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A new perspective on deferred statistical convergence of order α in fuzzy normed linear spaces

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Abstract. In this paper, we present the concepts of deferred statistically convergent sequences and deferred statistically Cauchy sequences in a fuzzy normed linear space (FNS) under order α. We derive essential properties of these sequences and explore their interdependencies. Additionally, we introduce the definitions of deferred statistical limit points and deferred statistical cluster points for sequences in FNS and examine their relationships in the Condition of order α.

1. Introduction

The concept of a fuzzy norm was initially introduced by Katsaras [26] while exploring fuzzy topological vector spaces. In 1992, Felbin[19] introduced the concept of a fuzzy norm on a linear space, building upon the notion of fuzzy numbers. This concept was rooted in the treatment of a fuzzy metric originally proposed by Kaleva and Seikkala[25]. Subsequent studies, such as those in [13, 14], delved into various topological properties of these fuzzy normed linear spaces (FNSs). Additionally, different types of FNSs were explored in works like [3, 8]. In this article, we adopt the methodology outlined in Felbin's[19] work. In the study of FNSs, the convergence of sequences of fuzzy numbers is a key method for defining ordinary convergence within these spaces. This paper seeks to utilize the concept of statistical convergence of fuzzy number sequences to explore a more extensive form of convergence, namely statistical convergence within an FNS. The objective is to establish foundational principles and essential insights in this context. For more on fuzzy set one may refer [40, 44].

In 1951, Fast [18] first introduced statistical convergence as a more general form of convergence for real sequences. This concept has since been extensively studied and applied by numerous researchers [5–7, 20, 36, 37]. In 1993, Fridy introduced the notions of statistical cluster points and statistical limit points of real sequences [21], with applications in turnpike theory, especially in the study of optimal paths [35]. Additionally, statistical convergence has been studied in broader abstract spaces like fuzzy number spaces [2, 33], locally convex spaces [32] and Banach spaces [12, 27, 34]. In 2010, Colak [9] introduced the

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notion of statistical convergence with order α. Later, it was generalized to λ-statistical convergence with a degree of order α by Colak [10] in 2011 and in 2014, Sengul introduced the notion of lacunary statistical convergence with a degree of order α [39]. For additional information regarding statistical convergence with a degree of order α and its generalizations, one may refer to [15, 16]. In 1932, Agnew [1] introduced the idea of the deferred cesàro mean for real or complex number sequences. Later, in 2016, Küçükaslan and Yılmaztürk [31] built upon this concept by presenting the notion of deferred statistical convergence. Gupta and Bhardwaj [23] further advanced this idea by incorporating modulus functions, which led to the development of strongly deferred cesàro convergence and its connection with deferred f-statistical convergence. Additional insights and studies on deferred statistical convergence can be found in [28–30]. Given the importance of sequence convergence in FNSs for fuzzy functional analysis, it is believed that extending the concept of statistical convergence to FNSs will significantly enhance this field.

As a preparatory step, this paper is organized into several sections to systematically present our research findings. In the second section, we provide essential groundwork by presenting preliminary definitions and results concerning fuzzy numbers, fuzzy normed linear spaces (FNSs) and statistical convergence. This sets the stage for a comprehensive understanding of the concepts under investigation. Moving to Section 3, we introduce novel notions within the context of FNSs, specifically focusing on deferred statistical convergence of order α and deferred statistically Cauchy sequences order α. We derive key results in this section, contributing to the foundational aspects of deferred statistical convergence order α in the context of fuzzy normed linear spaces. In the subsequent section, we delve into the definition of deferred statistical limit points order α and deferred statistical cluster points order α for sequences within an FNS. Our exploration includes an in-depth investigation into the relationships between these newly introduced concepts. Finally, we discuss potential application areas of this study and offer insights into future directions for research.

2. Definitions and Preliminaries

Definition 2.1. [4] *Let $A \subset \mathbb{N}$. If $m, n \in \mathbb{N}$, we denote by $A(m, n)$ the cardinality of the set $A \cap [m, n]$. Let $0 < \alpha \le 1$ be a real number. Define*

$$\underline{d}_\alpha(A) = \liminf_{n\to\infty} \frac{A(1,n)}{n^\alpha}, \quad \overline{d}_\alpha(A) = \limsup_{n\to\infty} \frac{A(1,n)}{n^\alpha}$$

called the lower and upper asymptotic density of order α of the set A, respectively. If the limit

$$\lim_{n\to\infty} \frac{A(1,n)}{n^\alpha}$$

exists, then $\underline{d}_\alpha(A) = \overline{d}_\alpha(A) = d_\alpha(A)$ is said to be the asymptotic density of the set A of order α.

Definition 2.2. [9] *A sequence (x_k) is to be statistically convergent of order α $(0 < \alpha \le 1)$ to the limit ℓ if, for any given $\varepsilon > 0$, the subsequent condition is satisfied:*

$$\lim_{n\to\infty} \frac{1}{n^\alpha} \Big| \{k \le n : |x_k - \ell| \ge \varepsilon\} \Big| = 0, n \in \mathbb{N}.$$

In this context, ℓ is referred to as the statistical limit of order α for the sequence (x_k), denoted as $x_k \xrightarrow{st^\alpha} \ell$.

Definition 2.3. [38] *Let $\mu : \mathbb{R} \to [0, 1]$ represent a fuzzy subset of the real numbers, $\mathbb{R}$. Then, for any κ belonging to the interval $[0, 1]$, the κ-level set of μ is represented by μ_κ and defined as the set measure μ defined on the real numbers $\mathbb{R}$, the notation $[\mu]_\kappa$ denotes the set of points t in $\mathbb{R}$ where the measure μ evaluates to at least κ, when $0 < \kappa \leqslant 1$. If $\kappa = 0$, $[\mu]_\kappa$ refers to the closure of the set of points t in $\mathbb{R}$ where μ evaluates to strictly greater than 0.*

Definition 2.4. [38] *A fuzzy set denoted by μ defined on $\mathbb{R}$ is termed a fuzzy number subject to the specified conditions:*

1. *μ is normal, meaning $\exists$ a specific point t_0 in $\mathbb{R}$ where μ attains its maximum membership grade of 1.*
2. *μ is fuzzy convex, indicating that for any pair of real numbers t_1 and t_2 and any λ in the interval $[0,1]$, $\mu(\lambda t_1 + (1-\lambda)t_2)$ is greater than or equal to the minimum of $\mu(t_1)$ and $\mu(t_1)$.*
3. *μ exhibits upper semi-continuous.*
4. *The set $[\mu]_0$ consisting of all t in $\mathbb{R}$ where $\mu(t)$ is greater than 0 is compact.*

Here r(a real number) can be represented and defined as follows, fuzzy number $\tilde{r}$, if t equals r, then $\tilde{r}(t)$ equals 1, if t is not equal to r, then $\tilde{r}(t)$ equals 0. If each κ-level set $[\mu]_\kappa$ forms a non-empty, bounded and closed interval, denoted as $[\mu]_\kappa = [\mu_\kappa^-, \mu_\kappa^+]$, then μ qualifies as a fuzzy number.

Definition 2.5. [38] *Consider $\mathcal{L}(\mathbb{R})$, the collection encompassing all fuzzy numbers. If a fuzzy number μ is a member of $\mathcal{L}(\mathbb{R})$ and its membership grade $\mu(t)$ is zero for t less than zero, it is referred to as a non-negative fuzzy number.*

The set $\mathcal{L}^*(\mathbb{R})$ denotes the group of all non-negative fuzzy numbers. We can assert that $\mu \in \mathcal{L}^*(\mathbb{R})$ iff $\mu_\kappa^- \geqslant 0$ for each $\kappa \in [0,1]$. Clearly, $\widetilde{0} \in \mathcal{L}^*(\mathbb{R})$.

$\preccurlyeq$ (a partial order) on $\mathcal{L}(\mathbb{R})$ is represented by

$$\mu \preccurlyeq \nu \text{ iff } \mu_\kappa^- \leqslant \nu_\kappa^- \quad \text{and} \quad \mu_\kappa^+ \leqslant \nu_\kappa^+ \text{for all } \kappa \in [0,1].$$

Then the inequality is represented by $\prec$ on $\mathcal{L}(\mathbb{R})$ is established as $\mu \prec \nu$ (or $\nu \succ \mu$) iff $\mu_\kappa^- < \nu_\kappa^-$ and $\mu_\kappa^+ < \nu_\kappa^+$ for all $\kappa \in [0,1]$.

