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On the positive cone of rings of measurable functions

Pronay Biswas[a], Sagarmoy Bag[b], Sujit Kumar Sardar[a,*]

[a]*Department of Mathematics, Jadavpur University, 188, Raja Subodh Chandra Mallick Rd, Kolkata 700032, West Bengal*
[b]*Department of Mathematics, Bankim Sardar College, Tangrakhali, West Bengal 743329, India*

Abstract. For a measurable space $(X, \mathcal{A})$, let $\mathcal{M}^+(X, \mathcal{A})$ be the commutative semiring of non-negative real-valued measurable functions with pointwise addition and pointwise multiplication. We show that there is a lattice isomorphism between the ideal lattice of $\mathcal{M}^+(X, \mathcal{A})$ and the ideal lattice of its ring of differences $\mathcal{M}(X, \mathcal{A})$. Moreover, we infer that each ideal of $\mathcal{M}^+(X, \mathcal{A})$ is a semiring z-ideal. We investigate the duality between cancellative congruences on $\mathcal{M}^+(X, \mathcal{A})$ and $Z_{\mathcal{A}}$-filters on X. We observe that every σ-algebra is a completely regular σ-frame, so compactness and pseudocompactness coincide in σ-algebras, and we provide a new characterization for compact measurable spaces via algebraic properties of $\mathcal{M}^+(X, \mathcal{A})$. It is shown that the space of (real) maximal congruences on $\mathcal{M}^+(X, \mathcal{A})$ is homeomorphic to the space of (real) maximal ideals of the $\mathcal{M}(X, \mathcal{A})$. We solve the isomorphism problem for the semirings of the form $\mathcal{M}^+(X, \mathcal{A})$ for compact and realcompact measurable spaces.

1. Introduction

In what follows, the pair $(X, \mathcal{A})$ stands for a nonempty set X with a σ-algebra $\mathcal{A}$ on X. We call $(X, \mathcal{A})$ a measurable space. A σ-algebra $\mathcal{A}$ is said to separate points if for any two distinct points $x, y \in X$, we get $A \in \mathcal{A}$ such that $x \in A$ and $y \notin A$. Unless otherwise stated, by a measurable space we shall always mean a *T-measurable space* ([11]), that is, $\mathcal{A}$ separates points of X. A function $f : X \to \mathbb{R}$ is said to be $\mathcal{A}$-measurable (or measurable) if $f^{-1}(\mathfrak{O}) \in \mathcal{A}$, where $\mathfrak{O}$ is any open set in $\mathbb{R}$. The collection of all real-valued measurable functions on $(X, \mathcal{A})$, denoted by $\mathcal{M}(X, \mathcal{A})$, with pointwise addition and pointwise multiplication, forms a commutative lattice-ordered ring with unity.

In this paper, we initiate a study of the *positive cone* (the set of all non-negative elements) of the ring $\mathcal{M}(X, \mathcal{A})$, which we denote by $\mathcal{M}^+(X, \mathcal{A})$. The set $\mathcal{M}^+(X, \mathcal{A})$ forms a commutative lattice-ordered semiring with the usual operations. One of the main objectives of this paper is to construct various bridges between the ideals and congruences of the ring $\mathcal{M}(X, \mathcal{A})$ and congruences of the semiring $\mathcal{M}^+(X, \mathcal{A})$.

In some recent papers like [1, 4, 11], the ring $\mathcal{M}(X, \mathcal{A})$ has been studied extensively. It is easy to show that $\mathcal{M}(X, \mathcal{A})$ is always a von Neumann regular ring. Therefore each ideal of $\mathcal{M}(X, \mathcal{A})$ is a z-ideal in the sense

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Communicated by Pratulananda Das
* Corresponding author: Sujit Kumar Sardar
Email addresses: pronayb.math.rs@jadavpuruniversity.in, pronaybiswas1729@gmail.com (Pronay Biswas), sagarmoy.bag01@gmail.com (Sagarmoy Bag), sksardarjumath@gmail.com, sujitk.sardar@jadavpuruniversity.in (Sujit Kumar Sardar)
ORCID iDs: https://orcid.org/0009-0008-2710-1748 (Pronay Biswas), https://orcid.org/0009-0007-7235-0180 (Sagarmoy Bag), https://orcid.org/0000-0001-7837-0835 (Sujit Kumar Sardar)

of Mason [13]. Estaji et. al. [11] gave a complete description of the maximal ideals in $\mathcal{M}(X, \mathcal{A})$ in terms of the lattice-theoretic aspects of $\mathcal{A}$. They have solved the isomorphism problem of rings like $\mathcal{M}(X, \mathcal{A})$ in the category of compact measurable spaces. Acharyya et. al. [4] provided an alternative method to describe the maximal ideals via the space of all $Z_{\mathcal{A}}$-ultrafilters on X. They showed that the structure space of the ring $\mathcal{M}(X, \mathcal{A})$ is zero-dimensional. In [1], a proof of the isomorphism problem for rings like $\mathcal{M}(X, \mathcal{A})$ is given, in the category of realcompact measurable spaces. In this paper, we try to embark on an alternative study of compact measurable spaces and realcompact measurable spaces in a more congruence-theoretic slant.

In section 3, we provide a complete description of ideals of the semiring $\mathcal{M}^+(X, \mathcal{A})$. A major achievement of this section is an isomorphism between the ideal lattices of the semiring $\mathcal{M}^+(X, \mathcal{A})$ and the ring $\mathcal{M}(X, \mathcal{A})$. Whence each ideal in $\mathcal{M}^+(X, \mathcal{A})$ is of the form $I \cap \mathcal{M}^+(X, \mathcal{A})$, for some ideal I of the ring $\mathcal{M}(X, \mathcal{A})$. Under this circumstance, each ideal of $\mathcal{M}^+(X, \mathcal{A})$ turns out to be a z-ideal in the sense of [7].

In section 4 we study the interplay between cancellative congruences on $\mathcal{M}^+(X, \mathcal{A})$ and $Z_{\mathcal{A}}$-filters on X. We define z-congruence on both $\mathcal{M}^+(X, \mathcal{A})$ and $\mathcal{M}(X, \mathcal{A})$, which are heavily related to the concept of zero-sets. As anticipated, it turns out that in the ring $\mathcal{M}(X, \mathcal{A})$, there is a one-one correspondence between z-ideals and z-congruences. Consequently, we give a purely algebraic description of z-congruences on $\mathcal{M}^+(X, \mathcal{A})$.

In section 5, by exploiting the duality of maximal congruences and $Z_{\mathcal{A}}$-ultrafilters we show that the structure space of the semiring $\mathcal{M}^+(X, \mathcal{A})$ is homeomorphic to the structure space of the ring $\mathcal{M}(X, \mathcal{A})$. In Theorem 5.7 we unify Theorem 2.13 of [4] and Proposition 4.11 of [11] by showing that $(X, \mathcal{A})$ is a compact measurable space if and only if $\mathcal{A}$ is a finite σ-algebra if and only if each maximal congruence on $\mathcal{M}^+(X, \mathcal{A})$ is fixed. In Remark 5.8 we observe an interesting fact that in the case of σ-algebras (viz. σ-frames), the concepts of compactness and pseudocompactness coincide.

In section 6 our purpose is twofold. First, we initiate a study on quotients of the semiring $\mathcal{M}^+(X, \mathcal{A})$. We show that $\mathcal{M}^+(X, \mathcal{A})/\rho$ is a totally ordered semiring if ρ is a maximal congruence and the quotient semiring $\mathcal{M}^+(X, \mathcal{A})/\rho$ is either isomorphic to or, it properly contains the semifield of non-negative reals. This leads us to the definition of real maximal congruences. We observe that the collection of all real maximal congruences, denoted by $\mathcal{RMC}ong(\mathcal{M}^+(X, \mathcal{A}))$, can be perceived dually as a topological space with the Stone topology and as T-measurable space. In both cases $\mathcal{RMC}ong(\mathcal{M}^+(X, \mathcal{A}))$ is homeomorphic to $RMax(X, \mathcal{A})$ (set of all real maximal ideals of $\mathcal{M}(X, \mathcal{A})$) as a topological space and as a measurable space. Lastly, we solve the isomorphism problem for the semiring of the form $\mathcal{M}^+(X, \mathcal{A})$ in the category of realcompact measurable spaces.

2. Preliminaries

To make this article self-contained, we recall some basics from semiring theory.

A *semiring* S is a non-empty set with two binary operations $+$ and $\cdot$ such that $(S, +)$ and $(S, \cdot)$ are commutative monoids and $(a + b) \cdot c = a \cdot c + b \cdot c$ and $a \cdot 0 = 0$, for all $a, b, c \in S$.

A semiring S is said to be an *additively cancellative semiring* (or simply *cancellative*) if $a + c = b + c$ implies $a = b$, for all $a, b, c \in S$.

For a cancellative semiring $(S, +\cdot)$, we define $D(S) = \{a - b \colon a, b \in S\}$. Then $(D(S), +, \cdot)$ forms a ring containing the formal differences of elements from S. We call $D(S)$ the *ring of differences* of the cancellative semiring S (cf. Chapter II, Theorem 5.11, [12]).

An *ideal* I of S is a submonoid of $(S, +)$ such that $s \cdot t \in I$, for all $s \in S$ and for all $t \in I$.

Semiring, being a more general algebraic structure than a ring, contains more classes of ideals than a ring. An ideal I is said to be a *k-ideal* if $a + b \in I$ and $b \in I$ implies $a \in I$. The class of k-ideals behaves more like ring ideals. Lastly, we call an ideal I a *strong ideal* if $a + b \in I$, then both $a \in I$ and $b \in I$.

Unlike rings, the factor objects of semirings are not determined by ideals. Instead of ideals, congruence plays an important role in the quotient of semirings.

Definition 2.1. *A congruence k is an equivalence relation on S, which is also a subsemiring of the product semiring $S \times S$.*

Equivalently, a congruence is an equivalence relation on S which is compatible with the binary operations. By compatibility, we mean:

1. $(a, b) \in k$ and $(c, d) \in k \implies (a + c, b + d) \in k$.
2. $(a, b) \in k$ and $(c, d) \in k \implies (a \cdot c, b \cdot d) \in k$.

A congruence k on S is said to be a *cancellative congruence* if $(a + c, b + c) \in k$ implies $(a, b) \in k$, for all $a, b, c \in S$.

A cancellative congruence ρ on S is said to be a *regular congruence* if there exist elements e_1 and e_2 in S such that for all $a \in S$, $(a + e_1a, e_2a) \in \rho$ and $(a + ae_1, ae_2) \in \rho$.

Evidently, if S is a commutative semiring with unity then the class of all regular congruences coincides with the class of all cancellative congruences.

Definition 2.2. *A semiring S with a partial order " $\leq$ " is called a partially ordered semiring if the following conditions are satisfied: for all $a, b, c, d \in S$,*

1) $a \leq b \Leftrightarrow a + c \leq b + c$.

