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(q, p)-mixing weighted holomorphic mappings

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Abstract. As a generalization of (q, p)-mixing linear operators, we introduce a new notion of (q, p)-mixing weighted holomorphic mappings. Such maps are characterized in terms of both inequalities of integral and summability type. Their structure as an injective Banach ideal of weighted holomorphic mappings is established, and known composition-type results for (q, p)-mixing linear operators are extended to the weighted holomorphic setting.

1. Introduction

Given Banach spaces E and F, let $\mathcal{L}(E, F)$ be the Banach space of all bounded linear operators from E into F, equipped with the canonical operator norm. As usual, E^* denotes the space $\mathcal{L}(E, \mathbb{K})$, and B_E the closed unit ball of E.

The theory of p-summing operators was developed by Pietsch in [15, Part 3, 17]. An operator $T \in \mathcal{L}(E, F)$ is said to be p-summing with $p \in [1, \infty]$ if there exists a constant $C \geq 0$ such that

$$\left(\sum_{i=1}^{n} \|T(x_i)\|^p\right)^{\frac{1}{p}} \leq C \sup_{x^* \in B_{E^*}} \left(\sum_{i=1}^{n} |x^*(x_i)|^p\right)^{\frac{1}{p}} \qquad (1 \leq p < \infty),$$

$$\max_{1 \leq i \leq n} \|T(x_i)\| \leq C \sup_{x^* \in B_{E^*}} \left(\max_{1 \leq i \leq n} |x^*(x_i)|\right) \qquad (p = \infty),$$

for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$. In such a case, the least of all the constants C satisfying the inequality above, denoted by $\pi_p(T)$, defines a norm on the linear space $\Pi_p(E, F)$ of all p-summing operators from E into F. Such operators generate a Banach operator ideal $[\Pi_p, \pi_p]$ in the Pietsch's sense [15].

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The concept of (q,p)-mixing operators between Banach spaces extends the notion of p-summing operators. A first study on such operators can be found in [15, Part 4, 20].

An operator $T \in \mathcal{L}(E,F)$ is called (q,p)-mixing with $p,q \in [1,\infty]$ if for all Banach spaces G and all operators $S \in \Pi_q(F,G)$, the composition operator $S \circ T$ belongs to $\Pi_p(E,G)$. The Banach operator ideal of (q,p)-mixing operators is denoted by $[\mathcal{M}_{(q,p)}, \mathfrak{m}_{(q,p)}]$, where

$$\mathfrak{m}_{(q,p)}(T) = \sup\left\{\pi_p(S \circ T) \colon S \in \Pi_q(F,G),\ \pi_q(S) \le 1\right\}.$$

A variant of the notion of (q,p)-mixing operators will be introduced in this paper in the setting of weighted holomorphic mappings acting on an open subset of a complex Banach space and taking values in a complex Banach space. The introduction and study of analogous concepts in both the Lipschitz and multilinear settings has garnered extensive attention as evidenced by various studies [1, 7, 11, 14, 16].

To describe our aims, we recall some basic facts from the theory of weighted holomorphic mappings, borrowed from [2, 3, 10]. Let E and F be complex Banach spaces and let U be an open subset of E. A weight v on U is a continuous function $v\colon U \to (0,+\infty)$. The linear space of all holomorphic mappings from U to F is denoted by $\mathcal{H}(U,F)$. The linear space formed by all mappings $f \in \mathcal{H}(U,F)$ such that

$$\left\|f\right\|_v := \sup\left\{v(x)\left\|f(x)\right\| : x \in U\right\} < \infty,$$

is denoted by $\mathcal{H}_v^\infty(U,F)$, and it becomes a Banach space under the weighted supremum norm $\|\cdot\|_v$. We write $\mathcal{H}_v^\infty(U)$ instead of $\mathcal{H}_v^\infty(U,\mathbb{C})$. In the case $v(x) = 1$ for all $x \in U$, we obtain the space of all bounded holomorphic mappings $\mathcal{H}^\infty(U,F)$, endowed with the supremum norm $\|\cdot\|_\infty$.

Let $\mathcal{G}_v^\infty(U)$ be the space of all linear functionals on $\mathcal{H}_v^\infty(U)$ whose restriction to $B_{\mathcal{H}_v^\infty(U)}$ is continuous for the compact-open topology. In fact, $\mathcal{G}_v^\infty(U)$ is a closed subspace of $\mathcal{H}_v^\infty(U)^*$, and the mapping $J_v\colon \mathcal{H}_v^\infty(U) \to \mathcal{G}_v^\infty(U)^*$, given by $J_v(g)(\phi) = \phi(g)$ for $\phi \in \mathcal{G}_v^\infty(U)$ and $g \in \mathcal{H}_v^\infty(U)$, is an isometric isomorphism. For each $x \in U$, the functional $\delta_x\colon \mathcal{H}_v^\infty(U) \to \mathbb{C}$, defined by $\delta_x(f) = f(x)$ for $f \in \mathcal{H}_v^\infty(U)$, is in $\mathcal{G}_v^\infty(U)$, and there exists $g_x \in B_{\mathcal{H}_v^\infty(U)}$ such that $g_x(x) = \|\delta_x\| := \sup_{g\in B_{\mathcal{H}_v^\infty(U)}}\left|g(x)\right|$. For complete information on weighted holomorphic mappings, we refer to the survey by Bonet [4].

The study of holomorphic mappings defined by a summability property was addressed by Matos [12] and Pellegrino [13] with the obtainment of holomorphic versions of known results of the linear theory. Apparently, in the weighted holomorphic case, the first class defined by a summability property was the injective Banach ideal of p-summing weighted holomorphic mappings, introduced by the second author, Cabrera-Padilla and Çopur in [6]. These mappings can be seen as a natural extension of p-summing operators, transferring some properties from the linear case to the weighted holomorphic setting.