Definition 2.6. [38] *We define the operations of addition($\oplus$), multiplication($\otimes$) and scalar multiplication on the set $\mathcal{L}(\mathbb{R})$ as follows:*

(i) The convolution of two functions μ and ν, denoted as $(\mu \oplus \nu)(t)$ is defined for any t in the real numbers $\mathbb{R}$ as the supremum of the minimum values obtained by shifting and overlapping the functions μ and ν.

(ii) The product convolution of two functions μ and ν, denoted as $(\mu \otimes \nu)(t)$ is defined for any t in the real numbers $\mathbb{R}$ as the supremum of the minimum values obtained by scaling and overlapping the functions μ and ν.

(iii) Scalar multiplication of a function μ by a scalar k is defined for any t in the real numbers $\mathbb{R}$ as μ evaluated at t/k, where k is a real number not equal to zero. Additionally when $k = 0$, the result is the zero function $\widetilde{0}(t)$.

Theorem 2.7. [38]. *Let μ, ν belongs to $\mathcal{L}(\mathbb{R})$ and κ belongs to $[0,1]$. Consequently,*

$$(i)\ [\mu \oplus \nu]_\kappa = [\mu_\kappa^- + \nu_\kappa^-, \mu_\kappa^+ + \nu_\kappa^+]$$
$$(ii)\ [\mu \otimes \nu]_\kappa = [\mu_\kappa^- \nu_\kappa^-, \mu_\kappa^+ \nu_\kappa^+]\,(\mu, \nu \in L^*(\mathbb{R}))$$
$$(iii)\ [k\mu]_\kappa = k[\mu]_\kappa = \begin{cases} [k\mu_\kappa^-, k\mu_\kappa^+] & \text{if } k \geqslant 0 \\ [k\mu_\kappa^+, k\mu_\kappa^-] & \text{if } k < 0. \end{cases}$$

Theorem 2.8. [22]. *Let μ be a fuzzy number in $\mathcal{L}(\mathbb{R})$, with κ-level sets denoted by $[\mu]_\kappa = [\mu_\kappa^-, \mu_\kappa^+]$. The theorem establishes the following:*

(i) μ_κ^- is a function that is bounded, non-decreasing on the interval $(0,1]$ and left-continuous,

(ii) μ_κ^+ is a function that is bounded, non-increasing function on $(0,1]$ and right-continuous,

(iii) at $\kappa = 0$, both μ_0^- and μ_0^+ are continuous,

(iv) μ_1^- is less than or equal to μ_1^+.

Conversely, if functions $a(\kappa)$ and $b(\kappa)$ meet conditions (i)-(iv), $\exists$ a unique fuzzy number $\mu \in \mathcal{L}(\mathbb{R})$ such that $[\mu]_\kappa = [a(\kappa), b(\kappa)]$ for all $\kappa \in [0,1]$.

Definition 2.9. [38]. *Considering μ and ν as elements belonging to the space $\mathcal{L}(\mathbb{R})$ define the discrepancy between two measures μ and ν, represented as $\mathcal{A}(\mu,\nu)$, is defined as the supremum over all possible values of κ in the interval $[0,1]$ of the maximum absolute differences between the lower and upper variations of μ and ν. The function $\mathcal{A}$ is referred to as the supremum metric on the set $\mathcal{L}(\mathbb{R})$. If (μ_n) is a sequence in $\mathcal{L}(\mathbb{R})$ and μ is an element of $\mathcal{L}(\mathbb{R})$, we say that the sequence (μ_n) converges to μ in the metric $\mathcal{A}$, denoted as $\mu_n \xrightarrow{\mathcal{A}} \mu$ or $(\mathcal{A})$-$\lim_{n\to\infty} \mu_n = \mu$, if the limit as n approaches infinity of the supremum metric $\mathcal{A}(\mu_n, \mu)$ is equal to zero.*

Definition 2.10. [19]. *Let V be a vector space in $\mathbb{R}$, equipped with a mapping $\|\cdot\| : V \to \mathcal{L}^*(\mathbb{R})$ and consider symmetric, non-decreasing mappings $\mathcal{L}, \mathcal{R} : [0,1]\times[0,1] \to [0,1]$, satisfying $\mathcal{L}(0,0) = 0$ and $\mathcal{R}(1,1) = 1$. This quadruple $(V, \|\cdot\|, \mathcal{L}, \mathcal{R})$ is termed an FNS, with $\|\cdot\|$ referred to as a fuzzy norm, provided it meets the following conditions:*

(i) The norm of $\mathcal{P}$ equals zero if and only if $\mathcal{P}$ is the zero vector 0.

(ii) The norm of a scalar multiple $r\mathcal{P}$ is equal to the absolute value of r multiplied by the norm of $\mathcal{P}$, for all vectors $\mathcal{P}$ in V and for all scalars r.

(iii) For any vectors $\mathcal{P}$ and $\mathcal{Q}$ in V

(a) The norm of their sum $\mathcal{P}+\mathcal{Q}$ is greater than or equal to the minimum of their norms.

(b) The norm of their sum $\mathcal{P}+\mathcal{Q}$ is less than or equal to the maximum of their norms.

They also define functions $\mathcal{L}(\mathcal{P},\mathcal{Q})$ and $\mathcal{R}(\mathcal{P},\mathcal{Q})$ as the minimum and maximum of $\mathcal{P}$ and $\mathcal{Q}$ respectively, when $\mathcal{P}$ and $\mathcal{Q}$ are in the interval $[0,1]$. The fuzzy normed space is denoted as $(V, \|\cdot\|)$ or simply $\mathcal{P}$ when $\mathcal{L}$ and $\mathcal{R}$ follow these definitions.

Lemma 2.11. [19]. *In a normed linear space, the norm of the sum of two vectors is less than or equal to the sum of their individual norms as defined in Definition 2.10 (iii)(a) (with $\mathcal{L}$ being the minimum function) is equivalent to the inequality $\|\mathcal{P}+\mathcal{Q}\|^-_\kappa \leqslant \|V\|^-_\kappa + \|\mathcal{Q}\|^-_\kappa$, holding $\mathcal{P},\mathcal{Q} \in V$ and for all $\kappa \in (0,1]$.*

Lemma 2.12. [19]. *The triangle inequality specified in Definition 2.10 (iii)(b) (with $\mathcal{R}$ being the maximum function) is equivalent to the inequality $|\mathcal{P}+\mathcal{Q}|^+_\kappa \leqslant |V|^+_\kappa + |\mathcal{Q}|^+_\kappa$ $\mathcal{P},\mathcal{Q} \in V$ and for all $\kappa \in (0,1]$.*

Remark 2.13. [19]. *From Theorem 2.8 (iii) and Lemma 2.11, we can deduce that the condition outlined in Definition 2.10 (iii)(a) (where $\mathcal{L}$ is the minimum function) implies that*

$$\lim_{\kappa\to 0} \|\mathcal{P}+\mathcal{Q}\|^-_\kappa \leqslant \lim_{\kappa\to 0} \|\mathcal{P}\|^-_\kappa + \lim_{\kappa\to 0} \|\mathcal{Q}\|^-_\kappa,$$

that is, $\|\mathcal{P}+\mathcal{Q}\|^-_0 \leqslant \|\mathcal{P}\|^-_0 + \|\mathcal{Q}\|^-_0$. Similarly, according to Definition 2.10 (iii)(b) (with $\mathcal{R}$ being the maximum function), it follows that the non-negative part of the sum of two elements, denoted by $\|\mathcal{P}+\mathcal{Q}\|^+_0$, is bounded above by the sum of their respective non-negative parts, $\|\mathcal{P}\|^+_0 + \|\mathcal{Q}\|^+_0$. Consequently, in a Fuzzy Normed Space FNS $(V, \|\cdot\|)$, the triangle inequality specified in Definition 2.10 (iii) suggests that the norm of the sum of two elements, denoted by $\|\mathcal{P}+\mathcal{Q}\|$, is less than or equal to the composition of the norms of $\mathcal{P}$ and $\mathcal{Q}$, denoted by $\|\mathcal{P}\| \oplus \|\mathcal{Q}\|$.

