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The zero-divisor graph of 2×2 matrix ring and its energies

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Abstract. Let $R = M_2(\mathbb{F})$ be a 2×2 matrix ring over a finite field $\mathbb{F}$. The zero-divisor graph of R, denoted by $\Gamma^t(R)$, is a simple undirected graph with the vertex set consisting of all nonzero left zero-divisors in R, and two vertices A and B being adjacent if and only if $AB^t = 0$, where B^t is a transpose of the matrix B. In this paper, we consider a subgraph of $\Gamma^t(R)$ denoted by $IdN(R)$ whose vertex set consists of all non-trivial idempotent and nonzero nilpotent elements in R. It has been established that the components of $IdN(R)$ are either complete graphs or complete bipartite graphs. Additionally, a necessary and sufficient condition for the regularity of $IdN(R)$ is obtained. We also analyze the adjacency and Laplacian spectra, as well as the energy and Laplacian energy of $IdN(R)$. Furthermore, it is proved that Beck's conjecture holds for $IdN(R)$.

1. Introduction

The energy of a graph is a spectral quantity introduced in the 1970s, having a chemical background [9]. Its mathematical theory is nowadays well elaborated [8, 16, 22] and it found diverse applications across fields like mathematics, chemistry, computer science, network analysis, but also social science, environmental analysis, machine learning, bioinformatics, etc [10]. In this paper, we present some graph-energy-related studies in the area of associative ring algebra.

Let R be an associative ring. A mapping $*$ defined on R is called an *involution* if $(a+b)^* = a^*+b^*$, $(ab)^* = b^*a^*$, and $(a^*)^* = a$ for all $a, b \in R$. An element e in a ring R is called an *idempotent* if $e^2 = e$. The elements 0 and 1 are called trivial idempotents. The idempotents other than 0 and 1 are called non-trivial idempotents. The set of idempotent elements in R is denoted by $Id(R)$. In [18], authors studied rings with involution.

I. Beck [3] introduced the concept of the zero-divisor graph of commutative rings, primarily for the purpose of studying graph coloring. He conjectured that $\chi(\Gamma(R)) = \omega(\Gamma(R))$. Patil and Waphare [21] introduced the zero-divisor graph of $*$-rings, denoted by $\Gamma^*(R)$. This is a simple undirected graph whose vertex set is the set of all nonzero zero-divisors in R, and two vertices x and y are adjacent if and only if $xy^* = 0$. Transpose of a matrix is an involution on a matrix ring $M_2(\mathbb{F})$. Hence the zero-divisor graph of R,

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$\Gamma^*(R)$ is denoted by $\Gamma^t(R)$, and it is a simple undirected graph with the vertex set of all nonzero zero-divisors in R, and two vertices A and B are adjacent if and only if $AB^t = 0$.

For details of the study of zero-divisor graphs see [1, 2, 5, 13, 15, 19, 23].

Let $\mathbb{F}$ be a finite field and $R = M_2(\mathbb{F})$, we consider the subgraph of $\Gamma^t(R)$ having vertices all non-trivial idempotents and nonzero nilpotent elements, it is denoted by $IdN(R)$ (as it contains only the idempotent and nilpotent elements). In $IdN(R)$ two vertices A and B are adjacent if and only if $AB^t = 0$. In this paper, we derive necessary and sufficient conditions for the regularity of $IdN(R)$. It is shown that $IdN(R)$ is disconnected, and its components form either complete bipartite graphs or complete graphs. Furthermore, we explicitly determine the adjacency spectrum, Laplacian spectrum, the energy and the Laplacian energy of $IdN(R)$. Further we prove that Beck's conjecture holds for $IdN(R)$.

2. Preliminaries

We begin by introducing definitions required for the subsequent discussion.

For a vertex $u \in V(G)$, let $C(u)$ represents the subgraph of G induced by the set of all vertices connected to u by a path. This subgraph $C(u)$ is called the *connected component* of G containing u. A graph G is connected if and only if number of connected components in G is 1. If the degree of every vertex in G is k, then G is called *k-regular* and k is called the valency of G.

