Filomat 39:22 (2025), 7849–7865
https://doi.org/10.2298/FIL2522849L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Cubic B-spline approach for solving fractional optimal control problems in the Caputo-Fabrizio sense

Mehrdad Lakestani[a,b,*], Remzi Tuntaş[b]

[a]*Faculty of Mathematics, Statistics and Computer Science, University of Tabriz, Tabriz, Iran*
[b]*Faculty of Engineering, Department of Computer Engineering, Van Yuzuncu Yil University, Van, Türkiye*

Abstract. In this paper, a numerical method is presented for solving fractional optimal control problems (FOCPs) in the Caputo-Fabrizio sense using cubic B-spline functions. Operational matrices for ordinary and fractional derivatives are constructed to facilitate the transformation of the FOCP into a nonlinear programming problem. The Lagrange multiplier method is applied to obtain the optimal solution. Additionally, the error bounds for the proposed approximations are derived to ensure the reliability of the method. The efficiency and accuracy of the approach are demonstrated through three numerical examples, confirming its effectiveness in solving fractional control problems.

1. Introduction

In recent years, fractional optimal control problems (FOCPs) have attracted increasing attention due to their ability to model complex dynamical systems with memory effects and nonlocal interactions [1, 13, 23, 33]. These problems arise in various fields, including engineering, physics, biology, and economics, particularly when classical integer-order models fail to accurately capture system dynamics [11, 22, 26].

Due to the inherent complexity of FOCPs, obtaining analytical solutions is often impractical, necessitating the development of efficient numerical methods. Significant research efforts have been devoted to this area, particularly following the introduction of the Caputo-Fabrizio (CF) fractional derivative, which employs a nonsingular kernel [3, 8, 9, 24, 28, 34]. This definition has proven useful in diverse applications, such as economics, chemistry, biology, physics, signal and image processing, engineering, and control theory [12, 29, 32, 36], further motivating its study.

Several numerical approaches have been proposed for solving FOCPs in the Caputo-Fabrizio sense. Dehestani and Ordokhani [7] utilized Gegenbauer polynomials, while Ghaderi et al. [16] applied Chebyshev cardinal functions to derive necessary optimality conditions. The Adams Bashforth method was used in [17] for an optimal control framework related to a Caputo-Fabrizio fractional model of the COVID-19

2020 *Mathematics Subject Classification*. Primary 49M99; Secondary 65D07.
Keywords. Optimal control, Fractional derivatives, Cubic B-spline functions, Operational matrix
Received: 07 February 2025; Revised: 21 May 2025; Accepted: 27 May 2025
Communicated by Miodrag Spalević
Research supported by TÜBİTAK (Scientific and Technological Research Institute of Türkiye).
* Corresponding author: Mehrdad Lakestani
Email addresses: lakestani@tabrizu.ac.ir, mehrdadlakestani@yyu.edu.tr (Mehrdad Lakestani), rtuntas@yyu.edu.tr (Remzi Tuntaş)
ORCID iDs: https://orcid.org/0000-0002-2752-0167 (Mehrdad Lakestani), https://orcid.org/0000-0001-7973-2412 (Remzi Tuntaş)

pandemic. Noori et al. [31] introduced a numerical approach based on Hermite spline functions for solving CF-FOCPs.

Other advanced techniques have also been explored. Singh et al. [35] combined the fractional integral operational matrix with fractional Legendre wavelets to address multi-dimensional CF-FOCPs efficiently. Kheyrandish et al. [25] proposed an artificial intelligence-based technique using a fractional power series neural network. Ghosh et al. [19] applied a gradient-based optimization algorithm, while Mortezaee et al. [30] introduced a fuzzy hyperbolic model supported by optimality conditions and a learning algorithm. Yildiz et al. [37] formulated a framework for time-fractional optimal control problems using the Caputo-Fabrizio derivative, transforming them into forward-backward fractional differential equations expressed as Volterra integral formulations.

Among the various numerical approaches, B-spline functionsparticularly cubic B-splineshave gained popularity due to their robustness, computational efficiency, and smooth approximation properties [2, 10, 27]. Their local support and flexibility make them especially effective for handling high-dimensional and multi-delay problems.

This paper introduces a novel numerical approach for solving FOCPs in the Caputo-Fabrizio sense using cubic B-spline functions. Operational matrices for both ordinary and fractional derivatives are constructed and integrated with dual basis functions to transform the original problem into a system of algebraic equations. This transformation allows for an efficient numerical solution, providing accurate approximations of the optimal control and state trajectories.

The fractional optimal control systems considered in this study are formulated as follows:

$$\min(\max)\mathcal{J}\left(x(t), u(t)\right) = \int_0^T \mathcal{V}\left(x(t), u(t), t\right) dt, \tag{1.1}$$

subject to the state equation

$$\mathfrak{F}\left(\dot{x}(t), {}_0^{CF}\mathrm{D}_t^{\alpha} x(t), x(t), u(t), t\right) = 0, \quad 0 \le t \le T, \quad 0 < \alpha \le 1, \tag{1.2}$$

with the initial condition

$$x(0) = x_0, \tag{1.3}$$

where $\mathcal{J}$ is the performance index, $u(t) = [u_1(t), u_2(t), \dots, u_s(t)]^\top$ and $x(t) = [x_1(t), x_2(t), \dots, x_r(t)]^\top$ are control and state vector functions, respectively. The function $\mathcal{V}$ and the vector functions $\mathfrak{F} = [\mathfrak{f}_1, \mathfrak{f}_2, \dots, \mathfrak{f}_r]^\top$ and $g(t) = [g_1(t), g_2(t), \dots, g_r(t)]^\top$ are generally nonlinear and smooth.

The structure of the paper is as follows:

Section 2 introduces the cubic B-spline basis and function expansion, along with the derivation of operational matrices for ordinary and fractional derivatives. Section 3 outlines the proposed numerical method for solving the problem. Section 4 presents an error analysis of the approximation method. Section 5 demonstrates the efficiency of the approach through numerical results. Lastly, the conclusion section provides a summary and discussion of the findings and potential directions for future research.

2. Cubic B-spline Functions

B-spline functions are fundamental tools for approximating functions in $L^2(\mathbb{R})$. The n-th order B-spline, represented as $\mathcal{N}_n(t)$, is constructed using a knot sequence $\{\dots, -1, 0, 1, \dots\}$ and consists of piecewise polynomials of degree $n-1$ defined within the intervals formed by these knots. The fundamental case, $\mathcal{N}_1(t)$, corresponds to the characteristic function $\chi_{[0,1]}(t)$, which is 1 for t in $[0, 1]$ and 0 otherwise. For any integer $n \ge 2$, the higher-order B-splines are formulated recursively as follows [6, 20]:

$$\mathcal{N}_n(t) = (\mathcal{N}_{n-1} * \mathcal{N}_1)(t) = \int_{-\infty}^{\infty} \mathcal{N}_{n-1}(t - x)\mathcal{N}_1(x)\, dx.$$

It has been established that for $n \geq 2$, the n-th order B-spline can be specified using the following expression [4]:

$$\mathcal{N}_n(t) = \frac{t}{n-1}\mathcal{N}_{n-1}(t) + \frac{n-t}{n-1}\mathcal{N}_{n-1}(t-1),$$

with the support of $\mathcal{N}_n(t)$ defined as $[0, n]$. For our purposes, we examine the case where $n = 4$, thereby utilizing cubic B-spline functions. The explicit formulation for $N_4(t)$ can be expressed as:

$$\mathcal{N}_4(t) = \begin{cases} \frac{1}{6}t^3, & t \in [0,1], \\ \frac{2}{3} - 2t + 2t^2 - \frac{1}{2}t^3, & t \in [1,2], \\ -\frac{22}{3} + 10t - 4t^2 + \frac{1}{2}t^3, & t \in [2,3], \\ \frac{32}{3} - 8t + 2t^2 - \frac{1}{6}t^3, & t \in [3,4], \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Note that the third derivative of $N_4(t)$ is

$$\mathcal{N}_4'''(t) = \begin{cases} 1, & t \in (0,1), \\ -3, & t \in (1,2), \\ 3, & t \in (2,3), \\ -1, & t \in (3,4), \\ 0 & t \notin [0,4]. \end{cases} \tag{2.2}$$

