-r,r])}\int_{-r}^{r}|c^\wedge(-t)c|\Big[L_1\Big\|\frac{1}{c}\psi_1(t+w)-\psi_1(t)\|+L_2\Big\|\frac{1}{c}\psi_2(t+w)-\psi_2(t)\Big\|\Big]d\mu(t)\\
\leq\ & \frac{L_1}{\mu([-r,r])}\int_{-r}^{r}\Big\|c^\wedge(-t)[\psi_1(t+w)-c\psi_1(t)]\Big\|d\mu(t)+\frac{L_2}{\mu([-r,r])}\int_{-r}^{r}\Big\|c^\wedge(-t)[\psi_2(t+w)-c\psi_2(t)]\Big\|d\mu(t)\\
& \to 0,(r\to+\infty).
\end{aligned}$$

As a result $J_2 \to 0$, as $r \to \infty$. □

Lemma 2.20. *Let $f \in C(\mathbb{R}\times X\times X, X)$ satisfy (A1)and the following condition*

(A3) *$c^\wedge(-t)f(t,c^\wedge(t)x,c^\wedge(t)y)$ is uniformly continuous for x,y in any bounded subset of X uniformly in $t\in\mathbb{R}$; that is, for any $\epsilon>0$ and any bounded subset $\mathcal{F}\subseteq X$, there exists $\delta>0$ such that $x,y\in\mathcal{F}$, $\|x-x'\|<\delta$ and $\|y-y'\|<\delta$ imply that*

$$\Big\|c^\wedge(-t)[f(t,c^\wedge(t)x,c^\wedge(t)y)-f(t,c^\wedge(t)x',c^\wedge(t)y')]\Big\|\leq\epsilon, t\in\mathbb{R}.$$

Then for each $\psi_1,\psi_2\in\mathcal{PSAP}_{w,c}(\mathbb{R},X,\mu)$, $f(.,\psi_1(.),\psi_2(.))\in\mathcal{PSAP}_{w,c}(\mathbb{R},X,\mu)$.

Proof. According to (**A1**)(a), we have $f(.,\psi(.))\in\mathcal{BC}_{w,c}(\mathbb{R},X)$ and $\exists M>0$ such that

$$\sup_{t\in\mathbb{R}}\|c^\wedge(-t)f(t,x,y)\|<M \text{ for all } x,y\in X.$$

Then

$$\begin{aligned}
& \frac{1}{\mu(r,\rho)}\int_{-r}^{r}\Big\|c^\wedge(-t)\Big[f(t+w,\psi_1(t+w),\psi_2(t+w))-cf(t,\psi_1(t),\psi_2(t))\Big]\Big\|d\mu(t)\\
\leq\ & \frac{1}{\mu(r,\rho)}\int_{-r}^{r}\Big\|c^\wedge(-t)\Big[f(t+w,\psi_1(t+w),\psi_2(t+w))-cf(t,\frac{1}{c}\psi_1(t+w),\frac{1}{c}\psi_2(t+w))\Big]\Big\|d\mu(t)\\
+\ & \frac{1}{\mu(r,\rho)}\int_{-r}^{r}\Big\|c^\wedge(-t)\Big[cf(t,\frac{1}{c}\psi_1(t+w),\frac{1}{c}\psi_2(t+w))-cf(t,\psi_1(t),\psi_2(t))\Big]\Big\|d\mu(t)\\
=\ & J_1+J_2.
\end{aligned}$$

According to (**A1**)(b), we obtain that $J_1\to 0$, as $r\to\infty$.
For J_2, we take a bounded subset $\mathcal{F}=\{c^\wedge(-t)\psi_i(t): t\in\mathbb{R}, i=1,2\}$.
If $\|c^\wedge(-t)[\psi_i(t+w)-c\psi_i(t)]\|<|c|\delta, i=1,2$. Then

$$\|c^\wedge(-t-w)\psi_i(t+w)-c^\wedge(-t)\psi_i(t)\|=\|\tfrac{1}{c}c^\wedge(-t)[\psi_i(t+w)-c\psi_i(t)]\|<\delta, i=1,2.$$