According to Definition 2.10, we have $\mathcal{P}$ is zero, iff $\|\mathcal{P}\| = \widetilde{0}$, iff $\|\mathcal{P}\|^-_\kappa = \|\mathcal{P}\|^+_\kappa$ is zero, for all $\kappa \in [0,1]$. Furthermore, we have $\|\mathcal{P}\|^-_0$ is greater than zero, whenever $\mathcal{P} \neq 0$. Now if r is zero,, then $[\|r\mathcal{P}\|]_\kappa = [\|\theta\|]_\kappa = [0,0] = [|r|\|\mathcal{P}\|]_\kappa$ for all $\kappa \in [0,1]$ and $\mathcal{P} \in V$. For $r \neq 0$, we have $[\|r\mathcal{P}\|]_\kappa = [|r|\|\mathcal{P}\|]_\kappa$ for each $\kappa \in [0,1]$, i.e., $\|r\mathcal{P}\|^-_\kappa = |r|\|\mathcal{P}\|^-_\kappa$ and $\|r\mathcal{P}\|^+_\kappa = |r|\|\mathcal{P}\|^+_\kappa$ for each $\kappa \in [0,1]$.

Example 2.14. [38]. *Given $(V, \|\cdot\|_C)$ as a standard normed linear space, a fuzzy norm $\|\cdot\|$ on V can be derived as follows:*

$$\|\mathcal{P}\|(t) = \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant \zeta\|\mathcal{P}\|_C \text{ or } t \geqslant \eta\|\mathcal{P}\|_C \\ \frac{t}{(1-\zeta)\|\mathcal{P}\|_C} - \frac{\zeta}{1-\zeta} & \text{if } \zeta\|\mathcal{P}\|_C \leqslant t \leqslant \|\mathcal{P}\|_C \\ \frac{-t}{(\eta-1)\|\mathcal{P}\|_C} + \frac{b}{\eta-1} & \text{if } \|\mathcal{P}\|_C \leqslant t \leqslant \eta\|\mathcal{P}\|_C \end{cases}$$

in the given context, $\|\mathcal{P}\|_C$ denotes the standard norm of $\mathcal{P}$ (excluding the zero vector), where $0 < \zeta < 1$ and $1 < \eta < \infty$. For the zero vector $\mathcal{P}$ is zero,, we define $\|\mathcal{P}\| = \widetilde{0}$. Consequently, $(V, \|\cdot\|)$ constitutes a (FNS). The specific fuzzy norm discussed here is referred to as the triangular fuzzy norm [24].

Definition 2.15. [17]. *In a fuzzy normed space (FNS) $(V, \|\cdot\|)$, a sequence $(\mathcal{P}_r)$ converges to $\mathcal{P} \in V$ with respect to the fuzzy norm, denoted as $\mathcal{P}_r \xrightarrow{FN} \mathcal{P}$, if the fuzzy norm of $\mathcal{P}_r - \mathcal{P}$ approaches zero as r tends to infinity. Specifically, this means that for any ε is greater than zero, $\exists$ a natural number $N(\varepsilon)$ such that for all $r \geq N(\varepsilon)$, the fuzzy norm $\|\mathcal{P}_r - \mathcal{P}\|$ is within ε of zero. Formally, this is expressed as:*

$$\sup_{\kappa\in[0,1]} \|\mathcal{P}_r - \mathcal{P}\|^+_\kappa = \|\mathcal{P}_r - \mathcal{P}\|^+_0 < \varepsilon.$$

Definition 2.16. [20]. *The natural density of a set K of positive integers is determined as consider counting how many elements of K are less than or equal to any large number r. Divide this count by r. As r becomes very large, this ratio tends to a limit, which is called the natural density $\delta(K)$. For sets K with finitely many elements, the natural density $\delta(K)$ is 0.*

Although the concept of natural density may not have a well-defined value for every set K, it is important to note that the upper density of K is always defined. The upper density provides a measure of the limiting the proportion of elements of a given set K about a larger set regardless of whether the natural density is defined for K or not,

$$\bar{\delta}(K) = \limsup_{r\to\infty} \frac{1}{r}|\{k \in K : k \leqslant r\}|.$$

Definition 2.17. [33]. *A sequence of real numbers $(\mathcal{P}_r)$ is considered to be statistically convergent to a real number a if for any positive value ε, $\exists$ a set,*

$$K(\varepsilon) = \{r \in \mathbb{N} : |\mathcal{P}_r - a| \geqslant \varepsilon\}$$

has a natural density equal to zero. In this scenario, we denote the convergence as $st - \lim \mathcal{P}_r = a$.

Theorem 2.18. [11]. *Let $(\mathcal{P}_r)$ and $(\mathcal{Q}_r)$ be sequences of real numbers that converge statistically and let $\kappa, \beta \in \mathbb{R}$. Then, the following holds:*

$$\mathrm{st} - \lim(\kappa\mathcal{P}_r + \beta\mathcal{Q}_r) = \kappa\, \mathrm{st} - \lim \mathcal{P}_r + \beta\, \mathrm{st} - \lim \mathcal{Q}_r.$$

Theorem 2.19. [41]. *Let $(\mathcal{P}_r)$, $(\mathcal{Q}_r)$ and $(\mathcal{Z}_r)$ be sequences of real numbers such that $\mathcal{P}_r \leqslant \mathcal{Q}_r \leqslant \mathcal{Z}_r$, for every natural number r belonging to the set K and under the condition that the function $\delta(K)$ evaluates to 1 and $\mathrm{st} - \lim \mathcal{P}_r = \mathrm{st} - \lim \mathcal{Z}_r = a$. Then $\mathrm{st} - \lim \mathcal{Q}_r = a$.*

Definition 2.20. [33]. *A sequence of fuzzy numbers (μ_r) is said to statistically converge to the fuzzy number μ, denoted as the statistical limit of μ_r being μ, if for every positive number ε, $\exists$ a positive integer N such that,*

$$\delta(\{r \in \mathbb{N} : \mathcal{A}(\mu_r, \mu) \geqslant \varepsilon\})$$

is zero.

Definition 2.21. [1]. *Consider $K \subseteq \mathbb{N}$ and let $K_{d,c}(n)$ denote the set of integers in the interval $[d(n) + 1, c(n)]$ that belong to K, where sequences $d = (d(n))$ and $c = (c(n))$ consist of non-negative integers that satisfy the following conditions:*

$$d(n) < c(n) \quad \textit{for all} \quad n \in \mathbb{N} \quad \textit{and} \quad \lim_{n\to\infty} c(n) = \infty. \tag{1}$$

The deferred density of K is represented and defined by

$$\delta_{d,c}(K) = \lim_{n\to\infty} \tfrac{1}{c(n)-d(n)}|K_{d,c}(n)|.$$

Definition 2.22. [31]. *A sequence $(\mathcal{P}_r)$ of real numbers is deferred statistically convergent to $l \in \mathbb{R}$ if, such that*

$$\lim_{n\to\infty} \frac{1}{c(n) - d(n)} |\{r \in \mathbb{N} \cap [d(n) + 1, c(n)] : |\mathcal{P}_r - l| \geq \varepsilon\}|$$

is zero, for every ε, is greater than zero then $\exists$ an integer $N(\varepsilon)$ and sequences $d = (d(n))$ and $c = (c(n))$ consist of non-negative integers satisfying the conditions specified in Equation (1).

3. Main Results

In this section, we explore the concepts of deferred statistically convergent sequences of order α and deferred statistically Cauchy sequences of order α within (FNSs). We establish fundamental properties related to these concepts and delve into the definition of deferred statistical cluster points of order α and deferred statistical limit points of order α for sequences in FNS. Our investigation focuses on exploring the relationships between these introduced notions, presenting key results that contribute to a deeper understanding of statistical convergence of order α in FNSs.

