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Stancu-Jakimovski-Leviatan operators involving confluent appell polynomials

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Abstract. The Stancu-type generalization of Jakimovski-Leviatan operators, involving confluent Appell polynomials, and their approximation features are the focus of this paper. Moreover, the modulus of continuity and Peetre$-\mathcal{K}$ functional are used to determine the rate of convergence of the confluent Jakimovski-Leviatan operators. Next, we demonstrate that the newly created operators diminish confluent Bernoulli polynomials and confluent Hermite polynomials, under specific choices of $A(\psi)$. Lastly, we provide a graphic comparison between the newly created operators and the Stancu-type Jakimovski-Leviatan operators.

1. Introduction

A polynomial set is referred to be an Appell set [3] if the following criterion is fulfilled: A formal power series exists

$$A(\psi) = \sum_{\eta=0}^{\infty} \theta_\eta \frac{\psi^\eta}{\eta!}, \quad (A(0) \neq 0),$$

which is known as the deciding function that allows us to have

$$A(\psi) e^{x\psi} = \sum_{\eta=0}^{\infty} P_\eta(x) \frac{\psi^\eta}{\eta!}. \tag{1}$$

The series in (1) is presummated to be convergent in $|\psi| < r$ for some $r > 0$. The Appell polynomials additionally be explained in terms of $P'_\eta(x) = \eta P_{\eta-1}(x)$ recurrence formulas, where $\eta = 1, 2, ...$

Theorem 1.1. *Take into consideration the polynomial sequence $\left\{P_\eta^{(u,v)}(x)\right\}_{\eta\geq 0}$, with $v \notin \{..., -2, -1, 0\}$. Consequently, the subsequent claims are similar.*

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i) A confluent Appell polynomial sequence is $\left\{P_{\eta}^{(u,v)}(x)\right\}_{\eta\geq 0}$.

ii) The polynomial set $\left\{P_{\eta}^{(u,v)}(x)\right\}_{\eta\geq 0}$*'s generating function is given by*

$$\sum_{\eta=0}^{\infty} P_{\eta}^{(u,v)}(x)\frac{\psi^{\eta}}{\eta!} = A(\psi)\,{}_1F_1(u;v;x\psi), \tag{2}$$

where $\{\theta_{\zeta}\}$ *is a predetermined sequence of real numbers with* $\theta_0 \neq 0$, $A(\psi)$ *is an analytical function with a power series expansion of*

$$A(\psi) = \sum_{\zeta=0}^{\infty} \theta_{\zeta}\frac{\psi^{\zeta}}{\zeta!}, \tag{3}$$

and the confluent hypergeometric function is denoted by

$${}_1F_1(u;v;s) = \sum_{\eta=0}^{\infty} \frac{(u)_{\eta}}{(v)_{\eta}}\frac{s^{\eta}}{\eta!}. \tag{4}$$

For all finite s, this function converges, presuming $v \notin \{..., -2, -1, 0\}$ *[11]. From here, the definition of the Pocchammer symbol [11] is denoted by*

$$\begin{cases} (u)_{\eta} = u(u+1)...(u+\eta-1) & ; \quad \eta \geq 1 \\ (u)_0 = 1 & ; \quad 1. \end{cases}$$

Jakimovski and Leviatan [8] define the operators as

$$T_{\eta}(\tau;x) = \frac{e^{-\eta x}}{A(1)}\sum_{\zeta=0}^{\infty} P_{\zeta}(\eta x)\,\tau\left(\frac{\zeta}{\eta}\right).$$

Sucu and Varma [12] define the operators as

$$Z_{\eta}(\tau;x) = \frac{e^{-\eta x}}{A(1)}\sum_{\zeta=0}^{\infty} P_{\zeta}(\eta x)\,\tau\left(\frac{\zeta+\alpha}{\eta+\beta}\right).$$

Özarslan and Çekim [10] construct confluent Jakimovski-Leviatan operators as follows

$$L_{\eta}(\tau;x) = \frac{1}{A(1)\,{}_1F_1(u;v;\eta x)}\sum_{\zeta=0}^{\infty} \frac{P_{\zeta}^{(u,v)}(\eta x)}{\zeta!}\tau\left(\frac{\zeta}{\eta}\right),$$

$v > u > 0$, $x \geq 0$ and $\eta \in \mathbb{N}$. Moreover, it is anticipated that these operators satisfy the following criteria: $A(1) \neq 0$, $\frac{\theta_{\zeta}}{A(1)} \geq 0$, $(\zeta = 0, 1, 2...)$, for $|\psi| < R\,(R > 1)$, the power series at (2) and (3) converge.