Given $p \in [1,\infty]$, a mapping $f \in \mathcal{H}(U,F)$ is said to be p-summing weighted holomorphic if there exists a constant $C \ge 0$ such that

$$\left(\sum_{i=1}^n |\lambda_i|^p\, v(x_i)^p \left\|f(x_i)\right\|^p\right)^{\frac{1}{p}} \le C \sup_{g\in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^n |\lambda_i|^p\, v(x_i)^p \left|g(x_i)\right|^p\right)^{\frac{1}{p}} \qquad (1 \le p < \infty),$$

$$\max_{1\le i\le n}|\lambda_i|\, v(x_i)\left\|f(x_i)\right\| \le C \sup_{g\in B_{\mathcal{H}_v^\infty(U)}}\left(\max_{1\le i\le n}|\lambda_i|\, v(x_i)\left|g(x_i)\right|\right) \qquad (p = \infty),$$

for any $n \in \mathbb{N}$, $\lambda_1,\dots,\lambda_n \in \mathbb{C}$ and $x_1,\dots,x_n \in U$. In such a case, $\pi_p^{\mathcal{H}_v^\infty}(f)$ denotes the infimum of all constants C satisfying the inequality above, and $\Pi_p^{\mathcal{H}_v^\infty}(U,F)$ stands for the linear space of all p-summing weighted holomorphic mappings from U into F.

We now extend the concept of (q,p)-mixing operators to the case of weighted holomorphic mappings as follows.

Definition 1.1. *Let U be an open subset of a complex Banach space E, v be a weight on U and F be a complex Banach space. For $p,q \in [1,\infty]$, a mapping $f \in \mathcal{H}(U,F)$ is said to be (q,p)-mixing weighted holomorphic if the composition*

$T \circ f$ belongs to $\Pi_p^{\mathcal{H}_v^\infty}(U, G)$ for all complex Banach spaces G and all operators $T \in \Pi_q(F, G)$. We set

$$\mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}(f) = \sup\left\{\pi_p^{\mathcal{H}_v^\infty}(T \circ f) \colon T \in \Pi_q(F, G),\ \pi_q(T) \leq 1\right\}.$$

The set of all (q, p)-mixing weighted holomorphic maps from U into F is denoted by $\mathcal{M}_{(q,p)}^{\mathcal{H}_v^\infty}(U, F)$.

Observe that a mapping $f \in \mathcal{H}(U, F)$ is (q, p)-mixing weighted holomorphic if for each complex Banach space G, the composition operator C_f with symbol f maps the operator space $\Pi_q(F, G)$ to the weighted holomorphic space $\Pi_p^{\mathcal{H}_v^\infty}(U, G)$. Composition operators between function spaces represent an important area in functional analysis and operator theory. Researchers have extensively explored these operators defined between spaces of holomorphic functions (see, for example, [8, 17]).

The objective of this paper is to examine the properties of (q, p)-mixing weighted holomorphic mappings. We want to answer the following question that arises naturally: *What results about (q, p)-mixing operators have analogs for (q, p)-mixing weighted holomorphic mappings?*

Following the model of the linear case, as studied in the monographs by Defant and Floret [9] and Pietsch [15], well-known results for the class of (q, p)-mixing operators between Banach spaces are extended in this paper to the class of (q, p)-mixing weighted holomorphic mappings addressing different properties: inclusion and coincidence between different classes of weighted holomorphic mappings (Propositions 2.2–2.5); Pietsch type composition (Proposition 2.6); injective Banach ideal property (Theorem 2.7 and Corollary 2.9); and characterizations in terms of integral and summability inequalities (Theorem 2.8).

2. The results

From now on, unless otherwise stated, E, F and G will be complex Banach spaces, U an open subset of E and v a weight on U. Moreover, we will assume that $p, q \in [1, \infty]$.

Recently, the concept of weighted holomorphic ideals was introduced by the second author, Cabrera-Padilla and Çopur in [5].

A Banach weighted holomorphic ideal is an assignment $\left[\mathcal{I}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}\right]$ which maps each (E, U, v, F) – where E is a complex Banach space, U is an open subset of E, v is a weight on U and F is a complex Banach space – to both a set $\mathcal{I}^{\mathcal{H}_v^\infty}(U, F) \subseteq \mathcal{H}_v^\infty(U, F)$ and a function $\|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} : \mathcal{I}^{\mathcal{H}_v^\infty}(U, F) \to [0, +\infty)$, that satisfy the following properties:

(P1) $\left(\mathcal{I}^{\mathcal{H}_v^\infty}(U, F), \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}\right)$ is a Banach space with $\left\|f\right\|_v \leq \left\|f\right\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}$ for all $f \in \mathcal{I}^{\mathcal{H}_v^\infty}(U, F)$,

(P2) Given $h \in \mathcal{H}_v^\infty(U)$ and $y \in F$, the map $h \cdot y \colon x \in U \mapsto h(x)y \in F$ is in $\mathcal{I}^{\mathcal{H}_v^\infty}(U, F)$ with $\left\|h \cdot y\right\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} = \|h\|_v \left\|y\right\|$,

(P3) If V is an open subset of E such that $V \subseteq U$, $h \in \mathcal{H}(V, U)$ with

$$c_v(h) := \sup\left\{\frac{v(x)}{v(h(x))} \colon x \in V\right\} < \infty,$$

$f \in \mathcal{I}^{\mathcal{H}_v^\infty}(U, F)$ and $S \in \mathcal{L}(F, H)$ where H is a complex Banach space, then $S \circ f \circ h$ is in $\mathcal{I}^{\mathcal{H}_v^\infty}(V, H)$ with $\|S \circ f \circ h\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} \leq \|S\| \, \|f\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} c_v(h)$.

A Banach weighted holomorphic ideal $[\mathcal{I}^{\mathcal{H}_v^\infty}, \| \cdot \|_{\mathcal{I}^{\mathcal{H}_v^\infty}}]$ is called:

(I) injective if for any map $f \in \mathcal{H}_v^\infty(U, F)$, any complex Banach space H and any into linear isometry $\iota \colon F \to H$, we have that f belongs to $\mathcal{I}^{\mathcal{H}_v^\infty}(U, F)$ with $\left\|f\right\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} = \left\|\iota \circ f\right\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}$ whenever $\iota \circ f \in \mathcal{I}^{\mathcal{H}_v^\infty}(U, H)$.

By [6, Proposition 1.6], the class $[\Pi_p^{\mathcal{H}_v^\infty}, \pi_p^{\mathcal{H}_v^\infty}]$ is an injective Banach weighted holomorphic ideal.