A graph is called *complete* if any two distinct vertices are adjacent. The complete graph with n vertices is denoted by K_n. A graph is called bipartite if its vertex set can be partitioned into two disjoint subsets U and V, such that there are no edges between vertices within U or within V. A bipartite graph is complete if every vertex in U is connected to every vertex in V by a path. Such a graph is denoted as $K_{m,n}$, where $m = |U|$ and $n = |V|$. Let G be a simple graph with vertices $v_1, v_2, \ldots, v_n$, the *adjacency matrix* of G denoted by $A(G)$, is an $n \times n$ matrix $[a_{ij}]$, whose rows and columns are indexed by $v_1, v_2, \ldots, v_n$, where $a_{ij} = 1$ if the vertices v_i and v_j are adjacent with each other, and 0 otherwise. The multiset of eigenvalues of $A(G)$ is called the *adjacency spectrum* of G. The *Laplacian matrix* of G is given by $L(G) = D - A(G)$, where D is the diagonal matrix with entries as degrees of vertices. The *Laplacian spectrum* of G is the multiset of eigenvalues of $L(G)$. The *energy* of a graph G denoted by $E(G)$ is the sum of the absolute values of the eigenvalues of the adjacency matrix of G. Let G be a graph with n vertices and m edges, and let $\mu_1, \mu_2, \ldots, \mu_n$ be the Laplacian eigenvalues of G, then the Laplacian energy of G is denoted by $LE(G)$ and is given by $LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|$ (see [11]).

The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number of colors required to color the vertices of G in such a way that no two adjacent vertices share the same color. The *clique number* of G, denoted by $\omega(G)$, is the size of the largest clique (a complete subgraph) within G.

For undefined terminology and notations, see [7, 12].

Let $\mathbb{F}$ be a finite field. Let $\mathbb{Z}(M_2(\mathbb{F}))$ denote the set of all non-trivial idempotent and nonzero nilpotent elements in $M_2(\mathbb{F})$. We use the following notations:

$E_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $E^0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_a = \begin{pmatrix} 0 & 0 \\ a & 1 \end{pmatrix}$, $E^a = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}$, $F^a = \begin{pmatrix} 0 & a \\ 0 & 1 \end{pmatrix}$, $F_a = \begin{pmatrix} 1 & 0 \\ a & 0 \end{pmatrix}$, $E_{ij} = \begin{pmatrix} i & j \\ i(1-i)j^{-1} & 1-i \end{pmatrix}$,
$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $M = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $N_k = \begin{pmatrix} 1 & k \\ -\frac{1}{k} & -1 \end{pmatrix}$, where $i,\ j,\ a,\ k \in \mathbb{F} \setminus \{0\}$, and $i \neq 1$.

In [14], authors studied the generalized projections in $\mathbb{Z}_n$. In [4] authors studied 2×2 operator matrices and their applications. Recently, in [17, 20], idempotent and nilpotent elements in $Z(M_2(\mathbb{F}))$ were studied. The classification of idempotent and nilpotent elements in the ring $Z(M_2(\mathbb{F}))$ is given as in the following result.

Lemma 2.1. *[17] Every idempotent element in* $Z(M_2(\mathbb{F}))$ *has one of the following forms.*

(*i*) $E_0, E^0, E_a, E^a, F_a, F^a$ *for some* $a \in \mathbb{F}\setminus\{0\}$,

(*ii*) E_{ij} *for some nonzero* $i \in \mathbb{F}\setminus\{0,1\}$, $j \in \mathbb{F}\setminus\{0\}$.

Lemma 2.2. *[17] Every nilpotent element in $Z(M_2(\mathbb{F}))$ has one of the following forms.*

(i) aN, aM *for some* $a \in \mathbb{F}\backslash\{0\}$,
(ii) aN_k *for some* $a, k \in \mathbb{F}\backslash\{0\}$.

Let n denote the number of elements in the field $\mathbb{F}$ i.e., $n = |\mathbb{F}|$. In what follows, we count the total number of non-trivial idempotent and nonzero nilpotent elements in $Z(M_2(\mathbb{F}))$.

Remark 2.3. *The non-trivial idempotent elements in $Z(M_2(\mathbb{F}))$ are given in Lemma 2.1. We count their cardinalities: $|E_0| = |E^0| = 1$, since $a \in \mathbb{F} \setminus \{0\}$, $|E_a| = |E^a| = |F_a| = |F^a| = n-1$. Now, $i \in \mathbb{F}\backslash\{0,1\}$, $j \in \mathbb{F}\backslash\{0\}$, therefore $|E_{ij}| = (n-1)(n-2)$. Thus, the total number of non-trivial idempotent elements in $Z(M_2(\mathbb{F}))$ is $2+4(n-1)+(n-1)(n-2) = n(n+1)$. The nonzero nilpotent elements in $Z(M_2(\mathbb{F}))$ are given in Lemma 2.2. For $a, k \in \mathbb{F} \setminus \{0\}$, $|aM| = |aN| = n-1$, $|N_k| = n-1$. Therefore, $|aN_k| = (n-1)^2$. Therefore, the number of nonzero nilpotent elements in $Z(M_2(\mathbb{F}))$ is $2(n-1) + (n-1)^2 = n^2 - 1$.*

3. The graph $IdN(R)$

Let $R = M_2(\mathbb{F})$. In this section, we study the subgraph $IdN(R)$ of $\Gamma^t(R)$ whose vertex set consists of all non-trivial idempotent and nonzero nilpotent elements in R.