There are different studies conducted in this area [13], [7], [1], [2] and [6].

Now, we define Stancu-type generalizations of confluent Jakimovski-Leviatan operators

$$J_{\eta}(\tau;x) = \frac{1}{A(1)\,{}_1F_1(u;v;\eta x)}\sum_{\zeta=0}^{\infty} \frac{P_{\zeta}^{(u,v)}(\eta x)}{\zeta!}\tau\left(\frac{\zeta+\alpha}{\eta+\beta}\right)$$

where $0 \leq \alpha \leq \beta$, $v > u > 0$, $\eta \in \mathbb{N}$, $x \geq 0$ and $P_{\zeta}^{(u,v)}$ is given in (2).

2. Approximation Properties

Lemma 2.1. *For any $x \in [0,\infty)$, we get*

$$
\begin{aligned}
J_\eta(1;x) &= 1,\\
J_\eta(\psi;x) &= \frac{u}{v}\frac{{}_1F_1(u+1;v+1;\eta x)}{{}_1F_1(u;v;\eta x)}\frac{\eta}{\eta+\beta}x+\frac{1}{\eta+\beta}\left(\frac{A'(1)}{A(1)}+\alpha\right),\\
J_\eta\left(\psi^2;x\right) &= \frac{u(u+1)}{v(v+1)}\frac{{}_1F_1(u+2;v+2;\eta x)}{{}_1F_1(u;v;\eta x)}\frac{\eta^2}{(\eta+\beta)^2}x^2\\
&\quad+\frac{u}{v}\frac{{}_1F_1(u+1;v+1;\eta x)}{{}_1F_1(u;v;\eta x)}\left(\frac{2A'(1)}{A(1)}+2\alpha\right)\frac{\eta}{(\eta+\beta)^2}x\\
&\quad+\frac{A''(1)+2\alpha A'(1)+\alpha^2A(1)}{A(1)(\eta+\beta)^2}.
\end{aligned}
$$

Proof. Utilizing the Eqn. (2)

$$
\begin{aligned}
\sum_{\zeta=0}^{\infty}\frac{P_\zeta^{(u,v)}(\eta x)}{\zeta!} &= A(1)\,{}_1F_1(u;v;\eta x),\\
\sum_{\zeta=0}^{\infty}\frac{P_{\zeta+1}^{(u,v)}(\eta x)}{\zeta!} &= A'(1)\,{}_1F_1(u;v;\eta x)+\frac{u}{v}\eta x A(1)\,{}_1F_1(u+1;v+1;\eta x),\\
\sum_{\zeta=0}^{\infty}\frac{P_{\zeta+2}^{(u,v)}(\eta x)}{\zeta!} &= A''(1)\,{}_1F_1(u;v;\eta x)+A'(1)\,{}_1F_1(u;v;\eta x)+2\frac{u}{v}\eta x A'(1)\,{}_1F_1(u+1;v+1;\eta x)\\
&\quad+\frac{u(u+1)}{v(v+1)}(\eta x)^2A(1)\,{}_1F_1(u+2;v+2;\eta x),
\end{aligned}
$$

the proof may be demonstrated. To obtain the intended outcome, we apply these equalities in the operator. □

Theorem 2.2. *[9] For $\tau \in C[0,\infty)$, then*

$$\lim_{\eta\to\infty} J_\eta(\tau;x)=\tau(x)$$

uniformly converge in each compact subset of $[0,\infty)$.

Proof. From Lemma 2.1 and (4), we get

$$\lim_{\eta\to\infty} J_\eta(e_\iota;x)=e_\iota(x),\quad \iota=0,1,2.$$

Thus, the proof is finished using the well-known Korovkin theorem [9]. □

Lemma 2.3. *We have*

$$
\begin{aligned}
J_\eta(\psi-x;x) &= \left(\frac{u}{v}\frac{{}_1F_1(u+1;v+1;\eta x)}{{}_1F_1(u;v;\eta x)}\frac{\eta}{\eta+\beta}-1\right)x+\frac{1}{\eta+\beta}\left(\frac{A'(1)}{A(1)}+\alpha\right),\\
J_\eta\left((\psi-x)^2;x\right) &= \left(\frac{u(u+1)}{v(v+1)}\frac{{}_1F_1(u+2;v+2;\eta x)}{{}_1F_1(u;v;\eta x)}\frac{\eta^2}{(\eta+\beta)^2}-2\frac{u}{v}\frac{{}_1F_1(u+1;v+1;\eta x)}{{}_1F_1(u;v;\eta x)}+1\right)x^2\\
&\quad+\left(\frac{u}{v}\frac{{}_1F_1(u+1;v+1;\eta x)}{{}_1F_1(u;v;\eta x)}\left(\frac{2A'(1)}{A(1)}+2\alpha\right)\frac{\eta}{(\eta+\beta)^2}-\frac{2}{\eta+\beta}\left(\frac{A'(1)}{A(1)}+\alpha\right)\right)x\\
&\quad+\frac{A''(1)+2\alpha A'(1)+\alpha^2A(1)}{A(1)(\eta+\beta)^2}.
\end{aligned}
$$