We may establish a connection between the spaces $\mathcal{M}_{(q,p)}^{\mathcal{H}_v^\infty}$ and $\mathcal{H}_v^\infty$, which justifies the terminology and notation used in Definition 1.1.

Lemma 2.1. *If $f \in \mathcal{M}^{\mathcal{H}^\infty_v}_{(q,p)}(U,F)$, then $f \in \mathcal{H}^\infty_v(U,F)$ with $\|f\|_v \le \mathfrak{m}^{\mathcal{H}^\infty_v}_{(q,p)}(f)$.*

Proof. Note first that $\Pi_q(F,\mathbb{C}) = F^*$ and $\pi_q(y^*) = \|y^*\|$ for all $y^* \in F^*$ since $[\Pi_q, \pi_q]$ is a Banach operator ideal. Let $f \in \mathcal{M}^{\mathcal{H}^\infty_v}_{(q,p)}(U,F)$. Then $y^* \circ f \in \Pi^{\mathcal{H}^\infty_v}_p(U,\mathbb{C})$ with

$$\pi^{\mathcal{H}^\infty_v}_p(y^* \circ f) \le \mathfrak{m}^{\mathcal{H}^\infty_v}_{(q,p)}(f)\pi_q(y^*) = \mathfrak{m}^{\mathcal{H}^\infty_v}_{(q,p)}(f)\left\|y^*\right\|,$$

for all $y^* \in F^*$. It follows that $y^* \circ f \in \mathcal{H}^\infty_v(U,\mathbb{C})$ with $\|y^* \circ f\|_v \le \mathfrak{m}^{\mathcal{H}^\infty_v}_{(q,p)}(f)\left\|y^*\right\|$ for all $y^* \in F^*$, because $[\Pi^{\mathcal{H}^\infty_v}_p, \pi^{\mathcal{H}^\infty_v}_p]$ is a Banach weighted holomorphic ideal. In particular, we have that

$$v(x)\left\|f(x)\right\| = v(x)\left|y^*_x(f(x))\right| \le \left\|y^*_x \circ f\right\|_v \le \mathfrak{m}^{\mathcal{H}^\infty_v}_{(q,p)}(f)\left\|y^*_x\right\| \le \mathfrak{m}^{\mathcal{H}^\infty_v}_{(q,p)}(f)$$

for all $x \in U$, where we have taken a functional $y^*_x \in B_{F^*}$ so that $\left|y^*_x(f(x))\right| = \left\|f(x)\right\|$ by applying the Hahn–Banach theorem. Therefore $f \in \mathcal{H}^\infty_v(U,F)$ with $\left\|f\right\|_v \le \mathfrak{m}^{\mathcal{H}^\infty_v}_{(q,p)}(f)$. $\square$

We now show that the concept of (q,p)-mixing weighted holomorphic mappings extends the notion of p-summing weighted holomorphic mappings.

Proposition 2.2. *Every p-summing weighted holomorphic mapping $f: U \to F$ is (q,p)-mixing weighted holomorphic with $\mathfrak{m}^{\mathcal{H}^\infty_v}_{(q,p)}(f) \le \pi^{\mathcal{H}^\infty_v}_p(f)$.*

Proof. Let $f \in \Pi^{\mathcal{H}^\infty_v}_p(U,F)$. For a complex Banach space G and $T \in \Pi_q(F,G)$, it follows that $T \circ f \in \Pi^{\mathcal{H}^\infty_v}_p(U,G)$ with $\pi^{\mathcal{H}^\infty_v}_p(T \circ f) \le \|T\|\pi^{\mathcal{H}^\infty_v}_p(f) \le \pi_q(T)\pi^{\mathcal{H}^\infty_v}_p(f)$ by the ideal properties of $[\Pi^{\mathcal{H}^\infty_v}_p, \pi^{\mathcal{H}^\infty_v}_p]$ and $[\Pi_q, \pi_q]$. Hence $f \in \mathcal{M}^{\mathcal{H}^\infty_v}_{(q,p)}(U,F)$ with $\mathfrak{m}^{\mathcal{H}^\infty_v}_{(q,p)}(f) \le \pi^{\mathcal{H}^\infty_v}_p(f)$. $\square$

In fact, the class of p-summing weighted holomorphic mappings coincides with the class of (q,p)-mixing weighted holomorphic mappings in the extreme case $q = \infty$.

Proposition 2.3. *$\mathcal{M}^{\mathcal{H}^\infty_v}_{(\infty,p)}(U,F) = \Pi^{\mathcal{H}^\infty_v}_p(U,F)$ and $\mathfrak{m}^{\mathcal{H}^\infty_v}_{(\infty,p)}(f) = \pi^{\mathcal{H}^\infty_v}_p(f)$ for all $f \in \mathcal{M}^{\mathcal{H}^\infty_v}_{(\infty,p)}(U,F)$.*

Proof. Note that the proof of Proposition 2.2 is valid for $q = \infty$, and thus $\Pi^{\mathcal{H}^\infty_v}_p(U,F) \subseteq \mathcal{M}^{\mathcal{H}^\infty_v}_{(\infty,p)}(U,F)$ and $\mathfrak{m}^{\mathcal{H}^\infty_v}_{(\infty,p)}(f) \le \pi^{\mathcal{H}^\infty_v}_p(f)$ for all $f \in \Pi^{\mathcal{H}^\infty_v}_p(U,F)$.

Conversely, take $f \in \mathcal{M}^{\mathcal{H}^\infty_v}_{(\infty,p)}(U,F)$. Since $\mathrm{id}_F \in \Pi_\infty(F,F) = \mathcal{L}(F,F)$ and $\pi_\infty(\mathrm{id}_F) = \|\mathrm{id}_F\| = 1$, it follows that $f = \mathrm{id}_F \circ f$ is in $\Pi^{\mathcal{H}^\infty_v}_p(U,F)$, and $\pi^{\mathcal{H}^\infty_v}_p(f) = \pi^{\mathcal{H}^\infty_v}_p(\mathrm{id}_F \circ f) \le \mathfrak{m}^{\mathcal{H}^\infty_v}_{(\infty,p)}(f)\pi_\infty(\mathrm{id}_F) = \mathfrak{m}^{\mathcal{H}^\infty_v}_{(\infty,p)}(f)$. $\square$

In the case $q \le p$, we will see that (q,p)-mixing weighted holomorphic mappings are exactly weighted holomorphic mappings.