2) $a \leq c, b \leq d \Rightarrow ad + bc \leq ab + cd$.

Definition 2.3. *A congruence ρ on a partially ordered semiring S is called convex if for all $a, b, c, d \in S, (a, b) \in \rho, a \leq c \leq d \leq b \Rightarrow (c, d) \in \rho$.*

The following theorem is noted in [3].

Theorem 2.4. *Let S be a partially ordered semiring and ρ be a regular congruence on S. Then S/ρ is a partially ordered semiring, according to the definition $\rho(a) \leq \rho(b)$ if and only if there exists $x, y \in S$ such that $(x, y) \in \rho$ and $a + x \leq b + y$, it is necessary and sufficient that ρ is convex.*

For any two elements (x_1, x_2) and (y_1, y_2) of the semiring $S \times S$, we define the *twisted product* $(x_1, x_2) \cdot_t (y_1, y_2)$ as follows:

$$(x_1, x_2) \cdot_t (y_1, y_2) = (x_1y_1 + x_2y_2, x_1y_2 + x_2y_1)$$

Definition 2.5. *A congruence ρ on S is called prime congruence if for all $a, b, c, d \in S$, $(x_1, x_2) \cdot_t (y_1, y_2) \in \rho$ implies $(x_1, x_2) \in \rho$ or $(y_1, y_2) \in \rho$.*

The family $Cong(S)$ of all congruences on a semiring S forms a complete lattice with the following operations:

1. For any nonempty family $\mathfrak{F}$ of congruences on S, $\wedge\mathfrak{F}$ is defined by $(a, b) \in \wedge\mathfrak{F}$ if and only if $(a, b) \in \rho$ for every ρ in $\mathfrak{F}$.
2. For any nonempty family $\mathfrak{F}$ of congruences on S, $\vee\mathfrak{F}$ is defined by $(a, b) \in \vee\mathfrak{F}$ if and only if there exist elements $a = c_0, c_1, \cdots, c_n = b$ of S and congruences $\rho_1, \rho_2, \cdots, \rho_n$ of $\mathfrak{F}$ such that $(c_{i-1}, c_i) \in \rho_i$ for all $1 \leq i \leq n$.

3. The Semiring $\mathcal{M}^+(X, \mathcal{A})$ and its ideals

In this section, we focus on the nature of the semiring $\mathcal{M}^+(X, \mathcal{A})$, which is the positive cone (set of all non-negative elements) of the ring $\mathcal{M}(X, \mathcal{A})$. It is easy to observe that whenever $f + g = f + h$, then $g = h$, for any f, g, h in $\mathcal{M}^+(X, \mathcal{A})$. Therefore $\mathcal{M}^+(X, \mathcal{A})$ is an additively cancellative semiring. Since $\mathcal{M}^+(X, \mathcal{A})$ is the positive cone of the von Neumann regular ring $\mathcal{M}(X, \mathcal{A})$, so the semiring $\mathcal{M}^+(X, \mathcal{A})$ is a von Neumann regular semiring.

For $f, g \in \mathcal{M}^+(X, \mathcal{A})$, define $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in X$. Then $\mathcal{M}^+(X, \mathcal{A})$ is a partially ordered semiring with respect to the relation $' \leq'$.

Also for $f, g \in \mathcal{M}^+(X, \mathcal{A})$, define $(f \vee g)(x) = max\{f(x), g(x)\}$ and $(f \wedge g)(x) = min\{f(x), g(x)\}$ for all $x \in X$. Then $f \vee g, f \wedge g \in \mathcal{M}^+(X, \mathcal{A})$. Therefore $\mathcal{M}^+(X, \mathcal{A})$ is a latticed ordered semiring with respect to $\vee$ and $\wedge$.

3.1. Ideals of $\mathcal{M}^+(X, \mathcal{A})$

For any $f \in \mathcal{M}(X, \mathcal{A})$, define

$$f^+(x) = \begin{cases} f(x), & f(x) \geq 0. \\ 0, & f(x) < 0. \end{cases}, \quad f^-(x) = \begin{cases} 0, & f(x) \geq 0. \\ f(x), & f(x) \leq 0. \end{cases},$$

then $f = f^+ + f^- = f^+ - (-f^-)$ and $|f| = f^+ - f^-$. Clearly f^+ and $-f^-$ belongs to $\mathcal{M}^+(X, \mathcal{A})$. Therefore $\mathcal{M}(X, \mathcal{A})$ is the ring of differences of the semiring $\mathcal{M}^+(X, \mathcal{A})$. The following lemma manifests divisibility in the semiring $\mathcal{M}^+(X, \mathcal{A})$.

Lemma 3.1. *Let $f, g \in \mathcal{M}^+(X, \mathcal{A})$ and $f \leq g^r$, for some $r \geq 1$. Then f is a multiple of g.*

Proof. If $f, g \in \mathcal{M}^+(X, \mathcal{A})$ be such that $f \leq g^r$, for some $r \geq 1$. Then clearly $Z(f) \supseteq Z(g)$. Define

$$h(x) = \begin{cases} \frac{f(x)}{g(x)}, & x \notin Z(g). \\ 0, & x \in Z(g). \end{cases},$$

then, both $h|_{Z(g)}$ and $h|_{X \setminus Z(g)}$ are measurable. Hence by pasting lemma, $h \in \mathcal{M}^+(X, \mathcal{A})$ and clearly $f = gh$. □

It is easy to observe that in a partially ordered semiring, the class of l-ideals coincides with the class of strong ideals.

Corollary 3.2. *Every ideal of $\mathcal{M}^+(X, \mathcal{A})$ is a strong ideal.*

Proof. Let I be an ideal of $\mathcal{M}^+(X, \mathcal{A})$ and let $f + g \in I$. Then $f \leq f + g$ and $g \leq f + g$. By Lemma 3.1, $f = (f + g)h$ and $g = (f + f)k$ for some $h, k \in \mathcal{M}^+(X, \mathcal{A})$. Therefore f and g are in I. Hence I is a strong ideal. □

Remark 3.3. *One of the contrasting features between the semiring $\mathcal{M}^+(X, \mathcal{A})$ and the semiring $C^+(X)$ (viz. the semiring of non-negative real-valued continuous functions on a topological space X) is the nature of their ideals. There can exist many non-k-ideals in $C^+(X)$. Moreover, each ideal of $C^+(X)$ is a strong ideal (equivalently a k-ideal) if and only if X is an F-space (cf. Theorem 2.1, [15]). Moreover, the lattice of ideals of $\mathcal{M}^+(X, \mathcal{A})$ is always modular (cf. Proposition 6, [5]), whereas the lattice of ideals of $C^+(X)$ is modular if and only if X is an F-space.*

By $(\mathfrak{L}(\mathcal{M}^+(X, \mathcal{A})), \vee, \wedge)$ we mean the lattice of all ideals of the semiring $\mathcal{M}^+(X, \mathcal{A})$ with $I \vee J = I + J$ and $I \wedge J = I \cap J$. Similarly $(\mathfrak{L}(\mathcal{M}(X, \mathcal{A})), \vee, \wedge)$ is the lattice of all ideals of $\mathcal{M}(X, \mathcal{A})$ with obvious join and meet. We define two maps $\alpha \colon \mathfrak{L}(\mathcal{M}(X, \mathcal{A})) \to \mathfrak{L}(\mathcal{M}^+(X, \mathcal{A}))$ and $\beta \colon \mathfrak{L}(\mathcal{M}^+(X, \mathcal{A})) \to \mathfrak{L}(\mathcal{M}(X, \mathcal{A}))$ as follows:

$$\alpha(I) = I \cap \mathcal{M}^+(X, \mathcal{A}) \qquad \text{and} \qquad \beta(I) = \{f - g \colon f, g \in I\}.$$

Lemma 3.4. *The following statements hold.*

1. *The map β is an onto lattice homomorphism.*
2. *The map α is an onto lattice homomorphism.*

Proof. 1. The equality $\beta(I + J) = \beta(I) + \beta(J)$ and the inclusion $\beta(I \cap J) \subseteq \beta(I) \cap \beta(J)$ easily follows. For the reverse inequality, let $f \in \beta(I) \cap \beta(J)$. Then $f = g_1 - h_1 \in \beta(I)$ and $f = g_2 - h_2 \in \beta(J)$, for some $g_1, h_1 \in I$ and $g_2, h_2 \in J$. Therefore $g_1 + g_2 = h_1 + h_2 \in I \cap J$ and from Corollary 3.2 $g_1, g_2, h_1, h_2 \in I \cap J$. Therefore $f \in \beta(I \cap J)$, so $\beta(I \cap J) \supseteq \beta(I) \cap \beta(J)$. We conclude that β is a lattice homomorphism.

Moreover, we show that every ideal of $\mathcal{M}(X, \mathcal{A})$ is a difference ideal. Let $f \in I$, where I is an ideal in $\mathcal{M}(X, \mathcal{A})$. Clearly $f = f^+ - (-f^-)$ with $f^+ \leq |f|$ and $-f^- \leq |f|$. Therefore $f^+, -f^- \in I \cap \mathcal{M}^+(X, \mathcal{A}) = \alpha(I)$ and $\beta(\alpha(I)) = I$. Hence β is an onto map.

2. The equality $\alpha(I \cap J) = \alpha(I) \cap \alpha(J)$ and $\alpha(I) + \alpha(J) \subseteq \alpha(I + J)$ are obvious. Now suppose $f \in \alpha(I + J)$. Then $f = g + h$ and $f \leq |g| + |h|$. Applying Riesz decomposition theorem (cf. Proposition 1.1.4 of [6]) we get $f = s + t$, for some $s, t \in \mathcal{M}^+(X, \mathcal{A})$ such that $0 \leq s \leq |g|$ and $0 \leq t \leq |h|$. Recall that each ideal of $\mathcal{M}(X, \mathcal{A})$ is

an l-ideal. Therefore $s \in I \cap \mathcal{M}(X,\mathcal{A}) = \alpha(I)$ and $t \in J \cap \mathcal{M}(X,\mathcal{A})$. Hence $f \in \alpha(I) + \alpha(J)$, which validates the equality $\alpha(I+J) = \alpha(I) + \alpha(J)$. Thus we have proved that α is a lattice homomorphism.