We define the function $\phi(t)$ as $\mathcal{N}_4(t)$, and examine the family of functions $\phi_{j,k}(t) = \phi(2^j t - k)$. It can be easily inferred that the support of $\phi_{j,k}(t)$ is represented by:

$$\operatorname{supp} \phi_{j,k}(t) = [2^{-j}k, 2^{-j}(4+k)].$$

To ensure that these functions are well-defined on the interval $[0, T]$, we introduce the adjustment:

$$\varphi_{j,k}(t) = \phi_{j,k}(t)\chi_{[0,T]}(t). \tag{2.3}$$

Let S_j denote the set of indices k for which the condition

$$\operatorname{supp} \phi_{j,k}(t) \cap (0, T) \neq \emptyset$$

holds. For instance, when $T \in \mathbb{N}$, it yields:

$$S_j = \{-3, -2, \ldots, 2^j T - 1\}.$$

2.1. Function Approximation

Given a natural number $M \in \mathbb{N}$, a function $f(t)$ defined on $L^2[0, T]$ can be approximated using cubic B-spline functions as demonstrated in [10, 27]:

$$f(t) \approx \sum_{k \in S_M} c_k \varphi_{M,k}(t) = C^T \Phi_M(t), \tag{2.4}$$

where C and $\Phi_M(t)$ represent vectors of dimension $S = |S_M|$, defined as follows:

$$C = \begin{bmatrix} c_{k_1} \\ c_{k_2} \\ \vdots \\ c_{k_{|S_M|}} \end{bmatrix} \quad \text{and} \quad \Phi_M(t) = \begin{bmatrix} \varphi_{M,k_1}(t) \\ \varphi_{M,k_2}(t) \\ \vdots \\ \varphi_{M,k_{|S_M|}}(t) \end{bmatrix}, \tag{2.5}$$

with $|S_M|$ denoting the count of elements in the set $S_M = \{k_1, k_2, \dots, k_S\}$.

The coefficients c_k can be determined as follows [10, 27]:

$$c_k = \int_0^T f(t)\tilde{\phi}_{M,k}(t)\, dt, \quad k \in S_M, \tag{2.6}$$

where $\tilde{\phi}_{M,k}(t)$ are the dual functions corresponding to $\varphi_{M,k}(t)$. These dual functions are expressible as linear combinations of the scaling functions $\varphi_{M,k}(t)$ for $k \in S_M$.

Let $\tilde{\Phi}_M$ represent the vector of dual functions associated with Φ_M:

$$\tilde{\Phi}_M = \begin{bmatrix} \tilde{\phi}_{M,k_1} \\ \tilde{\phi}_{M,k_2} \\ \vdots \\ \tilde{\phi}_{M,k_{|S_M|}} \end{bmatrix}.$$

Assume that $\tilde{\Phi}_M = P_M \Phi_M$. The duality property gives rise to the following equation:

$$\int_0^T \tilde{\Phi}_M(t)\Phi_M^{\top}(t)\, dt = I, \tag{2.7}$$

where I denotes the $S \times S$ identity matrix. This leads to:

$$P_M \int_0^T \Phi_M(t)\Phi_M^{\top}(t)\, dt = I.$$

Define

$$\tilde{P}_M = \int_0^T \Phi_M(t)\Phi_M^{\top}(t)\, dt, \tag{2.8}$$

where $\tilde{P}_M$ is a symmetric 7-diagonal matrix of size $S \times S$ as:

$$\tilde{P}_M = \frac{1}{2^{M+4} \times 315} \times \begin{bmatrix} 20 & 129 & 60 & 1 & & & & & \\ 129 & 1208 & 1062 & 120 & 1 & & & & \\ 60 & 1062 & 2396 & 1191 & 120 & 1 & & & \\ 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 \\ & & & 1 & 120 & 1191 & 2396 & 1062 & 60 \\ & & & & 1 & 120 & 1062 & 1208 & 129 \\ & & & & & 1 & 60 & 129 & 20 \end{bmatrix}.$$

Consequently, the matrix P_M can be determined as:

$$P_M = (\tilde{P}_M)^{-1}.$$

2.2. Operational Matrix of ordinary Derivative

The derivative of the vector Φ_M as defined in (2.5) can be expressed as

$$\Phi'_M \approx \mathfrak{D}\,\Phi_M, \tag{2.9}$$

where $\mathfrak{D}$ is the operational matrix of derivatives with dimensions $S\times S$ that corresponds to the cubic B-spline functions defined over the interval $[0,T]$.

To determine the matrix $\mathfrak{D}$, we consider the following expression:

$$\mathfrak{D} = \int_0^T \Phi'_M(t)\,\tilde{\Phi}_M^\top(t)\,dt = \left(\int_0^T \Phi'_M(t)\,\Phi_M^\top(t)\,dt\right)(P_M)^\top = \Omega P_M, \tag{2.10}$$

where

$$\Omega = \int_0^T \Phi'_M(t)\,\Phi_M^\top(t)\,dt. \tag{2.11}$$

In equation (2.11), Ω is a $S\times S$ matrix, and its entries can be computed as:

$$\Omega_{i,j} = \int_0^T \varphi_{M,k_j}(t)\,\frac{d}{dt}\varphi_{M,k_i}(t)\,dt, \quad i,j = 1,2,\ldots,S.$$

By straightforward calculation, the matrix Ω can be obtained as

$$\Omega = \frac{1}{720}\times\begin{bmatrix} -10 & -71 & -38 & -1 & & & & & \\ -9 & -160 & -254 & -56 & -1 & & & & \\ 18 & 174 & -10 & -245 & -56 & -1 & & & \\ 1 & 56 & 245 & 0 & -245 & -56 & -1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & 56 & 245 & 0 & -245 & -56 & -1 \\ & & & 1 & 56 & 245 & 10 & -174 & -18 \\ & & & & 1 & 56 & 254 & 160 & 9 \\ & & & & & 1 & 38 & 71 & 10 \end{bmatrix}. \tag{2.12}$$

2.3. Operational Matrix of Fractional Derivative

Definition 2.1. *Let f denote a function belonging to the Sobolev space $H^1(0,T)$. Furthermore, let $M(\alpha)$ represent a normalization function satisfying $M(0) = M(1) = 1$. The Caputo-Fabrizio fractional derivative of order $\alpha \in (0,1)$ is defined as follows [15, 31]:*

$${}_0^{CF}D_t^\alpha f(t) = \frac{M(\alpha)}{1-\alpha}\int_0^t e^{-\theta(t-\tau)} f'(\tau)d\tau, \tag{2.13}$$

where $\theta = \dfrac{\alpha}{1-\alpha}$. Here it is assumed $M(\alpha) = 1$.