Definition 3.1. *Let $(V, \|\cdot\|)$ represent a fuzzy normed space (FNS) and let $\{P_r\}$ be a sequence in V. Suppose $d = \{d(n)\}$ and $c = \{c(n)\}$ are sequences of non-negative integers satisfying the conditions specified in Equation (1). The sequence $\{P_r\}$ is said to be deferred statistically convergent of order α $(0 < \alpha \leq 1)$ to $P \in V$, denoted by*

$$P_r \xrightarrow{st^{\alpha}_{dc}(FN)} P,$$

if and only if

$$st^{\alpha}_{dc}\text{-}\lim \|P_r - P\|^{\alpha} = 0\ ,$$

i.e., for each $\varepsilon > 0$, we have

$$\delta^{\alpha}_{dc}(\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : A(\|P_r - P\|^{\alpha}, 0\ ) \geq \varepsilon\}) = 0.$$

$$\textit{Equivalently,}\quad \lim_{n\to\infty} \frac{1}{(c(n)-d(n))^{\alpha}} |\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : A(\|P_r - P\|^{\alpha}, 0\ ) \geq \varepsilon\}| = 0.$$

This implies that for each $\varepsilon > 0$, the density of the set

$$K^{\alpha}(\varepsilon) = \{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|P_r - P\|^{\alpha}_0 \geq \varepsilon\}$$

is zero. That is for every $\varepsilon > 0$,

$$\|P_r - P\|^{\alpha}_0 < \varepsilon, \quad \textit{for almost all } r.$$

The element $P \in V$ serves as the deferred statistical limit of order α of the sequence $\{P_r\}$. An equivalent interpretation of the definition above is:

$$P_r \xrightarrow{st^{\alpha}_{dc}(FN)} P \iff st^{\alpha}_{dc}\text{-}\lim \|P_r - P\|^{\alpha}_0 = 0.$$

Since, $st^{\alpha}_{dc}\text{-}\lim \|P_r - P\|^{\alpha}_0 = 0$, it follows that

$$st^{\alpha}_{dc}\text{-}\lim \|P_r - P\|^{-}_{\kappa}\alpha = st^{\alpha}_{dc}\text{-}\lim \|P_r - P\|^{+}_{\kappa}\alpha = 0,$$

for each $\kappa \in [0,1]$, as

$$0 \leq \|P_r - P\|^{-}_{\kappa}\alpha \leq \|P_r - P\|^{+}_{\kappa}\alpha \leq \|P_r - P\|^{\alpha}_0,$$

holds for every $n \in \mathbb{N}$ and $\kappa \in [0,1]$. Throughout this paper, whenever we say $\{P_r\}$ is deferred statistically convergent of order α to $P \in V$, it means that $\{P_r\}$ satisfies the definition above with respect to the fuzzy norm of order α on V.

Remark 3.2. *It is clear that,*

- *If $c(n)$ is n and $d(n)$ is zero and α is one, then deferred statistical convergence on fuzzy normed spaces (FNSs) coincide with the definition of statistical convergence on fuzzy normed spaces (FNSs) [38].*

- *If $c(n)$ is λ_n and $d(n)$ is zero and α is one, where λ is a non-decreasing sequence of positive integers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, λ_1 is zero, then deferred statistical convergence in (FNSs) coincides with λ-statistical convergence in (FNSs) [43].*

- *If $c(n)$ is k_n and $d(n)$ is k_{n-1} and α is one, for any lacunary sequence of non-negative integer satisfying $k_n - k_{n-1} \to \infty$ as $n \to \infty$, then deferred statistical convergence of (FNSs) coincides with lacunary statistical convergence of (FNSs) [42].*

Example 3.3. *Let $(\mathbb{R}, \|\cdot\|_{\mathbb{R}})$ is a normed linear space. Then the fuzzy norm $\|\cdot\|_F$ on $\mathbb{R}$ of order α can be obtained as*

$$\|\mathcal{P}\|^{\alpha}(t) = \begin{cases} \dfrac{t - |\mathcal{P}|^{\alpha}}{t + |\mathcal{P}|^{\alpha}} & t > |\mathcal{P}|^{\alpha} \\ 0, & t \leq |\mathcal{P}|^{\alpha} \end{cases},$$

and $(\mathbb{R}, \|\|_F)$ is a FNS.

Example 3.4. *Consider the fuzzy norm defined in Example 3.1 above and let $d = (d(n))$ and $c = (c(n))$ be sequences satisfying (1). Let us define a sequence $(\mathcal{P}_r)$ as*

$$\mathcal{P}_r = \begin{cases} k^2, & [\sqrt{c(n)}] - 1 < k \leq [\sqrt{c(n)}],\ n = 1, 2, 3, \ldots \\ 0, & otherwise \end{cases},$$

where the value of $d(n)$ is greater than zero and less than or equal to the integer part of the square root of $c(n)$ less than one. Let us verify that $(\mathcal{P}_r)$ is deferred statistically convergent of order α $(0 < \alpha \leq 1)$ to 0, i.e.,

$$\mathcal{P}_r \xrightarrow{st^{\alpha}_{d,c}(FN)} 0.$$

***Justification**: We have*

$$K^n_{\alpha}(\varepsilon) = \{r \in \mathbb{N} \cap [d(n) + 1, c(n)] : \|\mathcal{P}_r - 0\|^+_{\alpha} \geqslant \varepsilon\},$$

for every $0 < \varepsilon < 1$. Using the definition of the fuzzy norm for suitable $t > |\mathcal{P}_r|$, this implies that

$$\begin{aligned} K^n_{\alpha}(\varepsilon) &= \left\{r \in \mathbb{N} \cap [d(n) + 1, c(n)] : \frac{t - |\mathcal{P}_r|^{\alpha}}{t + |\mathcal{P}_r|^{\alpha}} \geqslant \varepsilon\right\} \\ &= \left\{r \in \mathbb{N} \cap [d(n) + 1, c(n)] : |\mathcal{P}_r|^{\alpha} \leqslant \frac{t(1 - \varepsilon)}{1 + \varepsilon}\right\}. \end{aligned}$$

For a particular value of $\alpha = 0.5$, let $t = 3k^2$ and $\varepsilon = 0.5$. Then,

$$K^n_{\alpha}(\varepsilon) \subseteq \left\{r \in \mathbb{N} \cap [d(n) + 1, c(n)] : \mathcal{P}_r = k^2\right\} = \{1, 4, 9, 16, \ldots\}.$$

To compute the deferred statistical density of K^n_{ε}, we have

$$\delta^{\alpha}_{d,c}(K^n_{\alpha}(\varepsilon)) = \lim_{n\to\infty} \frac{|K^n_{\varepsilon}|}{(c(n) - d(n))^{\alpha}}.$$

Since, the numerator $|K^n_{\varepsilon}|$ grows sublinearly compared to $(c(n) - d(n))^{\alpha}$ for $\alpha = 0.5$, the density is zero

$$\delta^{\alpha}_{d,c}(K^n_{\alpha}(\varepsilon)) = 0.$$

Hence, $\mathcal{P}_r \xrightarrow{st^{\alpha}_{d,c}(FN)} 0$. Every convergent sequence is deferred statistically convergent of order α. However, the converse is not always true.

Example 3.5. *Let $(\mathbb{R}^m, \|\cdot\|)$ denote an FNS and let $\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_m) \in \mathbb{R}^m$ be a fixed non-zero vector. In this context, the fuzzy norm on $\mathbb{R}$ is specified as in Example 2.1, ensuring that $\|\mathcal{P}\|_C = \left(\mathcal{P}_1^2 + \cdots + \mathcal{P}_m^2\right)^{1/2}$ and $d = (d(n))$*

and $c = (c(n))$ are sequences of non-negative integers satisfying the conditions specified in Equation (1). Now define a sequence $(\mathcal{P}_r)$ in $\mathbb{R}$ as

$$\mathcal{P}_r = \begin{cases} \sin n\pi, & \text{if } r \neq n^2, \\ n, & \text{if } r = n^2. \end{cases}$$

Where $r \in \mathbb{N}$. Further, let α be a parameter such that $0 < \alpha \leq 1$. Define the deferred statistical fuzzy norm convergence with the parameter α.