In addition, for any ideal I of $\mathcal{M}^+(X,\mathcal{A})$, $I \subseteq \alpha(\beta(I))$. Let $f \in \alpha(\beta(I))$. Then $f = g - h$, for some $f, g \in I$. Then $f \in I$. Indeed I is a strong ideal and hence a k-ideal. Therefore $\alpha(\beta(I)) = I$. Hence α is an onto map. $\square$

Proposition 3.5. *The lattice $\mathfrak{L}(\mathcal{M}^+(X,\mathcal{A}))$ is isomorphic to $\mathfrak{L}(\mathcal{M}(X,\mathcal{A}))$.*

Proof. For any two ideals I and J of $\mathcal{M}^+(X,\mathcal{A})$, if $\beta(I) = \beta(J)$, then by Lemma 3.4 $I = \alpha(\beta(I)) = \alpha(\beta(J)) = J$. Which shows β is injective. Similarly, α is also injective. $\square$

Corollary 3.6. *Each ideal of $\mathcal{M}^+(X,\mathcal{A})$ is of the form $I \cap \mathcal{M}^+(X,\mathcal{A})$ for some ideal I of $\mathcal{M}(X,\mathcal{A})$.*

Corollary 3.7. *Each prime ideal (maximal ideal) of $\mathcal{M}^+(X,\mathcal{A})$ is of the form $P \cap \mathcal{M}^+(X,\mathcal{A})$ for some prime ideal (maximal ideal) P of $\mathcal{M}(X,\mathcal{A})$.*

Remark 3.8. *As a direct consequence of the above discussion and Theorem 2.11 of [4], we achieve a complete description of maximal ideals in $\mathcal{M}^+(X,\mathcal{A})$. Each maximal ideal of $\mathcal{M}^+(X,\mathcal{A})$ is of the form $M_+^p = \{f \in \mathcal{M}^+(X,\mathcal{A}) \colon p \in cl_{\hat{X}} Z(f)\}$, where $\hat{X}$ is the space of all ultrafilters of the measurable space $(X,\mathcal{A})$ under Stone-topology.*

Definition 3.9. *([7]) An ideal I of a semiring $(S, +, \cdot, 0, 1)$ is said to be a z-ideal if $\mathcal{M}_a^+ \subseteq I$ for every $a \in I$.*

Here $\mathcal{M}_a^+ = \bigcap_{M \in \mathcal{M}^+(a)} M$ and $\mathcal{M}^+(a)$ is the set of all maximal ideals of S containing a.

Lemma 3.10. *For any $f \in \mathcal{M}^+(X,\mathcal{A})$, $\mathcal{M}_f^+ = \{g \in \mathcal{M}^+(X,\mathcal{A}) \colon Z(f) \subseteq Z(g)\}$.*

The proof relies on Lemma 3.1 and Remark 3.8. As an easy consequence of Lemma 3.10, we have the following.

Corollary 3.11. *Each ideal of $\mathcal{M}^+(X,\mathcal{A})$ is a z-ideal.*

Remark 3.12. *It is evident that for any $f \in \mathcal{M}(X,\mathcal{A})$, $Z(f) = Z(|f|)$. Therefore $Z[\mathcal{M}(X,\mathcal{A})] = Z[\mathcal{M}^+(X,\mathcal{A})]$. In other words, the collections of zero-sets of the ring $\mathcal{M}(X,\mathcal{A})$ and the semiring $\mathcal{M}^+(X,\mathcal{A})$ are the same. Moreover, let I be an ideal of the ring $\mathcal{M}(X,\mathcal{A})$, then $f \in I$ if and only if $|f| \in I$, so $Z[I] = Z[I \cap \mathcal{M}^+(X,\mathcal{A})]$. Now since each ideal of the semiring $\mathcal{M}^+(X,\mathcal{A})$ is of the form $I \cap \mathcal{M}^+(X,A)$, where I is an ideal of the ring $\mathcal{M}(X,\mathcal{A})$ (cf. Corollary 3.6). We conclude that $Z[I \cap \mathcal{M}^+(X,\mathcal{A})]$ is a $Z_{\mathcal{A}}$-filter on X. In the next section, we deal with the question of whether we can extend this ideal-filter connection to a congruence-filter connection between $Cong(\mathcal{M}^+(X,\mathcal{A}))$ and $Z_{\mathcal{A}}$-filters on X.*

4. Congruences on $\mathcal{M}^+(X,\mathcal{A})$

For any congruence ρ on $\mathcal{M}^+(X,\mathcal{A})$, we define

$$E(\rho) = \{E(f,g) \colon (f,g) \in \rho\},$$

where $E(f,g) = \{x \in X \colon f(x) = g(x)\}$, is the agreement set of f and g. It is clear that $E(f,g) = Z(f-g)$.

Theorem 4.1. *For a measurable space $(X,\mathcal{A})$, $A \in \mathcal{A}$ if and only if it is the agreement set of some functions f, g in $\mathcal{M}^+(X,\mathcal{A})$.*

Proof. Let $A \in \mathcal{A}$. Then $A = Z(f)$, where $f = \chi_{A^c} \in \mathcal{M}^+(X,\mathcal{A})$. Therefore $A = E(f, \mathbf{0})$.

Conversely, let A be an agreement set of f, g in $\mathcal{M}^+(X,\mathcal{A})$. Then $A = Z(f-g) \in \mathcal{A}$. $\square$

Likewise, in the case of ideals, it is customary to ask questions about the structure of congruences on $\mathcal{M}^+(X,\mathcal{A})$. The following generic example shows that not all congruences on $\mathcal{M}^+(X,\mathcal{A})$ are cancellative.

Example 4.2. *Let $S = \mathcal{M}^+(X,\mathcal{A}) \setminus \{\mathbf{0}\}$. Then $k = (S \times S) \cup \Delta_{\mathcal{M}^+(X,\mathcal{A})}$ is a non-trivial congruence on $\mathcal{M}^+(X,\mathcal{A})$, where $\Delta_{\mathcal{M}^+(X,\mathcal{A})} = \{(f,f) \colon f \in \mathcal{M}^+(X,\mathcal{A})\}$. Then $(f+\mathbf{1}, f) \in k$ for any $f \in S$, but $(\mathbf{1},\mathbf{0}) \notin k$ because zero-class of k is a singleton set, that is, $[0]_k = \{(\mathbf{0},\mathbf{0})\}$.*

An easy conclusion we can make from the above example is that, unlike the connection of ideals and $Z_{\mathcal{A}}$-filters, we cannot create congruences and $Z_{\mathcal{A}}$-filters connection. Indeed, $E(f+\mathbf{1}, f) = \phi \in E(k)$, where k_+ is the congruence defined in Example 4.2. We observe the following important correlations. Compare with the semiring $C^+(X)$; see Theorem 3.2 and Theorem 3.3 of [2].

Proposition 4.3. *The following statements hold for any measurable space $(X, \mathcal{A})$.*

1. *If ρ_+ is a cancellative congruence on $\mathcal{M}^+(X, \mathcal{A})$, then $E(\rho_+) = \{E(f, g) : (f, g) \in \rho_+\}$ is a $Z_{\mathcal{A}}$-filter on X.*
2. *If $\mathfrak{F}$ is a $Z_{\mathcal{A}}$-filter on X, then $E^{-1}(\mathfrak{F}) = \{(f, g) \in \mathcal{M}^+(X, \mathcal{A}) \times \mathcal{M}^+(X, \mathcal{A}) : E(f, g) \in \mathfrak{F}\}$ is a cancellative congruence on $\mathcal{M}^+(X, \mathcal{A})$.*

4.1. On z-congruences of $\mathcal{M}^+(X, \mathcal{A})$

Likewise in the case of the semiring $C^+(X)$, it is natural to consider z-congruences on $\mathcal{M}^+(X, \mathcal{A})$.

Definition 4.4. *A congruence ρ on $\mathcal{M}^+(X, \mathcal{A})$ is called* z-congruence *if for all f, g in $\mathcal{M}^+(X, \mathcal{A})$, $E(f, g) \in E(\rho)$ implies that $(f, g) \in \rho$.*

Therefore for each z-congruence ρ on $\mathcal{M}^+(X, \mathcal{A})$, $E^{-1}(E(\rho)) = \rho$. Each z-congruence is a cancellative congruence. The set of all z-congruences on $\mathcal{M}^+(X, \mathcal{A})$ is denoted by $\mathcal{Z}Cong$. Let us denote the collection of all $\mathcal{A}$-filters on X by $\mathcal{Z}_{\mathcal{A}}$. Both $\mathcal{Z}Cong$ and $\mathcal{Z}_{\mathcal{A}}$ are partially ordered by inclusions.

Theorem 4.5. *The map $E\colon (\mathcal{Z}Cong, \subseteq) \to (\mathcal{Z}_{\mathcal{A}}, \subseteq)$ is an order-isomorphism.*

The proof relies on the fact that both E and E^{-1} are order preserving maps and $E^{-1}(E(\rho) = \rho$ and $E(E^{-1}(\mathfrak{F})) = F$ for any z-congruence ρ and $Z_{\mathcal{A}}$-filter F respectively. The following class of z-congruences can be easily obtained by Theorem 4.5.

Corollary 4.6. *Every maximal congruence on $\mathcal{M}^+(X, \mathcal{A})$ is a z-congruence.*

Theorem 4.7. *An intersection of an arbitrary non-empty family of z-congruences on $\mathcal{M}^+(X, \mathcal{A})$ is a z-congruence.*

Therefore $(\mathcal{Z}Cong, \wedge)$ can be regarded as a complete $\wedge$-semilattice, where $\rho_1 \wedge \rho_2 = \rho_1 \cap \rho_2$. We define join of two z-congruences as

$$\rho_1 \vee_z \rho_2 = (\rho_1 \vee \rho_2)_z,$$

where $\rho_1 \vee \rho_2$ is the usual join of two congruences and $(\rho_1 \vee \rho_2)_z$ is the smallest z-congruence containing $\rho_1 \vee \rho_2$. In view of Theorem 4.7 and Theorem 2.31 of [10] we have the following result.

Corollary 4.8. *The lattice $(\mathcal{Z}Cong, \vee_z, \wedge)$ is a complete lattice.*

We have given a complete description of ideals of $\mathcal{M}^+(X, \mathcal{A})$ (cf. Corollary 3.6). Now we investigate if there is a complete description of cancellative congruences on $\mathcal{M}^+(X, \mathcal{A})$ in terms of the congruences on the ring $\mathcal{M}(X, \mathcal{A})$. For that we define $\mathcal{E}(f, g) = \{x \in X\colon f(x) = g(x)\}$, the agreement set of $f, g \in \mathcal{M}(X, \mathcal{A})$. There is a one-one correspondence between the set of the ideals of $\mathcal{M}(X, \mathcal{A})$ and the set of all congruences on $\mathcal{M}(X, \mathcal{A})$. Indeed, for a ring R, we know that there is a one-one correspondence between the class of congruences on R and the class of ideals in R (cf. Chapter I, Remark 7.6(iii), [12]). Therefore for any congruence k on $\mathcal{M}(X, \mathcal{A})$, $\mathcal{E}(k) = \{\mathcal{E}(f, g)\colon (f, g) \in k\} = Z[I_k]$, where $I_k = \{s - t\colon (s, t) \in k\}$ is the corresponding ideal of the congruence k. For any ideal I, the corresponding congruence is $k_I = \{(h, k)\colon h - k \in I\}$. Observe that $k_{I_k} = k$ for any congruence k.