The Caputo-Fabrizio fractional derivative of order $\alpha \in (0,1)$ for the vector Φ_M as defined in (2.5), can be formulated as

$${}_0^{C}\mathrm{D}_t^\alpha \Phi_M(t) \approx \mathfrak{D}_\alpha\,\Phi_M(t), \tag{2.14}$$

where $\mathfrak{D}_\alpha$ denotes the operational matrix of the fractional derivative, with dimension $S \times S$, corresponding to the cubic B-spline functions defined over the interval $[0, T]$. To compute the matrix $\mathfrak{D}_\alpha$, we first calculate the fractional derivative of the entries of vector Φ_M as follows:

$$ {}_0^{CF}\mathrm{D}_t^\alpha \varphi_{M,k_i}(t) = \frac{1}{1-\alpha}\int_0^t \varphi'_{M,k_i}(\tau)e^{-\theta(t-\tau)}d\tau, \quad \alpha \in (0,1), \quad i = 1, 2, \ldots, S. \tag{2.15}$$

By performing twice integration by parts, it yields:

$$\begin{aligned} {}_0^{CF}\mathrm{D}_t^\alpha \varphi_{M,k_i}(t) &= \frac{1}{1-\alpha}\left[\frac{\varphi'_{M,k_i}(t)}{\theta} - \frac{\varphi''_{M,k_i}(t)}{\theta^2} + \left(\frac{\varphi''_{M,k_i}(0)}{\theta^2} - \frac{\varphi'_{M,k_i}(0)}{\theta}\right)e^{-\theta t}\right] \\ &\quad + \frac{1}{(1-\alpha)\theta^2}\int_0^t \varphi'''_{M,k_i}(\tau)e^{-\theta(t-\tau)}d\tau, \quad \alpha \in (0,1), \quad i = 1, 2, \ldots, S. \end{aligned} \tag{2.16}$$

Using relation (2.2), the third derivative of $\varphi_{M,k_i}(t)$ will be a constant piecewise function as

$$\varphi'''_{M,k_i}(t) = 2^{3M} \times \begin{cases} 1, & t \in (\frac{k_i}{2^M}, \frac{k_i+1}{2^M}), \\ -3, & t \in (\frac{k_i+1}{2^M}, \frac{k_i+2}{2^M}), \\ 3, & t \in (\frac{k_i+2}{2^M}, \frac{k_i+3}{2^M}), \\ -1, & t \in (\frac{k_i+3}{2^M}, \frac{k_i+4}{2^M}), \\ 0 & t \notin [\frac{k_i}{2^M}, \frac{k_i+4}{2^M}]. \end{cases} \tag{2.17}$$

So the integral term in relation (2.16) can be found as following:
Case 1: if $t \le \frac{k_i}{2^M}$ then

$$\int_0^t \varphi'''_{M,k_i}(\tau)e^{-\theta(t-\tau)}d\tau = 0.$$

Case 2: if $t \in (\frac{k_i}{2^M}, \frac{k_i+1}{2^M})$ then

$$\int_0^t \varphi'''_{M,k_i}(\tau)e^{-\theta(t-\tau)}d\tau = 2^{3M}\int_{\frac{k_i}{2^M}}^t e^{-\theta(t-\tau)}d\tau = \frac{2^{3M}}{\theta}(1 - e^{-\theta(t-\frac{k_i}{2^M})}).$$

Case 3: if $t \in (\frac{k_i+1}{2^M}, \frac{k_i+2}{2^M})$ then

$$\begin{aligned} \int_0^t \varphi'''_{M,k_i}(\tau)e^{-\theta(t-\tau)}d\tau &= 2^{3M}\int_{\frac{k_i}{2^M}}^{\frac{k_i+1}{2^M}} e^{-\theta(t-\tau)}d\tau - 3\times 2^{3M}\int_{\frac{k_i+1}{2^M}}^t e^{-\theta(t-\tau)}d\tau \\ &= \frac{2^{3M}}{\theta}\left(4e^{-\theta(t-\frac{k_i+1}{2^M})} - e^{-\theta(t-\frac{k_i}{2^M})} - 3\right) \\ &= \frac{2^{3M}}{\theta}\left[\left(4e^{\theta(\frac{k_i+1}{2^M})} - e^{\theta(\frac{k_i}{2^M})}\right)e^{-\theta t} - 3\right]. \end{aligned}$$

Case 4: if $t \in (\frac{k_i+2}{2^M}, \frac{k_i+3}{2^M})$ then

$$\begin{aligned} \int_0^t \varphi'''_{M,k_i}(\tau)e^{-\theta(t-\tau)}d\tau &= 2^{3M}\int_{\frac{k_i}{2^M}}^{\frac{k_i+1}{2^M}} e^{-\theta(t-\tau)}d\tau - 3\times 2^{3M}\int_{\frac{k_i+1}{2^M}}^{\frac{k_i+2}{2^M}} e^{-\theta(t-\tau)}d\tau \\ &\quad + 3\times 2^{3M}\int_{\frac{k_i+2}{2^M}}^t e^{-\theta(t-\tau)}d\tau \\ &= \frac{2^{3M}}{\theta}\left(4e^{-\theta(t-\frac{k_i+1}{2^M})} - 6e^{-\theta(t-\frac{k_i+2}{2^M})} - e^{-\theta(t-\frac{k_i}{2^M})} + 3\right) \\ &= \frac{2^{3M}}{\theta}\left[\left(4e^{\theta(\frac{k_i+1}{2^M})} - 6e^{\theta(\frac{k_i+2}{2^M})} - e^{\theta(\frac{k_i}{2^M})}\right)e^{-\theta t} + 3\right] \end{aligned}$$

Case 5: if $t \in (\frac{k_i+3}{2^M}, \frac{k_i+4}{2^M})$ then

$$\begin{aligned}\int_0^t \varphi'''_{M,k_i}(\tau)e^{-\theta(t-\tau)}d\tau &= 2^{3M}\int_{\frac{k_i}{2^M}}^{\frac{k_i+1}{2^M}} e^{-\theta(t-\tau)}d\tau - 3\times 2^{3M}\int_{\frac{k_i+1}{2^M}}^{\frac{k_i+2}{2^M}} e^{-\theta(t-\tau)}d\tau \\ &+ 3\times 2^{3M}\int_{\frac{k_i+2}{2^M}}^{\frac{k_i+3}{2^M}} e^{-\theta(t-\tau)}d\tau - 2^{3M}\int_{\frac{k_i+3}{2^M}}^{t} e^{-\theta(t-\tau)}d\tau + \\ &= \frac{2^{3M}}{\theta}\left(4e^{-\theta(t-\frac{k_i+1}{2^M})} - 6e^{-\theta(t-\frac{k_i+2}{2^M})} + 4e^{-\theta(t-\frac{k_i+3}{2^M})} - e^{-\theta(t-\frac{k_i}{2^M})} - 1\right) \\ &= \frac{2^{3M}}{\theta}\left[\left(4e^{\theta(\frac{k_i+1}{2^M})} - 6e^{\theta(\frac{k_i+2}{2^M})} + 4e^{\theta(\frac{k_i+3}{2^M})} - e^{\theta(\frac{k_i}{2^M})}\right)e^{-\theta t} - 1\right].\end{aligned}$$

Case 6: if $t \geq \frac{k_i+4}{2^M}$ then

$$\begin{aligned}\int_0^t \varphi'''_{M,k_i}(\tau)e^{-\theta(t-\tau)}d\tau &= 2^{3M}\int_{\frac{k_i}{2^M}}^{\frac{k_i+1}{2^M}} e^{-\theta(t-\tau)}d\tau - 3\times 2^{3M}\int_{\frac{k_i+1}{2^M}}^{\frac{k_i+2}{2^M}} e^{-\theta(t-\tau)}d\tau \\ &+ 3\times 2^{3M}\int_{\frac{k_i+2}{2^M}}^{\frac{k_i+3}{2^M}} e^{-\theta(t-\tau)}d\tau - 2^{3M}\int_{\frac{k_i+3}{2^M}}^{\frac{k_i+4}{2^M}} e^{-\theta(t-\tau)}d\tau + \\ &= \frac{2^{3M}}{\theta}\left(4e^{-\theta(t-\frac{k_i+1}{2^M})} - 6e^{-\theta(t-\frac{k_i+2}{2^M})} + 4e^{-\theta(t-\frac{k_i+3}{2^M})} - e^{-\theta(t-\frac{k_i}{2^M})} - e^{-\theta(t-\frac{k_i+4}{2^M})}\right). \\ &= \frac{2^{3M}}{\theta}\left(4e^{\theta(\frac{k_i+1}{2^M})} - 6e^{\theta(\frac{k_i+2}{2^M})} + 4e^{\theta(\frac{k_i+3}{2^M})} - e^{\theta(\frac{k_i}{2^M})} - e^{\theta(\frac{k_i+4}{2^M})}\right)e^{-\theta t}.\end{aligned}$$