Proof. From Lemma 2.1 and the linearity of the operator

$$\begin{aligned} J_\eta(\psi - x; x) &= J_\eta(e_1; x) - xJ_\eta(e_0; x), \\ J_\eta\left((\psi - x)^2; x\right) &= J_\eta(e_2; x) - 2xJ_\eta(e_1; x) + x^2 J_\eta(e_0; x). \end{aligned}$$

The proof is finished using these equalities. □

3. Rate of Convergence

$$\omega(\tau, \delta) := \sup_{x\in[0,\infty)} \sup_{|\psi-x|\le\delta} \left|\tau(\psi) - \tau(x)\right|, \ \ \tau \in C[0,\infty), \ \ \delta > 0$$

is expression of the modulus of continuity. That results from the modulus of continuity's property listed below

$$\left|\tau(\psi) - \tau(x)\right| \le \left(\frac{|\psi - x|}{\delta} + 1\right)\omega(\tau, \delta).$$

Theorem 3.1. $\tau \in C[0,\infty)$ *and for every* $x \in [0,\infty)$,

$$\left|J_\eta(\tau; x) - \tau(x)\right| \le 2\omega\left(\tau, \delta_\eta\right).$$

Here,

$$\begin{aligned} \delta_\eta(x) &= \left\{\left(\frac{u(u+1)}{v(v+1)} \frac{{}_1F_1(u+2; v+2; \eta x)}{{}_1F_1(u; v; \eta x)} \frac{\eta^2}{(\eta+\beta)^2} - 2\frac{u}{v}\frac{{}_1F_1(u+1; v+1; \eta x)}{{}_1F_1(u; v; \eta x)} + 1\right)x^2 \right. \\ &\quad + \left(\frac{u}{v}\frac{{}_1F_1(u+1; v+1; \eta x)}{{}_1F_1(u; v; \eta x)}\left(\frac{2A'(1)}{A(1)} + 2\alpha\right)\frac{\eta}{(\eta+\beta)^2} - \frac{2}{\eta+\beta}\left(\frac{A'(1)}{A(1)} + \alpha\right)\right)x \\ &\quad \left. + \frac{A''(1) + 2\alpha A'(1) + \alpha^2 A(1)}{A(1)(\eta+\beta)^2}\right\}^{\frac{1}{2}}. \end{aligned} \tag{5}$$

Proof. Utilizing linearity of the operators J_η, we get

$$\begin{aligned} J_\eta\left(\left|\tau(\psi) - \tau(x)\right|; x\right) &\le \left\{J_\eta(1; x) + \frac{1}{\delta}J_\eta\left(|\psi - x|; x\right)\right\}\omega(\tau, \delta), \\ &\le \left\{1 + \frac{1}{\delta}J_\eta\left(|\psi - x|; x\right)\right\}\omega(\tau, \delta), \\ &\le \left\{1 + \frac{1}{\delta}\left(J_\eta(1; x)\right)^{\frac{1}{2}}\left(J_\eta\left((\psi - x)^2; x\right)\right)^{\frac{1}{2}}\right\}\omega(\tau, \delta), \\ &= \left\{\frac{1}{\delta}\left(\delta_\eta(x)\right)^{\frac{1}{2}} + 1\right\}\omega(\tau, \delta) \end{aligned} \tag{6}$$

where $\delta_\eta(x)$ is given by (5). Taking into account this inequality in (6) leads to

$$\left|J_\eta(\tau; x) - \tau(x)\right| \le \left\{\frac{1}{\delta}\sqrt{\delta_\eta(x)} + 1\right\}\omega(\tau, \delta).$$

We get the intended outcome if we select $\delta = \sqrt{\delta_\eta(x)}$. □

Lemma 3.2. $\tau \in C_B[0,\infty)$ *and for* $x \in [0,\infty)$, *we have*

$$\left|J_\eta(\tau; x)\right| \le \|\tau\|.$$

Proof. For J_η, we get

$$\begin{aligned}
\left|J_\eta(\tau;x)\right| &= \left|\frac{1}{A(1)\,{}_1F_1(u;v;\eta x)}\sum_{\zeta=0}^{\infty}\frac{P_\zeta^{(u,v)}(\eta x)}{\zeta!}\tau\left(\frac{\zeta+\alpha}{\eta+\beta}\right)\right| \\
&\le \frac{1}{A(1)\,{}_1F_1(u;v;\eta x)}\sum_{\zeta=0}^{\infty}\frac{P_\zeta^{(u,v)}(\eta x)}{\zeta!}\left|\tau\left(\frac{\zeta+\alpha}{\eta+\beta}\right)\right| \\
&\le \|\tau\|\,J_\eta(1;x) \\
&\le \|\tau\|.
\end{aligned}$$