Definition 4.9. *A congruence k on $\mathcal{M}(X, \mathcal{A})$ is said to be a z-congruence if $\mathcal{E}(f, g) \in \mathcal{E}(k)$ implies $(f, g) \in k$.*

Theorem 4.10. *Each congruence k on $\mathcal{M}(X, \mathcal{A})$ is a z-congruence.*

Corollary 4.11. *There is a one-one correspondence between z-ideals and z-congruences in the ring $\mathcal{M}(X, \mathcal{A})$.*

Since $\mathcal{M}(X,\mathcal{A})$ is the difference ring of the cancellative semiring $\mathcal{M}^+(X,\mathcal{A})$, we can define a map ∇ as follows

$$\begin{aligned} \nabla\colon Cong(\mathcal{M}(X,\mathcal{A})) &\to Cong(\mathcal{M}^+(X,\mathcal{A})) \\ \rho &\longmapsto \rho^{\nabla} \end{aligned}$$

where $\rho^{\nabla} = \rho \cap (\mathcal{M}^+(X,\mathcal{A}) \times \mathcal{M}^+(X,\mathcal{A}))$. Also, there exists a map Δ in the opposite direction, defined as follows

$$\begin{aligned} \Delta\colon Cong(\mathcal{M}^+(X,\mathcal{A})) &\to Cong(\mathcal{M}(X,\mathcal{A})) \\ \rho_+ &\longmapsto \rho_+^{\Delta} \end{aligned}$$

where $\rho_+^{\Delta} = \{(f,g)\colon f-g = h-k \text{ for some } (h,k) \in \rho_+\}$.

Proposition 4.12. *Each z-congruence of $\mathcal{M}^+(X,\mathcal{A})$ is of the form $k \cap (\mathcal{M}^+(X,\mathcal{A}) \times \mathcal{M}^+(X,\mathcal{A}))$ for some congruence k on $\mathcal{M}(X,\mathcal{A})$.*

Proof. Since every congruence $\mathcal{M}(X,\mathcal{A})$ is a z-congruence, it is easy to show that ρ^{∇} is a z-congruence on $\mathcal{M}^+(X,\mathcal{A})$ for any congruence ρ on $\mathcal{M}(X,\mathcal{A})$. Now suppose ρ_+ is a z-congruence on $\mathcal{M}^+(X,\mathcal{A})$. Let $\mathcal{E}(f,g) \in \mathcal{E}(\rho_+^{\Delta})$. Then define $h = f-(f\wedge g)$ and $k = g-(g\wedge f)$. Clearly, $h,k \in \mathcal{M}^+(X,\mathcal{A})$. Also, we have $\mathcal{E}(f,g) = E(h,k) \in \rho_+$ and $(h,k) \in \rho_+$. But from the construction of h and k, it is clear that $f-g = h-k$. Therefore $(f,g) \in \rho_+^{\Delta}$ and hence ρ_+^{Δ} is a z-congruence on $\mathcal{M}(X,\mathcal{A})$. Also, since every z-congruence is a cancellative congruence on $\mathcal{M}^+(X,\mathcal{A})$, we have $\rho_+^{\Delta\nabla} = \rho_+$ by Chapter II, Theorem 7.1 of [12]. This completes the proof. □

Due to Corollary 3.11, Theorem 4.10 and Corollary 4.11, we arrive at the following remarkable correspondence theorem for z-ideals and z-congruences in $\mathcal{M}^+(X,\mathcal{A})$.

Proposition 4.13. *There is a one-one correspondence between z-ideals and z-congruences in the semiring $\mathcal{M}^+(X,\mathcal{A})$.*

Proof. Let k be a z-congruence on $\mathcal{M}^+(X,\mathcal{A})$. Then the corresponding ideal (zeroth class), denoted by $I_k = \{f\colon (f,\mathbf{0}) \in k\}$ is again a z-ideal of $\mathcal{M}^+(X,\mathcal{A})$. We denote this map from $\mathcal{Z}Cong(\mathcal{M}^+(X,\mathcal{A}))$ to $\mathfrak{L}(\mathcal{M}^+(X,\mathcal{A}))$ by $\mathcal{I}_+$. For any z-congruence k we can easily observe that $\mathcal{I}_+(k) = \alpha(\mathcal{I}(k^{\Delta}))$. The following diagram captures the essence of our goal.

$$\begin{tikzcd}
\mathfrak{L}(\mathcal{M}(X,\mathcal{A})) \arrow[rr, bend left, "\mathcal{K}(I)=k_I"] \arrow[d, bend right, "\alpha"'] & & Cong(\mathcal{M}(X,\mathcal{A})) \arrow[ll, bend left, "\mathcal{I}(k)=I_k"] \arrow[d, bend right, "\nabla"'] \\
\mathfrak{L}(\mathcal{M}^+(X,\mathcal{A})) \arrow[rr, "\mathcal{K}_+"] \arrow[u, bend right, "\beta"'] & & \mathcal{Z}Cong(\mathcal{M}^+(X,\mathcal{A})) \arrow[ll, bend left, "\mathcal{I}_+"] \arrow[u, bend right, "\Delta"']
\end{tikzcd}$$

Now we define a map $\mathcal{K}_+\colon \mathfrak{L}(\mathcal{M}^+(X,\mathcal{A})) \mapsto \mathcal{Z}Cong(\mathcal{M}^+(X,\mathcal{A}))$ by $\mathcal{K}_+(J) = \mathcal{K}_{\beta(I)}^{\nabla}$. Since each ideal J of $\mathcal{M}^+(X,\mathcal{A})$ is a strong ideal, we immediately have $\mathcal{K}_{\beta(I)}^{\nabla} = \{(f,g)\colon f+s = g+t, \text{ for some } s,t \in I\}$, which is again a z-congruence on $\mathcal{M}^+(X,\mathcal{A})$ corresponding to the z-ideal I. Moreover $\mathcal{I}_+(\mathcal{K}_+(I)) = I$ for any ideal I and $\mathcal{K}_+(I_+(k)) = k$ for any z-congruence k on $\mathcal{M}^+(X,\mathcal{A})$. This completes the proof.

□

4.2. Prime z-congruences

Definition 4.14. *A $Z_{\mathcal{A}}$-filter $\mathfrak{F}$ on X is said to be prime if $A, B \in \mathcal{A}$, $A \cup B \in \mathfrak{F}$ implies $A \in \mathfrak{F}$ or $B \in \mathfrak{F}$.*

Lemma 4.15. *For any f_1, f_2, g_1 and g_2 in $\mathcal{M}^+(X, \mathcal{A})$, $E(f_1, g_1) \cup E(f_2, g_2) = E(f_1 f_2 + g_1 g_2, f_1 g_2 + f_2 g_1)$.*

Theorem 4.16. *If ρ is a prime z-congruence on $\mathcal{M}^+(X, \mathcal{A})$, then $E(\rho)$ is a prime z-filter on X.*

The next theorem gives us a picture of the relation between prime congruences and z-congruences to some extent.

Theorem 4.17. *For a z-congruence ρ on $\mathcal{M}^+(X, \mathcal{A})$ the following are equivalent:*

1. *ρ is prime.*
2. *ρ contains a prime congruence.*
3. *For all $f_1, f_2, g_1, g_2 \in \mathcal{M}^+(X, \mathcal{A})$, $f_1 f_2 + g_1 g_2 = f_2 g_2 + f_2 g_1$ implies that either $(f_1, g_1) \in \rho$ or $(f_2, g_2) \in \rho$.*
4. *For all $f, g \in \mathcal{M}^+(X, \mathcal{A})$ there exists $A \in E(\rho)$ such that either $f \geq g$ or $g \geq f$ on A.*

Proof. (1) $\Rightarrow$ (2) : Trivial.

(2) $\Rightarrow$ (3) : Let ρ contain a prime congruence σ and $f_1, f_2, g_1, g_2 \in \mathcal{M}^+(X, \mathcal{A})$, such that $f_1 f_2 + g_1 g_2 = f_2 g_2 + f_2 g_1$. Then $(f_1 f_2 + g_1 g_2, f_2 g_2 + f_2 g_1)$ is a member of the diagonal congruence and hence a member of σ. Since σ is prime, then either $(f_1, g_1) \in \rho_1$ or $(f_2, g_2) \in \sigma$. Since $\sigma \subseteq \rho$, so either $(f_1, g_1) \in \rho$ or $(f_2, g_2) \in \rho$.

(3) $\Rightarrow$ (4) : Let $f, g \in \mathcal{M}^+(X, \mathcal{A})$. We define $h_1 = f - (f \wedge g)$ and $h_2 = g - (g \wedge f)$. Then $h_1, h_2 \in \mathcal{M}^+(X, \mathcal{A})$ and $h_1 h_2 = \mathbf{0}$. Thus $(h_1, \mathbf{0}) \in \rho$ or $(\mathbf{0}, h_2) \in \rho$ by (3). This implies that $E(h_1, \mathbf{0}) \in E(\rho)$ or $E(\mathbf{0}, h_2) \in E(\rho)$. Clearly, $f \geq g$ on $E(h_1, \mathbf{0})$ and $g \geq f$ on $E(h_2, \mathbf{0})$.

(4) $\Rightarrow$ (1) : Let $f_1, f_2, g_1, g_2 \in \mathcal{M}^+(X, \mathcal{A})$ such that $(f_1 f_2 + g_1 g_2, f_1 g_2 + f_2 g_1) \in \rho$. Let $A = E(f_1 f_2 + g_1 g_2, f_1 g_2 + f_2 g_1) \in E(\rho)$. Set $h_1 = |f_1 - g_1|$ and $h_2 = |f_2 - g_2|$. Then $h_1, h_2 \in \mathcal{M}^+(X, \mathcal{A}), E(f_1, g_1) = E(h_1, \mathbf{0})$ and $E(f_2, g_2) = E(h_2, \mathbf{0})$.

By (4) there exists $A_1 \in E(\rho)$ such that $h_1 \leq h_2$ or $h_2 \leq h_1$ on A_1. We assume that $h_1 \leq h_2$ on A_1. Then $h_1 \leq h_2$ on $A \cap A_1$. Now $E(f_1 f_2 + g_1 g_2, f_1 g_2 + f_2 g_1) = E(f_1, g_1) \cup E(f_2, g_2)$. Therefore $A \cap A_1 \subseteq E(h_1, \mathbf{0}) = E(f_1, g_1)$. Then $E(f_1, g_1) \in E(\rho)$ as $A \cap A_1 \in E(\rho)$ and $E(\rho)$ is an $\mathcal{A}$-filter on X. Thus $(f_1, g_1) \in \rho$ as ρ is a z-congruence.