Proposition 2.4. *If $p, q \in [1,\infty]$ and $q \le p$, then $\mathcal{M}^{\mathcal{H}^\infty_v}_{(q,p)}(U,F) = \mathcal{H}^\infty_v(U,F)$ and $\mathfrak{m}^{\mathcal{H}^\infty_v}_{(q,p)}(f) = \|f\|_v$ for all $f \in \mathcal{M}^{\mathcal{H}^\infty_v}_{(q,p)}(U,F)$.*

Proof. Let $f \in \mathcal{H}^\infty_v(U,F)$. Let G be a complex Banach space and $T \in \Pi_q(F,G)$. Then $T \in \Pi_p(F,G)$ with $\pi_p(T) \le \pi_q(T)$ by [15, Proposition 17.3.9]. Assume first $p < \infty$. For any $n \in \mathbb{N}$, $\lambda_i \in \mathbb{C}$ and $x_i \in U$ for $i = 1,\dots,n$, one has that

$$\left(\sum_{i=1}^n |\lambda_i|^p\, v(x_i)^p \left\|(T \circ f)(x_i)\right\|^p\right)^{\frac{1}{p}} \le \pi_p(T) \sup_{y^* \in B_{F^*}} \left(\sum_{i=1}^n |\lambda_i|^p\, v(x_i)^p \left|y^*(f(x_i))\right|^p\right)^{\frac{1}{p}}.$$

Moreover, given any $y^* \in B_{F^*}$, it holds that

$$\left(\sum_{i=1}^{n} |\lambda_i|^p \, v(x_i)^p \left|y^*(f(x_i))\right|^p\right)^{\frac{1}{p}} \leq \left\|f\right\|_v \sup_{g \in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^{n} |\lambda_i|^p \, v(x_i)^p \left|g(x_i)\right|^p\right)^{\frac{1}{p}}$$

since $y^* \circ f \in \mathcal{H}_v^\infty(U)$ with $\left\|y^* \circ f\right\|_v \leq \left\|y^*\right\| \left\|f\right\|_v \leq \left\|f\right\|_v$, and thus

$$\left(\sum_{i=1}^{n} |\lambda_i|^p \, v(x_i)^p \left|y^*(f(x_i))\right|^p\right)^{\frac{1}{p}} \leq \left\|f\right\|_v \sup_{g \in B_{\mathcal{H}_v^\infty}} \left(\sum_{i=1}^{n} |\lambda_i|^p \, v(x_i)^p \left|g(x_i)\right|^p\right)^{\frac{1}{p}}.$$

Consequently, we have that

$$\left(\sum_{i=1}^{n} |\lambda_i|^p \, v(x_i)^p \left\|(T \circ f)(x_i))\right\|^p\right)^{\frac{1}{p}} \leq \pi_q(T) \left\|f\right\|_v \sup_{g \in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^{n} |\lambda_i|^p \, v(x_i)^p \left|g(x_i)\right|^p\right)^{\frac{1}{p}}$$

whenever $p < \infty$. Similarly, we obtain

$$\max_{1 \leq i \leq n} |\lambda_i| \, v(x_i) \left\|(T \circ f)(x_i)\right\| \leq \pi_q(T) \left\|f\right\|_v \sup_{g \in B_{\mathcal{H}_v^\infty}} \left(\max_{1 \leq i \leq n} |\lambda_i| \, v(x_i) \left|g(x_i)\right|\right).$$

Hence $T \circ f \in \Pi_p^{\mathcal{H}_v^\infty}(U, G)$ and $\pi_p^{\mathcal{H}_v^\infty}(T \circ f) \leq \pi_q(T) \left\|f\right\|_v$. Thus $f \in \mathcal{M}_{(q,p)}^{\mathcal{H}_v^\infty}(U, F)$ and $\mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}(f) \leq \left\|f\right\|_v$. This completes the proof in view of Lemma 2.1. $\square$

We establish the following inclusion between two classes of mixing weighted holomorphic mappings with different parameters. Compare to [15, Proposition 20.1.9].

Proposition 2.5. *Let $1 \leq p_2 \leq p_1 \leq q_1 \leq q_2 \leq \infty$. Then $\mathcal{M}_{(q_2,p_2)}^{\mathcal{H}_v^\infty}(U, F) \subseteq \mathcal{M}_{(q_1,p_1)}^{\mathcal{H}_v^\infty}(U, F)$ and $\mathfrak{m}_{(q_1,p_1)}^{\mathcal{H}_v^\infty}(f) \leq \mathfrak{m}_{(q_2,p_2)}^{\mathcal{H}_v^\infty}(f)$ for all $f \in \mathcal{M}_{(q_2,p_2)}^{\mathcal{H}_v^\infty}(U, F)$.*

Proof. Let $f \in \mathcal{M}_{(q_2,p_2)}^{\mathcal{H}_v^\infty}(U, F)$. Let G be a complex Banach space and $T \in \Pi_{q_1}(F, G)$. Then $T \in \Pi_{q_2}(F, G)$ and $\pi_{q_2}(T) \leq \pi_{q_1}(T)$ by [15, Proposition 17.3.9]. It follows that $T \circ f \in \Pi_{p_2}^{\mathcal{H}_v^\infty}(U, G)$ and $\pi_{p_2}^{\mathcal{H}_v^\infty}(T \circ f) \leq \mathfrak{m}_{(q_2,p_2)}^{\mathcal{H}_v^\infty}(f)\pi_{q_2}(T)$. Hence $T \circ f \in \Pi_{p_1}^{\mathcal{H}_v^\infty}(U, G)$ with $\pi_{p_1}^{\mathcal{H}_v^\infty}(T \circ f) \leq \pi_{p_2}^{\mathcal{H}_v^\infty}(T \circ f)$ by [6, Proposition 1.9]. Therefore $f \in \mathcal{M}_{(q_1,p_1)}^{\mathcal{H}_v^\infty}(U, F)$. Moreover, we have that $\pi_{p_1}^{\mathcal{H}_v^\infty}(T \circ f) \leq \mathfrak{m}_{(q_2,p_2)}(f)\pi_{q_1}(T)$, and passing to the supremum on all $T \in \Pi_{q_2}(F, G)$ with $\pi_{q_2}(T) \leq 1$, we arrive at $\mathfrak{m}_{(q_1,p_1)}^{\mathcal{H}_v^\infty}(f) \leq \mathfrak{m}_{(q_2,p_2)}^{\mathcal{H}_v^\infty}(f)$. $\square$

We now state the following Pietsch type multiplication formula (see [15, Theorem 20.1.8]).