Justification: *For a given ε, satisfying $0 < \varepsilon \leqslant b\left(\mathcal{P}_1^2 + \cdots + \mathcal{P}_m^2\right)^{1/2}$, it follows that*

$$K_\alpha^n(\varepsilon) = \left\{ r \in \mathbb{N} \cap [d(n)+1, c(n)] : \frac{t - \|\mathcal{P}_r - \theta\|_0^+}{t + \|\mathcal{P}_r - \theta\|_0^+} \geq \varepsilon^\alpha \right\}.$$

Case 1: If $r = n^2$, then $\mathcal{P}_r = n$, we have

$$\|\mathcal{P}_r - \theta\|_0^+ = \|n - 0\|_0^+ = n.$$

Thus, the inequality becomes

$$\frac{t-n}{t+n} \geq \varepsilon^\alpha.$$

Solving for n, we have

$$\frac{t(1-\varepsilon^\alpha)}{1+\varepsilon^\alpha} \geq n.$$

Case 2: If $r \neq n^2$, then $\mathcal{P}_r = \sin n\pi = 0$, we have

$$\|\mathcal{P}_r - \theta\|_0^+ = 0.$$

For any $\varepsilon^\alpha > 0$, this condition is always satisfied the calculation for $\alpha = 0.5$ Let $\alpha = 0.5$ and $t = 3n^2$. For $\varepsilon = 0.5$:

$$\frac{3n^2(1-0.5^{0.5})}{1+0.5^{0.5}} \geq n.$$

$$\implies \frac{3n^2(1-\sqrt{0.5})}{1+\sqrt{0.5}} \geq n.$$

Solving this inequality determines the set $K_\alpha^n(\varepsilon)$. For that we have,

$$\delta_{d,c}^\alpha(K_\alpha^n(\varepsilon)) = \lim_{n\to\infty} \frac{|K_\alpha^n(\varepsilon)|}{(c(n)-d(n))^\alpha}.$$

For $\alpha = 0.5$, $\delta_{d,c}^\alpha(K_\alpha^n(\varepsilon)) = 0$, as the terms $K_\alpha^n(\varepsilon)$ become sparse for large n. Thus, $\mathcal{P}_r$ converges in the sense of deferred statistical fuzzy norm convergence for $\alpha = 0.5$. Then the general observation the sequence $\mathcal{P}_r$ converges in the sense of deferred statistical fuzzy norm convergence for all $0 < \alpha \leq 1$. However, since $\{1, 4, 9, 16, \ldots\}$ contains infinitely many terms, $\mathcal{P}_r$ is not convergent in the classical sense.

The following summarizes some fundamental properties of statistical limits.

Theorem 3.6. *Let $(\mathcal{P}_r)$ and $(\mathcal{Q}_r)$ be sequences in $(V, \|\cdot\|)$ and let $d = (d(n))$ and $c = (c(n))$ be sequences satisfying Equation (1). Suppose $\mathcal{P}_r \xrightarrow{st_{d,c}^\alpha(FN)} \mathcal{P}$ and $\mathcal{Q}_r \xrightarrow{st_{d,c}^\alpha(FN)} \mathcal{Q}$, where $\mathcal{P}, \mathcal{Q} \in V$. Then:*

(i) $\mathcal{P}_r + \mathcal{Q}_r \xrightarrow{st_{d,c}^\alpha(FN)} \mathcal{P} + \mathcal{Q}$,

(ii) $t\mathcal{P}_r \xrightarrow{st_{d,c}^\alpha(FN)} t\mathcal{P}$ for all $t \in \mathbb{R}$.

Proof. (i) Assume $\mathcal{P}_r \xrightarrow{st^{\alpha}_{d,c}(FN)} \mathcal{P}$ and $Q_r \xrightarrow{st^{\alpha}_{d,c}(FN)} Q$. Using the norm properties, we have

$$\|(\mathcal{P}_r + Q_r) - (\mathcal{P} + Q)\|^{\alpha} \leq \|\mathcal{P}_r - \mathcal{P}\|^{\alpha} + \|Q_r - Q\|^{\alpha}. \tag{2}$$

For all $\varepsilon > 0$, define the sets:

$$K^{\alpha}_n(\varepsilon) = \{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|(\mathcal{P}_r + Q_r) - (\mathcal{P} + Q)\|^{\alpha} \geq \varepsilon\},$$
$$K^{\alpha}_{n,1}(\varepsilon) = \{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{P}\|^{\alpha} \geq \varepsilon/2\},$$
$$K^{\alpha}_{n,2}(\varepsilon) = \{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|Q_r - Q\|^{\alpha} \geq \varepsilon/2\}.$$

From Equation (2), we have $K^{\alpha}(\varepsilon) \subseteq K^{\alpha}_{n,1}(\varepsilon) \cup K^{\alpha}_{n,2}(\varepsilon)$. Since $\mathcal{P}_r \xrightarrow{st^{\alpha}_{d,c}(FN)} \mathcal{P}$ and $Q_r \xrightarrow{st^{\alpha}_{d,c}(FN)} Q$, it follows that:

$$\delta^{\alpha}_{d,c}(K^{\alpha}_{n,1}(\varepsilon)) = \delta^{\alpha}_{d,c}(K^{\alpha}_{n,2}(\varepsilon)) = 0.$$

Thus, $\delta^{\alpha}_{d,c}(K^{\alpha}_n(\varepsilon)) = 0$, proving that $\mathcal{P}_r + Q_r \xrightarrow{st^{\alpha}_{d,c}(FN)} \mathcal{P} + Q$.

(ii) Let $t \in \mathbb{R}$. Assume $\mathcal{P}_r \xrightarrow{st^{\alpha}_{d,c}(FN)} \mathcal{P}$. For all $\varepsilon > 0$, we have

$$\begin{aligned}\delta^{\alpha}_{d,c}\left(\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|t\mathcal{P}_r - t\mathcal{P}\|^{\alpha} \geq \varepsilon\}\right) &= \delta^{\alpha}_{d,c}\left(\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : |t|^{\alpha}\|\mathcal{P}_r - \mathcal{P}\|^{\alpha} \geq \varepsilon\}\right) \\ &= \delta^{\alpha}_{d,c}\left(\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{P}\|^{\alpha} \geq \varepsilon/|t|^{\alpha}\}\right).\end{aligned}$$

Since $\mathcal{P}_r \xrightarrow{st^{\alpha}_{d,c}(FN)} \mathcal{P}$, the density above is zero. Therefore, $t\mathcal{P}_r \xrightarrow{st^{\alpha}_{d,c}(FN)} t\mathcal{P}$. □

Example 3.7. *Consider the fuzzy normed space $(R, \|\cdot\|)$, where the fuzzy norm of order α is defined as:*

$$A(\|x\|^{\alpha}, t) = \begin{cases} 1, & \text{if } |x|^{\alpha} \leq t, \\ 0, & \text{if } |x|^{\alpha} > t, \end{cases}$$

for $x \in R$, $t \geq 0$ and $0 < \alpha \leq 1$. Let the sequences $d(n) = n$, $c(n) = 2n$ and $\{P_r\}$ be given by

$$P_r = (-1)^r r^{\beta}, \quad \text{for some } \beta > \frac{1}{\alpha}.$$

We aim to show that $\{P_r\}$ is deferred statistically convergent of order α to $P = 0$.
Verification: *Compute the difference in fuzzy norm, for the sequence $\{P_r\}$, we have*

$$\|P_r - 0\|^{\alpha} = |P_r|_{\alpha} = |(-1)^r r^{\beta}|_{\alpha} = r^{\beta\alpha}.$$

Set up the density condition, for any $\varepsilon > 0$, consider the set

$$K^{\alpha}_n(\varepsilon) = \{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|P_r - 0\|_{\alpha} \geq \varepsilon\}.$$

Substituting $\|P_r - 0\|_{\alpha} = r^{\beta\alpha}$, the condition becomes

$$K^{\alpha}_n(\varepsilon) = \{r \in \mathbb{N} \cap [n+1, 2n] : r^{\beta\alpha} \geq \varepsilon\}.$$

$$\Longrightarrow K^{\alpha}_n(\varepsilon) = \{r \in \mathbb{N} \cap [n+1, 2n] : r \leq \varepsilon^{-\frac{1}{\beta\alpha}}\}.$$

Calculate the deferred density, the number of terms in $K_{\alpha}(\varepsilon)$ is approximately $|K^{\alpha}_n(\varepsilon)| \leq \min\{2n - n, \varepsilon^{-\frac{1}{\beta\alpha}}\}$.. Hence, the density of $K^{\alpha}_n(\varepsilon)$ is

$$\delta^{\alpha}_{dc}(K^{\alpha}_n(\varepsilon)) = \lim_{n\to\infty} \frac{|K^{\alpha}_n(\varepsilon)|}{(c(n) - d(n))^{\alpha}} = \lim_{n\to\infty} \frac{\min\{n, \varepsilon^{-\frac{1}{\beta\alpha}}\}}{n^{\alpha}}.$$

Since $\beta\alpha > 1$, the term $\varepsilon^{-\frac{1}{\beta\alpha}}$ becomes negligible as $n \to \infty$ and thus $\delta^{\alpha}_{dc}(K^{\alpha}_n(\varepsilon)) = 0$. Since the deferred density of $K_{\alpha}(\varepsilon)$ is zero for any $\varepsilon > 0$, we conclude that

$$P_r \xrightarrow{st^{\alpha}_{dc}} (FN)0.$$

Thus, the sequence $\{P_r\}$ is deferred statistically convergent of order α to $P = 0$.