□

The space of functions τ, for which τ, τ', and τ'' are continuous on $[0,\infty)$, is denoted by $C_B^2[0,\infty)$.

$$\|\kappa\|_{C_B^2[0,\infty)} := \|\kappa\|_{C_B[0,\infty)} + \|\kappa'\|_{C_B[0,\infty)} + \|\kappa''\|_{C_B[0,\infty)}$$

gives the norm on the space $C_B^2[0,\infty)$ [5]. At this point, we describe the Peetre-$\mathcal{K}$ functional as

$$\mathcal{K}(\tau,\lambda) := \inf_{\kappa\in C_B^2[0,\infty)}\left\{\|\tau-\kappa\| + \lambda\,\|\kappa''\|\right\}, \qquad \lambda > 0.$$

Theorem 3.3. *Let $\tau \in C_B[0,\infty)$ and $x \in [0,\infty)$. Next for all $\eta \in \mathbb{N}$, we have*

$$\left|J_\eta(\tau;x) - \tau(x)\right| \le 2\mathcal{K}\left(\tau;\lambda_\eta(x)\right),$$

where

$$\lambda_\eta(x) = \left(\frac{A''(1) + 2(\eta+\alpha)A'(1) + \left((\eta+\alpha)^2 + 2\eta(\eta+\beta)\right)A(1)}{2A(1)(\eta+\beta)^2}\right).$$

Proof. For a given function $\kappa \in C_B^2[0,\infty)$, we get the following Taylor expansion

$$\kappa(\psi) = \kappa(x) + (\psi-x)\kappa'(x) + \int_x^\psi (\psi-\xi)\kappa''(\xi)d\xi. \tag{7}$$

Applying J_η operator to the equation (7), we get

$$\begin{aligned}
\left|J_\eta(\kappa;x) - \kappa(x)\right| &= \left|J_\eta\left((\psi-x)\kappa'(x);x\right)\right| + \left|J_\eta\left(\int_x^\psi(\psi-\xi)\kappa''(\xi)d\xi;x\right)\right| \\
&\le \|\kappa'\|_{C_B[0,\infty)}\left|J_\eta(\psi-x;x)\right| + \|\kappa''\|_{C_B[0,\infty)}\left|J_\eta\left(\int_x^\psi(\psi-\xi)d\xi;x\right)\right| \\
&\le \|\kappa'\|_{C_B[0,\infty)}\left|J_\eta(\psi-x;x)\right| + \frac{1}{2}\|\kappa''\|_{C_B[0,\infty)}J_\eta\left((\psi-x)^2;x\right).
\end{aligned}$$

So,

$$\left|J_\eta(\kappa;x) - \kappa(x)\right| \le \lambda_\eta\|\kappa\|_{C_B^2[0,\infty)}.$$

Using the disparity mentioned above, we get

$$\begin{aligned}
\left|J_\eta(\tau;x) - \tau(x)\right| &= \left|J_\eta(\tau;x) - \tau(x) + J_\eta(\kappa;x) - J_\eta(\kappa;x) + \kappa(x) - \kappa(x)\right| \\
&\le \|\tau-\kappa\|_{C_B[0,\infty)}\left|J_\eta(1;x)\right| + \|\tau-\kappa\|_{C_B[0,\infty)} + \left|J_\eta(\kappa;x) - \kappa(x)\right| \\
&\le 2\left(\|\tau-\kappa\|_{C_B[0,\infty)} + \lambda_\eta\,\|\kappa\|_{C_B^2[0,\infty)}\right) \\
&\le 2\mathcal{K}\left(\tau;\lambda_\eta\right).
\end{aligned}$$

As a result,

$$\left|J_\eta(\tau;x)-\tau(x)\right|\leq 2\mathcal{K}\left(\tau;\lambda_\eta\right). \tag{8}$$

Thus the proof is completed. □

The second modulus of continuity of the function is defined by

$$\omega_2(\tau,\delta)=\sup_{0<\psi\leq\delta}\left\|\tau(.+2\psi)-2\tau(.+\psi)+\tau(.)\right\|_{C_B[0,\infty)},\quad \tau\in C_B[0,\infty).$$

The second modulus of continuity and the Peetre-$\mathcal{K}$ functional are related in the following way:

$$\mathcal{K}(\tau;\delta)\leq M\left(\min(1,\delta)\|\tau\|_{C_B[0,\infty)}+\omega_2\left(\tau,\sqrt{\delta}\right)\right), \tag{9}$$

where the constant M is independent of the values of δ and τ from [4].