Similarly $h_2 \leq h_1$ on A_1 implies that $(f_2, g_2) \in \rho$. Hence, ρ is a prime congruence. □

Corollary 4.18. *If ρ is a prime congruence on $\mathcal{M}^+(X, \mathcal{A})$, then $E(\rho)$ is a prime $\mathcal{A}$-filter on X.*

Proof. Let $\sigma = E^{-1}(E(\rho))$. Then σ is a z-congruence and $\rho \subseteq \rho$. So σ is prime by Theorem 4.17. Hence $E(\rho) = E(\sigma)$ is a prime $Z_{\mathcal{A}}$-filter on X by Theorem 4.17. □

Theorem 4.19. *If $\mathfrak{F}$ is a prime $Z_{\mathcal{A}}$-filter on X, then $E^{-1}(\mathfrak{F})$ is a prime congruence on $\mathcal{M}^+(X, \mathcal{A})$.*

Proof. $E^{-1}(\mathfrak{F})$ is a congruence on $\mathcal{M}^+(X, \mathcal{A})$ by Proposition 4.3. Let $f_1, g_1, f_2, g_2 \in \mathcal{M}^+(X, \mathcal{A})$ such that $(f_1 g_1 + f_2 g_2, f_1 g_2 + f_2 g_1) \in E^{-1}(\mathfrak{F})$. Then $E(f_1 g_1 + f_2 g_2, f_1 g_2 + f_2 g_1) = E(f_1, g_1) \cup E(f_2, g_2) \in \mathfrak{F}$. Since $\mathfrak{F}$ is a prime $\mathcal{A}$-filter, therefore either $E(f_1, g_1) \in \mathfrak{F}$ or $E(f_2, g_2) \in \mathfrak{F}$. Thus either $(f_1, g_1) \in E^{-1}(\mathfrak{F})$ or $(f_2, g_2) \in E^{-1}(\mathfrak{F})$. Hence $E^{-1}(\mathfrak{F})$ is a prime congruence on $\mathcal{M}^+(X, \mathcal{A})$. □

We have the following result as an easy consequence of Proposition 4.3 and Theorem 4.5.

Theorem 4.20. *The following statements hold for a measurable space $(X, \mathcal{A})$.*

a) *If ρ is a maximal congruence on $\mathcal{M}^+(X, \mathcal{A})$ then $E(\rho)$ is a $Z_{\mathcal{A}}$-ultrafilter on X.*

b) *If $\mathcal{U}$ is a $Z_{\mathcal{A}}$-ultrafilter on X then $E^{-1}(\mathcal{U})$ is a maximal congruence on $\mathcal{M}^+(X, \mathcal{A})$.*

Theorem 4.21. *Every maximal congruence on $\mathcal{M}^+(X, \mathcal{A})$ is a prime congruence.*

Theorem 4.22. *Every prime $Z_{\mathcal{A}}$-filter on X is a $Z_{\mathcal{A}}$-ultrafilter.*

Theorem 4.23. *Let ρ be a z-congruence on $\mathcal{M}^+(X,\mathcal{A})$. Then ρ is maximal if and only if it is prime.*

Proof. If ρ is a maximal congruence, then it is prime by Theorem 4.21.

Conversely let, ρ be prime. Then $E(\rho)$ is a prime $Z_{\mathcal{A}}$-filter by Theorem 4.18. By Theorem 4.22, $E(\rho)$ is an $\mathcal{A}$-ultrafilter. So $E^{-1}(E(\rho))$ is a maximal congruence on $\mathcal{M}^+(X,\mathcal{A})$ by Theorem 4.20. Since ρ is a z-congruence, hence $E^{-1}(E(\rho)) = \rho$ is maximal. □

We denote the intersection of all maximal congruences containing (f,g) as $\mathfrak{M}(f,g)$. clearly, $\mathfrak{M}(f,g)$ is a z-congruence, for any $f,g \in \mathcal{M}^+(X,\mathcal{A})$ Next we give an algebraic characterization of z-congruences on $\mathcal{M}^+(X,\mathcal{A})$.

Proposition 4.24. *A congruence ρ on $\mathcal{M}^+(X,\mathcal{A})$ is a z-congruence if and only if $\mathfrak{M}(f,g) \subseteq \rho$ for every $(f,g) \in \rho$.*

Proof. First, we observe that if (h,k) belongs to every maximal congruence that (f,g) belongs to, then $E(f,g) \subseteq E(h,k)$. Indeed, if we have $x \in E(f,g)$ and $x \notin E(h,k)$, then for this fixed point x, consider the fixed maximal ideal M_x of $\mathcal{M}(X,\mathcal{A})$. Which forces $E^{-1}(Z[M_x])$ to be a maximal congruence (cf. Theorem 4.20), where $Z[M_x]$ is a $Z_{\mathcal{A}}$-ultrafilter (cf. Theorem 2.7, [4]). Then clearly $(f,g) \in E^{-1}(Z[M_x])$ but $(h,k) \notin E^{-1}(Z[M_x])$. Which is a contradiction. Therefore we have $E(f,g) \subseteq E(h,k)$.

Now let ρ be a z-congruence on $\mathcal{M}^+(X,\mathcal{A})$ and $(f,g) \in \rho$. For any $(h,k) \in \mathfrak{M}(f,g)$ we have $E(f,g) \subseteq E(h,k)$, which implies $(h,k) \in \rho$. Indeed, this follows from that fact that $E(\rho)$ is a $Z_{\mathcal{A}}$-filter and ρ is a z-congruence. Therefore $\mathfrak{M}(f,g) \subseteq \rho$.

Conversely, it follows easily since each maximal congruence is a z-congruence. □

Corollary 4.25. *Any z-congruence ρ on $\mathcal{M}^+(X,\mathcal{A})$ is of the form $\rho = \bigvee_{z\,(f,g)\in\rho} \mathfrak{M}(f,g)$.*

5. Structure space of $\mathcal{M}^+(X,\mathcal{A})$

Let $Max(X,\mathcal{A})$ be the set of all maximal ideals of $\mathcal{M}(X,\mathcal{A})$ and for $f \in \mathcal{M}(X,\mathcal{A})$ set $\mathcal{M}_f = \{M \in Max(X,\mathcal{A}) : f \in M\}$. Then $\{\mathcal{M}_f : f \in \mathcal{M}(X,\mathcal{A})\}$ is base for closed sets for some topology on $Max(X,\mathcal{A})$. $Max(X,\mathcal{A})$ with this topology is called structure space of the ring $\mathcal{M}(X,\mathcal{A})$; See [4].

Let S be a semiring and $\mathcal{M}Cong(S)$ be the set of all maximal congruences on S. For $a,b \in S$, set $\mathfrak{m}(a,b) = \{\rho \in \mathcal{M}Cong(S) : (a,b) \in \rho\}$. For $a,b,c,d \in S$, $\mathfrak{m}(a,b) \cup \mathfrak{m}(c,d) \subseteq \mathfrak{m}(ac+bd, ad+bc)$. If every maximal congruence on S is prime, then the equality holds. Using this fact, we have the following theorem.

Theorem 5.1 (Theorem 2.9, [2])**.** *If each maximal congruence on S is prime, then $\{\mathfrak{m}(a,b) : (a,b) \in S \times S\}$ is a base for closed sets of some topology on $\mathcal{M}Cong(S)$.*

The set $\mathcal{M}Cong(S)$ with this topology is said to be the structure space of S, defined in [14]. We now show that $Max(X,\mathcal{A})$ can be achieved via the positive cone of the ring $\mathcal{M}(X,\mathcal{A})$.

Theorem 5.2. *$\mathcal{M}Cong(\mathcal{M}^+(X,\mathcal{A}))$ is homeomorphic to $Max(X,\mathcal{A})$.*

Proof. Let $\rho \in \mathcal{M}Cong(\mathcal{M}^+(X,\mathcal{A}))$. Then $E(\rho)$ is a $Z_{\mathcal{A}}$-ultrafilter on X (cf. Theorem 4.20). Then $Z^{-1}[E(\rho)]$ is a maximal ideal in $\mathcal{M}(X,\mathcal{A})$ by Theorem 2.7 of [4]. Define $\eta : \mathcal{M}Cong(\mathcal{M}^+(X,\mathcal{A})) \to Max(X,\mathcal{A})$ by $\eta(\rho) = Z^{-1}[E(\rho)]$ for every $\rho \in \mathcal{M}Cong(\mathcal{M}^+(X,\mathcal{A}))$.

Let $\rho_1, \rho_2 \in \mathcal{M}Cong(\mathcal{M}^+(X,\mathcal{A}))$ such that $\eta(\rho_1) = \eta(\rho_2)$. Then $Z^{-1}[E(\rho_1)] = Z^{-1}[E(\rho_2)]$. For any $Z_{\mathcal{A}}$-filter $\mathfrak{F}$, $ZZ^{-1}[\mathfrak{F}] = \mathfrak{F}$, so we have $E(\rho_1) = E(\rho_2)$. Also ρ_1, ρ_2 are z-congruences, therefore we have $\rho_1 = \rho_2$. Hence η is injective.

Let M be a maximal ideal of $\mathcal{M}(X,\mathcal{A})$. Then $Z[M]$ is a $Z_{\mathcal{A}}$-ultrafilter by Theorem 2.7 of [4]. By Theorem 4.20, $E^{-1}Z[M]$ is a maximal congruence on $\mathcal{M}^+(X,\mathcal{A})$. Let $\rho = E^{-1}Z[M]$. Then $\eta(\rho) = Z^{-1}E(\rho) = M$. Hence η is onto.

Let $\mathfrak{m}(f,g)$ be a basic closed set in $\mathcal{M}Cong(\mathcal{M}^+(X,\mathcal{A}))$, where $f,g \in \mathcal{M}^+(X,\mathcal{A})$. Then $M \in \eta(\mathfrak{m}(f,g)) \Leftrightarrow M = \eta(\rho), \rho \in \mathfrak{m}(f,g) \Leftrightarrow M = Z^{-1}E(\rho), (f,g) \in \rho \Leftrightarrow Z[M] = E(\rho), (f,g) \in \rho \Leftrightarrow E(f,g) = Z(f-g) \in E(\rho) = Z[M] \Leftrightarrow f-g \in M$ as M is a z-ideal $\Leftrightarrow M \in \mathcal{M}_{f-g}$. Hence $\eta(\mathfrak{m}(f,g)) = \mathcal{M}_{f-g}$ is a basic closed set in $Max(X,\mathcal{A})$.