Definition 3.1. *Let $R = M_2(\mathbb{F})$. The graph $IdN(R)$ is a simple undirected graph with vertex set consisting of all non-trivial idempotent and nonzero nilpotent elements in R, and two distinct vertices A and B are adjacent if and only if $AB^t = 0$.*

Recall the following result from Number Theory. A positive integer a is said to be a quadratic residue modulo n if there is an integer x such that $x^2 \equiv a \pmod{n}$. Let p be a prime. Then -1 is quadratic residue modulo p if and only if $a^2 = -1$ for some $a \in \mathbb{F} \setminus \{0\}$, where $\mathbb{F}$ is a field with p^k elements.

Remark 3.2. *Let $\mathbb{F}_{p^k}$ be a field with p^k elements, where p is a prime and k is a positive integer. Then:*

1. *If $p \equiv 1 \pmod{4}$, then there are exactly 2 elements $a \in \mathbb{F}$ with $a^2 = -1$.*
2. *If $p \equiv 3 \pmod{4}$, then there is no element $a \in \mathbb{F}$ with $a^2 = -1$.*
3. *If $p = 2$, then $(a+1)^2 = a^2 + 1 = 0$ implies $a = -1$, since for $p = 2$, we have $-1 = 1$, hence there is exactly one element $a \in \mathbb{F}$ with $a^2 = -1$.*

Let $\mathcal{N}(A)$ denote the set of vertices adjacent to A in G. That is $\mathcal{N}(A) = \{B \in V(G)\backslash\{A\} : AB^t = 0\}$. Henceforth, we assume that $\mathbb{F}$ is a finite field with $n = |\mathbb{F}| = p^k$, where p is a prime and k is a positive integer.
Let $R = M_2(\mathbb{F})$ and aM, aN, aN_k are nilpotent elements in $Z(M_2(\mathbb{F}))$. Observe the following:

1. For all $a, b \in \mathbb{F} \setminus \{0\}$, $bM(aN)^t = 0$. Hence every element from aM is adjacent to each element in bN. Since $(aM)(bM)^t \neq 0$ for all $a, b \in \mathbb{F} \setminus \{0\}$. Hence no element of type aM is adjacent to any other element bM i.e., $\{aM : a \in \mathbb{F} \setminus \{0\}\}$ is an independent set. Similarly, $\{aN : a \in \mathbb{F} \setminus \{0\}\}$ is an independent set.
2. Since $bN_k(aN_{-1/k})^t = 0$ for all $a, b \in \mathbb{F} \setminus \{0\}$. Hence every element of the form bN_k is adjacent to each element of the form $aN_{-1/k}$. And $(aN_k)(bN_k)^t \neq 0$ for all $a, b \in \mathbb{F} \setminus \{0\}$. Hence no element of the from aN_k is adjacent to any other element in bN_k i.e., $\{aN_k : a \in \mathbb{F} \setminus \{0\}\}$ is an independent set.

In [20], it is proved that $GId(R)$ is regular. We characterize the rings $M_2(\mathbb{F})$ for which $IdN(R)$ is regular. The following result shows that $IdN(R)$ is regular for $p \equiv 3 \pmod{4}$.

Lemma 3.3. *Let $R = M_2(\mathbb{F})$ and $p \equiv 3 \pmod{4}$. Then $IdN(R)$ is a $2n-1$ regular graph.*

Proof. Suppose $p \equiv 3 \pmod{4}$. Therefore $a^2 \neq -1$ for each $a \in \mathbb{F}$. First we determine the degrees of the vertices in $IdN(R)$.