From (8) and (9),

$$\left|J_\eta(\tau;x)-\tau(x)\right|\leq 2M\left(\omega_2\left(\tau,\sqrt{\lambda_\eta}\right)+\min\left(1,\lambda_\eta\right)\|\tau\|_{C_B[0,\infty)}\right).$$

4. Special Cases

In this section, confluent Hermite polynomials and confluent Bernoulli polynomials are obtained by selecting $A(\psi)=e^{-\frac{\psi^2}{2}}$ and $A(\psi)=\frac{\psi}{e^\psi-1}$ in (2).

4.1. The Confluent Bernoulli Polynomials

Choosing $A(\psi)=\frac{\psi}{e^\psi-1}$ in (2) we get confluent Bernoulli polynomials. The generating function of the confluent Bernoulli polynomials is

$$\frac{\psi}{e^\psi-1}{}_1F_1(u;v;x\psi)=\sum_{\zeta=0}^{\infty}\mathcal{B}_\zeta^{(u,v)}(x)\frac{\psi^\zeta}{\zeta!},\quad \left|\psi\right|<\pi,\ \ v\notin\{...,-2,-1,0\}.$$

The Stancu-Jakimovski-Leviatan operators involving confluent Bernoulli polynomials are represented as

$$J_\eta^{\mathcal{B}}(\tau;x)=\frac{(e-1)}{{}_1F_1(u;v;\eta x)}\sum_{\zeta=0}^{\infty}\frac{\mathcal{B}_\zeta^{(u,v)}(\eta x)}{\zeta!}\tau\left(\frac{\zeta+\alpha}{\eta+\beta}\right).$$

Now, we give approximation properties of our operator.

Lemma 4.1. *For any $x\in[0,\infty)$, we get*

$$\begin{aligned}
J_\eta^{\mathcal{B}}(1;x) &= 1,\\
J_\eta^{\mathcal{B}}(\psi;x) &= \frac{u}{v}\frac{{}_1F_1(u+1;v+1;\eta x)}{{}_1F_1(u;v;\eta x)}\frac{\eta}{\eta+\beta}x+\frac{1}{\eta+\beta}\left(\alpha-\frac{1}{e-1}\right),\\
J_\eta^{\mathcal{B}}\left(\psi^2;x\right) &= \frac{u(u+1)}{v(v+1)}\frac{{}_1F_1(u+2;v+2;\eta x)}{{}_1F_1(u;v;\eta x)}\frac{\eta^2}{(\eta+\beta)^2}x^2\\
&\quad+\frac{u}{v}\frac{{}_1F_1(u+1;v+1;\eta x)}{{}_1F_1(u;v;\eta x)}\left(2\alpha-\frac{2}{e-1}\right)\frac{\eta}{(\eta+\beta)^2}x\\
&\quad+\frac{-e^2+3e-2\alpha(e-1)+\alpha^2(e-1)^2}{(\eta+\beta)^2(e-1)^2}.
\end{aligned}$$

Lemma 4.2. *We have*

$$\begin{aligned} J_{\eta}^{\mathcal{B}}\left(\psi-x;x\right) &= \left(\frac{u}{v}\frac{{}_1F_1\left(u+1;v+1;\eta x\right)}{{}_1F_1\left(u;v;\eta x\right)}\frac{\eta}{\eta+\beta}-1\right)x+\frac{1}{\eta+\beta}\left(\alpha-\frac{1}{e-1}\right), \\ J_{\eta}^{\mathcal{B}}\left(\left(\psi-x\right)^2;x\right) &= \left(\frac{u\left(u+1\right)}{v\left(v+1\right)}\frac{{}_1F_1\left(u+2;v+2;\eta x\right)}{{}_1F_1\left(u;v;\eta x\right)}\frac{\eta^2}{\left(\eta+\beta\right)^2}-2\frac{u}{v}\frac{{}_1F_1\left(u+1;v+1;\eta x\right)}{{}_1F_1\left(u;v;\eta x\right)}\frac{\eta}{\eta+\beta}+1\right)x^2 \\ &\quad+\left(\frac{u}{v}\frac{{}_1F_1\left(u+1;v+1;\eta x\right)}{{}_1F_1\left(u;v;\eta x\right)}\left(2\alpha-\frac{2}{e-1}\right)\frac{\eta}{\left(\eta+\beta\right)^2}-\frac{2}{\eta+\beta}\left(\alpha-\frac{1}{e-1}\right)\right)x \\ &\quad+\frac{-e^2+3e-2\alpha\left(e-1\right)+\alpha^2\left(e-1\right)^2}{\left(\eta+\beta\right)^2\left(e-1\right)^2}. \end{aligned}$$

