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Coincidence points for KKM type maps and maps with a upper semicontinuous selection property

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Abstract. In this paper we present coincidence results between *KKM* type maps and maps which have a upper semicontinuous selection property.

1. Introduction

In this paper using a fixed point theorem from the literature for *KKM* maps [5] we establish several collectively coincidence results between two classes of set–valued maps defined on Hausdorff topological vector spaces. One class includes *KKM* type maps while the other class considers maps which have an upper semicontinuous selection type property. To obtain our coincidence results we need to present some properties (see below) of *KKM*, *W*, *HLPY* and *KLU* maps which will then be used with various selection theorems and a fixed point result for *KKM* maps defined on convex admissible subsets of Hausdorff topological vector spaces

First we describe the maps considered in this paper. We begin with the Kautani maps. An upper semicontinuous map $\phi : X \to CK(Y)$ is said to be Kakutani (and we write $\phi \in Kak(X, Y)$); here X is a Hausdorff topological space, Y is a Hausdorff topological vector space and $CK(Y)$ denotes the family of nonempty, convex, compact subsets of Y. Next we define a very general class of maps considered in the literature [3, 4, 13]. Let X be a convex subset of a Hausdorff topological vector space and Y a Hausdorff topological space. If $S, T : X \to 2^Y$ (nonempty subsets of Y) are two set valued maps such that $T(co\,(A)) \subseteq S(A)$ for each finite subset A of X then we call S a generalized *KKM* mapping w.r.t. T. Now the set valued map $T : X \to 2^Y$ is said to have the *KKM* property if for any generalized *KKM* map $S : X \to 2^Y$ w.r.t. T the family $\{\overline{S(x)} : x \in X\}$ has the finite intersection property (the intersection of each finite subfamily is nonempty). We let

$$KKM(X, Y) = \{T : X \to 2^Y \,|\, T \text{ has the } KKM \text{ property}\}.$$

Next we recall the following result [4].

Theorem 1.1. *Let X be a convex subset of a Hausdorff topological vector space and Y, Z be Hausdorff topological spaces.*

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(i). $T \in KKM(X,Y)$ iff $T|_{\triangle} \in KKM(\triangle, Y)$ for each polytope $\triangle$ in X;
(ii). if $T \in KKM(X,Y)$ and $f \in C(Y,Z)$ then $f\,T \in KKM(X,Z)$;
(iii). if Y is a normal space, $\triangle$ a polytope of X and if $T : \triangle \to 2^Y$ is a set valued map such that for each $f \in C(Y, \triangle)$ we have that $f\,T$ has a fixed point in $\triangle$, then $T \in KKM(\triangle, Y)$.

We also note the following properties.

Let C and X be convex subsets of a Hausdorff topological vector space E with $C \subseteq X$ and Y a Hausdorff topological space.

(i). If $T \in KKM(X,Y)$ then $G \equiv T|_C \in KKM(C,Y)$.

This can be seen from Theorem 1.1 (i). Note $T \in KKM(X,Y)$ so $T|_{\triangle} \in KKM(\triangle, Y)$ for each polytope $\triangle$ in X from Theorem 1.1 (i). Thus in particular for any polytope $\triangle$ in C we have $T|_{\triangle} \in KKM(\triangle, Y)$ so from Theorem 1.1 (i) we have $T|_C \in KKM(C,Y)$.

(ii). If $T \in KKM(X,Y)$, $T(X) \subseteq Z \subseteq Y$ and Z is closed in Y then $T \in KKM(X,Z)$.

Let $S : X \to 2^Z$ be a generalized KKM map w.r.t. T i.e. $T(co\,(A)) \subseteq S(A)$ for each finite subset A of X. We must show $\{\,\overline{S(x)}^Z : x \in X\}$ has the finite intersection property. Note since $S : X \to 2^Y$ is a generalized KKM map w.r.t. T then since $T \in KKM(X,Y)$ we have that $\{\,\overline{S(x)}\,(= \overline{S(x)}^Y) : x \in X\}$ has the finite intersection property. However note for $x \in X$ that

$$\overline{S(x)}^Z = \overline{S(x)}^Y \cap Z = \overline{S(x)}^Y\,(= \overline{S(x)})$$

since Z is closed in Y (note $S(X) \subseteq Z$ so $\overline{S(x)}^Y \subseteq Z$). Thus $\{\,\overline{S(x)}^Z : x \in X\} = \{\,\overline{S(x)}\,(= \overline{S(x)}^Y) : x \in X\}$ has the finite intersection property.

Next we recall the following fixed point result for KKM maps [3]. Recall a nonempty subset W of a Hausdorff topological vector space E is said to be admissible if for any nonempty compact subset K of W and every neighborhood V of 0 in E there exists a continuous map $h : K \to W$ with $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace of E (for example every nonempty convex subset of a locally convex space is admissible).

Theorem 1.2. *Let X be an admissible convex set in a Hausdorff topological vector space E and $T \in KKM(X,X)$ be a closed compact map. Then T has a fixed point in X.*

Next we recall some composition results from the literature [14, 15].

Theorem 1.3. *Let X be a convex admissible subset of a Hausdorff topological vector space, Y a subset of a Hausdorff topological vector space, $T \in KKM(X,Y)$ a upper semicontinuous compact map with compact values and $G \in Kak(Y,X)$. Then $G\,T \in KKM(X,X)$.*

Theorem 1.4. *Let X be a convex admissible subset of a Hausdorff topological vector space, Y a convex subset of a Hausdorff topological vector space and Y normal, $T \in KKM(X,Y)$ a upper semicontinuous compact map with compact values and $G \in Kak(Y,X)$. Then $T\,G \in KKM(Y,Y)$.*

Theorem 1.5. *Let X be an admissible convex set in a Hausdorff topological vector space and Y be an admissible convex set in a Hausdorff topological vector space and suppose Y is a normal space. Let Z be a subset of a Hausdorff topological space with Z a normal space. Also assume $T \in KKM(X,Y)$ is an upper semicontinuous compact map with compact values and $H \in KKM(Y,Z)$ is an upper semicontinuous map with compact values. Then $H\,T \in KKM(X,Z)$.*

Next we describe the maps due to Wu [17]. Let X be a subset of a Hausdorff topological space and Y a subset of a Hausdorff topological vector space. We say $\Phi \in W(X,Y)$ if $\Phi : X \to 2^Y$ and there exists a lower semicontinuous map $\theta : X \to 2^Y$ with $\overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$. Alternatively (see Remark 1.7) we could say $\Phi \in W(X,Y)$ if there exists a lower semicontinuous map $\theta : X \to 2^Y$ with closed convex values such that $\theta(x) \subseteq \Phi(x)$ for $x \in X$.

Next we recall a selection theorem [1] (see the proof in Theorem 1.1) for Wu maps.