Theorem 3.8. *Consider a sequence $(\mathcal{P}_r)$ in $(V, \|\cdot\|)$ and let $d = (d(n))$ and $c = (c(n))$ be sequences satisfying* (1). *Then $(\mathcal{P}_r)$ is a deferred statistical convergence sequence of order α in V iff $(\mathcal{P}_r)$ is a sequence for which $\exists$ a convergent sequence $(\mathcal{Q}_r)$ such that $\mathcal{P}_r = \mathcal{Q}_r$ for a.a.r (almost all r)*

Proof. Let $\{A_i : i \in I\}$ be a countable collection of subsets of the natural numbers such that each A_i has density 1, then $\exists$ a subset A of the natural numbers with density 1, such that the set difference $A \setminus A_i$ is finite for every $i \in I$. We assume that $\mathcal{P}_r \xrightarrow{st_{d,c}(FN)} \mathcal{P}$, meaning that the sequence $\mathcal{P}_r$ converges to $\mathcal{P}$ in a deferred manner with respect to the fuzzy norm. For each $i \in \mathbb{N}$, define the set

$$A_i = \left\{ r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{P}\|_0^\alpha \leqslant \frac{1}{i} \right\}.$$

Here, each set A_i has density 1, since the sequence $(\mathcal{P}_r)$ is α-deferred statistically convergent. Let A be defined as the set with density 1, as stated in the proof. It follows that $\mathcal{P}_r$ converges to $\mathcal{P}$ within A. In simpler terms, for any $\varepsilon > 0$, there exists a specific index $N(\varepsilon)$ such that if r exceeds this index and belongs to set A, then the distance between $\mathcal{P}_r$ and $\mathcal{P}$ under the α-norm, denoted $\|\cdot\|_0^\alpha$, is less than ε. Next, we construct a new sequence $(\mathcal{Q}_r)$ such that

$$\mathcal{Q}_r = \mathcal{P}_r \quad \text{if} \quad r \in A, \quad \mathcal{Q}_r = \mathcal{P} \quad \text{if} \quad r \notin A.$$

This construction ensures that the sequence $(\mathcal{Q}_r)$ converges to $\mathcal{P}$ with respect to the fuzzy norm on V and $\mathcal{P}_r$ is equal to $\mathcal{Q}_r$ for almost all r. Now assume that $\mathcal{P}_r = \mathcal{Q}_r$ for almost all r and that $\mathcal{Q}_r \xrightarrow{FN} \mathcal{P}$, where FN denotes convergence in the fuzzy norm. For $\varepsilon > 0$, consider each $s \in \mathbb{N} \cap [d(n)+1, c(n)]$ and we can write the following

$$\left\{ r \leqslant s : \|\mathcal{P}_r - \mathcal{P}\|_0^\alpha \geqslant \varepsilon \right\} \subseteq \{ r \leqslant s : \mathcal{P}_r \neq \mathcal{Q}_r \} \cup \{ r \leqslant s : \|\mathcal{Q}_r - \mathcal{P}\|_0^\alpha > \varepsilon \}. \tag{3}$$

The collection of integers in the second set on the right-hand side of equation (3) has a fixed count, which is denoted as $p = p(\varepsilon)$. Thus, we have

$$\lim_{s\to\infty} \frac{1}{s} \left| \left\{ r \leqslant s : \|\mathcal{P}_r - \mathcal{P}\|_0^\alpha \geqslant \varepsilon \right\} \right| \leqslant \lim_{s\to\infty} \frac{1}{s} \left| \{ r \leqslant s : \mathcal{P}_r \neq \mathcal{Q}_r \} \right| + \lim_{s\to\infty} \frac{p}{s}.$$

Since $\mathcal{P}_r = \mathcal{Q}_r$ for almost all r, we have

$$\lim_{s\to\infty} \frac{1}{s} \left| \left\{ r \leqslant s : \|\mathcal{P}_r - \mathcal{P}\|_0^\alpha \geqslant \varepsilon \right\} \right| = 0.$$

This implies that $\|\mathcal{P}_r - \mathcal{P}\|_0^\alpha \geqslant \varepsilon$ for almost all r, which means that $(\mathcal{P}_r)$ is α-deferred statistically convergent to $\mathcal{P}$. □

Definition 3.9. *Let $(\mathcal{P}_r)$ be a sequence in $(V, \|\cdot\|)$ with $d = (d(n))$ and $c = (c(n))$ being sequences satisfying* (1). *We say that the sequence $(\mathcal{P}_r)$ in V is α-deferred statistically Cauchy with respect to the fuzzy norm on V if for every $\varepsilon > 0$, there exists $\mathbb{N}$ such that the density of the set*

$$\left\{ r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{P}_N\|_0^\alpha \geqslant \varepsilon \right\}$$

is zero as $n \to \infty$.

Example 3.10. *Consider the sequence $(\mathcal{P}_r)$ in the normed space $(\mathbb{R}, |\cdot|)$ given by $\mathcal{P}_r = \frac{1}{r}$. Let $d(n) = n$ and $c(n) = 2n$. These sequences $d(n)$ and $c(n)$ satisfy the conditions specified in Equation* (1). *Then it is shown that the sequence $\mathcal{P}_r = \frac{1}{r}$ is α-deferred statistically Cauchy with respect to the fuzzy norm on V.*

Theorem 3.11. *Let $(\mathcal{P}_r)$ be a sequence in $(V, \|\cdot\|)$ with $d = (d(n))$ and $c = (c(n))$ being sequences satisfying* (1). *Then every α-deferred statistically convergent sequence is also a α-deferred statistically Cauchy sequence.*

Proof. Let $\mathcal{P}_r \overset{st_{d,c}(FN)}{\longrightarrow} \mathcal{P}$ and $\varepsilon > 0$. Then we have:

$$\lim_{n\to\infty} \frac{1}{(c(n)-d(n))^{\alpha}} \left|\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{P}\|_0^{\alpha} \geqslant \frac{\varepsilon}{2}\right\}\right| = 0.$$

Choose $N \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \frac{1}{(c(n)-d(n))^{\alpha}} \left|\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P} - \mathcal{P}_N\|_0^{\alpha} \geqslant \frac{\varepsilon}{2}\right\}\right| = 0.$$

Therefore,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{(c(n)-d(n))^{\alpha}} \left|\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{P}_N\|_0^{\alpha} \geqslant \varepsilon\right\}\right| \\
&= \lim_{n\to\infty} \frac{1}{(c(n)-d(n))^{\alpha}} \left|\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|(\mathcal{P}_r - \mathcal{P}) + (\mathcal{P} - \mathcal{P}_N)\|_0^{\alpha} \geqslant \varepsilon\right\}\right| \\
&\leq \lim_{n\to\infty} \frac{1}{(c(n)-d(n))^{\alpha}} \left|\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{P}\|_0^{\alpha} \geqslant \frac{\varepsilon}{2}\right\}\right| \\
&+ \lim_{n\to\infty} \frac{1}{(c(n)-d(n))^{\alpha}} \left|\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P} - \mathcal{P}_N\|_0^{\alpha} \geqslant \frac{\varepsilon}{2}\right\}\right| = 0.
\end{aligned}$$