So the Caputo-Fabrizio fractional derivative of the functions $\varphi_{M,k_i}(t)$ can be found explicitly. Assume

$$\omega_{i,\alpha}(t) = {}_0^{CF}\mathrm{D}_t^{\alpha}\varphi_{M,k_i}(t).$$

Now we can expand these fractional derivaties using relation (2.4) as:

$$\omega_{i,\alpha}(t) = \sum_{k\in S_M} d_{k,k_i}\varphi_{M,k}(t), \quad i = 1,2,\ldots,S, \tag{2.18}$$

where d_{k,k_i} are the entries of matrix $\mathfrak{D}_\alpha$ and they can be obtained as follows.

$$\mathfrak{D}_\alpha = \int_0^T \omega_\alpha\,\tilde{\Phi}_M^\top(t)\,dt = \left(\int_0^T \omega_\alpha\,\Phi_M^\top(t)\,dt\right)(P_M)^\top = \Omega_\alpha P_M,$$

where

$$\omega_\alpha = [\omega_{k_1,\alpha}, \omega_{k_2,\alpha}, \ldots, \omega_{k_S,\alpha}]^\top,$$

and

$$\Omega_\alpha = \int_0^T \omega_\alpha\,\Phi_M^\top(t)\,dt.$$

3. Description of Method

Each state function x_i and control function u_j over the interval $[0,T]$ can be approximated as follows:

$$\begin{aligned} x_i &\approx X_i^\top\Phi_M(t), \quad i = 1,2,\ldots,r, \\ u_j &\approx U_j^\top\Phi_M(t), \quad j = 1,2,\ldots,s, \end{aligned} \tag{3.1}$$

where $X_i, i = 1, 2, \ldots, r$ and $U_j, j = 1, 2, \ldots, s$ are vectors of dimensions S.

The derivative of the state function $x_i(t)$, using relations (2.9) and (3.1), can be approximated as:

$$\dot{x}_i(t) = X_i^\top \mathfrak{D}\Phi_M(t), \quad i = 1, 2, \ldots, r. \tag{3.2}$$

Similarly, the CF fractional derivative of the state function $x_i(t)$ of order α, using relations (2.14) and (3.1), can be approximated as:

$$ {}_0^C\mathrm{D}_t^\alpha x_i(t) = X_i^\top \mathfrak{D}_\alpha \Phi_M(t), \quad i = 1, 2, \ldots, r. \tag{3.3}$$

Consider the expression

$$\hat{\Phi}_{M,k}(t) = I_k \otimes \Phi_M(t),$$

where I_k represents the identity matrix of size $k \times k$, and $\otimes$ indicates the Kronecker product. Utilizing equation (3.1), we can approximate the state and control vectors $x(t)$ and $u(t)$ as follows:

$$\begin{aligned} x(t) &\approx X^\top \hat{\Phi}_{M,r}(t), \\ u(t) &\approx U^\top \hat{\Phi}_{M,s}(t). \end{aligned} \tag{3.4}$$

Here, X and U are vectors of dimensions rS and sS, respectively, defined by:

$$X = \left[X_1^\top, X_2^\top, \ldots, X_r^\top\right]^\top,$$

and

$$U = \left[U_1^\top, U_2^\top, \ldots, U_s^\top\right]^\top.$$

Define

$$\hat{\Phi}'_{M,r}(t) = I_r \otimes \mathfrak{D}\Phi_M(t),$$

and

$$\hat{\Phi}^{(\alpha)}_{M,r}(t) = I_r \otimes \mathfrak{D}_\alpha \Phi_M(t).$$

Using the relationships (3.2), (3.3), and (3.4), the ordinary and CF fractional derivatives of the vector function $x(t)$ can be approximated as follows:

$$\dot{x}(t) \approx X^\top \hat{\Phi}'_{M,r}(t), \tag{3.5}$$

and

$$ {}_0^C\mathrm{D}_t^\alpha x(t) \approx X^\top \hat{\Phi}^{(\alpha)}_{M,r}(t). \tag{3.6}$$

Substituting equations (3.4), (3.5), and (3.6) into the problem defined by (1.1) - (1.3) transforms it into the following optimization problem:

$$\min(\max)\mathcal{J}(X, U) = \int_0^T \mathcal{V}\left(X^\top \hat{\Phi}_{M,r}(t), U^\top \hat{\Phi}_{M,r}(t), t\right) dt, \tag{3.7}$$

subject to

$$\mathfrak{F}\left(X^\top \hat{\Phi}'_{M,r}(t), X^\top \hat{\Phi}^{(\alpha)}_{M,r}(t), X^\top \hat{\Phi}_{M,r}(t), U^\top \hat{\Phi}_{M,r}(t), t\right) = 0, \tag{3.8}$$

and

$$X^\top \hat{\Phi}_{M,r}(0) = x_0. \tag{3.9}$$

Applying the Bole integration technique over the interval $[0, T]$, the value of $\mathcal{J}(X, U)$ in relation (3.7) is approximated as follows:

$$\min(\max)\mathcal{J}(X, U) \approx \sum_{\ell=0}^{n} \omega_\ell \mathcal{V}\left(X^\top \hat{\Phi}_{M,r}(\tau_\ell), U^\top \hat{\Phi}_{M,r}(\tau_\ell), \tau_\ell\right), \tag{3.10}$$

where ω_ℓ and τ_ℓ for $\ell = 1, 2, \ldots, n$ represent the weights and nodes of the Bole integration method.

By collocating the vector function $\mathfrak{F}$ at the points $t = s_i = \frac{i}{S}T$ for $i = 1, 2, \ldots, S$, we obtain:

$$\mathfrak{F}_i(X, U) = \mathfrak{F}\left(X^\top \hat{\Phi}'_{M,r}(s_i), X^\top \hat{\Phi}^{(\alpha)}_{M,r}(s_i), X^\top \hat{\Phi}_{M,r}(s_i), U^\top \hat{\Phi}_{M,r}(s_i), s_i\right) = 0, \quad i = 1, 2, \ldots, S. \tag{3.11}$$

Let Λ and $\Lambda_i \in \mathbb{R}^r$ for $i = 1, 2, \ldots, S$. Define the function

$$\tilde{\mathcal{J}}(X, U, \Lambda_1, \ldots, \Lambda_{sS}, \Lambda) = \mathcal{J}(X, U) + \sum_{i=1}^{S} \Lambda_i^\top \mathfrak{F}_i(X, U) + \Lambda^\top \left(X^\top \hat{\Phi}_{M,r}(t_0) - x_0\right).$$

The Lagrange multipliers associated with the optimal solution to the problem defined in (3.10), under the constraints specified in (3.11) and (3.9), yield the following conditions:

$$\begin{aligned} &\frac{\partial \tilde{\mathcal{J}}}{\partial X} = 0, \\ &\frac{\partial \tilde{\mathcal{J}}}{\partial U} = 0, \\ &\frac{\partial \tilde{\mathcal{J}}}{\partial \Lambda} = 0, \\ &\frac{\partial \tilde{\mathcal{J}}}{\partial \Lambda_i} = 0, \quad i = 1, 2, \ldots, S. \end{aligned} \tag{3.12}$$

The system of equations resulting from (3.12) can be solved to find the vectors X and U. As a result, the state and control functions $x(t)$ and $u(t)$ can be established using the relationships provided in (3.4).