Theorem 1.6. *Let X be a paracompact subset of a Hausdorff topological space and Y a metrizable complete subset of a Hausdorff locally convex linear topological space. Suppose $\Phi \in W(X,Y)$ and let $\theta : X \to 2^Y$ be a lower semicontinuous map with $\overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$. Then there exists an upper semicontinuous map $\Psi : X \to CK(Y)$ (collection of nonempty convex compact subsets of Y) with $\Psi(x) \subseteq \overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$.*

Remark 1.7. *(i). Let X be paracompact and Y a metrizable subset of a complete Hausdorff locally convex linear topological space E and $\Phi \in W(X,Y)$ with $\theta : X \to 2^Y$ a lower semicontinuous map and $\overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$. Note [12] that $\overline{co}\,\theta : X \to 2^Y$ (since $\overline{co}\,(\theta(x)) \subseteq \Phi(x) \subseteq Y$ for $x \in X$) is lower semicontinuous, so from Michael's selection theorem there exists a continuous (single valued) map $f : X \to Y$ with $f(x) \in \overline{co}\,(\theta(x))$ for $x \in X$, so consequently $f(x) \in \overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$.*

(ii). We could rephrase Theorem 1.6 (without mentioning θ) as: Let X be a paracompact subset of a Hausdorff topological space, Y a metrizable complete subset of a Hausdorff locally convex linear topological space and $\Phi \in W(X,Y)$. Then there exists an upper semicontinuous map $\Psi : X \to CK(Y)$ with $\Psi(x) \subseteq \Phi(x)$ for $x \in X$.

We now note two properties for W maps.

(i). Let $F \in W(X,Y)$ and $Z \subseteq X$. Then $F \in W(Z,Y)$.

To see this note there exists a lower semicontinuous map $\theta : X \to 2^Y$ with $\overline{co}\,(\theta(x)) \subseteq F(x)$ for $x \in X$. Let Ω be a closed subset of Y. Then $\{x \in Z : \theta(x) \subseteq \Omega\} = Z \cap \{x \in X : \theta(x) \subseteq \Omega\}$ which is closed in Z since $\theta : X \to 2^Y$ is lower semicontinuous. Thus $F \in W(Z,Y)$.

(ii). Let $F \in W(X,Y)$ and $F(X) \subseteq W \subseteq Y$. Then $F \in W(X,W)$.

To see this note there exists a lower semicontinuous map $\theta : X \to 2^Y$ with $\overline{co}\,(\theta(x)) \subseteq F(x)$ for $x \in X$. Let Ω be a closed subset of W. Then $\Omega = W \cap C$ for some closed set C of Y. Now since $F(X) \subseteq W$ then $\{x \in X : \theta(x) \subseteq \Omega\} = \{x \in X : \theta(x) \subseteq C\}$ which is closed in X since $\theta : X \to 2^Y$ is lower semicontinuous. Thus $F \in W(X,W)$.

Let Z be a subset of a Hausdorff topological space Y_1 and W a subset of a Hausdorff topological vector space Y_2 and G a multifunction. We say $F \in HLPY(Z,W)$ [11] if W is convex and there exists a map $S : Z \to W$ (i.e. $S : Z \to P(W)$ (collection of subsets of W)) with $co\,(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and $Z = \bigcup\{\, int\, S^{-1}(w) : w \in W\}$; here $S^{-1}(w) = \{z \in Z : w \in S(z)\}$ and note $S(x) \neq \emptyset$ for each $x \in Z$ is redundant since if $z \in Z$ then there exists a $w \in W$ with $z \in int\, S^{-1}(w) \subseteq S^{-1}(w)$ so $w \in S(z)$ i.e. $S(z) \neq \emptyset$. For the selection theorem below see [11].

Theorem 1.8. *Let X be a paracompact subset of a Hausdorff topological space, Y a convex subset of a Hausdorff topological vector space and $F \in HLPY(X,Y)$ (let $S : X \to 2^Y$ with $co\,(S(x)) \subseteq F(x)$ for $x \in X$ and $X = \bigcup\{\, int\, S^{-1}(w) : w \in Y\}$). Then there exists a continuous (single–valued) map $f : X \to Y$ with $f(x) \in co\, S(x) \subseteq F(x)$ for all $x \in X$.*

Remark 1.9. *(i). These maps are related to the DKT maps in the literature and $F \in DKT(Z,W)$ [5] if W is convex and there exists a map $S : Z \to W$ with $co\,(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and the fibre $S^{-1}(w)$ is open (in Z) for each $w \in W$. Note these maps were motivated from the Fan maps.*

(ii). We could rephrase Theorem 1.8 (without mentioning S) as: Let X be a paracompact subset of a Hausdorff topological space, Y a convex subset of a Hausdorff topological vector space and $F \in HLPY(X,Y)$. Then there exists a continuous map $f : X \to Y$ with $f(x) \in F(x)$ for all $x \in X$.

We now note two properties for $HLPY$ maps (here Y is convex).

(i). Let $F \in HLPY(X,Y)$ and $Z \subseteq X$. Then $F \in HLPY(Z,Y)$.

To see this note exists a map $S : X \to Y$ with $co\,(S(x)) \subseteq F(x)$ for $x \in X$ and $X = \bigcup\{\, int\, S^{-1}(y) : y \in Y\}$. Let S also denote the restriction of S to Z. Notice

$$Z = Z \cap X = Z \cap \left(\bigcup\{\, int\, S^{-1}(y) : y \in Y\}\right) = \bigcup\{Z \cap\, int\, S^{-1}(y) : y \in Y\},$$

so $Z \subseteq \bigcup\{\, int_Z\, S^{-1}(y) : y \in Y\}$ since for each $y \in Y$ we have that $Z \cap int\, S^{-1}(y)$ is open in Z. On the other hand clearly $\bigcup\{\, int_Z\, S^{-1}(y) : y \in Y\} \subseteq Z$. Thus $Z = \bigcup\{\, int_Z\, S^{-1}(y) : y \in Y\}$ so $F \in HLPY(Z,Y)$.

(ii). Let $F \in HLPY(X,Y)$ and $F(X) \subseteq W \subseteq Y$ with W convex. Then $F \in HLPY(X,W)$.

To see this note exists a map $S : X \to Y$ with $co\,(S(x)) \subseteq F(x)$ for $x \in X$ and $X = \bigcup\{\,int\, S^{-1}(y) :\ y \in Y\}$. Now note for any $x \in X$ there exists a $y \in Y$ with $x \in int\, S^{-1}(y) \subseteq S^{-1}(y)$ so $y \in S(x) \subseteq co\,(S(x)) \subseteq F(x) \subseteq W$. Thus $X = \bigcup\{\,int\, S^{-1}(y) :\ y \in W\}$, so $F \in HLPY(X,W)$.

Let X be a subset of a Hausdorff topological space and Y a subset of a Hausdorff topological vector space. We say $T : X \to 2^Y$ has the strong continuous inclusion property (SCIP) [10] at $x \in X$ if there exists an open set $U(x)$ in X containing x and a $F^x : U(x) \to 2^Y$ such that $F^x(w) \subseteq T(w)$ for all $w \in U(x)$ and $co\,F^x : U(x) \to 2^Y$ is compact valued and upper semicontinuous. We write $T \in KLU(X,Y)$ if T has the SCIP at every $x \in X$.