Proposition 2.6. *Let $1 \leq p \leq q \leq r \leq \infty$. If $f \in \mathcal{M}_{(q,p)}^{\mathcal{H}_v^\infty}(U, F)$ and $S \in \mathcal{M}_{(r,q)}(F, G)$, then $S \circ f \in \mathcal{M}_{(r,p)}^{\mathcal{H}_v^\infty}(U, G)$ and $\mathfrak{m}_{(r,p)}^{\mathcal{H}_v^\infty}(S \circ f) \leq \mathfrak{m}_{(r,q)}(S)\mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}(f)$.*

Proof. Let H be a complex Banach space and $T \in \Pi_r(G, H)$. Then $T \circ S \in \Pi_q(F, H)$ with $\pi_q(T \circ S) \leq \pi_r(T)\mathfrak{m}_{(r,q)}(S)$ by [15, Theorem 20.1.8]. Hence $T \circ (S \circ f) = (T \circ S) \circ f \in \Pi_p^{\mathcal{H}_v^\infty}(U, H)$ with $\pi_p(T \circ (S \circ f)) \leq \pi_q(T \circ S)\mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}(f)$. Therefore $S \circ f \in \mathcal{M}_{(r,p)}^{\mathcal{H}_v^\infty}(U, G)$ and, since $\pi_p(T \circ (S \circ f)) \leq \pi_r(T)\mathfrak{m}_{(r,q)}(S)\mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}(f)$, taking the supremum over all $T \in \Pi_r(G, H)$ with $\pi_r(T) \leq 1$, we conclude that $\mathfrak{m}_{(r,p)}^{\mathcal{H}_v^\infty}(S \circ f) \leq \mathfrak{m}_{(r,q)}(S)\mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}(f)$. $\square$

Motivated by [15, Theorem 20.1.2], we now show that (q, p)-mixing weighted holomorphic mappings enjoy a Banach ideal property.

Theorem 2.7. $\left[\mathcal{M}_{(q,p)}^{\mathcal{H}_v^\infty}, \mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}\right]$ *is a Banach weighted holomorphic ideal.*

Proof. Let U be an open subset of a complex Banach space E, let v be a weight on U, and let F be a complex Banach space.

(P1) Clearly, the zero function $\mathbf{0}$ on U is in $\mathcal{M}_{(q,p)}^{\mathcal{H}_v^\infty}(U,F)$. By Lemma 2.1, if $f \in \mathcal{M}_{(q,p)}^{\mathcal{H}_v^\infty}(U,F)$ and $\mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}(f) = 0$, then $f = \mathbf{0}$. Let $f, g \in \mathcal{M}_{(q,p)}^{\mathcal{H}_v^\infty}(U,F)$. Let G be a complex Banach space and $T \in \Pi_q(F,G)$. By the ideal property of $[\Pi_p^{\mathcal{H}_v^\infty}, \pi_p^{\mathcal{H}_v^\infty}]$, we have that $T \circ (f+g) = T \circ f + T \circ g \in \Pi_p^{\mathcal{H}_v^\infty}(U,G)$ and

$$\pi_p^{\mathcal{H}_v^\infty}(T \circ (f+g)) \leq \pi_p^{\mathcal{H}_v^\infty}(T \circ f) + \pi_p^{\mathcal{H}_v^\infty}(T \circ g) \leq \pi_q(T)\left(\mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}(f) + \mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}(g)\right).$$

Hence $f + g \in \mathcal{M}_{(q,p)}^{\mathcal{H}_v^\infty}(U,F)$. By taking the supremum over all $T \in \Pi_q(F,G)$ with $\pi_q(T) \leq 1$ in the preceding inequality, we conclude that $\mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}(f+g) \leq \mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}(f) + \mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}(g)$.

By a similar argument, for $\alpha \in \mathbb{C}$ and $f \in \mathcal{M}_{(q,p)}^{\mathcal{H}_v^\infty}(U,F)$, we have $T \circ (\alpha f) = \alpha \cdot (T \circ f) \in \Pi_p^{\mathcal{H}_v^\infty}(U,G)$ and

$$\pi_p^{\mathcal{H}_v^\infty}(T \circ (\alpha f)) = |\alpha|\, \pi_p^{\mathcal{H}_v^\infty}(T \circ f) \leq |\alpha|\, \pi_q(T)\mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}(f).$$

Hence $\alpha f \in \mathcal{M}_{(q,p)}^{\mathcal{H}_v^\infty}(U,F)$ and $\mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}(\alpha f) \leq |\alpha|\, \mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}(f)$. This inequality becomes an equality to zero if $\alpha = 0$ by Lemma 2.1, while if $\alpha \neq 0$, we have that $\mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}(f) = \mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}(\alpha^{-1}(\alpha f)) \leq |\alpha|^{-1}\mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}(\alpha f)$ and thus $|\alpha|\, \mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}(f) \leq \mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}(\alpha f)$. Thus, we have shown that $(\mathcal{M}_{(q,p)}^{\mathcal{H}_v^\infty}(U,F), \mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty})$ is a normed space.