According to (A3), we have

$$\begin{aligned}
& \frac{1}{\mu([-r,r])}\int_{-r}^{r}\Big\|c^{\wedge}(-t)\Big[cf(t,\frac{1}{c}\psi_1(t+w),\frac{1}{c}\psi_2(t+w))-cf(t,\psi_1(t),\psi_2(t))\Big]\Big\|d\mu(t)\\
= \; & |c|\frac{1}{\mu([-r,r])}\int_{-r}^{r}\Big\|c^{\wedge}(-t)\Big[f(t,c^{\wedge}(t)c^{\wedge}(-t-w)\psi_1(t+w),c^{\wedge}(t)c^{\wedge}(-t-w)\psi_2(t+w))\\
- \; & f(t,c^{\wedge}(t)c^{\wedge}(-t)\psi_1(t),c^{\wedge}(t)c^{\wedge}(-t)\psi_2(t))\Big]\Big\|d\mu(t)\\
= \; & |c|\frac{1}{\mu([-r,r])}\int_{[-r,r]\setminus \mathrm{M}_{r,|c|\delta}(\psi_i)}\Big\|c^{\wedge}(-t)\Big[f(t,c^{\wedge}(t)c^{\wedge}(-t-w)\psi_1(t+w),c^{\wedge}(t)c^{\wedge}(-t-w)\psi_2(t+w))\\
- \; & f(t,c^{\wedge}(t)c^{\wedge}(-t)\psi_1(t),c^{\wedge}(t)c^{\wedge}(-t)\psi_2(t))\Big]\Big\|d\mu(t)\\
+ \; & |c|\frac{1}{\mu([-r,r])}\int_{\mathrm{M}_{r,|c|\delta}(\psi_i)}\|c^{\wedge}(-t)\Big[f(t,c^{\wedge}(t)c^{\wedge}(-t-w)\psi_1(t+w),c^{\wedge}(t)c^{\wedge}(-t-w)\psi_2(t+w))\\
- \; & f(t,c^{\wedge}(t)c^{\wedge}(-t)\psi_1(t),c^{\wedge}(t)c^{\wedge}(-t)\psi_2(t))\Big]\Big\|d\mu(t)\\
\leq \; & \Big(1-\frac{1}{\mu([-r,r])}\int_{\mathrm{M}_{r,|c|\delta}(\psi_i)}d\mu(t)\Big)|c|\varepsilon+2M|c|\frac{1}{\mu([-r,r])}\int_{\mathrm{M}_{r,|c|\delta}(\psi_i)}d\mu(t).
\end{aligned}$$

Using lemma 2.12 and the fact $\psi_1,\psi_2\in\mathcal{PSAP}_{w,c}(\mathbb{R},X,\mu)$, we obtained $J_2\to 0$, as $r\to\infty$. Hence

$$\lim_{|t|\to+\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\Big\|c^{\wedge}(-t)\Big[f(t+w,\psi_1(t+w),\psi_2(t+w))-cf(t,\psi_1(t),\psi_2(t))\Big]\Big\|d\mu(t)=0.$$

□

3. Main Results:

In this article, we consider the following hypothesis:

(**H1**) The functions $\sigma_1,\sigma_2,\tau_1,\tau_2$ are w-periodic.