This shows that $(\mathcal{P}_r)$ is α-deferred statistically Cauchy. □

Theorem 3.12. *Let $(\mathcal{P}_r)$ be a sequence in $(V, \|\cdot\|)$, where $d = (d(n))$ and $c = (c(n))$ are sequences of non-negative integers satisfying the conditions specified in Equation* (1). *Define $E_N(\varepsilon)$ as the set $\{r \in \mathbb{N} : \|\mathcal{P}_r - \mathcal{P}_N\|_0^{\alpha} \geqslant \varepsilon\}$ for any $N \in \mathbb{N}$. If $(\mathcal{P}_r)$ is α-deferred statistically Cauchy, then for any $\varepsilon > 0$, there exists a set $A \subset \mathbb{N}$ such that $\|\mathcal{P}_m - \mathcal{P}_r\|_0^{\alpha} < \varepsilon$ for all $m, r \notin A$ and $\delta_{d,c}(A) = 0$.*

Proof. Let $(\mathcal{P}_r)$ be α-deferred statistically Cauchy. By definition, for any $\varepsilon > 0$, we have that for large enough n, the density of the set

$$\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{P}_N\|_0^{\alpha} \geqslant \varepsilon\}$$

is zero, for every $N \in \mathbb{N}$. Consider the set $E_N(\varepsilon)$

$$E_N(\varepsilon) = \{r \in \mathbb{N} : \|\mathcal{P}_r - \mathcal{P}_N\|_0^{\alpha} \geqslant \varepsilon\}.$$

By the condition that the sequence is α-deferred statistically Cauchy, for any $\varepsilon > 0$, we know that the density of $E_N(\varepsilon)$ tends to zero as $n \to \infty$. This means that for sufficiently large n, the set $E_N(\varepsilon)$ becomes sparse, i.e., most of the terms in the sequence $\mathcal{P}_r$ are within ε of $\mathcal{P}_N$ in the α-norm. Now, define the set A as

$$A = \{r \in \mathbb{N} : \|\mathcal{P}_r - \mathcal{P}_N\|_0^{\alpha} < \varepsilon \text{ for all } N\}.$$

For all $m, r \notin A$, we have

$$\|\mathcal{P}_m - \mathcal{P}_r\|_0^{\alpha} < \varepsilon.$$

Since $E_N(\varepsilon)$ is sparse for large n, the set A will have the property that the density of A with respect to the sequence $(d(n))$ and $(c(n))$ is zero, i.e.,

$$\delta_{d,c}^{\alpha}(A) = 0.$$

Thus, for any $\varepsilon > 0$, there exists a set $A \subset \mathbb{N}$ such that for all $m, r \notin A$, we have $\|\mathcal{P}_m - \mathcal{P}_r\|_0^{\alpha} < \varepsilon$ and the density of A is zero, i.e., $\delta_{d,c}^{\alpha}(A) = 0$. This completes the proof. □

Theorem 3.13. *Let $(\mathcal{P}_r)$ be a sequence in $(V, \|\cdot\|)$, where $d = (d(n))$ and $c = (c(n))$ are sequences of non-negative integers satisfying the conditions specified in Equation* (1). *Define $E_N(\varepsilon)$ as the set $\{r \in \mathbb{N} : \|\mathcal{P}_r - \mathcal{P}_N\|_0^{\alpha} \geqslant \varepsilon\}$ for any $N \in \mathbb{N}$. If $(\mathcal{P}_r)$ is α-deferred statistically Cauchy, then for any $\varepsilon > 0$, there exists a set $A \subset \mathbb{N}$ such that $\|\mathcal{P}_m - \mathcal{P}_r\|_0^{\alpha} < \varepsilon$ for all $m, r \notin A$ and $\delta_{d,c}(A) = 0$.*

Proof. Let $(\mathcal{P}_r)$ be α-deferred statistically Cauchy. By definition, for any $\varepsilon > 0$, we have that for large enough n, the density of the set

$$\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{P}_N\|_0^\alpha \geqslant \varepsilon\}$$

is zero, for every $N \in \mathbb{N}$. Consider the set $E_N(\varepsilon)$:

$$E_N(\varepsilon) = \{r \in \mathbb{N} : \|\mathcal{P}_r - \mathcal{P}_N\|_0^\alpha \geqslant \varepsilon\}.$$

By the condition that the sequence is α-deferred statistically Cauchy, for any $\varepsilon > 0$, we know that the density of $E_N(\varepsilon)$ tends to zero as $n \to \infty$. This means that for sufficiently large n, the set $E_N(\varepsilon)$ becomes sparse, i.e., most of the terms in the sequence $\mathcal{P}_r$ are within ε of $\mathcal{P}_N$ in the α-norm. Now, define the set A as:

$$A = \{r \in \mathbb{N} : \|\mathcal{P}_r - \mathcal{P}_N\|_0^\alpha < \varepsilon \text{ for all } N\}.$$

For all $m, r \notin A$, we have that

$$\|\mathcal{P}_m - \mathcal{P}_r\|_0^\alpha < \varepsilon.$$

Since $E_N(\varepsilon)$ is sparse for large n, the set A will have the property that the density of A with respect to the sequence $(d(n))$ and $(c(n))$ is zero, i.e.,

$$\delta_{d,c}^\alpha(A) = 0.$$

Thus, for any $\varepsilon > 0$, there exists a set $A \subset \mathbb{N}$ such that for all $m, r \notin A$, we have $\|\mathcal{P}_m - \mathcal{P}_r\|_0^\alpha < \varepsilon$ and the density of A is zero, i.e., $\delta_{d,c}^\alpha(A) = 0$. This completes the proof. □

Example 3.14. *For $\alpha = \frac{1}{2}$, consider the sequence $\mathcal{P}_r = \frac{1}{r^2}$ in $\mathbb{R}$ and let $d(n) = n$ and $c(n) = 2n$. The interval $[d(n)+1, c(n)] = [n+1, 2n]$ grows without bound as $n \to \infty$. Define the set $E_N(\varepsilon) = \{r \in \mathbb{N} : \|\mathcal{P}_r - \mathcal{P}_N\|_0^{\frac{1}{2}} \geqslant \varepsilon\}$, where $\|\mathcal{P}_r - \mathcal{P}_N\|_0 = \left|\frac{1}{r^2} - \frac{1}{N^2}\right|$. For large r and N, this difference becomes small, meaning that the set $E_N(\varepsilon)$ becomes sparse. Define the set $A = \{r \in \mathbb{N} : \|\mathcal{P}_r - \mathcal{P}_N\|_0^{\frac{1}{2}} < \varepsilon \text{ for all } N\}$, which consists of indices r where the terms $\mathcal{P}_r$ are close to each other. The density of A with respect to d(n) and c(n) is zero, i.e., $\delta_{d,c}^\alpha(A) = 0$. Thus, the sequence $\mathcal{P}_r = \frac{1}{r^2}$ is α-deferred statistically Cauchy for $\alpha = \frac{1}{2}$, satisfying the conditions of the theorem.*

Remark 3.15. *Suppose $(\mathcal{P}_r)$ is a sequence in V and $\left(\mathcal{P}_{r_j}\right)$ is a subsequence of $(\mathcal{P}_r)$. Define K as the set $\left\{r_j : j \in \mathbb{N}\right\}$. If $\delta^\alpha(K) = 0$, we call $\left(\mathcal{P}_{r_j}\right)$ a thin subsequence of $(\mathcal{P}_r)$. Conversely, $\left(\mathcal{P}_{r_j}\right)$ is referred to as a non-thin subsequence if $\delta^\alpha(K) > 0$ or if $\delta^\alpha(K)$ is undefined and $\bar{\delta^\alpha}(K) > 0$.*

Definition 3.16. *Let $(\mathcal{P}_r)$ be a sequence in $(X, \|\cdot\|)$ and let $d = (d(n))$ and $c = (c(n))$ be sequences satisfying* (1)*. An element $\mathcal{Z}$ belongs to X is termed a deferred statistical cluster point of $(\mathcal{P}_r)$ if, for every $\varepsilon > 0$, the measure $\bar{\delta}_{d,c}^\alpha$ applied to the set*

$$\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{Z}\|_0^+ < \varepsilon\right\}$$

is greater than zero. The set of all deferred statistical cluster points of $(\mathcal{P}_r)$ is represented by $\Gamma_{FN}^\alpha(\mathcal{P}_r)$. It is worth noting that being in $\Gamma_{FN}^\alpha(\mathcal{P}_r)$ implies that $\bar{\delta}_{d,c}^\alpha$ applied to the sets

$$\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{Z}\|_\kappa^+ < \varepsilon\right\}$$

and

$$\left\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{Z}\|_\kappa^- < \varepsilon\right\}$$

is greater than zero for every $\varepsilon > 0$ and each $\kappa \in [0, 1]$.