Theorem 4.3. *For $\tau \in C[0,\infty)$ and every $x \in [0,\infty)$,*

$$\left|J_{\eta}^{\mathcal{B}}\left(\tau;x\right)-\tau\left(x\right)\right| \leq 2\omega\left(\tau,\vartheta_{\eta}\right).$$

Here,

$$\begin{aligned} \vartheta_{\eta}\left(x\right) &= \left\{\left(\frac{u\left(u+1\right)}{v\left(v+1\right)}\frac{{}_1F_1\left(u+2;v+2;\eta x\right)}{{}_1F_1\left(u;v;\eta x\right)}\frac{\eta^2}{\left(\eta+\beta\right)^2}-2\frac{u}{v}\frac{{}_1F_1\left(u+1;u+1;\eta x\right)}{{}_1F_1\left(u;v;\eta x\right)}\frac{\eta}{\eta+\beta}+1\right)x^2\right. \\ &\quad+\left(\frac{u}{v}\frac{{}_1F_1\left(u+1;v+1;\eta x\right)}{{}_1F_1\left(u;v;\eta x\right)}\left(2\alpha-\frac{2}{e-1}\right)\frac{\eta}{\left(\eta+\beta\right)^2}-\frac{2}{\eta+\beta}\left(\alpha-\frac{1}{e-1}\right)\right)x \\ &\quad\left.+\frac{-e^2+3e-2\alpha\left(e-1\right)+\alpha^2\left(e-1\right)^2}{\left(\eta+\beta\right)^2\left(e-1\right)^2}\right\}^{\frac{1}{2}}. \end{aligned} \tag{10}$$

Proof. By utilizing linearity of the operators, $J_{\eta}^{\mathcal{B}}$, we get

$$\begin{aligned} \left|J_{\eta}^{\mathcal{B}}\left(\tau\left(\psi\right)-\tau\left(x\right);x\right)\right| &\leq J_{\eta}^{\mathcal{B}}\left(\left|\tau\left(\psi\right)-\tau\left(x\right)\right|;x\right) \\ &\leq \left\{J_{\eta}^{\mathcal{B}}\left(1;x\right)+\frac{1}{\delta}J_{\eta}^{\mathcal{B}}\left(\left|\psi-x\right|;x\right)\right\}\omega\left(\tau,\delta\right), \\ &\leq \left\{1+\frac{1}{\delta}J_{\eta}^{\mathcal{B}}\left(\left|\psi-x\right|;x\right)\right\}\omega\left(\tau,\delta\right), \\ &\leq \left\{1+\frac{1}{\delta}\left(J_{\eta}^{\mathcal{B}}\left(\left(\psi-x\right)^2;x\right)\right)^{\frac{1}{2}}\left(J_{\eta}^{\mathcal{B}}\left(1;x\right)\right)^{\frac{1}{2}}\right\}\omega\left(\tau,\delta\right), \\ &= \left\{1+\frac{1}{\delta}\left(\vartheta_{\eta}\left(x\right)\right)^{\frac{1}{2}}\right\}\omega\left(\tau,\delta\right). \end{aligned} \tag{11}$$

where $\vartheta_{\eta}(x)$ is given by (10). Taking into account this inequality in (11) leads to

$$\left|J_{\eta}^{\mathcal{B}}\left(\tau;x\right)-\tau\left(x\right)\right| \leq \left\{1+\frac{1}{\delta}\sqrt{\vartheta_{\eta}\left(x\right)}\right\}\omega\left(\tau,\delta\right).$$

If we select $\delta = \sqrt{\vartheta_{\eta}(x)}$, we obtain the desired consequence. □

4.2. The Confluent Hermite Polynomials

Choosing $A(\psi) = e^{-\frac{\psi^2}{2}}$ in (2), then we get confluent Hermite polynomials. The generating function of the confluent Hermite polynomials is

$$e^{-\frac{\psi^2}{2}}{}_1F_1\left(u;v;x\psi\right)=\sum_{\zeta=0}^{\infty}\mathcal{H}_{\zeta}^{(u,v)}\left(x\right)\frac{\psi^{\zeta}}{\zeta!},$$

where $v \notin \{..., -2, -1, 0\}$.
The Stancu-Jakimovski-Leviatan operators involving confluent Hermite polynomials are presented as

$$J_{\eta}^{\mathcal{H}}(\tau; x) = \frac{e^{\frac{1}{2}}}{{}_1F_1(u; v; \eta x)} \sum_{\zeta=0}^{\infty} \frac{\mathcal{H}_{\zeta}^{(u,v)}(\eta x)}{\zeta!} \tau\left(\frac{\zeta+\alpha}{\eta+\beta}\right).$$

Now, we give approximation properties of our operator.