(**H2**) There exists $L^1_{f_1},L^2_{f_1},L^1_{f_2},L^2_{f_2}>0$ such that $\forall t,x_1,x_2,y_1,y_2\in\mathbb{R}$, we have:

$$\|f_1(t,x_1,y_1)-f_1(t,x_2,y_2)\|\leq L^1_{f_1}\|x_1-x_2\|+L^2_{f_1}\|y_1-y_2\|,$$

$$\|f_2(t,x_1,y_1)-f_2(t,x_2,y_2)\|\leq L^1_{f_2}\|x_1-x_2\|+L^2_{f_2}\|y_1-y_2\|.$$

(**H3**) Uniformly for $x,y\in X$: $\begin{cases} \sup_{t\in\mathbb{R}}\|c^{\wedge}(-t)f_1(t,x,y)\|<+\infty,\\ \sup_{t\in\mathbb{R}}\|c^{\wedge}(-t)f_2(t,x,y)\|<+\infty.\end{cases}$

(**H4**) Uniformly for $x,y\in X$: $\begin{cases} \lim_{r\to+\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\Big\|c^{\wedge}(-t)[f_1(t+w,cx,cy)-cf_1(t,x,y)]\Big\|d\mu(t)=0,\\ \lim_{r\to+\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\Big\|c^{\wedge}(-t)[f_2(t+w,cx,cy)-cf_2(t,x,y)]\Big\|d\mu(t)=0.\end{cases}$

Theorem 3.1. *Let $\mu\in\mathcal{M}$ satisfies (**H0**)and the conditions (**H1**),..., (**H4**) are hold then the system* (1) *admits a unique solution (x,y) in $\mathcal{PSAP}_{w,c}(\mathbb{R},X,\mu)^2$.*

Proof. We put the operator ζ defined on $\mathcal{PSAP}_{w,c}(\mathbb{R}, X, \mu)^2$ such that $\forall (x, y) \in \mathcal{PSAP}_{w,c}(\mathbb{R}, X, \mu)^2$:

$$\zeta(x,y) = \Big(\sigma_1(t)x(t-\rho_1) + \int_{t-\tau_1(t)}^{t} f_1(s, x(s-\lambda_1), y(s-\lambda_2))ds; \sigma_2(t)y(t-\rho_2) + \int_{t-\tau_2(t)}^{t} f_2(s, x(s-\lambda_1), y(s-\lambda_2))ds\Big)$$

according to Lemmas 2.11, 2.13, 2.14 and 2.15, we have $\zeta : \mathcal{PSAP}_{w,c}(\mathbb{R}, X, \mu)^2 \to \mathcal{PSAP}_{w,c}(\mathbb{R}, X, \mu)^2$.
We define the following norm:

$$\|(f_1, f_2)\|_{bwc'} = \|f_1\|_{bwc} + \|f_2\|_{bwc}$$

$\forall x_1, y_1, x_2, y_2 \in \mathcal{PSAP}_{w,c}(\mathbb{R}, X, \mu)$, we have:

$$\begin{aligned}
& \zeta(x_1,y_1)(t) - \zeta(x_2,y_2)(t) \\
= \;& \Big(\sigma_1(t)\Big[x_1(t-\rho_1) - x_2(t-\rho_1)\Big] + \int_{t-\tau_1(t)}^{t} \Big[f_1(s, x_1(s-\lambda_1), y_1(s-\lambda 2)) - f_1(s, x_2(s-\lambda_1), y_2(s-\lambda_2))\Big]ds \\
; \;& \sigma_2(t)\Big[y_1(t-\rho_2) - y_2(t-\rho_2)\Big] + \int_{t-\tau_2(t)}^{t} \Big[f_2(s, x_1(s-\lambda_1), y_1(s-\lambda_2)) - f_2(s, x_2(s-\lambda_1), y_2(s-\lambda_2))\Big]ds\Big).
\end{aligned}$$

∗ First of all:

$$\begin{aligned}
& \Big\|c^\wedge(-t)\Big[\sigma_1(t)\Big(x_1(t-\rho_1) - x_2(t-\rho_1)\Big) + \int_{t-\tau_1(t)}^{t} \Big(f_1(s, x_1(s-\lambda_1), y_1(s-\lambda_2)) - f_1(s, x_2(s-\lambda_1), y_2(s-\lambda_2))\Big)ds\Big]\Big\| \\
\leq \;& \|\sigma_1\|_\infty |c^\wedge(-\rho_1)| \Big\|c^\wedge(-t+\rho_1)\Big[x_1(t-\rho_1) - x_2(t-\rho_1)\Big]\Big\| \\
+ \;& \Big\| \int_{-\tau_1(t)}^{0} c^\wedge(-s-t)c^\wedge(s)\Big[f_1(s+t, x_1(s+t-\lambda_1), y_1(s+t-\lambda_2)) - f_1(s+t, x_2(s+t-\lambda_1), y_2(s+t-\lambda_2))\Big]ds\Big\| \\
\leq \;& \|\sigma_1\|_\infty |c^\wedge(-\rho_1)| \|x_1 - x_2\|_{bwc} + L^1_{f_1}|c^\wedge(-\lambda_1)| \int_{-\|\tau_1\|_\infty}^{0} |c^\wedge(s)| \|c^\wedge(-s-t+\lambda_1)\| \|x_1(s+t-\lambda_1) - x_2(s+t-\lambda_1)\| ds \\
+ \;& L^2_{f_1}|c^\wedge(-\lambda_2)| \int_{-\|\tau_1\|_\infty}^{0} |c^\wedge(s)| \|c^\wedge(-s-t+\lambda_2)\| \|y_1(s+t-\lambda_2) - y_2(s+t-\lambda_2)\| ds \\
\leq \;& \|\sigma_1\|_\infty |c^\wedge(-\rho_1)| \|x_1 - x_2\|_{bwc} + M_1 L^1_{f_1}|c^\wedge(-\lambda_1)| \|x_1 - x_2\|_{bwc} + M_1 L^2_{f_1}|c^\wedge(-\lambda_2)| \|y_1 - y_2\|_{bwc}.
\end{aligned}$$

∗ Next:

$$\begin{aligned}
& \Big\|c^\wedge(-t)\Big[\sigma_2(t)\Big(y_1(t-\rho_2) - y_2(t-\rho_2)\Big) + \int_{t-\tau_2(t)}^{t} \Big(f_2(s, x_1(s-\lambda_1), y_1(s-\lambda_2)) - f_2(s, x_2(s-\lambda_1), y_2(s-\lambda_2))\Big)ds\Big]\Big\| \\
\leq \;& \|\sigma_2\|_\infty |c^\wedge(-\rho_2)| \|y_1 - y_2\|_{bwc} + M_2 L^1_{f_2}|c^\wedge(-\lambda_1)| \|x_1 - x_2\|_{bwc} + M_2 L^2_{f_2}|c^\wedge(-\lambda_2)| \|y_1 - y_2\|_{bwc}
\end{aligned}$$

with $M_i = \int_{-\|\tau_i\|_\infty}^{0} |c^\wedge(s)| ds$, $i = 1, 2$.
We find that

$$\begin{aligned}
\|\zeta(x_1,y_1) - \zeta(x_2,y_2)\|_{bwc'} \leq \;& \Big[\|\sigma_1\|_\infty |c^\wedge(-\rho_1)| + (M_1 L^1_{f_1} + M_2 L^1_{f_2})|c^\wedge(-\lambda_1)|\Big] \|x_1 - x_2\|_{bwc} \\
+ \;& \Big[\|\sigma_2\|_\infty |c^\wedge(-\rho_2)| + (M_1 L^2_{f_1} + M_2 L^2_{f_2})|c^\wedge(-\lambda_2)|\Big] \|y_1 - y_2\|_{bwc}.
\end{aligned}$$

We pose $K = \max(\Big[\|\sigma_1\|_\infty |c^\wedge(-\rho_1)| + (M_1 L^1_{f_1} + M_2 L^1_{f_2})|c^\wedge(-\lambda_1)|\Big], \Big[\|\sigma_2\|_\infty |c^\wedge(-\rho_2)| + (M_1 L^2_{f_1} + M_2 L^2_{f_2})|c^\wedge(-\lambda_2)|\Big])$ as a result

$$\begin{aligned}
\|\zeta(x_1,y_1) - \zeta(x_2,y_2)\|_{bwc'} \leq \;& K\Big(\|x_1 - x_2\|_{bwc} + \|y_1 - y_2\|_{bwc}) \\
\leq \;& K\|(x_1,y_1) - (x_2,y_2)\|_{bwc'}.
\end{aligned}$$