In order to prove that it is in fact a Banach space, we consider a Cauchy sequence $(f_n)_n$ in $(\mathcal{M}_{(q,p)}^{\mathcal{H}_v^\infty}(U,F), \mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty})$. Using Lemma 2.1, given $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\left\|f_n - f_k\right\|_v \leq \mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}(f_n - f_k) < \varepsilon,$$

for every $n, k \geq n_\varepsilon$. Therefore $(f_n)_n$ is a Cauchy sequence in the Banach space $(\mathcal{H}_v^\infty(U,F), \|\cdot\|_v)$. Thus, there is an $f \in \mathcal{H}_v^\infty(U,F)$ such that $\left\|f_n - f\right\|_v \to 0$ as $n \to +\infty$, and so $\left\|T \circ f_n - T \circ f\right\|_v \to 0$ as $n \to +\infty$. On the other hand, for every $n, k \geq n_\varepsilon$, we have the following inequalities

$$\pi_p^{\mathcal{H}_v^\infty}(T \circ f_n - T \circ f_k) \leq \pi_q(T)\mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}(f_n - f_k) < \pi_q(T)\varepsilon,$$

which yield that $(T \circ f_n)_n$ is a Cauchy sequence in the Banach space $(\Pi_p^{\mathcal{H}_v^\infty}(U,G), \pi_p^{\mathcal{H}_v^\infty})$. Hence there exists $g \in \Pi_p^{\mathcal{H}_v^\infty}(U,G)$ so that $\pi_p^{\mathcal{H}_v^\infty}(T \circ f_n - g) \to 0$ as $n \to +\infty$. Now, using the fact that $\|\cdot\|_v \leq \pi_p^{\mathcal{H}_v^\infty}$ on $\Pi_p^{\mathcal{H}_v^\infty}(U,G)$, we conclude that $g = T \circ f$, and therefore $f \in \mathcal{M}_{(q,p)}^{\mathcal{H}_v^\infty}(U,F)$. To show that $(f_n)_n$ converges to f in $(\mathcal{M}_{(q,p)}^{\mathcal{H}_v^\infty}(U,F), \mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty})$, taking limit as $k \to +\infty$ in the aforementioned inequalities, we obtain that for every $n \geq n_\varepsilon$,

$$\pi_p^{\mathcal{H}_v^\infty}(T \circ f_n - T \circ f) < \pi_q(T)\varepsilon,$$

and we arrive at the desired result by taking the supremum over all $T \in \Pi_q(F,G)$ with $\pi_q(T) \leq 1$.

(P2) Let $h \in \mathcal{H}_v^\infty(U)$ and $y \in F$. Let G be a complex Banach space and $T \in \Pi_q(F,G)$. By the Banach ideal property of $[\Pi_p^{\mathcal{H}_v^\infty}, \pi_p^{\mathcal{H}_v^\infty}]$, we have that $T \circ (h \cdot y) = h \cdot T(y) \in \Pi_p^{\mathcal{H}_v^\infty}(U,G)$ and $\pi_p^{\mathcal{H}_v^\infty}(T \circ (h \cdot y)) = \|h\|_v \left\|T(y)\right\|$. Therefore $h \cdot y \in \mathcal{M}_{(q,p)}^{\mathcal{H}_v^\infty}(U,F)$ with

$$\mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}(h \cdot y) = \|h\|_v \sup_{\pi_q(T) \leq 1} \left\|T(y)\right\| \leq \|h\|_v \sup_{\pi_q(T) \leq 1} \left(\pi_q(T)\left\|y\right\|\right) = \|h\|_v \left\|y\right\|.$$

The reverse inequality results from the fact that $\left\|y\right\| \|h\|_v = \left\|h \cdot y\right\|_v \leq \mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}(h \cdot y)$ by Lemma 2.1.

(P3) Let V, h, H and S be as in (P3), which is mentioned at the beginning of this section. Let $f \in \mathcal{M}_{(q,p)}^{\mathcal{H}_v^\infty}(U,F)$. Let G be a complex Banach space and $T \in \Pi_q(H,G)$. Since $T \circ S \in \Pi_q(F,G)$ and $\pi_q(T \circ S) \leq \pi_q(T)\,\|S\|$, we deduce that $T \circ S \circ f \in \Pi_p^{\mathcal{H}_v^\infty}(U,G)$ with

$$\pi_p^{\mathcal{H}_v^\infty}(T \circ S \circ f) \leq \pi_q(T)\,\|S\|\,\mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}(f).$$

By the Banach ideal property of $[\Pi_p^{\mathcal{H}_v^\infty}, \pi_p^{\mathcal{H}_v^\infty}]$, we have that $T \circ S \circ f \circ h \in \Pi_p^{\mathcal{H}_v^\infty}(V,G)$ and

$$\pi_p^{\mathcal{H}_v^\infty}(T \circ S \circ f \circ h) \leq \pi_q(T)\,\|S\|\,\mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}(f) c_v(h).$$

Consequently, $S \circ f \circ h \in \mathcal{M}_{(q,p)}^{\mathcal{H}_v^\infty}(V,H)$, and by taking the supremum over all $T \in \Pi_q(H,G)$ with $\pi_q(T) \leq 1$, we obtain that $\mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}(S \circ f \circ h) \leq \|S\|\,\mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}(f) c_v(h)$. $\square$

Since $\mathcal{H}_v^\infty(U)$ is isometrically isomorphic to $\mathcal{G}_v^\infty(U)^*$, $\mathcal{P}(B_{\mathcal{H}_v^\infty(U)})$ will denote the set of all Borel regular probability measures η on $(B_{\mathcal{H}_v^\infty(U)}, w^*)$, where w^* denotes the weak* topology. Similarly, $\mathcal{P}(B_{F^*})$ stands for the set of all Borel regular probability measures μ on (B_{F^*}, w^*).

We now characterize (q,p)-mixing weighted holomorphic mappings in terms of an integral inequality, a summability inequality, and a mixture inequality of both types. In this way we present variants for (q,p)-mixing weighted holomorphic mappings of some known criterions for (q,p)-mixing operators (see [15, Theorems 20.1.4 and 20.1.7]).

Theorem 2.8. *For $1 \leq p < q < \infty$ and $f \in \mathcal{H}_v^\infty(U,F)$, the following statements are equivalent:*

(i) f is (q,p)-mixing weighted holomorphic.

(ii) There exists a constant $C \geq 0$ such that for each measure $\mu \in \mathcal{P}(B_{F^*})$, we can find a measure $\eta \in \mathcal{P}(B_{\mathcal{H}_v^\infty})$ such that

$$\left(\int_{B_{F^*}} \left|y^*(f(x))\right|^q d\mu(y^*)\right)^{\frac{1}{q}} \leq C\left(\int_{B_{\mathcal{H}_v^\infty(U)}} \left|g(x)\right|^p d\eta(g)\right)^{\frac{1}{p}},$$

for all $x \in U$.