Let $\mathcal{M}_f$ be a basic closed set in $Max(X,\mathcal{A})$, where $f \in \mathcal{M}(X,\mathcal{A})$. Then $\rho \in \eta^{-1}(\mathcal{M}_f) \Leftrightarrow \eta(\rho) \in \mathcal{M}_f \Leftrightarrow f \in \eta(\rho) = Z^{-1}E(\rho) \Leftrightarrow Z(f) = Z(|f|) = E(|f|,\mathbf{0}) \in E(\rho) \Leftrightarrow (|f|,\mathbf{0}) \in \rho$ as ρ is a z-congruence $\Leftrightarrow \rho \in m(|f|,\mathbf{0})$. Therefore $\eta^{-1}(\mathcal{M}_f) = m(|f|,\mathbf{0})$ is a basic closed set in $\mathcal{M}Cong(\mathcal{M}^+(X,\mathcal{A}))$. Hence η is a homeomorphism. □

Definition 5.3. *A congruence ρ on $\mathcal{M}^+(X,\mathcal{A})$ is called* fixed *if $\bigcap\{E(f,g) : (f,g) \in \rho\}$ is nonempty and* free *otherwise.*

Theorem 5.4. *For a measurable space $(X,\mathcal{A})$, the set of all fixed maximal congruences of $\mathcal{M}^+(X,\mathcal{A})$ is the set $\{\rho_x : x \in X\}$, where $\rho_x = \{(f,g) \in \mathcal{M}^+(X,\mathcal{A}) \times \mathcal{M}^+(X,\mathcal{A}) : f(x) = g(x)\}$. Moreover, for two distinct points $x, y \in X$ we have $\rho_x \neq \rho_y$.*

Next, we try to characterize the measurable space $(X,\mathcal{A})$ for which each maximal congruence on $\mathcal{M}^+(X,\mathcal{A})$ is fixed. We need the following definitions.

Definition 5.5. *(cf. Definition 7.15 of [10]) Let L be a complete lattice, and let a be an element of L. Then a is called* compact *if $a \leq \bigvee X$, for some $X \subseteq L$, implies that $a \leq \bigvee S$ for some finite $S \subseteq X$. In particular, if the top element of L is compact, then we call L a* compact lattice.

Definition 5.6. *([11]) A measurable space $(X,\mathcal{A})$ is said to be a* compact measurable space *if $\mathcal{A}$ is a compact lattice.*

Theorem 5.7. *Let $(X,\mathcal{A})$ be a measurable space. Then the following are equivalent.*

1) Each maximal ideal of $\mathcal{M}(X,\mathcal{A})$ is fixed.

2) Each maximal congruence on $\mathcal{M}^+(X,\mathcal{A})$ is fixed.

3) $\mathcal{A}$ is a finite σ-algebra on X.

4) $(X,\mathcal{A})$ is a compact measurable space.

5) $\mathcal{M}(X,\mathcal{A}) = \mathcal{M}^*(X,\mathcal{A}) = \{f \in \mathcal{M}(X,\mathcal{A})\colon f$ is bounded on $X\}$.

Proof. (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5) : Follows from Theorem 2.13 of [4].

(1) $\Leftrightarrow$ (4) Follows from Proposition 4.11 of [11].

(1) $\Rightarrow$ (2) : Let ρ be a maximal congruence on $\mathcal{M}^+(X,\mathcal{A})$. Then $E(\rho)$ is an $Z_{\mathcal{A}}$-ultrafilter on X. So $Z^{-1}[E(\rho)]$ is a maximal ideal of $\mathcal{M}(X,\mathcal{A})$. Thus $Z^{-1}[E(\rho)] = \{h \in \mathcal{M}(X,\mathcal{A}) : h(x) = 0\}$ for some $x \in X$. Let $(f,g) \in \rho$. Then $E(f,g) = Z(f-g) \in E(\rho)$. Therefore $f-g \in Z^{-1}[E(\rho)]$. So $(f-g)(x) = 0$ i.e., $x \in E(f,g)$. Hence ρ is a fixed congruence on $\mathcal{M}^+(X,\mathcal{A})$.

(2) $\Rightarrow$ (1) : Let M be any maximal ideal of $\mathcal{M}(X,\mathcal{A})$. Then $Z[M]$ is a $Z_{\mathcal{A}}$-ultrafilter on X. Thus $E^{-1}(Z[M])$ is a maximal congruence on $\mathcal{M}^+(X,\mathcal{A})$. By (2) $E^{-1}(Z[M])$ is fixed. So by Theorem 5.4 $E^{-1}(Z[M]) = \rho_x$ for some $x \in X$. Let $f \in M$. Then $Z(f) = E(|f|,\mathbf{0}) \in Z[M]$. Thus $(|f|,\mathbf{0}) \in E^{-1}(Z[M]) = \rho_x$. Therefore $f(x) = |f|(x) = \mathbf{0}(x) = 0$. Hence M is a fixed maximal ideal of $\mathcal{M}(X,\mathcal{A})$.

(3) $\Rightarrow$ (4) Trivially true.

(4) $\Rightarrow$ (3) Suppose $\mathcal{A}$ is an infinite σ-algebra on X. For each $x \in X$, we define $x^\vee = \bigcap\{B \in \mathcal{A}\colon x \in B\}$. Since $\mathcal{A}$ is a complete lattice, we have $x^\vee \in \mathcal{A}$ for all $x \in X$. It is easy to see that $x^\vee \cap y^\vee = \emptyset$ or $x^\vee = y^\vee$, for two distinct points $x, y \in X$. Let $\mathfrak{X} = \{x^\vee : x \in X\}$. It is clear that any $B \in \mathcal{A}$ can be written as unions of elements of $\mathfrak{X}$. Since $\mathcal{A}$ is infinite, then $\mathfrak{X}$ has infinite cardinality. Now let, $\{x_1^\vee, x_2^\vee, \cdots\}$ be a countable infinite subset of $\mathfrak{X}$. Let $A = \bigcup_i x_i^\vee$. Then $A \in \mathcal{A}$. By our hypothesis, $(X,\mathcal{A})$ is a compact measurable space. Then there exists a finite subcollection $\{x_{i_j}^\vee\}_j$ of $\{x_i^\vee\}_i$ such that $A = \bigcup_j x_{i_j}^\vee$. This contradicts the fact that any two members of the family $\{x_i^\vee\}_i$ is pairwise disjoint. □

Remark 5.8. *From the definition of a σ-algebra, it is clear that a σ-algebra is a σ-frame (A σ-frame L is a lattice with countable joins $\bigvee_n$, finite meets $\wedge$, a top element $\top$ and a bottom element $\bot$ such that $x \wedge \bigvee_n x_n = \bigvee_n (x \wedge x_n)$, for $n \in \mathbb{N}$, for all $x, x_n \in L$). A σ-frame S is said to be* pseudocompact *if every σ-frame maps $\phi \colon \mathfrak{L}(\mathbb{R}) \to S$ is a bounded map (cf. Definition 3, [9]). Here $\mathcal{L}(\mathbb{R})$ is the σ-frame (frame) of real numbers, which is isomorphic to $\mathcal{O}(\mathbb{R})$, the lattice of open sets of $\mathbb{R}$. Therefore a σ-algebra $\mathcal{A}$ is pseudocompact if every $f \colon \mathcal{O}(\mathbb{R}) \to \mathcal{A}$ is bounded. Unlike the classical case of rings of continuous functions, that is, a space X is pseudocompact if and only if $C(X) = C^*(X)$, Theorem 5.7 is unable to capture pseudocompactness of $(X, \mathcal{A})$. Now, we argue why this phenomenon occurs. We show that any σ-algebra is a regular σ-frame (for definition, see the Background section of [9]). Consider any nonempty element B of $\mathcal{A}$. If B is an atom, then vacuously B is a regular element of the σ-frame $\mathcal{A}$. Now let B be a non-atom of $\mathcal{A}$. Then there exists $\{E_n \colon n \in \mathbb{N}\}$, a pairwise disjoint countable nonempty members of $\mathcal{A}$ defined as in Lemma 2.12 of [4]. It is easy to see that $E_{i+1} \prec E_i$ and $B = \bigvee_i E_i$, $i = 1, 2, \cdots$. Therefore every non-atom is also a regular element. We conclude that $\mathcal{A}$ is a regular σ-frame. Now under the Axiom of countable dependent choice, any regular σ-frame is completely regular. Then by Corollary 2 of [9], compactness and pseudocompactness coincide in $\mathcal{A}$.*

Proposition 5.9. *For every compact measurable space $(X, \mathcal{A})$,*

$$(\mathcal{M}Cong(\mathcal{M}^+(X, \mathcal{A}), \{\mathfrak{m}(f, g) \colon f, g \in \mathcal{M}^+(X, \mathcal{A})\})$$

is T-measurable.

Proof. We know from Theorem 5.7 that each maximal congruence on $\mathcal{M}^+(X, \mathcal{A})$ is of the form ρ_x for each $x \in X$ (cf. Theorem 5.4). It follows that $\mathfrak{m}(f, g) = \{\rho_x \colon (f, g) \in \rho_x\} = \{\rho_x \colon x \in E(f, g)\}$. Therefore $\mathfrak{m}(f, g)^c = \{\rho_x \colon x \in E^c(f, g)\} = \{\rho_x \colon x \in E(\chi_{E(f,g)}, \mathbf{0})\} = \mathfrak{m}(\chi_{E(f,g)}, \mathbf{0})$. Moreover $\bigcup_{n\in\mathbb{N}} \mathfrak{m}(f_n, g_n) = \bigcup_{n\in\mathbb{N}} \{\rho_x \colon x \in E(f_n, g_n)\} = \{\rho_x \colon x \in \bigcup_{n\in\mathbb{N}} E(f_n, g_n)\} = \mathfrak{m}(\chi_{\cap_{n\in\mathbb{N}} E^c(f_n, g_n)}, \mathbf{0})$. Therefore

$$(\mathcal{M}Cong(\mathcal{M}^+(X, \mathcal{A}), \{\mathfrak{m}(f, g) \colon f, g \in \mathcal{M}^+(X, \mathcal{A})\})$$

is a measurable space and by Theorem 5.4 it is a T-measurable space. □

Definition 5.10. *([11]) Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be two measurable spaces. We say that $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are homeomorphic if there exists a one-one and onto function $f \colon X \to Y$ such that $A \in \mathcal{A}$ if and only if $f(A) \in \mathcal{B}$, for every $A \in \mathcal{A}$.*

When $(X, \mathcal{A})$ is homeomorphic to $(Y, \mathcal{B})$ we will simply write $X \cong Y$.

Theorem 5.11. *For every compact measurable space $(X, \mathcal{A})$, $X \cong \mathcal{M}Cong(\mathcal{M}^+(X, \mathcal{A}))$ as measurable spaces.*

Proof. We define $\phi \colon X \to \mathcal{M}Cong(\mathcal{M}^+(X, \mathcal{A}))$ by $\phi(x) = \rho_x$. From Theorem 5.7 it is clear that ϕ is one-one and onto, Moreover $\phi(E(f, g)) = \{\rho_x \colon x \in E(f, g)\} = \mathfrak{m}(f, g)$ and $\phi^{-1}(\mathfrak{m}(f, g)) = E(f, g)$. Therefore, ϕ is a homeomorphism. □

The following corollary is a direct consequence of Theorem 5.2 and Theorem 5.11.