In this paper our map T will usually be a compact map so T has the SCIP is equivalent (see [2, pp 465]) to T has the CIP [9].

We now note two properties for KLU maps which we will use in Section 2.

(i). Let $F \in KLU(X,Y)$ and $Z \subseteq X$. Then $F \in KLU(Z,Y)$.

To see this let $x \in Z$. Then $x \in X$ and since $F \in KLU(X,Y)$ then there exists an open set $U(x)$ in X containing x and a $\Phi^x : U(x) \to 2^Y$ such that $\Phi^x(w) \subseteq F(w)$ for all $w \in U(x)$ and $co\,\Phi^x : U(x) \to 2^Y$ is compact valued and upper semicontinuous. Let $V(x) = Z \cap U(x)$. Note $V(x)$ is open in Z and $co\,\Phi^x : V(x) \to 2^Y$ is upper semicontinuous (consider $co\,\Phi^x \circ i$ where $i : V(x) \to U(x)$ is the inclusion) and compact valued. Thus $F \in KLU(Z,Y)$.

(ii). Let $F \in KLU(X,Y)$ and $F(X) \subseteq W \subseteq Y$. Then $F \in KLU(X,W)$.

Let $x \in X$. Then there exists an open set $U(x)$ in X containing x and a $\Phi^x : U(x) \to 2^Y$ such that $\Phi^x(w) \subseteq F(w)$ for all $w \in U(x)$ and $co\,\Phi^x : U(x) \to 2^Y$ is compact valued and upper semicontinuous. Let $\Psi^x : U(x) \to 2^W$ be obtained by restricing the range of Φ^x and let Ω be open in W. Then $\Omega = W \cap V$ for some open set V of Y. Now since $F(X) \subseteq W$ then $\{y \in U(x) :\ co\,\Psi^x(y) \subseteq \Omega\} = \{y \in U(x) :\ co\,\Phi^x(y) \subseteq V\}$ which is open in $U(x)$. Thus $co\,\Psi^x : U(x) \to 2^W$ is upper semicontinuous so $F \in KLU(X,W)$.

Next we recall a selection theorem [10].

Theorem 1.10. *Let X be a paracompact subset of a Hausdorff topological space, Y a subset of a Hausdorff topological vector space and $T \in KLU(X,Y)$. Then there exists an upper semicontinuous map $G : X \to CK(Y)$ with $G(w) \subseteq co\,T(w)$ for all $w \in X$.*

2. Coincidence result.

Let X be a subset of a Hausdorff topological space and Y a subset of a Hausdorff topological vector space. We say $F \in HYKKM(X,Y)$ if $F : X \to 2^Y$ and there exists an upper semicontinuous map $\Psi \in KKM(X,Y)$ with compact values and with $\Psi(x) \subseteq co\,(F(x))$ for $x \in X$. We say $F \in CKKM(X,Y)$ if $F : X \to 2^Y$ and there exists an upper semicontinuous map $\Psi \in KKM(X,Y)$ with compact values and with $\Psi(x) \subseteq F(x)$ for $x \in X$ (in this case we only need Y to be a subset of a Hausdorff topological space). We begin with coincidence results between KKM type maps and KLU maps.

Theorem 2.1. *Let X be a subset of a Hausdorff topological vector space and Y a convex admissible subset of a Hausdorff topological vector space and let X be paracompact. Suppose $F \in KLU(X,Y)$ with $co\,F$ a compact map and $G \in HYKKM(Y,X)$. Then there exists a $x \in X$ with $co\,F(x) \cap \Lambda^{-1}(x) \neq \emptyset$; here $\Lambda(w) = co\,G(w)$ for $w \in Y$.*

Proof. From Theorem 1.10 there exists an upper semicontinuous map $\Psi : X \to CK(Y)$ (i.e. $\Psi \in Kak(X,Y)$) with $\Psi(x) \subseteq co\,(F(x))$ for $x \in X$ and note Ψ is a compact map since $co\,F$ is a compact map. Also by definition there exists an upper semicontinuous map $\Phi \in KKM(Y,X)$ with compact values and with $\Phi(y) \subseteq co\,(G(y))$ for $y \in Y$. From Theorem 1.3 note $\Psi\,\Phi \in KKM(Y,Y)$ is an upper semicontinuous compact map (note Ψ is a compact map) with compact values, so a closed map [2]. Now Theorem 1.2 guarantees a $y \in Y$ with $y \in \Psi\,\Phi(y)$. Let $x \in \Phi(y)$ with $y \in \Psi(x)$, so $y \in \Psi(x) \subseteq co\,(F(x))$ and $x \in \Phi(y) \subseteq co\,G(y) = \Lambda(y)$ (so $y \in \Lambda^{-1}(x)$). $\square$

Remark 2.2. *(i). One could also consider $\Phi\,\Psi$ in the proof of Theorem 2.1 if X is a convex admissible subset of a Hausdorff topological vector space, X is a normal space, X is paracompact and Y is a convex admissible subset of a Hausdorff topological vector space. To see this note from Theorem 1.4 that $\Phi\,\Psi \in KKM(X,X)$ is an upper semicontinuous compact map (note Ψ is a compact map and Φ is upper semicontinuous with compact values). Now apply Theorem 1.2.*

(ii). We could replace "$co\,F$ is a compact map" with "$co\,G$ is a compact map" in the statement of Theorem 2.1. To see this notice in the proof we just replace "Ψ is a compact map" with "Φ is a compact map". In fact we can improve this result (see Theorem 2.4).

(iii). In the proof of Theorem 2.1 note from Section 1 that $\Psi \in Kak(X, \overline{co}\,(F(X)))$ and $\Phi \in KKM(\overline{co}\,(F(X)), X)$. Thus $\Psi\,\Phi \in KKM(\overline{co}\,(F(X)), \overline{co}\,(F(X)))$ is an upper semicontinuous compact map. We can now apply Theorem 1.2 if we assume $\overline{co}\,(F(X))$ is admissible instead of assuming Y is admissible.

There are obvious analogues of Theorem 2.1 and Remark 2.2 when the *HYKKM* map is replaced by a *CKKM* map. Here is the analogue of Theorem 2.1.

Theorem 2.3. *Let X be a subset of a Hausdorff topological space and Y a convex admissible subset of a Hausdorff topological vector space and let X be paracompact. Suppose $F \in KLU(X,Y)$ with $co\,F$ a compact map and $G \in CKKM(Y,X)$. Then there exists a $x \in X$ with $co\,F(x) \cap G^{-1}(x) \neq \emptyset$.*

Proof. Let Ψ be as in Theorem 2.1. Now by definition there exists an upper semicontinuous map $\Phi \in KKM(Y,X)$ with compact values and with $\Phi(y) \subseteq G(y)$ for $y \in Y$. From Theorem 1.3 note $\Psi\,\Phi \in KKM(Y,Y)$ is an upper semicontinuous compact map, so Theorem 1.2 guarantees a $y \in Y$ with $y \in \Psi\,\Phi(y)$. Let $x \in \Phi(y)$ with $y \in \Psi(x)$, so $y \in \Psi(x) \subseteq co\,(F(x))$ and $x \in \Phi(y) \subseteq G(y)$. □

Our next result improves Remark 2.2 (ii) (i.e. we remove the assumption that X is paracompact).