Lemma 4.4. *For $x \in [0, \infty)$, we get*

$$\begin{aligned}
J_{\eta}^{\mathcal{H}}(1; x) &= 1, \\
J_{\eta}^{\mathcal{H}}(\psi; x) &= \frac{u}{v} \frac{{}_1F_1(u+1; v+1; \eta x)}{{}_1F_1(u; v; \eta x)} \frac{\eta}{\eta+\beta} x + \frac{\alpha-1}{\eta+\beta}, \\
J_{\eta}^{\mathcal{H}}\left(\psi^2; x\right) &= \frac{u(u+1)}{v(v+1)} \frac{{}_1F_1(u+2; v+2; \eta x)}{{}_1F_1(u; v; \eta x)} \frac{\eta^2}{(\eta+\beta)^2} x^2 \\
&\quad + (2\alpha-2) \frac{u}{v} \frac{{}_1F_1(u+1; v+1; \eta x)}{{}_1F_1(u; v; \eta x)} \frac{\eta}{(\eta+\beta)^2} x + \frac{\alpha^2-2\alpha}{(\eta+\beta)^2}.
\end{aligned}$$

Lemma 4.5. *For every $x \in [0, \infty)$, the following identities verify*

$$\begin{aligned}
J_{\eta}^{\mathcal{H}}(\psi - x; x) &= \left(\frac{u}{v} \frac{{}_1F_1(u+1; v+1; \eta x)}{{}_1F_1(u; v; \eta x)} \frac{\eta}{\eta+\beta} - 1\right) x + \frac{\alpha-1}{\eta+\beta}, \\
J_{\eta}^{\mathcal{H}}\left((\psi - x)^2; x\right) &= \left(\frac{u(u+1)}{v(v+1)} \frac{{}_1F_1(u+2; v+2; \eta x)}{{}_1F_1(u; v; \eta x)} \frac{\eta^2}{(\eta+\beta)^2} - 2\frac{u}{v} \frac{{}_1F_1(u+1; v+1; \eta x)}{{}_1F_1(u; v; \eta x)} \frac{\eta}{\eta+\beta} + 1\right) x^2 \\
&\quad + \left(\frac{u}{v} \frac{{}_1F_1(u+1; v+1; \eta x)}{{}_1F_1(u; v; \eta x)} (2\alpha-2) \frac{\eta}{(\eta+\beta)^2} - \frac{2\alpha-2}{\eta+\beta}\right) x + \frac{\alpha^2-2\alpha}{(\eta+\beta)^2}.
\end{aligned}$$

Theorem 4.6. *For $\tau \in C[0, \infty)$ and every $x \in [0, \infty)$,*

$$\left|J_{\eta}^{\mathcal{H}}(\tau; x) - \tau(x)\right| \leq 2\omega\left(\tau, \gamma_{\eta}\right).$$

Here,

$$\begin{aligned}
\gamma_{\eta}(x) &= \left(\frac{u(u+1)}{v(v+1)} \frac{{}_1F_1(u+2; v+2; \eta x)}{{}_1F_1(u; v; \eta x)} \frac{\eta^2}{(\eta+\beta)^2} - 2\frac{u}{v} \frac{{}_1F_1(u+1; v+1; \eta x)}{{}_1F_1(u; v; \eta x)} \frac{\eta}{\eta+\beta} + 1\right) x^2 \\
&\quad + \left(\frac{u}{v} \frac{{}_1F_1(u+1; v+1; \eta x)}{{}_1F_1(u; v; \eta x)} (2\alpha-2) \frac{\eta}{(\eta+\beta)^2} - \frac{2\alpha-2}{\eta+\beta}\right) x + \frac{\alpha^2-2\alpha}{(\eta+\beta)^2}. \qquad (12)
\end{aligned}$$