Corollary 5.12. *If $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are two compact measurable spaces, then $X \cong Y$ as measurable spaces if and only if $\mathcal{M}^+(X, \mathcal{A}) \cong \mathcal{M}^+(Y, \mathcal{B})$ as semirings.*

6. Real maximal congruences on $\mathcal{M}^+(X, \mathcal{A})$

Here we initiate a study of quotients of $\mathcal{M}^+(X, \mathcal{A})$ via some important class of congruences on it. Our goal is to give an alternative description of realcompact measurable spaces, in view of $\mathcal{M}^+(X, \mathcal{A})$. The following lemma infers that the class of z-congruences is ideal to consider while sculpting quotients of $\mathcal{M}^+(X, \mathcal{A})$.

Lemma 6.1. *Every z-congruence on $\mathcal{M}^+(X, \mathcal{A})$ is convex.*

Proof. Let ρ be a z-congruence on $\mathcal{M}^+(X, \mathcal{A})$. Let $f, g, f_1, g_1 \in \mathcal{M}^+(X, \mathcal{A})$ such that $f \leq f_1 \leq g_1 \leq g$ and $(f, g) \in \rho$. Then $E(f, g) \subseteq E(f_1, g_1)$, $E(f, g) \in E(\rho)$ and $E(\rho)$ is a $Z_{\mathcal{A}}$-filter $\Rightarrow E(f_1, g_1) \in E(\rho)$. Also $E(\rho)$ is a z-congruence. Therefore $(f_1, g_1) \in \rho$. Hence ρ is a convex congruence. □

Theorem 6.2. *Let ρ be a z-congruence on $\mathcal{M}^+(X,\mathcal{A})$ and $f,g \in \mathcal{M}^+(X,\mathcal{A})$. Then $\rho(f) \leq \rho(g)$ if and only if $f \leq g$ on some member of $E(\rho)$.*

Proof. Let $f,g \in \mathcal{M}^+(X,\mathcal{A})$ such that $\rho(f) \leq \rho(g)$. Then by Theorem 2.4 and Lemma 6.1, there exists $(h,k) \in \rho$ such that $f + h \leq g + k$. We have $f \leq g$ on $E(h,k) \in E(\rho)$.

Conversely, let $f \leq g$ on some $A \in E(\rho)$. Then $A = E(f_1, g_1)$ for some $(f_1, g_1) \in \rho$. Set $h = (f - g) \vee \mathbf{0}$. Then $h \in \mathcal{M}^+(X,\mathcal{A})$ and $A \subseteq E(h,\mathbf{0})$. Since $E(\rho)$ is a $Z_{\mathcal{A}}$-filter, therefore $E(h,\mathbf{0}) \in E(\rho)$. This implies that $(h,\mathbf{0}) \in \rho$ as ρ is a z-congruence. Also $f + \mathbf{0} \leq g + h$. Hence $\rho(f) \leq \rho(g)$. □

Theorem 6.3. *Let $f,g \in \mathcal{M}^+(X,\mathcal{A})$ and ρ be a maximal congruence on $\mathcal{M}^+(X,\mathcal{A})$. Then $\rho(f) < \rho(g)$ if and only if there exists $Z \in E(\rho)$ such that $f < g$ on Z.*

Proof. Let $f < g$ on some $Z \in E(\rho)$. Then $E(f,g) \cap Z = \emptyset$. Therefore $E(f,g) \notin E(\rho)$. This implies that $(f,g) \notin \rho$. So, $\rho(f) \neq \rho(g)$. By the above theorem, we get $\rho(f) < \rho(g)$.

Conversely let, $\rho(f) < \rho(g)$. Since every maximal congruence is a z-congruence, then by Theorem 6.2 we have $f \leq g$ on some $Z_1 \in E(\rho)$. Again $\rho(f) \neq \rho(g)$, this implies $(f,g) \notin \rho$. Therefore $E(f,g) \notin E(\rho)$ as ρ is a z-congruence. As $E(\rho)$ is an $Z_{\mathcal{A}}$-ultrafilter then there exists $Z_2 \in E(\rho)$ such that $E(f,g) \cap Z_2 = \emptyset$. Then $f < g$ on $Z_1 \cap Z_2 \in E(\rho)$. This completes the proof. □

Theorem 6.4. *Let ρ be a congruence on $\mathcal{M}^+(X,\mathcal{A})$. Then the quotient semiring $\mathcal{M}^+(X,\mathcal{A})/\rho$ is totally ordered if and only if ρ is maximal.*

Proof. First suppose ρ is a maximal congruence. Let $f,g \in \mathcal{M}^+(X,\mathcal{A})$. Set $A_1 = \{x \in X : f(x) \leq g(x)\}$, $A_2 = \{x \in X : g(x) \leq f(x)\}$. Then $A_1, A_2 \in \mathcal{A}$ and $A_1 \cup A_2 = X$. Since ρ is a maximal congruence, it is prime. Therefore $E(\rho)$ is a prime $Z_{\mathcal{A}}$-filter. Thus either $A_1 \in E(\rho)$ or $A_2 \in E(\rho)$. Now $f \leq g$ on A_1 and $g \leq f$ on A_2. Then by Theorem 6.2 either $\rho(f) \leq \rho(g)$ or $\rho(g) \leq \rho(f)$. Hence $\mathcal{M}^+(X,\mathcal{A})/\rho$ is totally ordered semiring.

Conversely, let the quotient semiring $\mathcal{M}^+(X,\mathcal{A})/\rho$ be a totally ordered semiring. Then it follows from Chapter II, Theorem 7.1 of [12] that the difference ring of $\mathcal{M}^+(X,\mathcal{A})/\rho$ is of the form $\mathcal{M}(X,\mathcal{A})/k$, where k is a congruence on $\mathcal{M}(X,\mathcal{A})$ and $k^{\nabla} = \rho$. Again $\mathcal{M}(X,\mathcal{A})/k$ is a totally ordered ring and $\mathcal{M}(X,\mathcal{A})/k \cong \mathcal{M}(X,\mathcal{A})/I_k$, where I_k is the corresponding ideal to the congruence k in $\mathcal{M}(X,\mathcal{A})$. Then it follows that $\mathcal{M}(X,\mathcal{A})/I_k$ is a totally ordered ring. Hence, I_k is a maximal ideal by Theorem 3.5 of [4]. Therefore, the corresponding congruence k of I_k on $\mathcal{M}(X,\mathcal{A})$ is a maximal congruence, so the congruence ρ on $\mathcal{M}^+(X,\mathcal{A})$ is a maximal congruence. □

We can easily prove the following result.

Theorem 6.5. *Let ρ be maximal congruence on $\mathcal{M}^+(X,\mathcal{A})$. then the mapping $\phi : \mathbb{R}_+ \to \mathcal{M}^+(X,\mathcal{A})/\rho$, defined by $\phi(r) = \rho(\boldsymbol{r})$ is an order preserving isomorphism from $\mathbb{R}_+$ into $\mathcal{M}^+(X,\mathcal{A})/\rho$.*

Definition 6.6. *Let ρ be a maximal congruence on $\mathcal{M}^+(X,\mathcal{A})$. Then ρ is called real if ϕ is onto where ϕ is defined in Theorem 6.5. A maximal congruence is said to be hyper-real if it is not real.*

Theorem 6.7. *A maximal congruence ρ on $\mathcal{M}^+(X,\mathcal{A})$ is real if and only if $\mathcal{M}^+(X,\mathcal{A})/\rho$ is isomorphic to $\mathbb{R}_+$.*

Proof. If ρ is real then by the definition of real congruence $\mathcal{M}^+(X,\mathcal{A})/\rho$ is isomorphic to $\mathbb{R}_+$.

Conversely let, $\mathcal{M}^+(X,\mathcal{A})/\rho$ be isomorphic to $\mathbb{R}_+$ and ψ is an isomorphism form $\mathcal{M}^+(X,\mathcal{A})/\rho$ onto $\mathbb{R}_+$. Then $\phi \circ \psi$ is an isomorphism from $\mathbb{R}_+$ into $\mathbb{R}_+$. But only non-zero isomorphism from $\mathbb{R}_+$ into $\mathbb{R}_+$ is the identity map. Therefore $\phi \circ \psi$ is the identity map. So ϕ is onto. Hence ρ is a real maximal congruence. □

Theorem 6.8. *For each $x \in X$, the fixed congruence $\rho_x = \{(f,g) \in \mathcal{M}^+(X,\mathcal{A}) \times \mathcal{M}^+(X,\mathcal{A}) : f(x) = g(x)\}$ on $\mathcal{M}^+(X,\mathcal{A})$ is real.*

Proof. Follows from Theorem 5.4 and Theorem 6.7. □

Lemma 6.9, Theorem 6.10, Theorem 6.11, 6.12 follows arguing similarly as in the proof of Lemma 4.9, Theorem 4.8, Theorem 4.10, and Theorem 4.11 respectively in [3].

Lemma 6.9. *For any maximal congruence ρ on $\mathcal{M}^+(X,\mathcal{A})$ each non-zero element in $\mathcal{M}^+(X,\mathcal{A})/\rho$ has multiplicative inverse.*

Theorem 6.10. *A maximal congruence ρ on $\mathcal{M}^+(X,\mathcal{A})$ is real if and only if the set $\{\rho(\boldsymbol{n}) : n \in \mathbb{N}\}$ is cofinal in $\mathcal{M}^+(X,\mathcal{A})/\rho$.*

An element a in a totally ordered semiring S is called *infinitely large* if $a \geq n$ for all $n \in \mathbb{N}$.

Theorem 6.11. *Let ρ be a maximal congruence on $\mathcal{M}^+(X,\mathcal{A})$ and $f \in \mathcal{M}^+(X,\mathcal{A})$. Then the following statements are equivalent:*

1) *$\rho(f)$ is infinitely large.*
2) *For all $n \in \mathbb{N}$, $Z_n = \{x \in X : f(x) \geq n\} \in E(\rho)$.*
3) *For all $n \in \mathbb{N}$, $(f \wedge \boldsymbol{n}, \boldsymbol{n}) \in \rho$.*
4) *f is unbounded on each member of $E(\rho)$.*

Theorem 6.12. *A maximal congruence ρ is real if and only if $E(\rho)$ is closed under countable intersection.*

Theorem 6.13. *A maximal congruence ρ on $\mathcal{M}^+(X,\mathcal{A})$ is real if and only if $Z^{-1}[E(\rho)]$ is a real maximal ideal in $\mathcal{M}(X,\mathcal{A})$.*

Proof. A maximal ideal M in $\mathcal{M}(X,\mathcal{A})$ is real if and only if $Z[M]$ is closed under countable intersection by Theorem 2.2[1]. Therefore the maximal ideal $Z^{-1}[E(\rho)]$ is real if and only if $ZZ^{-1}E(\rho) = E(\rho)$ is closed under countable intersection if and only if ρ is real by Theorem 6.12. This completes the proof. $\square$

Let $\mathcal{RMC}ong(\mathcal{M}^+(X,\mathcal{A})) = \{\rho \in \mathcal{MC}ong(\mathcal{M}^+(X,\mathcal{A})) : \rho$ is a real maximal congruence$\}$ and $RMax(X,\mathcal{A})$ be the set of all real maximal ideals of $\mathcal{M}(X,\mathcal{A})$). Topologically, $\mathcal{RMC}ong(\mathcal{M}^+(X,\mathcal{A}))$ is a subspace of $\mathcal{MC}ong(\mathcal{M}^+(X,\mathcal{A}))$ with basic closed sets
$\mathfrak{m}^R(f,g) = \{\rho \in \mathcal{RMC}ong(\mathcal{M}^+(X,\mathcal{A}))\colon (f,g) \in \rho\}$.