Theorem 2.4. *Let X be a subset of a Hausdorff topological vector space and Y a convex admissible subset of a Hausdorff topological vector space. Suppose $F \in KLU(X,Y)$ and $G \in HYKKM(Y,X)$ with $co\,G$ a compact map. Then there exists a $x \in X$ with $co\,F(x) \cap \Lambda^{-1}(x) \neq \emptyset$; here $\Lambda(w) = co\,G(w)$ for $w \in Y$.*

Proof. By definition there exists an upper semicontinuous map $\Phi \in KKM(Y,X)$ with compact values and with $\Phi(y) \subseteq co\,(G(y))$ for $y \in Y$ and note Φ is a compact map since $co\,G$ is a compact map. Also note from Secton 1 that $\Phi \in KKM(Y, \overline{co}\,(G(Y)))$. Next since $F \in KLU(X,Y)$ we have from Section 1 that $F \in KLU(\overline{co}\,(G(Y)), Y)$. Now $\overline{co}\,G(Y)$ is compact (since $co\,G$ is a compact map) so paracompact, and Theorem 1.10 guarantees that there exists an upper semicontinuous map $\Psi : \overline{co}\,(G(Y)) \to CK(Y)$ (i.e. $\Psi \in Kak(\overline{co}\,(G(Y)), Y)$) with $\Psi(x) \subseteq co\,(F(x))$ for $x \in \overline{co}\,(G(Y))$. From Theorem 1.3 note $\Psi\,\Phi \in KKM(Y,Y)$ is an upper semicontinuous compact map (note Φ is a compact map and Ψ is upper semicontinuous with compact values). Now Theorem 1.2 guarantees a $y \in Y$ with $y \in \Psi\,\Phi(y)$. □

We now present the analogue of Theorem 2.4 when the *HYKKM* map is replaced by a *CKKM* map.

Theorem 2.5. *Let X be a subset of a Hausdorff topological space and Y a convex admissible subset of a Hausdorff topological vector space. Suppose $F \in KLU(X,Y)$ and $G \in CKKM(Y,X)$ with G a compact map. Then there exists a $x \in X$ with $co\,F(x) \cap G^{-1}(x) \neq \emptyset$.*

Proof. By definition there exists an upper semicontinuous map $\Phi \in KKM(Y,X)$ with compact values and with $\Phi(y) \subseteq G(y)$ for $y \in Y$ and note Φ is a compact map since G is a compact map. Also note from Secton 1 that $\Phi \in KKM(Y, \overline{G(Y)})$ and $F \in KLU(\overline{G(Y)}, Y)$. Now $\overline{G(Y)}$ is compact so paracompact, and Theorem 1.10 guarantees that there exists an upper semicontinuous map $\Psi : \overline{G(Y)} \to CK(Y)$ with $\Psi(x) \subseteq co\,(F(x))$ for $x \in \overline{G(Y)}$. From Theorem 1.3 note $\Psi\,\Phi \in KKM(Y,Y)$ is an upper semicontinuous compact map. Now apply Theorem 1.2. □

Our next result, motivated in part from Theorem 2.4, considers Theorem 2.1 and the paracompactness condition.

Theorem 2.6. *Let X and Y be subsets of a Hausdorff topological vector space E and Y a convex admissible subset of E. Suppose $F \in KLU(X,Y)$ and $G \in HYKKM(Y,X)$ and assume there exists a compact subset K of Y with $co\,(F(X)) \subseteq K$. Let $L(K)$ be the linear span of K (i.e. the smallest linear subspace of E that contains K) and suppose $co\,G(Y) \subseteq X \cap L(K)$ and $X \cap L(K)$ is closed both in $L(K)$ and in X. Then there exists a $x \in X$ with $co\,F(x) \cap \Lambda^{-1}(x) \neq \emptyset$; here $\Lambda(w) = co\,G(w)$ for $w \in Y$.*

Proof. From Section 1 note $F \in KLU(X \cap L(K), Y)$. Recall $L(K)$ is Lindelöf so paracompact [6, 7, 8] and since $X \cap L(K)$ is closed in $L(K)$ then $X \cap L(K)$ is paracompact. Now Theorem 1.10 guarantees there exists an upper semicontinuous map $\Psi : X \cap L(K) \to CK(Y)$ with $\Psi(x) \subseteq co\,(F(x))$ for $x \in X \cap L(K)$ and note Ψ is a compact map since $co\,(F(X \cap L(K))) \subseteq co\,(F(X)) \subseteq K$. Since $G \in HYKKM(Y,X)$ there exists an upper semicontinuous map $\Phi \in KKM(Y,X)$ with compact values and with $\Phi(y) \subseteq co\,(G(y))$ for $y \in Y$. Now since $co\,G(Y) \subseteq X \cap L(K)$ and $X \cap L(K)$ is closed in X then from Section 1 we have that $\Phi \in KKM(Y, X \cap L(K))$ is an upper semicontinuous map. Thus $\Psi\,\Phi \in KKM(Y,Y)$ is an upper semicontinuous compact map. Now apply Theorem 1.2. □

Theorem 2.7. *Let X and Y be subsets of a Hausdorff topological vector space E and Y a convex admissible subset of E. Suppose $F \in KLU(X,Y)$ and $G \in CKKM(Y,X)$ and assume there exists a compact subset K of Y with $co\,(F(X)) \subseteq K$. Let $L(K)$ be the linear span of K and suppose $G(Y) \subseteq X \cap L(K)$ and $X \cap L(K)$ is closed both in $L(K)$ and in X. Then there exists a $x \in X$ with $co\,F(x) \cap G^{-1}(x) \neq \emptyset$.*

Proof. Let Ψ be as in Theorem 2.6. Since $G \in CKKM(Y,X)$ there exists an upper semicontinuous map $\Phi \in KKM(Y,X)$ with compact values and with $\Phi(y) \subseteq G(y)$ for $y \in Y$. Now since $G(Y) \subseteq X \cap L(K)$ and $X \cap L(K)$ is closed in X then from Section 1 we have that $\Phi \in KKM(Y, X \cap L(K))$ is an upper semicontinuous map. Thus $\Psi\,\Phi \in KKM(Y,Y)$ is an upper semicontinuous compact map. Now apply Theorem 1.2. □