4. Error Analysis

This section focuses on establishing the error bounds for the expansions discussed earlier.

Theorem 4.1. *[5] Let $f(t) \in C^4(0, T)$. The discrepancy between the original function and its cubic B-spline approximation given in* (2.4) *can be bounded as follows:*

$$|f(t) - \sum_{k \in S_M} c_k \varphi_{M,k}(t)| \leq C 2^{-4M} \|f^{(4)}\|_\infty = O(2^{-4M}),$$

where $\|f^{(4)}\|_\infty$ represents the maximum absolute value of the fourth derivative of $f(t)$ across the interval $(0, T)$, and C is a constant.

Additionally, the derivative of the error can be controlled by:

$$\left|\frac{d}{dt}\left(f(t)-\sum_{k\in S_M} c_k\varphi_{M,k}(t)\right)\right| \leq C'2^{-3M}\|f^{(4)}\|_\infty,$$

where C' is a constant.

Lemma 4.1. *Let $x(t) \in C^4(0,T)$. Assume that $x(t)$ is approximated by cubic B-splines on the interval $[0,T]$ as in relation (3.4). Then, the error in the CF fractional derivative of $x(t)$ of order $\alpha \in (0,1)$ can be bounded as follows:*

$$\left|{}_0^{CF}D_t^\alpha x(t) - {}_0^{CF}D_t^\alpha X^\top \hat{\Phi}_{M,r}(t)\right| \leq C_2 2^{-4M}\|x^{(4)}\|_\infty = O(2^{-4M}),$$

where C_2 is a constant.

Proof. Using integration by parts in (2.13), it yields:

$$ {}_0^{CF}\mathrm{D}_t^\alpha f(t) = \frac{1}{1-\alpha}\left(f(t) - e^{-\theta t}f(0) - \theta\int_0^t f(\tau)e^{-\theta(t-\tau)}d\tau\right). \tag{4.1}$$

So we have

$$\begin{aligned}\left|{}_0^{CF}\mathrm{D}_t^\alpha x(t) - {}_0^{CF}\mathrm{D}_t^\alpha \mathrm{X}^\top\hat{\Phi}_{M,r}(t)\right| &= \frac{1}{1-\alpha}\left|\left(x(t)-\mathrm{X}^\top\hat{\Phi}_{M,r}(t)\right) - \left(e^{-\theta t}x(0) - \mathrm{X}^\top\hat{\Phi}_{M,r}(0)\right)\right.\\ &\left.-\theta\int_0^t\left(x(\tau)-\mathrm{X}^\top\hat{\Phi}_{M,r}(t)\right)e^{-\theta(t-\tau)}d\tau\right|, \quad t\in[0,T].\end{aligned}$$

Using the Theorem 4.1 and initial condition (3.9) in the above relation, it yields

$$\begin{aligned}\left|{}_0^{CF}\mathrm{D}_t^\alpha x(t) - {}_0^{CF}\mathrm{D}_t^\alpha \mathrm{X}^\top\hat{\Phi}_{M,r}(t)\right| &\leq \frac{1}{1-\alpha}\left(C2^{-4M}\|x^{(4)}\|_\infty + C2^{-4M}\|x^{(4)}\|_\infty\theta\int_0^t e^{-\theta(t-\tau)}d\tau,\right)\\ &= \frac{1}{1-\alpha}\left(C2^{-4M}\|x^{(4)}\|_\infty + C2^{-4M}\|x^{(4)}\|_\infty(1-e^{-\theta t})\right)\\ &\leq \frac{2C}{1-\alpha}2^{-4M}\|x^{(4)}\|_\infty.\end{aligned}$$

This concludes the proof.
□

Theorem 4.2. *Let $x(t) \in C^4(0,T)$ and it is approximated by cubic B-splines on the interval $[0,T]$. Then, the error of the operational matrix of CF fractional derivative for $\alpha \in (0,1)$ is as follows:*

$$\left|{}_0^{CF}D_t^\alpha x(t) - X^\top\mathfrak{D}_\alpha\hat{\Phi}_{M,r}(t)\right| = O(2^{-4M}).$$

Proof. In order to prove the theorem one can write:

$$\begin{aligned}\left|{}_0^{CF}\mathrm{D}_t^\alpha x(t) - \mathrm{X}^\top\mathfrak{D}_\alpha\hat{\Phi}_{M,r}(t)\right| &= \left|{}_0^{CF}\mathrm{D}_t^\alpha x(t) - {}_0^{CF}\mathrm{D}_t^\alpha \mathrm{X}^\top\hat{\Phi}_{M,r}(t) + {}_0^{CF}\mathrm{D}_t^\alpha \mathrm{X}^\top\hat{\Phi}_{M,r}(t) - \mathrm{X}^\top\mathfrak{D}_\alpha\hat{\Phi}_{M,r}(t)\right|\\ &\leq \left|{}_0^{CF}\mathrm{D}_t^\alpha x(t) - {}_0^{CF}\mathrm{D}_t^\alpha \mathrm{X}^\top\hat{\Phi}_{M,r}(t)\right| + \left|{}_0^{CF}\mathrm{D}_t^\alpha \mathrm{X}^\top\hat{\Phi}_{M,r}(t) - \mathrm{X}^\top\mathfrak{D}_\alpha\hat{\Phi}_{M,r}(t)\right|\end{aligned}$$

Using the Lemma 4.1, for the first absolute value on the right-hand side of the above relation, one can obtain:

$$\left|{}_0^{CF}\mathrm{D}_t^\alpha x(t) - {}_0^{CF}\mathrm{D}_t^\alpha \mathrm{X}^\top\hat{\Phi}_{M,r}(t)\right| = O(2^{-4M}). \tag{4.2}$$

Also, using the relation (4.1), one can find that if $f \in C^4(0,T)$, then ${}_0^{CF}\mathrm{D}_t^\alpha f \in C^4(0,T)$. Thus, using Theorem 4.1, it yields:

$$\left|{}_0^{CF}\mathrm{D}_t^\alpha \mathrm{X}^\top\hat{\Phi}_{M,r}(t) - \mathrm{X}^\top\mathfrak{D}_\alpha\hat{\Phi}_{M,r}(t)\right| = O(2^{-4M}). \tag{4.3}$$

Relations (4.2) and (4.3) conclude the proof. □

5. Numerical Experiments

In this section, we provide numerical examples to demonstrate the implementation of the method outlined in Section 3. The computations are performed using Maple 2024 software on a personal computer. To highlight the efficiency of our approach, we present three numerical examples and compare the obtained results with existing findings from the literature.

Example 5.1. *Consider the subsequent FOCP [14, 31]:*

$$Min\ \mathcal{J}(x(t),u(t)) = \frac{1}{2}\int_0^2 \left(x^2(t) + u^2(t)\right)dt$$

subject to dynamical system and initial condition

$$\frac{3}{4}{}_0^{CF}D_t^{\alpha}x(t) + \frac{1}{4}\dot{x}(t) = x(t) - u(t), \quad t \in [0,1], \quad 0 < \alpha \leq 1,$$
$$x(0) = 1.$$

The precise solutions for this problem when $\alpha = 1$ are

$$x(t) = \frac{3e^{2t} + e^4e^{-2t}}{3+e^4}, \quad u(t) = \frac{3e^4e^{-2t} - 3e^{2t}}{3+e^4}.$$

This case was examined in [14, 21, 31]. In [14], the authors employed shifted Legendre polynomials, while in [21], the focus was on generalized shifted Chebyshev polynomials. Additionally, [31] utilized Hermite spline functions to address the problem. Table 1 presents a comparison of the error values of $\mathcal{J}$ derived from the proposed method for $M = 2, 3, \ldots, 7$ with those obtained using the methods outlined in [14, 21, 31]. Figures 1 and 2 illustrate the approximate solutions for the state function $x(t)$ and the control function $u(t)$, respectively, at different values of $\alpha = 0.6, 0.7, 0.8, 0.9, 0.95, 0.99, 1$ with $J = 4$. It is evident from the figures that as α approaches 1, the solutions tend to align with the solution corresponding to $\alpha = 1$.