(iii) There exists a constant $C \geq 0$ such that for all finite families of vectors $x_1, \ldots, x_m \in U$, scalars $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ and functionals $y_1^*, \ldots, y_n^* \in F^*$, we have

$$\left(\sum_{i=1}^m \left(\sum_{j=1}^n |\lambda_i|^q\, v(x_i)^q \left|y_j^*(f(x_i))\right|^q\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} \leq C\left(\sum_{j=1}^n \left\|y_j^*\right\|^q\right)^{\frac{1}{q}} \sup_{g \in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^m |\lambda_i|^p\, v(x_i)^p \left|g(x_i)\right|^p\right)^{\frac{1}{p}}.$$

(iv) There exists a constant $C \geq 0$ such that for all finite families of elements $x_1, \ldots, x_m \in U$, scalars $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$, functionals $y_1^*, \ldots, y_n^* \in F^*$ and measure $\mu \in \mathcal{P}(B_{F^*})$, we have

$$\left(\sum_{i=1}^m |\lambda_i|^p\, v(x_i)^p \left(\int_{B_{F^*}} \left|y^*(f(x_i))\right|^q d\mu(y^*)\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} \leq C \sup_{g \in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^m |\lambda_i|^p\, v(x_i)^p \left|g(x_i)\right|^p\right)^{\frac{1}{p}}.$$

In this case, $\mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}(f)$ is the minimum of all constants C satisfying (ii) or all satisfying (iii) or all satisfying (iv).

Proof. (i) $\Rightarrow$ (ii): Let $f \in \mathcal{M}_{(q,p)}^{\mathcal{H}_v^\infty}(U,F)$. Let $\mu \in \mathcal{P}(B_{F^*})$ and consider the operator $\iota_F \in \mathcal{L}(F, L_q(\mu))$ assigning to each $y \in F$ the function $\iota_F(y)\colon B_{F^*} \to \mathbb{C}$ defined by $\iota_F(y)(y^*) = y^*(y)$ for all $y^* \in B_{F^*}$. It is known that $\iota_F \in \Pi_q(F, L_q(\mu))$ and $\pi_q(\iota_F) = 1$. This means that $\iota_F \circ f \in \Pi_p^{\mathcal{H}_v^\infty}(U, L_q(\mu))$ and $\pi_p^{\mathcal{H}_v^\infty}(\iota_F \circ f) \leq \mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}(f)$. According to Pietsch's domination theorem for p-summing weighted holomorphic mappings (see [6, Theorem 1.7]), there exists a measure $\eta \in \mathcal{P}(B_{\mathcal{H}_v^\infty(U)})$ such that

$$\left\|\iota_F(f(x))\right\| \leq \pi_p^{\mathcal{H}_v^\infty}(\iota_F \circ f)\left(\int_{B_{\mathcal{H}_v^\infty(U)}} \left|g(x)\right|^p d\eta(g)\right)^{\frac{1}{p}},$$

for all $x \in U$, and therefore

$$\left(\int_{B_{F^*}} \left|y^*(f(x))\right|^q d\mu(y^*)\right)^{\frac{1}{q}} \leq \mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}(f)\left(\int_{B_{\mathcal{H}_v^\infty(U)}} \left|g(x)\right|^p d\eta(g)\right)^{\frac{1}{p}},$$

for all $x \in U$.

$(ii) \Rightarrow (iii)$: Let $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$, $x_1, \ldots, x_m \in U$ and $y_1^*, \ldots, y_n^* \in F^*$. We may assume that $y_j^* \neq 0$ for some $j \in \{1, \ldots, n\}$. Let $\mu \in \mathcal{P}(B_{F^*})$ be the measure defined by

$$\mu = \sum_{j=1}^n \frac{\|y_j^*\|^q}{\sum_{j=1}^n \|y_j^*\|^q}\delta_j,$$

where δ_j is the Dirac measure at $y_j^*/\|y_j^*\|$. By (ii) there is a constant $C \geq 0$ and a measure $\eta \in \mathcal{P}(B_{\mathcal{H}_v^\infty(U)})$ so that

$$\begin{aligned}
\left(\sum_{i=1}^m\left(\sum_{j=1}^n |\lambda_i|^q\, v(x_i)^q \left|y_j^*(f(x_i))\right|^q\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} &= \left(\sum_{j=1}^n \left\|y_j^*\right\|^q\right)^{\frac{1}{q}}\left(\sum_{i=1}^m |\lambda_i|^p\, v(x_i)^p\left(\int_{B_{F^*}} \left|y^*(f(x_i))\right|^q d\mu(y^*)\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} \\
&\leq C\left(\sum_{j=1}^n \left\|y_j^*\right\|^q\right)^{\frac{1}{q}}\left(\sum_{i=1}^m |\lambda_i|^p\, v(x_i)^p \int_{B_{\mathcal{H}_v^\infty(U)}} \left|g(x_i)\right|^p d\eta(g)\right)^{\frac{1}{p}} \\
&\leq C\left(\sum_{j=1}^n \left\|y_j^*\right\|^q\right)^{\frac{1}{q}} \sup_{g\in B_{\mathcal{H}_v^\infty(U)}}\left(\sum_{i=1}^m |\lambda_i|^p\, v(x_i)^p \left|g(x_i)\right|^p\right)^{\frac{1}{p}}.
\end{aligned}$$

$(iii) \Rightarrow (iv)$: Condition (iii) means that all finitely supported measures $\mu \in \mathcal{P}(B_{F^*})$ satisfy the inequality in (iii). Since the set of all such measures is dense in $\mathcal{P}(B_{F^*})$ for the topology $\sigma(C(B_{F^*})^*, C(B_{F^*}))$, it follows that the inequality in (iv) holds for all $\mu \in \mathcal{P}(B_{F^*})$.