Theorem 6.14. *The map $\tilde{\eta} : \mathcal{RMC}ong(\mathcal{M}^+(X,\mathcal{A})) \to RMax(X,\mathcal{A})$ is a homeomorphism, where $\tilde{\eta}$ is the restriction map $\eta|_{\mathcal{RMC}ong(\mathcal{M}^+(X,\mathcal{A}))}$ (cf. Theorem 5.2).*

Proof. For $\rho \in \mathcal{RMC}ong(\mathcal{M}^+(X,\mathcal{A}))$, $\tilde{\eta}(\rho) \in RMax(X,\mathcal{A})$ by Theorem 6.13. In view of Theorem 5.2 it is only to show that $\tilde{\eta}$ is onto. Let $M \in RMax(X,\mathcal{A})$. Then $Z[M]$ is an $Z_{\mathcal{A}}$-ultrafilter. Then $Z[M] = E(\rho)$ for a unique maximal congruence ρ on $\mathcal{M}^+(X,\mathcal{A})$. Since M is real, $Z[M] = E(\rho)$ is closed under countable intersection. Then by Theorem 6.12 ρ is real i.e., $\rho \in \mathcal{RMC}ong(\mathcal{M}^+(X,\mathcal{A}))$. Also $\tilde{\eta}(\rho) = Z^{-1}E(\rho) = Z^{-1}Z[M] = M$. Thus $\tilde{\eta}$ is onto. Moreover if $\mathcal{M}^R_f$ is a basic closed set in $RM(X,\mathcal{A})$, then $\tilde{\eta}(\mathfrak{m}^R(f,g)) = \mathcal{M}^R_{f-g}$ for $f,g \in \mathcal{M}^+(X,\mathcal{A})$ and $\tilde{\eta}^{-1}(\mathcal{M}^R_h) = \mathfrak{m}^R(|h|,\mathbf{0})$ for $h \in \mathcal{M}(X,\mathcal{A})$. This completes the proof. $\square$

The collection $\mathcal{RMC}ong(\mathcal{M}^+(X,\mathcal{A}))$ can be made into a measurable space also.

Theorem 6.15. *For a measurable space $(X,\mathcal{A})$,*

$$(\mathcal{RMC}ong(\mathcal{M}^+(X,\mathcal{A})), \{\mathfrak{m}^R(f,g)\colon f,g \in \mathcal{M}^+(X,\mathcal{A})\})$$

forms a T-measurable space.

Proof. To show that $\{\mathfrak{m}^R(f,g)\colon f,g \in \mathcal{M}^+(X,\mathcal{A})\}$ is a σ-algebra on $\mathcal{RMC}ong(\mathcal{M}^+(X,\mathcal{A})$. First we observe that $\mathfrak{m}^R(\mathbf{1},\mathbf{0}) = \emptyset$. Indeed, no proper congruence contains the identity pair $(\mathbf{1},\mathbf{0})$. Next, for any $\mathfrak{m}^R(f,g)$, the complement is
$\mathfrak{m}^R(f,g)^c = \mathcal{RMC}ong(\mathcal{M}^+(X,\mathcal{A})) \setminus \{\mathfrak{m}^R(f,g)\} = \mathfrak{m}^R(\chi_{Z(f)\cup Z(g)},\mathbf{0})$. Indeed the twisted product $(f,g) \cdot_t (\chi_{Z(f)\cup Z(g)},\mathbf{0}) \in \rho$, for all $\rho \in \mathfrak{m}^R(f,g)^c$ and ρ does not contain (f,g). Since every maximal congruence is prime, $(\chi_{Z(f)\cup Z(g)},\mathbf{0}) \in \rho$. Hence $\mathfrak{m}^R(f,g)^c \subseteq \mathfrak{m}^R(\chi_{Z(f)\cup Z(g)},\mathbf{0})$ and reverse inclusion follows easily. Finally, let $\rho \in \bigcup_{n\in\mathbb{N}} \mathfrak{m}^R(f_n,g_n)$. Then $\rho \in \mathfrak{m}^R(f_n,g_n)$ for some $n \in \mathbb{N}$, so $(f_n,g_n) \in \rho$. Therefore $E(\chi_{\bigcap_{k\in\mathbb{N}} E^c(f_k,g_k)},\mathbf{0}) \supseteq E(f_n,g_n)$. Hence $\bigcup_{n\in\mathbb{N}} \mathfrak{m}^R(f_n,g_n) \subseteq \mathfrak{m}^R(\chi_{\bigcap_{k\in\mathbb{N}} E^c(f_k,g_k)},\mathbf{0})$. The reverse inclusion follows easily. Therefore $\{\mathfrak{m}^R(f,g)\colon f,g \in \mathcal{M}^+(X,\mathcal{A})\}$ is a σ-algebra on $\mathcal{RMC}ong(\mathcal{M}^+(X,\mathcal{A})$. Moreover, since all fixed maximal congruences are real (cf. Theorem 6.8), the σ-algebra $\{\mathfrak{m}^R(f,g)\colon f,g \in \mathcal{M}^+(X,\mathcal{A})\}$ separates points (cf. Theorem 5.4). $\square$

In view of Theorem 6.14, we can define realcompactness as follows: A measurable space $(X, \mathcal{A})$ is said to be realcompact if every real maximal congruence on $\mathcal{M}^+(X, \mathcal{A})$ is fixed.

Let us denote the set of all fixed maximal congruences as $\mathcal{F}\mathcal{M}Cong(\mathcal{M}^+(X, \mathcal{A}))$.Then by Theorem 6.8, $\mathcal{F}\mathcal{M}Cong(\mathcal{M}^+(X, \mathcal{A})) \subseteq \mathcal{R}\mathcal{M}Cong(\mathcal{M}^+(X, \mathcal{A}))$. Equality holds for realcompact spaces.

Lemma 6.16. *For any measurable space $(X, \mathcal{A})$, we have $\mathcal{F}\mathcal{M}Cong(\mathcal{M}^+(X, \mathcal{A}) \cong X$.*

Proof. Follows from Proposition 5.9 and Theorem 5.11. □

Theorem 6.17. *For a realcompact space $(X, \mathcal{A})$ the measurable spaces $(RMax(X, \mathcal{A}), \{\mathcal{B}_f^R : f \in f \in \mathcal{M}(X, \mathcal{A})\})$ (cf. Theorem 2.4, [1]) and $(\mathcal{R}\mathcal{M}Cong(\mathcal{M}^+(X, \mathcal{A})), \{\mathfrak{m}^R(f, g) : f, g \in \mathcal{M}^+(X, \mathcal{A})\})$ are homeomorphic.*

Proof. Since $(X, \mathcal{A})$ is a realcompact space, $\mathcal{R}\mathcal{M}Cong(\mathcal{M}^+(X, \mathcal{A})) = \mathcal{F}\mathcal{M}Cong(\mathcal{M}^+(X, \mathcal{A}))$ and $RMax(X, \mathcal{A}) = FMax(X, \mathcal{A})$, where $FMax(X, \mathcal{A})$ is the set of all fixed maximal ideals of $\mathcal{M}(X, \mathcal{A})$. By Lemma 6.16 and Theorem 2.6 of [1], we conclude that $\mathcal{R}\mathcal{M}Cong(\mathcal{M}^+(X, \mathcal{A})) \cong X \cong RMax(X, \mathcal{A})$. □

Theorem 6.18. *Two realcompact measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are homeomorphic if and only if $\mathcal{M}^+(X, \mathcal{A})$ and $\mathcal{M}^+(Y, \mathcal{B})$ are isomorphic as semirings.*

Proof. If $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are homeomorphic, then $\mathcal{M}(X, \mathcal{A})$ and $\mathcal{M}(Y, \mathcal{B})$ are isomorphic as semirings. Since every ring homomorphism ϕ between $\mathcal{M}(X, \mathcal{A})$ and $\mathcal{M}(Y, \mathcal{B})$ is also a lattice homomorphism, the homomorphism preserves order. Indeed, if $f \leq g$ in $\mathcal{M}(X, \mathcal{A})$, then $g - f$ is a square and so $\phi(g - f)$ is a square in $\mathcal{M}(Y, \mathcal{B})$. Hence $\phi(f) \leq \phi(g)$. Therefore any isomorphism between $\mathcal{M}(X, \mathcal{A})$ and $\mathcal{M}(Y, \mathcal{B})$ gives rise to an isomorphism of the positive cones $\mathcal{M}^+(X, \mathcal{A})$ and $\mathcal{M}^+(Y, \mathcal{B})$.

Conversely, if there is a semiring isomorphism $\phi : \mathcal{M}^+(X, \mathcal{A}) \to \mathcal{M}^+(Y, \mathcal{B})$, then there is a homeomorphism $\psi : (\mathcal{R}\mathcal{M}Cong(\mathcal{M}^+(X, \mathcal{A})), \{\mathfrak{m}^R(f, g) : f, g \in \mathcal{M}^+(X, \mathcal{A})\}) \to (\mathcal{R}\mathcal{M}Cong(\mathcal{M}^+(Y, \mathcal{B})), \{\mathfrak{m}^R(h, k) : h, k \in \mathcal{M}^+(Y, \mathcal{B})\})$ defined as $\psi(\mathfrak{m}^R(f, g)) = \mathfrak{m}^R(\phi(f), \phi(g))$. For realcompact measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ we have, $\mathcal{R}\mathcal{M}Cong(\mathcal{M}^+(X, \mathcal{A}) \cong X \cong RMax(X, \mathcal{A})$ and $\mathcal{R}\mathcal{M}Cong(\mathcal{M}^+(Y, \mathcal{B})) \cong Y \cong RMax(Y, \mathcal{B})$ (cf. Theorem 6.17). Therefore $X \cong Y$ as measurable spaces. □

By **Meas**, we denote the category of measurable spaces and measurable functions. In particular, we are interested in the subcategory **RCTMeas**, consisting of realcompact T-measurable spaces. Theorem 6.18 solves the isomorphism problem for the semirings of the form $\mathcal{M}^+(X, \mathcal{A})$. We arrive at the following easy proposition.

Proposition 6.19. *The category **RCTMeas** is dual to the full subcategory of **CRig** (the category of commutative semirings and semiring homomorphisms) consisting of the semirings of the form $\mathcal{M}^+(X, \mathcal{A})$.*

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