Table 1: Approximate error value of $\mathcal{J}$ for $\alpha = 1$, for Example 5.1

Method	Absolute error of $\mathcal{J}$
Shifted Legendre polynomial method [14]	
M=6	6.75×10^{-11}
M=8	1.55×10^{-15}
generalized shifted Chebyshev polynomial method [21]	
$m_1 = m_2 = 6$	3.09×10^{-12}
$m_1 = m_2 = 8$	4.64×10^{-17}
Hermite spline method [31]	
$J = 6$	2.12×10^{-15}
$J = 7$	3.26×10^{-17}
Present method	
$M = 2$	8.21×10^{-9}
$M = 3$	3.51×10^{-11}
$M = 4$	1.53×10^{-13}
$M = 5$	6.46×10^{-16}
$M = 6$	2.63×10^{-18}
$M = 7$	1.05×10^{-20}

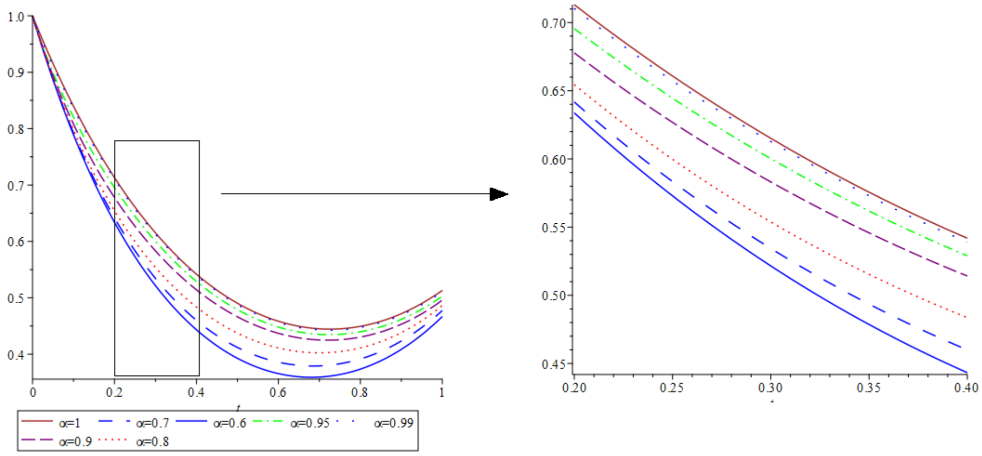

Figure 1: Plot of state function $x(t)$ for $\alpha = 0.8, 0.9, 0.95, 0.99, 1$ in Example 5.1

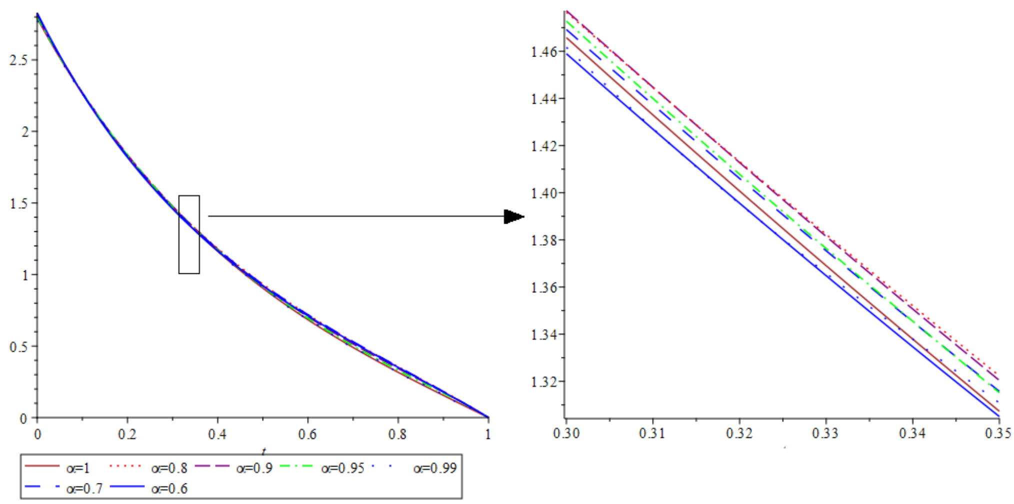

Figure 2: Plot of control function $u(t)$ for $\alpha = 0.8, 0.9, 0.95, 0.99, 1$ in Example 5.1

Example 5.2. *The dynamic equation governing a fractional order spring-mass-viscous damper system is presented in [18] as follows:*

$$M\ddot{x} + B_0 D_t^{\alpha} x + Kx = u, \quad 0 < \alpha < 1, \tag{5.1}$$

where M, B, and K denote the mass, damping coefficient, and stiffness, respectively. Here, x represents the displacement relative to a defined reference frame, and u signifies the external force applied. By setting $x_1 = x$, $x_2 = \dot{x}$, and normalizing B, M, and K to unity, Equation (5.1) *simplifies to the following form:*

$$\dot{X}(t) + P\, {}_0^{CF}D_t^{\alpha} X(t) = AX(t) + Bu(t), \quad 0 < \alpha < 1, \tag{5.2}$$

where

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, P = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We will now examine the optimization challenge defined as follows. The objective is to determine the optimal control u that minimizes the performance index given by:

$$\mathcal{J} = \frac{1}{2} \int_0^1 \left(x_1^2(t) + x_2^2(t) + u^2(t) \right) dt.$$

This is subject to the dynamic constraint specified in (5.2) *and the initial condition $X(0) = [1, 0]^{\top}$, where the final state is unconstrained.*

Tables 2 and 3 present the results obtained from the methods described, comparing them with the approach outlined in [15, 31]. Figure 3 illustrates the plots of x_1, x_2, and u derived from the present method for values of $\alpha = 0.7$, 0.8, and 0.9.

Table 2: Approximate solutions of x_1 for $a = 0.7$, 0.8, and 0.9 for Example 5.2

t	Method [15]	Method [31]	Present method
$\alpha = 0.7$			
0.2	0.97853919	0.97856232	0.97856206
0.4	0.92118394	0.92111669	0.92111851
0.6	0.83878727	0.83873176	0.83873164
0.8	0.74141242	0.74144543	0.74144664
1.0	0.63682516	0.63682688	0.63682826
$\alpha = 0.8$			
0.2	0.97849372	0.97852801	0.97852801
0.4	0.92161605	0.92149909	0.92150257
0.6	0.84085916	0.84076232	0.84076388
0.8	0.74640627	0.74644330	0.74644679
1.0	0.64544454	0.64543344	0.64543716
$\alpha = 0.9$			
0.2	0.97866088	0.97868398	0.97868686
0.4	0.92317218	0.92297994	0.92299204
0.6	0.84523603	0.84508932	0.84510166
0.8	0.75422193	0.75421750	0.75423367
1.0	0.65606892	0.65599923	0.65601553