$(iv) \Rightarrow (i)$: Let G be a complex Banach space and $T \in \Pi_q(F, G)$. The Pietsch domination theorem [15, Theorem 17.3.2] provides a measure $\mu \in \mathcal{P}(B_{F^*})$ for which

$$\left\|T(y)\right\|^p \leq \pi_q(T)^p\left(\int_{B_{F^*}} \left|y^*(y)\right|^q d\mu(y^*)\right)^{\frac{p}{q}},$$

for all $y \in F$. Take $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ and $x_1, \ldots, x_m \in U$. Using the preceding inequality and (iv), there exists a constant $C \geq 0$ that fulfills

$$\begin{aligned}
\left(\sum_{i=1}^m |\lambda_i|^p\, v(x_i)^p \left\|(T\circ f)(x_i)\right\|^p\right)^{\frac{1}{p}} &\leq \pi_q(T)\left(\sum_{i=1}^m |\lambda_i|^p\, v(x_i)^p\left(\int_{B_{F^*}} \left|y^*(f(x_i)))\right|^q d\mu(y^*)\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} \\
&\leq C\pi_q(T) \sup_{g\in B_{\mathcal{H}_v^\infty(U)}}\left(\sum_{i=1}^m |\lambda_i|^p\, v(x_i)^p \left|g(x_i)\right|^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Hence $T \circ f \in \Pi_p^{\mathcal{H}_v^\infty}(U, G)$ and $\pi_p^{\mathcal{H}_v^\infty}(T \circ f) \leq C\pi_q(T)$. Consequently, $f \in \mathcal{M}_{(q,p)}^{\mathcal{H}_v^\infty}(U, F)$ and $\mathfrak{m}_{(q,p)}^{\mathcal{H}_v^\infty}(f) \leq C$. □

We are now ready to easily prove the injectivity of the ideal of (q,p)-mixing weighted holomorphic maps. See [15, Theorem 20.1.6] for the linear case.

Corollary 2.9. *The Banach weighted holomorphic ideal* $\left[\mathcal{M}^{\mathcal{H}^\infty_v}_{(q,p)}, \mathfrak{m}^{\mathcal{H}^\infty_v}_{(q,p)}\right]$ *is injective.*

Proof. In the cases $q \geq p$ or $q = \infty$, the result is known in view of Propositions 2.3 and 2.4. Assume $1 \leq p < q < \infty$. Let $f \in \mathcal{H}^\infty_v(U,F)$, let H be a complex Banach space and let $\iota\colon F \to H$ be an into linear isometry such that $\iota \circ f \in \mathcal{M}^{\mathcal{H}^\infty_v}_{(q,p)}(U,H)$. Let $x_i \in U$ and $\lambda_i \in \mathbb{C}$ for $i = 1,\ldots,m$, and $y^*_j \in F^*$ for $j = 1,\ldots,n$. Applying the Hahn–Banach theorem, for each $j = 1,\ldots,n$ there exists $z^*_j \in H^*$ such that $z^*_j \circ \iota = y^*_j$ and $\left\|z^*_j\right\| = \left\|y^*_j\right\|$. Applying statement (iii) in Theorem 2.8, we deduce that

$$\begin{aligned}\left(\sum_{i=1}^m\left(\sum_{j=1}^n |\lambda_i|^q\,v(x_i)^q\left|y^*_j(f(x_i))\right|^q\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} &= \left(\sum_{i=1}^m\left(\sum_{j=1}^n |\lambda_i|^q\,v(x_i)^q\left|z^*_j((\iota\circ f)(x_i))\right|^q\right)^{\frac{p}{q}}\right)^{\frac{1}{p}}\\ &\leq \mathfrak{m}^{\mathcal{H}^\infty_v}_{(q,p)}(\iota\circ f)\left(\sum_{j=1}^n\left\|y^*_j\right\|^q\right)^{\frac{1}{q}}\sup_{g\in B_{\mathcal{H}^\infty_v(U)}}\left(\sum_{i=1}^m|\lambda_i|^p\,v(x_i)^p\left|g(x_i)\right|^p\right)^{\frac{1}{p}}.\end{aligned}$$

This means that $f \in \mathcal{M}^{\mathcal{H}^\infty_v}_{(q,p)}(U,F)$ and $\mathfrak{m}^{\mathcal{H}^\infty_v}_{(q,p)}(f) \leq \mathfrak{m}^{\mathcal{H}^\infty_v}_{(q,p)}(\iota\circ f)$. The reverse inequality follows from the Banach ideal property of $[\Pi^{\mathcal{H}^\infty_v}_p, \pi^{\mathcal{H}^\infty_v}_p]$. □

For $p,q,r \in [1,\infty)$ such that $1/p = 1/q + 1/r$, Theorem 20.1.10 in [15] states that

$$[\Pi_r, \pi_r] \leq [\mathcal{M}_{(q,p)}, m_{(q,p)}]$$

(this notation and the following should be self explanatory), and since

$$[\Pi_q, \pi_q] \circ [\mathcal{M}_{(q,p)}, m_{(q,p)}] \leq [\Pi_p, \pi_p]$$

by [15, Proposition 20.2.1], then the known Pietsch's multiplication formula [15, Theorem 20.2.4] is derived:

$$[\Pi_q, \pi_q] \circ [\Pi_r, \pi_r] \leq [\Pi_p, \pi_p].$$

We raise as open questions that analogous relations are valid for (q,p)-mixing weighted holomorphic mappings. To be more precise, are the following statements true?:

(i) If $f \in \Pi^{\mathcal{H}^\infty_v}_r(U,F)$ and $T \in \Pi_q(F,G)$, then $T\circ f \in \Pi^{\mathcal{H}^\infty_v}_p(U,G)$ and $\pi^{\mathcal{H}^\infty_v}_p(T\circ f) \leq \pi_q(T)\pi^{\mathcal{H}^\infty_v}_r(f)$.

(ii) If $f \in \Pi^{\mathcal{H}^\infty_v}_r(U,F)$, then $f \in \mathcal{M}^{\mathcal{H}^\infty_v}_{(q,p)}(U,F)$ and $\mathfrak{m}^{\mathcal{H}^\infty_v}_{(q,p)}(f) \leq \pi^{\mathcal{H}^\infty_v}_r(f)$.

Note that (ii) follows immediately from (i).

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