Theorem 2.8. *Let X and Y be subsets of a Hausdorff topological vector space E. Suppose $F \in KLU(X,E)$ and $G \in HYKKM(E,X)$ and assume there exists a compact subset K of E with $co\,(F(X)) \subseteq K$. Let $L(K)$ be the linear span of K and suppose $co\,G(L(K)) \subseteq X \cap L(K)$ and $X \cap L(K)$ is closed both in $L(K)$ and in X and $L(K)$ is an admissible subset of E. Then there exists a $x \in X$ with $co\,F(x) \cap \Lambda^{-1}(x) \neq \emptyset$; here $\Lambda(w) = co\,G(w)$ for $w \in Y$.*

Proof. Since $co\,(F(X)) \subseteq K$ and $F \in KLU(X,E)$ from Section 1 we have $F \in KLU(X \cap L(K), L(K))$ and since $X \cap L(K)$ is paracompact from Theorem 1.10 there exists an upper semicontinuous map $\Psi : X \cap L(K) \to CK(L(K))$ with $\Psi(x) \subseteq co\,(F(x))$ for $x \in X \cap L(K)$ and note Ψ is a compact map. Now since $G \in HYKKM(E,X)$ there exists an upper semicontinuous map $\Phi \in KKM(E,X)$ with compact values and with $\Phi(y) \subseteq co\,(G(y))$ for $y \in E$. Now since $co\,G(L(K)) \subseteq X \cap L(K)$ and $X \cap L(K)$ is closed in X then from Section 1 we have that $\Phi \in KKM(L(K), X \cap L(K))$ is an upper semicontinuous map. Thus $\Psi\,\Phi \in KKM(L(K), L(K))$ is an upper semicontinuous compact map. Now apply Theorem 1.2. □

In addition we have the following result (minor adjustments in the proof of Theorem 2.8).

Theorem 2.9. *Let X and Y be subsets of a Hausdorff topological vector space E. Suppose $F \in KLU(X,E)$ and $G \in CKKM(E,X)$ and assume there exists a compact subset K of E with $co\,(F(X)) \subseteq K$. Let $L(K)$ be the linear span of K and suppose $G(L(K)) \subseteq X \cap L(K)$ and $X \cap L(K)$ is closed both in $L(K)$ and in X and $L(K)$ is an admissible subset of E. Then there exists a $x \in X$ with $co\,F(x) \cap G^{-1}(x) \neq \emptyset$.*

We now present coincidence results between KKM type maps and W maps.

Theorem 2.10. *Let X be a subset of a Hausdorff topological vector space and Y a convex metrizable complete subset of a Hausdorff locally convex linear topological space and let X be paracompact. Suppose $F \in W(X,Y)$ with F a compact map and $G \in HYKKM(Y,X)$. Then there exists a $x \in X$ with $F(x) \cap \Lambda^{-1}(x) \neq \emptyset$; here $\Lambda(w) = co\,G(w)$ for $w \in Y$.*

Proof. From Theorem 1.6 there exists an upper semicontinuous map $\Psi : X \to CK(Y)$ (i.e. $\Psi \in Kak(X, Y)$) with $\Psi(x) \subseteq F(x)$ for $x \in X$ and note Ψ is a compact map since F is a compact map. Also by definition there exists an upper semicontinuous map $\Phi \in KKM(Y, X)$ with compact values and with $\Phi(y) \subseteq co\,(G(y))$ for $y \in Y$. From Theorem 1.3 (recall a convex subset of a locally convex space is admissible) note $\Psi\,\Phi \in KKM(Y, Y)$ is an upper semicontinuous compact map. Now Theorem 1.2 guarantees a $y \in Y$ with $y \in \Psi\,\Phi(y)$. Let $x \in \Phi(y)$ with $y \in \Psi(x)$, so $y \in \Psi(x) \subseteq F(x)$ and $x \in \Phi(y) \subseteq co\,G(y) = \Lambda(y)$. □

Remark 2.11. *There is an obvious analogue of Remark 2.2 in this setting also.*

Theorem 2.12. *Let X be a subset of a Hausdorff topological space and Y a convex metrizable complete subset of a Hausdorff locally convex linear topological space and let X be paracompact. Suppose $F \in W(X, Y)$ with F a compact map and $G \in CKKM(Y, X)$. Then there exists a $x \in X$ with $F(x) \cap G^{-1}(x) \neq \emptyset$.*

Proof. Let Ψ be as in Theorem 2.10. Now by definition there exists an upper semicontinuous map $\Phi \in KKM(Y, X)$ with compact values and with $\Phi(y) \subseteq G(y)$ for $y \in Y$. From Theorem 1.3 note $\Psi\,\Phi \in KKM(Y, Y)$ is an upper semicontinuous compact map, so Theorem 1.2 guarantees a $y \in Y$ with $y \in \Psi\,\Phi(y)$. Let $x \in \Phi(y)$ with $y \in \Psi(x)$, so $y \in \Psi(x) \subseteq F(x)$ and $x \in \Phi(y) \subseteq G(y)$. □

Theorem 2.13. *Let X be a subset of a Hausdorff topological vector space and Y a convex metrizable complete subset of a Hausdorff locally convex linear topological space. Suppose $F \in W(X, Y)$ and $G \in HYKKM(Y, X)$ with $co\,G$ a compact map. Then there exists a $x \in X$ with $F(x) \cap \Lambda^{-1}(x) \neq \emptyset$; here $\Lambda(w) = co\,G(w)$ for $w \in Y$.*

Proof. By definition there exists an upper semicontinuous map $\Phi \in KKM(Y, X)$ with compact values and with $\Phi(y) \subseteq co\,(G(y))$ for $y \in Y$ and note Φ is a compact map. Also note from Secton 1 that $\Phi \in KKM(Y, \overline{co}\,(G(Y)))$. Next since $F \in W(X, Y)$ we have from Section 1 that $F \in W(\overline{co}\,(G(Y)), Y)$. Now $\overline{co}\,G(Y)$ is compact so paracompact, and Theorem 1.6 guarantees that there exists an upper semicontinuous map $\Psi : \overline{co}\,(G(Y)) \to CK(Y)$ (i.e. $\Psi \in Kak(\overline{co}\,(G(Y)), Y)$) with $\Psi(x) \subseteq F(x)$ for $x \in \overline{co}\,(G(Y))$. From Theorem 1.3 note $\Psi\,\Phi \in KKM(Y, Y)$ is an upper semicontinuous compact map so the result follows from Theorem 1.2. □

Theorem 2.14. *Let X be a subset of a Hausdorff topological space and Y a convex metrizable complete subset of a Hausdorff locally convex linear topological space. Suppose $F \in W(X, Y)$ and $G \in CKKM(Y, X)$ with G a compact map. Then there exists a $x \in X$ with $F(x) \cap G^{-1}(x) \neq \emptyset$.*

Proof. By definition there exists an upper semicontinuous map $\Phi \in KKM(Y, X)$ with compact values and with $\Phi(y) \subseteq G(y)$ for $y \in Y$ and note Φ is a compact map. Also note from Secton 1 that $\Phi \in KKM(Y, \overline{G(Y)})$ and $F \in W(\overline{G(Y)}, Y)$. Now $\overline{G(Y)}$ is compact so paracompact, and Theorem 1.6 guarantees that there exists an upper semicontinuous map $\Psi : \overline{G(Y)} \to CK(Y)$ with $\Psi(x) \subseteq F(x)$ for $x \in \overline{G(Y)}$. From Theorem 1.3 note $\Psi\,\Phi \in KKM(Y, Y)$ is an upper semicontinuous compact map. Now apply Theorem 1.2. □

There are also obvious analogues of Theorem 2.6, Theorem 2.7, Theorem 2.8, and Theorem 2.9 with $F \in KLU(X, Y)$ replaced by $F \in W(X, Y)$. For completeness we present the analogue of Theorem 2.6 and Theorem 2.7.