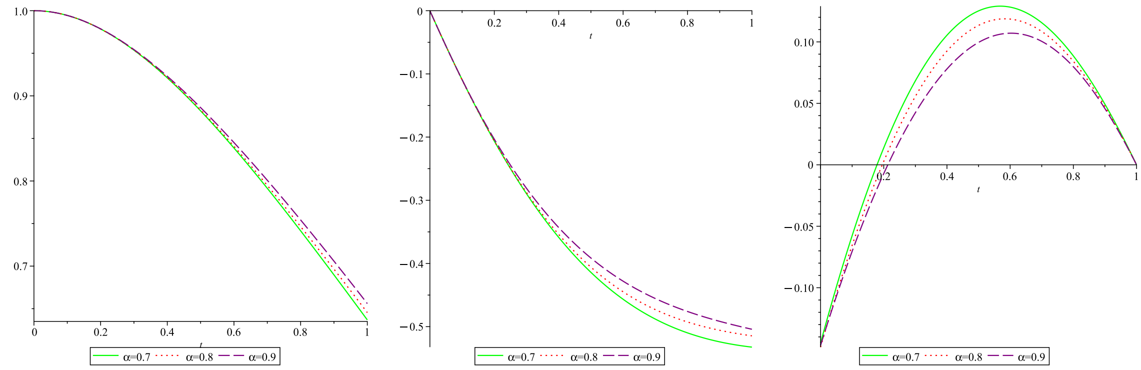

Figure 3: Plots of x_1 (left), x_2 (middle), and u (right) for $\alpha = 0.7, 0.8, 0.9$ with $M = 4$ in Example 5.2

Table 3: Approximate solutions of x_2 for $a = 0.7$, 0.8, and 0.9 for Example 5.2

t	Method [15]	Method [31]	Present method
$\alpha = 0.7$			
0.2	-0.20603646	-0.20627474	-0.20626893
0.4	-0.35834621	-0.35870747	-0.35871876
0.6	-0.45710883	-0.45667346	-0.45665892
0.8	-0.51004250	-0.50985911	-0.50986323
1.0	-0.53240402	-0.53240382	0.53240302
$\alpha = 0.8$			
0.2	-0.20566034	-0.20610149	-0.20609048
0.4	-0.35339900	-0.35397345	-0.35398529
0.6	-0.44554326	-0.44485639	-0.44483677
0.8	-0.49285573	-0.49259949	-0.49260267
1.0	-0.51453427	-0.51454358	-0.51454289
$\alpha = 0.9$			
0.2	-0.20251882	-0.20337756	-0.20333582
0.4	-0.34252493	-0.34315214	-0.34315192
0.6	-0.42893714	-0.42807526	-0.42804696
0.8	-0.47616318	-0.47599650	-0.47599180
1.0	-0.50409967	-0.50412524	-0.50412517

Example 5.3. *Examine the subsequent problem involving linear time-invariant systems:*

$$\min \mathcal{J} = \frac{1}{2}\int_0^1 \Big((x(t) - x_d(t))^2 + u^2(t)\Big)\,dt,$$

subject to

$${}_0^{CF}D_t^{\alpha} x(t) + x(t) - u(t) = f(t), \quad t \in (0,1], \quad x(0) = 2,$$

where

$$f(t) = -2 \cdot \frac{-e^{-\theta t} + e^{-2t}}{(1-\alpha)(\theta - 2)} + 1 + e^{-2t} + e^{t} - e$$

and

$$x_d(t) = \frac{e^{t} - e^{\theta t - \theta + 1}}{(1-\alpha)(\theta - 1)} + 1 + e^{-2t} - e^{t} + e$$

The precise solution to this problem is given by $x(t) = 1 + e^{-2t}$ and $u(t) = e - e^t$. Tables 4 and 5 show the maximum values of errors for $x(t)$ and $u(t)$, respectively, with $\alpha = 0.6, 0.7, 0.8, 0.9$ for different values of M and compare the results with those obtained by [30, 37]. These tables show that the presented method provides more accurate results than the others. Table 6 shows the absolute values of errors for $\mathcal{J}$ obtained for $M = 2, 3, 4, 5$ with different values of $\alpha = 0.6, 0.7, 0.8, 0.9$.

Table 4: Maximum values of the errors for $x(t)$, for Example 5.3

Methods	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$
Method [37]				
M=400	1.1×10^{-3}	1.1×10^{-3}	1.8×10^{-3}	6.4×10^{-3}
M=800	5.6×10^{-4}	5.3×10^{-4}	8.9×10^{-4}	3.2×10^{-3}
Method [30]	8.4×10^{-4}	3.0×10^{-4}	3.3×10^{-4}	3.5×10^{-4}
Present method				
M=1	6.2×10^{-4}	6.6×10^{-4}	6.8×10^{-4}	7.3×10^{-4}
M=2	4.5×10^{-5}	4.7×10^{-5}	4.9×10^{-5}	5.1×10^{-5}
M=3	3.8×10^{-6}	3.9×10^{-6}	3.9×10^{-6}	4.1×10^{-6}
M=4	2.9×10^{-7}	2.9×10^{-7}	2.8×10^{-7}	2.8×10^{-7}
M=5	2.0×10^{-8}	2.0×10^{-8}	2.1×10^{-8}	2.0×10^{-8}

Table 5: Maximum values of the errors for $u(t)$, for Example 5.3

Methods	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$
Method [37]				
M=400	3.0×10^{-3}	5.1×10^{-3}	9.5×10^{-3}	2.3×10^{-2}
M=800	1.5×10^{-3}	2.6×10^{-3}	4.7×10^{-3}	1.2×10^{-2}
Method [30]	2.9×10^{-4}	1.7×10^{-4}	1.7×10^{-3}	4.1×10^{-4}
Present method				
M=1	3.2×10^{-3}	3.8×10^{-3}	5.3×10^{-3}	8.4×10^{-3}
M=2	1.2×10^{-4}	1.6×10^{-4}	2.0×10^{-4}	3.3×10^{-4}
M=3	6.1×10^{-6}	7.3×10^{-6}	9.5×10^{-6}	1.4×10^{-5}
M=4	3.1×10^{-7}	3.7×10^{-7}	5.2×10^{-7}	8.5×10^{-7}
M=5	1.6×10^{-8}	2.0×10^{-8}	2.5×10^{-8}	4.6×10^{-8}

Table 6: Absolute errors of $\mathcal{J}$ obtained for Example 5.3

α	$M = 2$	$M = 3$	$M = 4$	$M = 5$	Exact value of $\mathcal{J}$
0.6	2.79×10^{-9}	1.50×10^{-11}	5.95×10^{-14}	2.74×10^{-16}	4.4151171302848869118
0.7	2.33×10^{-8}	1.36×10^{-10}	6.27×10^{-13}	2.63×10^{-15}	4.6517758226956862035
0.8	3.20×10^{-7}	2.37×10^{-9}	1.25×10^{-11}	5.54×10^{-14}	4.7683500427567263086
0.9	2.62×10^{-5}	1.87×10^{-7}	1.43×10^{-9}	7.87×10^{-12}	4.6467453161908688040

Conclusion

In this study, a numerical framework was developed for solving CF-FOCPs using cubic B-spline functions. The problem was discretized by constructing operational matrices for ordinary and fractional derivatives, allowing its transformation into a nonlinear programming problem. The Lagrange multiplier method was employed to determine the optimal solution. Furthermore, the error bounds ware established to assess the accuracy of the approximations. The effectiveness of the proposed method was verified through numerical experiments, demonstrating its capability in solving fractional control problems. Future research can focus on extending this approach to FOCPs involving other definitions of fractional derivatives.

Acknowlegments

The authors are supporetd by TÜBİTAK (Scientific and Technological Research Institute of Türkiye) with Grant No. E-21514107-115.02-613617 during the first author's stay in Van Yuzuncu Yil University from September 2024.

Conflicts of interest

The authors declare that there are no conflicts of interest.