Since $K < 1$ ζ is a contraction. Consequently, using the Banach fixed point theorem ζ admits a unique fixed point $(x, y) \in \mathcal{PSAP}_{w,c}(\mathbb{R}, X, \mu)$ that is $\zeta(x, y) = (x, y)$. Hence, (x, y) is the unique μ-pseudo S-asymptotically (w, c)-periodic solution of (1). $\square$

Corollary 3.2. *Let $\mu \in \mathcal{M}$ satisfies (**H0**) and we assume that the conditions (**H1**),..., (**H4**) are verified so the following equation:*

$$x(t) = \sigma_1(t)x(t - \rho_1) + \int_{t-\tau_1(t)}^{t} f_1(s, x(s))ds \tag{2}$$

have a unique μ-pseudo S-asymptotically (w, c)-periodic solution.

4. Example :

In this branch, we provide numerical example of (1) for various settings. We work on the following system:

$$\begin{cases} x(t) = \sin(2t)x(t-1) + \int_{t-|\sin(t)|}^{t} f_1(s, x(s-1), y(s-2))ds, \\ y(t) = \cos(2t)y(t-2) + \int_{t-|\cos(t)|}^{t} f_2(s, x(s-1), y(s-2))ds, \end{cases} \tag{3}$$

with $f_1(t, x, y) = c^\wedge(t)(\sin(|c^\wedge(-t)x|) + \cos(|c^\wedge(-t)y|)), f_2(t, x, y) = c^\wedge(t)(\cos(|c^\wedge(-t)x|) + \sin(|c^\wedge(-t)y|))$.
We have f_1, f_2 verify (**H2**), (**H3**) and (**H4**).
We take $w = \pi$ and $c = 1 + i$: we have $\sigma_1, \sigma_2, \tau_1, \tau_2$ are π-periodic and

$$\begin{cases} L^1_{f_1} = L^2_{f_1} = L^1_{f_2} = L^2_{f_2} = 1, \\ \|\sigma_1\|_\infty = \|\sigma_2\|_\infty = \|\tau_1\|_\infty = \|\tau_2\|_\infty = 1, \\ M_1 = M_2 = \int_{-1}^{0} |c^\wedge(s)|ds = \int_{-1}^{0} |1+i|^{\frac{s}{\pi}} ds = \int_{-1}^{0} \sqrt{2}^{\frac{s}{\pi}} ds = \ln(\sqrt[2\pi]{2})(1 - 2^{-\frac{1}{2\pi}}), \\ |c^\wedge(-\rho_1)| = |c^\wedge(-\lambda_1)| = |c^\wedge(-1)| = 2^{-\frac{1}{2\pi}}, \\ |c^\wedge(-\rho_2)| = |c^\wedge(-\lambda_2)| = |c^\wedge(-2)| = 2^{-\frac{1}{\pi}}. \end{cases}$$

We obtain

$$\begin{aligned} K &= \max\Big(\Big[\|\sigma_1\|_\infty|c^\wedge(-\rho_1)| + (M_1L^1_{f_1} + M_2L^1_{f_2})|c^\wedge(-\lambda_1)|\Big], \Big[\|\sigma_2\|_\infty|c^\wedge(-\rho_2)| + (M_1L^2_{f_1} + M_2L^2_{f_2})|c^\wedge(-\lambda_2)|\Big]\Big) \\ &= 0.916187872. \end{aligned}$$

We have $K < 1$ so for any measure $\mu \in \mathcal{M}$ satisfies (**H0**) the system (3) have a unique μ-pseudo S-asymptotically $(\pi, 1 + i)$-periodic solution.

Conflict of interest statement:
The authors have no competing interests.

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