Theorem 2.15. *Let X and Y be subsets of a Hausdorff locally convex linear topological space E with Y a convex metrizable complete subset of E. Suppose $F \in W(X, Y)$ and $G \in HYKKM(Y, X)$ and assume there exists a compact subset K of Y with $F(X) \subseteq K$. Let $L(K)$ be the linear span of K and suppose $co\,G(Y) \subseteq X \cap L(K)$ and $X \cap L(K)$ is closed both in $L(K)$ and in X. Then there exists a $x \in X$ with $F(x) \cap \Lambda^{-1}(x) \neq \emptyset$; here $\Lambda(w) = co\,G(w)$ for $w \in Y$.*

Proof. Note $F \in W(X \cap L(K), Y)$ and $X \cap L(K)$ is paracompact (see Theorem 2.6). Now Theorem 1.6 guarantees there exists an upper semicontinuous map $\Psi : X \cap L(K) \to CK(Y)$ with $\Psi(x) \subseteq F(x)$ for $x \in X \cap L(K)$ and note Ψ is a compact map. Since $G \in HYKKM(Y, X)$ there exists an upper semicontinuous map $\Phi \in KKM(Y, X)$ with compact values and with $\Phi(y) \subseteq co\,(G(y))$ for $y \in Y$. Now since $co\,G(Y) \subseteq X \cap L(K)$ and $X \cap L(K)$ is closed in X then from Section 1 we have that $\Phi \in KKM(Y, X \cap L(K))$ is an upper semicontinuous map. Thus $\Psi\,\Phi \in KKM(Y, Y)$ is an upper semicontinuous compact map. Now apply Theorem 1.2. □

Similarly we have the following analogue of Theorem 2.7.

Theorem 2.16. *Let X and Y be subsets of a Hausdorff locally convex linear topological space E with Y a convex metrizable complete subset of E. Suppose $F \in W(X,Y)$ and $G \in CKKM(Y,X)$ and assume there exists a compact subset K of Y with $F(X) \subseteq K$. Let $L(K)$ be the linear span of K and suppose $G(Y) \subseteq X \cap L(K)$ and $X \cap L(K)$ is closed both in $L(K)$ and in X. Then there exists a $x \in X$ with $F(x) \cap G^{-1}(x) \neq \emptyset$.*

Next we present coincidence results between KKM type maps and $HLPY$ maps.

Theorem 2.17. *Let X be a subset of a Hausdorff topological vector space and Y a convex admissible subset of a Hausdorff topological vector space and let X be paracompact. Suppose $F \in HLPY(X,Y)$ with F a compact map and $G \in HYKKM(Y,X)$. Then there exists a $x \in X$ with $F(x) \cap \Lambda^{-1}(x) \neq \emptyset$; here $\Lambda(w) = co\, G(w)$ for $w \in Y$.*

Proof. From Theorem 1.8 there exists a continuous (single valued) map $f : X \to Y$ with $f(x) \in F(x)$ for $x \in X$ and note f is a compact map since F is a compact map. Also by definition there exists an upper semicontinuous map $\Phi \in KKM(Y,X)$ with compact values and with $\Phi(y) \subseteq co\,(G(y))$ for $y \in Y$. From Theorem 1.1 (ii) note $f\,\Phi \in KKM(Y,Y)$ is an upper semicontinuous compact map. Now Theorem 1.2 guarantees a $y \in Y$ with $y \in f\,\Phi(y)$. Let $x \in \Phi(y)$ with $y = f(x)$, so $y = f(x) \subseteq F(x)$ and $x \in \Phi(y) \subseteq co\, G(y) = \Lambda(y)$. □

Similarly we obtain the following result (analogue of Theorem 2.3 with $F \in KLU(X,Y)$ with $F \in HLPY(X,Y)$).

Theorem 2.18. *Let X be a subset of a Hausdorff topological space and Y a convex admissible subset of a Hausdorff topological vector space and let X be paracompact. Suppose $F \in HLPY(X,Y)$ with F a compact map and $G \in CKKM(Y,X)$. Then there exists a $x \in X$ with $F(x) \cap G^{-1}(x) \neq \emptyset$.*

There are also analogues of Theorem 2.4 and Theorem 2.5 for $HLPY$ maps but in fact here we can obtain a more general result.

Theorem 2.19. *Let X be a subset of a Hausdorff topological vector space and Y a convex admissible subset of a Hausdorff topological vector space. Suppose $F \in HLPY(X,Y)$ and $G \in HYKKM(Y,X)$ with $co\, G$ a compact map. Then there exists a $x \in X$ with $F(x) \cap \Lambda^{-1}(x) \neq \emptyset$; here $\Lambda(w) = co\, G(w)$ for $w \in Y$.*

Proof. Since $F \in HLPY(X,Y)$ then from Section 1 we have $F \in HLPY(\overline{co}\,(G(Y)),Y)$. In addition since $\overline{co}\,(G(Y))$ is compact then Theorem 1.8 guarantees that there exists a continuous (single-valued) map $f : \overline{co}\,(G(Y)) \to Y$ with $f(x) \in F(x)$ for $y \in \overline{co}\,(G(Y))$. In fact [11] there exists a finite subset A of Y with $f(\overline{co}\,(G(Y))) \subseteq co\,(A)$, so $f \in C(\overline{co}\,(G(Y)), co\,(A))$. Also by definition there exists an upper semicontinuous map $\Phi \in KKM(Y,X)$ with compact values and with $\Phi(y) \subseteq co\,(G(y))$ for $y \in Y$ and note Φ is a compact map. From Section 1 note $\Phi \in KKM(co\,(A), \overline{co}\,(G(Y)))$ so from Theorem 1.1 (ii) note $f\,\Phi \in KKM(co\,(A), co\,(A))$ is a upper semicontinuous map. Now apply Theorem 1.2 (note $co\,(A)$ is a compact convex set in a finite dimensional subspace of the Hausdorff topological vector space associated with Y) so there exists a $y \in Y$ with $y \in f\,\Phi(y)$. □

Theorem 2.20. *Let X be a subset of a Hausdorff topological space and Y a convex admissible subset of a Hausdorff topological vector space. Suppose $F \in HLPY(X,Y)$ and $G \in CKKM(Y,X)$ with G a compact map. Then there exists a $x \in X$ with $F(x) \cap G^{-1}(x) \neq \emptyset$.*