Theorem 3.17. *For any given sequence $(\mathcal{P}_r)$ in $(V, \|\cdot\|^\alpha)$ and $d = (d(n))$ and $c = (c(n))$ being sequences satisfying* (1)*, then we have*

$$\Lambda_{FN}^\alpha(\mathcal{P}_r) \subseteq \Gamma_{FN}^\alpha(\mathcal{P}_r).$$

Proof. Suppose Q belongs to $\Lambda^{\alpha}_{FN}(\mathcal{P}_r)$. Consequently, there exists a non-thin subsequence $\left(\mathcal{P}_{r_j}\right)$ of $(\mathcal{P}_r)$ that converges to Q, denoted by

$$\bar{\delta}^{\alpha}_{d,c}\left(\left\{r_j : j \in \mathbb{N}\right\}\right) > 0.$$

Since

$$\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - Q\|^{\alpha} < \varepsilon\} \supseteq \left\{r_j \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_{r_j} - Q\|^{\alpha} < \varepsilon\right\},$$

for every $\varepsilon > 0$, we have

$$\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - Q\|^{\alpha} < \varepsilon\} \supseteq \left\{r_j : j \in \mathbb{N}\right\} \setminus \left\{r_j \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_{r_j} - Q\|^{\alpha} \geq \varepsilon\right\}.$$

Since $\left(\mathcal{P}_{r_j}\right) \xrightarrow{\mathrm{st}^{\alpha}_{d,c}(FN)} Q$, the set

$$\left\{r_j \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_{r_j} - Q\|^{\alpha} \geq \varepsilon\right\}$$

is finite for any $\varepsilon > 0$. Hence, we have $\bar{\delta}^{\alpha}_{d,c}\left(\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - Q\|^{\alpha} < \varepsilon\}\right) \geq \bar{\delta}^{\alpha}_{d,c}(\{r_j : j \in \mathbb{N}\}) - \bar{\delta}^{\alpha}_{d,c}(\{r_j \in \mathbb{N} \cap [d(n)+1, c(n)] : |\mathcal{P}_{r_j} - Q|^{\alpha} \geq \varepsilon\}) > 0$. Thus,

$$\bar{\delta}^{\alpha}_{d,c}\left(\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - Q\|^{\alpha} < \varepsilon\}\right) > 0$$

for every $\varepsilon > 0$. Therefore, Q belongs to $\Gamma^{\alpha}_{FN}(\mathcal{P}_r)$. □

Example 3.18. *Let $P_r = \frac{1}{r}$ for $r \in \mathbb{N}$, the space $(V, \|\cdot\|^{\alpha})$ be $(\mathbb{R}, |\cdot|^{\alpha})$ and $\alpha = \frac{1}{2}$. Let the sequences $d(n) = n^2$ and $c(n) = n^2 + n$. It satisfies the theorem $\Lambda^{\alpha}_{FN}(P_r) \subseteq \Gamma^{\alpha}_{FN}(P_r)$, showing that any $Q \in \Lambda^{\alpha}_{FN}$ also belongs to Γ^{α}_{FN}.*

Verification: *The sequence $P_r = \frac{1}{r}$ converges to $Q = 0$ as $r \to \infty$. Define the set of indices:*

$$S^{\alpha}_{n,\varepsilon} = \{r \in \mathbb{N} \cap [d(n)+1, c(n)] : |P_r - Q|^{\alpha} < \varepsilon\}.$$

Substituting $d(n) = n^2$, $c(n) = n^2 + n$ and $P_r = \frac{1}{r}$, we have, $S^{\alpha}_{n,\varepsilon} = \{r \in \mathbb{N} \cap [n^2+1, n^2+n] : \left|\frac{1}{r}\right|^{\alpha} < \varepsilon\}$. For the non-thin subsequence. For any subsequence $r_j = n^2 + j$, where $1 \leq j \leq n$, we have,

$$P_{r_j} = \frac{1}{r_j} = \frac{1}{n^2 + j}.$$

Since $r_j \to \infty$ as $j \to n$, this subsequence converges to $Q = 0$. Moreover, the non-thin condition is satisfied because:

$$\bar{\delta}^{\alpha}_{d,c}(\{r_j\}) > 0,$$

for large n. Thus $\Lambda^{\alpha}_{FN}(P_r) \subseteq \Gamma^{\alpha}_{FN}(P_r)$

Theorem 3.19. *Let $(\mathcal{P}_r)$ be a sequence in $(V, \|\cdot\|^{\alpha})$ and $d = (d(n))$ and $c = (c(n))$ be sequences satisfying (1). If $\mathcal{P}_r \xrightarrow{st_{d,c}(FN)} \mathcal{P}$, then*

$$\Lambda^{\alpha}_{FN}(\mathcal{P}_r) = \Gamma^{\alpha}_{FN}(\mathcal{P}_r) = \{\mathcal{P}\}.$$

Proof. Let $\mathcal{P}_r \xrightarrow{st^{\alpha}_{d,c}(FN)} \mathcal{P}$. Referring to Definitions 3.1 and 3.16, we conclude that $\mathcal{P}$ belongs to $\Gamma^{\alpha}_{FN}(\mathcal{P}_k)$. If $\exists$ at least one $\mathcal{Z}$ belonging to $\Gamma^{\alpha}_{FN}(\mathcal{P}_k)$ such that $\mathcal{Z} \neq \mathcal{P}$, we can infer the existence of $\varepsilon > 0$ such that

$$\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{P}\|^{\alpha} \geqslant \varepsilon\} \supseteq \{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{Z}\|^{\alpha} < \varepsilon\}$$

holds. Hence we get

$$\bar{\delta}^{\alpha}_{d,c}\left(\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{P}\|^{\alpha} \geqslant \varepsilon\}\right) \geqslant \bar{\delta}^{\alpha}_{d,c}\left(\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{Z}\|^{\alpha} < \varepsilon\}\right).$$

Since, $\mathcal{P}_r \xrightarrow{st^{\alpha}_{d,c}(FN)} \mathcal{P}$, we have $\delta^{\alpha}_{d,c}\left(\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{P}\|^{\alpha} \geqslant \varepsilon\}\right)$ is zero, which indicates that

$$\bar{\delta}^{\alpha}_{d,c}\left(\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{P}\|^{\alpha} \geqslant \varepsilon\}\right)$$

is zero. Thus, we get

$$\bar{\delta}^{\alpha}_{d,c}\left(\{r \in \mathbb{N} \cap [d(n)+1, c(n)] : \|\mathcal{P}_r - \mathcal{Z}\|^{\alpha} < \varepsilon\}\right)$$

is zero, a contradiction to $\mathcal{Z}$ belonging to $\Gamma^{\alpha}_{FN}(\mathcal{P}_r)$. Therefore, we should have $\Gamma^{\alpha}_{FN}(\mathcal{P}_r) = \{\mathcal{P}\}$. Also, since $\mathcal{P}_r \xrightarrow{st^{\alpha}_{d,c}(FN)} \mathcal{P}$. By Theorem 3.8 we conclude that $\mathcal{P} \in \Lambda^{\alpha}_{FN}(\mathcal{P}_r)$. Subsequently, Theorem 3.17 provides further insights $\Lambda^{\alpha}_{FN}(\mathcal{P}_r) = \Gamma^{\alpha}_{FN}(\mathcal{P}_r) = \{\mathcal{P}\}$. □

4. Conclusion

In this paper, we introduced the concepts of deferred statistically convergent sequences and deferred statistically Cauchy sequences within a fuzzy normed linear space (FNS) with respect to the norm $\|\cdot\|^{\alpha}$, where $0 < \alpha \leq 1$, establishing fundamental results and examining their basic properties. We defined deferred statistical limit points for sequences in FNS with the norm $\|\cdot\|^{\alpha}$ and investigated their relationships and implications. Additionally, we introduced deferred statistical cluster points for sequences in FNS with $\|\cdot\|^{\alpha}$ and examined their implications. Future research could explore the extension of these concepts to intuitionistic fuzzy normed spaces with norms raised to the power of α.

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