References

[1] M. Awadalla, J. Alahmadi, K.R. Cheneke, S. Qureshi, *Fractional optimal control model and bifurcation analysis of human syncytial respiratory virus transmission dynamics*, Fractal Fract. **8** (2024), 44.
[2] A. Babu, N. Asharaf, *Numerical solution of nonlinear Sine-Gordon equation using modified cubic B-spline-based differential quadrature method*, Comput. Methods Differ. Equ. **11** (2023), 369–386.
[3] M. Caputo, M. Fabrizio, *A new definition of fractional derivative without singular Kernel*, Progr. Fract. Differ. Appl. **2** (2015), 73–85.
[4] C. De Boor, *A practical guide to splines*, Springer-Verlag, 1978.
[5] C.A. Hall, W.W. Meyer, *Optimal error bounds for cubic spline interpolation*, J. Approx. Theory **16** (1976), 105–122.
[6] C.K. Chui, *An introduction to wavelets*, Elsevier, 2016.
[7] H. Dehestani, Y. Ordokhani, *A numerical study on fractional optimal control problems described by Caputo-Fabrizio fractional integro-differential equation*, Optim. Control Appl. Methods **44** (2023), 1873–1892.
[8] H. Dehestani, Y. Ordokhani, *Pell Lucas discretization method for finding the solution of CaputoFabrizio time-fractional diffusion equations*, Vietnam J. Math. **52** (2024), 235–254.
[9] H. Dehestani, Y. Ordokhani, *An efficient approach based on LegendreGaussLobatto quadrature and discrete shifted Hahn polynomials for solving CaputoFabrizio fractional Volterra partial integro-differential equations*, J. Comput. Appl. Math. **403** (2022), 113851.
[10] M. Dehghan, M. Lakestani, *The use of cubic B-spline scaling functions for solving the one-dimensional hyperbolic equation with a nonlocal conservation condition*, Numer. Methods Partial Differ. Equ. **23** (2007), 1277–1289.
[11] K. Diethelm, N.J. Ford, *The analysis of fractional differential equations*, Lecture Notes in Mathematics, Vol. 2004, Springer, 2010.
[12] M.A. Dokuyucu, E. Celik, H. Bulut, H.M. Baskonus, *Cancer treatment model with the Caputo-Fabrizio fractional derivative*, Eur. Phys. J. Plus **133** (2018), 1–6.
[13] K.T. Elgindy, *FourierGegenbauer pseudospectral method for solving periodic fractional optimal control problems*, Math. Comput. Simul. **225** (2024), 148–164.
[14] S.S. Ezz-Eldien, E.H. Doha, D. Baleanu, A.H. Bhrawy, *A numerical approach based on Legendre orthonormal polynomials for numerical solutions of fractional optimal control problems*, J. Vib. Control **23** (2017), 16–30.
[15] S. Ghaderi, A. Heydari, S. Effati, *Solving the fractional optimal control of a spring-mass-viscodamper system with Caputo-Fabrizio fractional operator*, Iran J. Sci. Technol. Trans. Sci. **45** (2021), 247–257.
[16] S. Ghaderi, S. Effati, A. Heydari, *A new numerical approach for solving fractional optimal control problems with the CaputoFabrizio fractional operator*, J. Math. **2022** (2022), 6680319.
[17] A. Hanif, A.I.K. Butt, W. Ahmad, *Numerical approach to solve Caputo-Fabrizio-fractional model of corona pandemic with optimal control design and analysis*, Math. Methods Appl. Sci. **46** (2023), 9751–9782.
[18] T.T. Hartley, C.F. Lorenzo, *A frequency-domain approach to optimal fractional-order damping*, Nonlinear Dyn. **38** (2004), 69–84.
[19] I. Ghosh, H.T. Cheong, K.L. Teo, *A novel gradient-based discrete time-delayed optimization algorithm for optimal control problems with CaputoFabrizio fractional derivative*, J. Comput. Appl. Math. (2025), 116526.
[20] J.C. Goswami, A.K. Chan, *Fundamentals of wavelets: theory, algorithms, and applications*, John Wiley & Sons, 2011.
[21] H. Hassani, J.A. Tenreiro Machado, E. Naraghirad, *Generalized shifted Chebyshev polynomials for fractional optimal control problems*, Commun. Nonlinear Sci. Numer. Simul. **75** (2019), 50–61.
[22] R. Hifler, *Applications of fractional calculus in physics*, World Scientific Publishing, 2000.
[23] S. Karami, A.F. Jahromi, M.H. Heydari, *A cardinal-based numerical method for fractional optimal control problems with CaputoKatugampola fractional derivative in a large domain*, Int. J. Syst. Sci. **55** (2024), 1719–1736.
[24] M.A. Khan, Z. Hammouch, D. Baleanu, *Modeling the dynamics of hepatitis E via the Caputo-Fabrizio derivative*, Math. Model. Nat. Phenom. **14** (2019), 56–85.
[25] F. Kheyrinataj, A. Nazemi, *On delay optimal control problems with a combination of conformable and Caputo-Fabrizio fractional derivatives via a fractional power series neural network*, Netw. Comput. Neural Syst. **33** (2022), 62–94.
[26] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, 2006.

[27] M. Lakestani, M. Dehghan, *Numerical solution of Fokker-Planck equation using the cubic B-spline scaling functions*, Numer. Methods Partial Differ. Equ. **25** (2009), 418–429.
[28] J. Losada, J.J. Nieto, *Properties of a new fractional derivative without singular kernel*, Progr. Fract. Differ. Appl. **1** (2015), 87–92.
[29] E.J. Moore, S. Sirisubtawee, S. Koonprasert, *A CaputoFabrizio fractional differential equation model for HIV/AIDS with treatment compartment*, Adv. Differ. Equ. **2019** (2019), 1–20.
[30] M. Mortezaee, M. Ghovatmand, A. Nazemi, *An application of generalized fuzzy hyperbolic model for solving fractional optimal control problems with CaputoFabrizio derivative*, Neural Process. Lett. **52** (2020), 1997–2020.
[31] A.N. Dalawi, M. Lakestani, E. Ashpazzadeh, *Solving fractional optimal control problems involving CaputoFabrizio derivative using Hermite spline functions*, Iran. J. Sci. **47** (2023), 545–566.
[32] P. Pandey, J.F. G-Aguilar, M.K. Kaabar, Z. Siri, A.M. Abd Allah, *Mathematical modeling of COVID-19 pandemic in India using Caputo-Fabrizio fractional derivative*, Comput. Biol. Med. **145** (2022), 105518.
[33] S. Sabermahani, Y. Ordokhani, *Solving distributed-order fractional optimal control problems via the Fibonacci wavelet method*, J. Vib. Control **30** (2024), 418–432.
[34] M. Sattari, M. Arab Ameri, *Compact finite difference scheme for numerical solution of Caputo-Fabrizio fractional Riccati differential equations*, Comput. Methods Differ. Equ. (In Press), DOI: 10.22034/cmde.2025.60832.2602.
[35] A. Singh, A. Kanaujiya, J. Mohapatra, *Fractional Legendre wavelet approach resolving multi-scale optimal control problems involving Caputo-Fabrizio derivative*, Numer. Algorithms (2024), 1–32.
[36] H.G. Sun, Y. Zhang, D. Baleanu, W. Chen, Y.Q. Chen, *A new collection of real-world applications of fractional calculus in science and engineering*, Commun. Nonlinear Sci. Numer. Simul. **64** (2018), 213–231.
[37] T.A. Yildiz, A. Jajarmi, B. Yildiz, D. Baleanu, *New aspects of time fractional optimal control problems within operators with nonsingular kernel*, Discrete Contin. Dyn. Syst. Ser. S **13** (2020), 407–428.