Proof. Since $F \in HLPY(X,Y)$ then from Section 1 we have $F \in HLPY(\overline{G(Y)}, Y)$ so Theorem 1.8 guarantees that there exists a continuous (single-valued) map $f : \overline{G(Y)} \to Y$ with $f(x) \in F(x)$ for $y \in \overline{G(Y)}$. Also there exists a finite subset A of Y with $f(\overline{G(Y)}) \subseteq co\,(A)$, so $f \in C(\overline{G(Y)}, co\,(A))$. Also by definition there exists an upper semicontinuous map $\Phi \in KKM(Y,X)$ with compact values and with $\Phi(y) \subseteq G(y)$ for $y \in Y$ and note Φ is a compact map. From Section 1 note $\Phi \in KKM(co\,(A), \overline{G(Y)})$ so from Theorem 1.1 (ii) note $f\,\Phi \in KKM(co\,(A), co\,(A))$ is a upper semicontinuous map. Now apply Theorem 1.2. □

There are also obvious analogues of Theorem 2.6, Theorem 2.7, Theorem 2.8, and Theorem 2.9 with $F \in KLU(X,Y)$ replaced by $F \in HLPY(X,Y)$. For completeness we present the analogue of Theorem 2.6 and Theorem 2.7.

Theorem 2.21. *Let X and Y be subsets of a Hausdorff topological vector space E with Y a convex admissible subset of E. Suppose $F \in HLPY(X,Y)$ and $G \in HYKKM(Y,X)$ and assume there exists a compact subset K of Y with $F(X) \subseteq K$. Let $L(K)$ be the linear span of K and suppose $co\,G(Y) \subseteq X \cap L(K)$ and $X \cap L(K)$ is closed both in $L(K)$ and in X. Then there exists a $x \in X$ with $F(x) \cap \Lambda^{-1}(x) \neq \emptyset$; here $\Lambda(w) = co\,G(w)$ for $w \in Y$.*

Proof. Note $F \in HLPY(X \cap L(K), Y)$ and $X \cap L(K)$ is paracompact (see Theorem 2.6). Now Theorem 1.8 guarantees there exists a continuous (single-valued) map $f : X \cap L(K) \to Y$ with $f(x) \in F(x)$ for $x \in X \cap L(K)$ and note f is a compact map. Since $G \in HYKKM(Y,X)$ there exists an upper semicontinuous map $\Phi \in KKM(Y,X)$ with compact values and with $\Phi(y) \subseteq co\,(G(y))$ for $y \in Y$. Next note $\Phi \in KKM(Y, X \cap L(K))$ is an upper semicontinuous map. Thus $f\,\Phi \in KKM(Y,Y)$ is an upper semicontinuous compact map. Now apply Theorem 1.2. □

Similarly we have the following analogue of Theorem 2.7.

Theorem 2.22. *Let X and Y be subsets of a Hausdorff topological vector space E with Y a convex admissible subset of E. Suppose $F \in HLPY(X,Y)$ and $G \in CKKM(Y,X)$ and assume there exists a compact subset K of Y with $F(X) \subseteq K$. Let $L(K)$ be the linear span of K and suppose $G(Y) \subseteq X \cap L(K)$ and $X \cap L(K)$ is closed both in $L(K)$ and in X. Then there exists a $x \in X$ with $F(x) \cap G^{-1}(x) \neq \emptyset$.*

Next we present coincidence results between KKM type maps and KKM type maps.

Theorem 2.23. *Let X be a convex admissible subset of a Hausdorff topological vector space, Y a convex admissible subset of a Hausdorff topological vector space and let X and Y be normal spaces. Suppose $F \in HYKKM(X,Y)$ with $co\,F$ a compact map and $G \in HYKKM(Y,X)$. Then there exists a $x \in X$ with $co\,F(x) \cap \Lambda^{-1}(x) \neq \emptyset$; here $\Lambda(w) = co\,G(w)$ for $w \in Y$.*

Proof. By definition there exists an upper semicontinuous map $\Psi \in KKM(X,Y)$ with compact values and with $\Psi(x) \subseteq co\,(F(x))$ for $x \in X$ and note Ψ is a compact map, and also by definition there exists an upper semicontinuous map $\Phi \in KKM(Y,X)$ with compact values and with $\Phi(y) \subseteq co\,(G(y))$ for $y \in Y$. From Theorem 1.5 note $\Psi\,\Phi \in KKM(Y,Y)$ is an upper semicontinuous compact map. Now apply Theorem 1.2. □

Similarly (with minor adjustments) we have the following results.

Theorem 2.24. *Let X be a convex admissible subset of a Hausdorff topological vector space, Y a convex admissible subset of a Hausdorff topological vector space and let X and Y be normal spaces. Suppose $F \in CKKM(X,Y)$ with F a compact map and $G \in HYKKM(Y,X)$. Then there exists a $x \in X$ with $F(x) \cap \Lambda^{-1}(x) \neq \emptyset$; here $\Lambda(w) = co\,G(w)$ for $w \in Y$.*

Theorem 2.25. *Let X be a convex admissible subset of a Hausdorff topological vector space, Y a convex admissible subset of a Hausdorff topological vector space and let X and Y be normal spaces. Suppose $F \in HYKKM(X,Y)$ with $co\,F$ a compact map and $G \in CKKM(Y,X)$. Then there exists a $x \in X$ with $co\,F(x) \cap G^{-1}(x) \neq \emptyset$.*

Theorem 2.26. *Let X be a convex admissible subset of a Hausdorff topological vector space, Y a convex admissible subset of a Hausdorff topological vector space and let X and Y be normal spaces. Suppose $F \in CKKM(X,Y)$ with F a compact map and $G \in CKKM(Y,X)$. Then there exists a $x \in X$ with $F(x) \cap G^{-1}(x) \neq \emptyset$.*

Finally we discuss a general class of maps [16] related to the KKM class. Let X be a convex subset of a Hausdorff topological vector space and Y a Hausdorff topological space. Now $F \in BPK(X,Y)$ if $F : X \to 2^Y$

and for any polytope P in X and any continuous map $f : F(P) \to P$ we have that $f(F|_P) : P \to 2^P$ has a fixed point.

We also note the following properties.

Let C and X be convex subsets of a Hausdorff topological vector space E with $C \subseteq X$ and Y a Hausdorff topological space.

(i). If $F \in BPK(X,Y)$ then $G \equiv F|_C \in BPK(C,Y)$.

Consider any polytope P in C and any continuous map $f : G(P) \to P$. Now since P is a polytope in X and (note $G(P) = F(P)$ since $P \subseteq C$) $f : F(p) \to P$ is a continuous map then since $F \in BPK(X,Y)$ there exists a $x \in P$ with $x \in f\, F|_P\,(x)$ i.e. $x \in f\, G(x)$ since $x \in P \subseteq C$. Thus $G = F|_C \in BPK(C,Y)$.