Proof. Using linearity of the operators $J_{\eta}^{\mathcal{H}}$, we obtain

$$\begin{aligned}
\left|J_{\eta}^{\mathcal{H}}(\tau(\psi) - \tau(x); x)\right| &\leq J_{\eta}^{\mathcal{H}}\left(\left|\tau(\psi) - \tau(x)\right|; x\right) \\
&\leq \left\{J_{\eta}^{\mathcal{H}}(1; x) + \frac{1}{\delta} J_{\eta}^{\mathcal{H}}\left(\left|\psi - x\right|; x\right)\right\} \omega(\tau, \delta), \\
&\leq \left\{1 + \frac{1}{\delta} J_{\eta}^{\mathcal{H}}\left(\left|\psi - x\right|; x\right)\right\} \omega(\tau, \delta), \\
&\leq \left\{1 + \frac{1}{\delta}\left(J_{\eta}^{\mathcal{H}}\left((\psi - x)^2; x\right)\right)^{\frac{1}{2}} \left(J_{\eta}^{\mathcal{H}}(1; x)\right)^{\frac{1}{2}}\right\} \omega(\tau, \delta), \\
&= \left\{1 + \frac{1}{\delta}\left(\gamma_{\eta}(x)\right)^{\frac{1}{2}}\right\} \omega(\tau, \delta). \qquad (13)
\end{aligned}$$

where $\gamma_\eta(x)$ is given by (12). Taking into account this inequality in (13) leads to

$$\left|J_\eta^{\mathcal{H}}(\tau;x)-\tau(x)\right|\leq\left\{1+\frac{1}{\delta}\sqrt{\vartheta_\eta(x)}\right\}\omega(\tau,\delta).$$

We get the intended outcome if we select $\delta=\sqrt{\vartheta_\eta(x)}$. □

5. Graphical Analysis

This section we will study at the approximations of the newly constructed confluent Stancu-Jakimovski-Leviatan operators and Jakimovski-Leviatan operators to a function τ.
Allow the function τ to be

$$\tau(x)=e^{-2x}.$$

For $A(\psi)=1$, we conspire the convergence of T_η Jakimovski-Leviatan operators and the newly constructed J_η confluent Stancu-Jakimovski-Leviatan operators and to the function τ in Figure 1. In Figure 1, we give illustration for choosed values $u=1$, $v=1.2$, $\alpha=1$, $\beta=1.5$ and $\eta=45$.

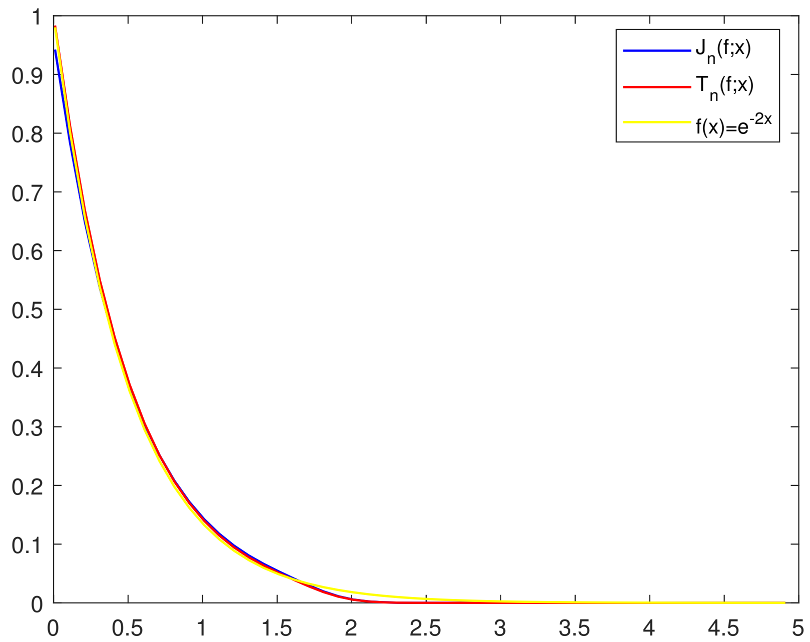

Figure 1: An illustration of the function $\tau(x)=e^{-2x}$ approximation for selected values $\alpha=1$, $\beta=1.5$, $u=1$, $v=1.2$ and $\eta=45$.

6. Conclusion

In this paper, Stancu-type generalization of confluent Jakimovski-Leviatan operators is defined. The central moments of the newly created operators J_η are acquired. Moreover, the rate of convergence is researched by utilizing Peetre$-\mathcal{K}$ functional and the modulus of continuity. The newly constructed operators' correspondence with $\mathcal{B}_\zeta^{(u,v)}$ and $\mathcal{H}_\zeta^{(u,v)}$ are given, respectively.
Eventually, the convergence of the confluent Stancu-Jakimovski-Leviatan J_η and the classical Jakimovski-Leviatan operators T_η to the choosed functions are visualized. Numerical examples provide a comparison of convergence.

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