(ii). If $F \in BPK(X,Y)$ with $F(X) \subseteq Z \subseteq Y$ then $F \in BPK(X,Z)$.

Note $F : X \to 2^Y$ and let $G : X \to 2^Z$ be the map obtained by restricting the range of F. Consider any polytope P in X and any continuous map $f : G(P) \to P$. Now since $G(P) = F(P)$ then $f : F(P) \to P$ is continuous and since $F \in BPK(X,Y)$ there exists a $x \in P$ with $x \in f\, F|_P\,(x) = f\, G|_P\,(x)$. Thus $F \in BPK(X,Z)$.

(iii). If $F \in BPK(X,Y)$ and $f \in C(Y,X)$ then $f\,F \in BPK(X,X)$.

Note $f\,F : X \to 2^X$. Consider any polytope P in X and any continuous map $g : fF(P) \to P$. We must show there exists a $x \in P$ with $x \in g\,(f\,F)|_P(x)$. To see this note $g\,f\,F|_P = h\,F|_P : P \to 2^P$ where $h = g\,f : F(p) \to P$ is a continuous map (note $h(F(P)) = g\,(f\,F(P)) \subseteq P$). Since $F \in BPK(X,Y)$ there exists a $x \in P$ with $x \in h\,F|_P\,(x)$ i.e. $x \in (g\,f)\,F|_P(x) = g\,(f\,F)|_P(x)$. Thus $f\,F \in BPK(X,X)$.

From Theorem 1.1 (parts (i) and (iii)) note $BPK(X,Y) \subseteq KKM(X,Y)$ when Y is normal. Also from Theorem 1.1 (parts (i), (ii)) and Theorem 1.2 note if $F \in KKM(X,Y)$ is closed then $F \in BPK(X,Y)$ and so the two classes coincide for closed compact maps. Now we recall the following fixed point result [16] for BPK maps.

Theorem 2.27. *Let X be an admissible convex set in a Hausdorff topological vector space and $F \in BPK(X,X)$ be a closed compact map. Then F has a fixed point in X.*

Let X be a convex subset of a Hausdorff topological vector space and Y a subset of a Hausdorff topological vector space. We say $F \in HYBPK(X,Y)$ if $F : X \to 2^Y$ and there exists an upper semicontinuous map $\Psi \in BPK(X,Y)$ with compact values and with $\Psi(x) \subseteq co\,(F(x))$ for $x \in X$. We say $F \in CBPK(X,Y)$ if $F : X \to 2^Y$ and there exists an upper semicontinuous map $\Psi \in BPK(X,Y)$ with compact values and with $\Psi(x) \subseteq F(x)$ for $x \in X$ (in this case we only need Y to be a subset of a Hausdorff topological space). Now we present coincidence results between BPK type maps and $HLPY$ maps. We will just consider the analogue of Theorem 2.17 and Theorem 2.18.

Theorem 2.28. *Let X be a subset of a Hausdorff topological vector space and Y a convex admissible subset of a Hausdorff topological vector space and let X be paracompact. Suppose $F \in HLPY(X,Y)$ with F a compact map and $G \in HYBPK(Y,X)$. Then there exists a $x \in X$ with $F(x) \cap \Lambda^{-1}(x) \neq \emptyset$; here $\Lambda(w) = co\,G(w)$ for $w \in Y$.*

Proof. From Theorem 1.8 there exists a continuous (single valued) map $f : X \to Y$ with $f(x) \in F(x)$ for $x \in X$ and note f is a compact map since F is a compact map. Also by definition there exists an upper semicontinuous map $\Phi \in BPK(Y,X)$ with compact values and with $\Phi(y) \subseteq co\,(G(y))$ for $y \in Y$. From the above note $f\,\Phi \in BPK(Y,Y)$ is an upper semicontinuous compact map. Apply Theorem 2.27 to $f\,\Phi$. □

Similarly we have the following result .

Theorem 2.29. *Let X be a subset of a Hausdorff topological space and Y a convex admissible subset of a Hausdorff topological vector space and let X be paracompact. Suppose $F \in HLPY(X,Y)$ with F a compact map and $G \in CBPK(Y,X)$. Then there exists a $x \in X$ with $F(x) \cap G^{-1}(x) \neq \emptyset$.*

Remark 2.30. *There are also obvious analogues of Theorem 2.19 (respectively, Theorem 2.20) with $G \in HYKKM(Y,X)$ (respectively, $G \in CKKM(Y,X)$) replaced by $G \in HYBPK(Y,X)$ (respectively, $G \in CBPK(Y,X)$). One only needs to make minor adjustments in the proof of Theorem 2.19 (respectively, Theorem 2.20) or alternatively one could deduce* *it immediately from Theorem 2.19 (respectively, Theorem 2.20) if one notes that G is a compact map and consider the comment before Theorem 2.27.*

There are also obvious analogues of Theorem 2.21 and Theorem 2.22 (and also Theorem 2.8 and Theorem 2.9). For completeness we present the analogue of Theorem 2.21 and Theorem 2.22.

Theorem 2.31. *Let X and Y be subsets of a Hausdorff topological vector space E with Y a convex admissible subset of E. Suppose $F \in HLPY(X, Y)$ and $G \in HYBPK(Y, X)$ and assume there exists a compact subset K of Y with $F(X) \subseteq K$. Let $L(K)$ be the linear span of K and suppose $co\, G(Y) \subseteq X \cap L(K)$ and $X \cap L(K)$ is closed in $L(K)$. Then there exists a $x \in X$ with $F(x) \cap \Lambda^{-1}(x) \neq \emptyset$; here $\Lambda(w) = co\, G(w)$ for $w \in Y$.*

Proof. Note $F \in HLPY(X \cap L(K), Y)$ and $X \cap L(K)$ is paracompact so from Theorem 1.8 there exists a continuous (single-valued) map $f : X \cap L(K) \to Y$ with $f(x) \in F(x)$ for $x \in X \cap L(K)$ and note f is a compact map. Since $G \in HBPK(Y, X)$ there exists an upper semicontinuous map $\Phi \in BPK(Y, X)$ with compact values and with $\Phi(y) \subseteq co\,(G(y))$ for $y \in Y$. Next note $\Phi \in BPK(Y, X \cap L(K))$ is an upper semicontinuous map. Thus $f\,\Phi \in BPK(Y, Y)$ is an upper semicontinuous compact map. Now apply Theorem 2.27. □

Similarly we have the following result

Theorem 2.32. *Let X and Y be subsets of a Hausdorff topological vector space E with Y a convex admissible subset of E. Suppose $F \in HLPY(X, Y)$ and $G \in CBPK(Y, X)$ and assume there exists a compact subset K of Y with $F(X) \subseteq K$. Let $L(K)$ be the linear span of K and suppose $G(Y) \subseteq X \cap L(K)$ and $X \cap L(K)$ is closed in $L(K)$. Then there exists a $x \in X$ with $F(x) \cap G^{-1}(x) \neq \emptyset$.*

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