1. $\mathcal{N}(E_0) = \{A \in Z(M_2(\mathbb{F}))\colon E_0A^t = 0 \text{ i.e., } AE_0 = 0\}$. Therefore $\mathcal{N}(E_0) = \left\{E^0, F_a, aM\colon a \in \mathbb{F} \setminus \{0\}\right\}$. Hence $|\mathcal{N}(E_0)| = 2n - 1$.
2. $\mathcal{N}(E^0) = \left\{A \in Z(M_2(\mathbb{F}))\colon E^0A^t = 0 \text{ i.e.,} AE^0 = 0\right\}$. Therefore $\mathcal{N}(E^0) = \{E_0, F^a, aN\colon a \in \mathbb{F} \setminus \{0\}\}$. Hence $|\mathcal{N}(E^0)| = 2n - 1$.
3. $\mathcal{N}(E_a) = \{A \in Z(M_2(\mathbb{F}))\colon E_aA^t = 0 \text{ i.e.,} A(E_a)^t = AF^a = 0\}$. Observe that $E^{-a}F^a = 0$, $E_{-1/a}F^a = 0$. Therefore $E_{-1/a} \in \mathcal{N}(E_a)$ if and only if $E_{-1/a} \neq E_a$ i.e. if and only if $a^2 \neq -1$ in $\mathbb{F}$.
   Next, if $E_{b,c} \in \mathcal{N}(E_a)$, then $E_{b,c}F^a = 0$. This gives $\begin{bmatrix} b & c \\ b(1-b)c^{-1} & 1-b \end{bmatrix}\begin{bmatrix} 0 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, which yields $ba + c = 0$. That is $c = -ba$. Now, $N_cF^a = 0$ if and only if $\begin{bmatrix} 1 & c \\ -1/c & -1 \end{bmatrix}\begin{bmatrix} 0 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, which yields $a + c = 0$. That is $c = -a$ i.e., $N_{-a} \in \mathcal{N}(E_a)$. Therefore $\mathcal{N}(E_a) = \{E^{-a}, E_{-1/a}, E_{b,-ba}, kN_{-a}\colon b \in \mathbb{F} \setminus \{0,1\}, k \in \mathbb{F} \setminus \{0\}\}$, for $a^2 \neq -1$. Hence $|\mathcal{N}(E_a)| = 2n - 1$.
4. $\mathcal{N}(E^a) = \{A \in Z(M_2(\mathbb{F}))\colon E^aA^t = 0 \text{ i.e.,} A(E^a)^t = AF_a = 0\}$. Note that $E_{-a}F_a = 0$, $E^{-1/a}F_a = 0$. Therefore $E^{-1/a} \in \mathcal{N}(E_a)$ if and only if $E^{-1/a} \neq E^a$ i.e., if and only if $a^2 \neq -1$ in $\mathbb{F}$. Next, if $E_{b,c} \in \mathcal{N}(E_a)$, then $E_{b,c}F_a = 0$, which yields $b + ac = 0$. That is $c = -ba^{-1}$. Now, $N_cF_a = 0$ gives $1 + ac = 0$. This implies $c = -a^{-1}$ i.e., $N_{-a^{-1}} \in \mathcal{N}(E^a)$. Therefore $\mathcal{N}(E^a) = \{E_{-a}, E^{-a^{-1}}, E_{b,-ba^{-1}}, kN_{-a^{-1}}\colon b \in \mathbb{F} \setminus \{0,1\}, k \in \mathbb{F} \setminus \{0\}\}$, for $a^2 \neq -1$. Hence $|\mathcal{N}(E^a)| = 2n - 1$.
5. $\mathcal{N}(F^a) = \{A \in Z(M_2(\mathbb{F}))\colon F^aA^t = 0 \text{ i.e.,} A(F^a)^t = AE_a = 0\}$. Note that $ME_a = E^0E_a = 0$ and $F_cE_{a_j} = 0$ for $c \in \mathbb{F} \setminus \{0\}$. If $E_{b,c}E_a = 0$, then we get $c = 0$, which is a contradiction. Hence $E_{b,c}E_a \neq 0$ for any b, c. Therefore $\mathcal{N}(F^a) = \{E^0, F_c, cM\colon c \in \mathbb{F} \setminus \{0\}\}$. Hence $|\mathcal{N}(F^a)| = 2n - 1$.
6. $\mathcal{N}(F_a) = \{A \in Z(M_2(\mathbb{F}))\colon F_aA^t = 0 \text{ i.e.,} A(F_a)^t = AE^a = 0\}$. Note that $NE^a = E_0E^a = 0$ and $F^cE^a = 0$ for $c \in \mathbb{F} \setminus \{0\}$. If $E_{b,c}E^a = 0$, then we get $b = 0$, which is a contradiction. Hence $E_{b,c}E^a \neq 0$ for any b, c. Therefore $\mathcal{N}(F_a) = \{E_0, F^c, cN\colon c \in \mathbb{F} \setminus \{0\}\}$. Hence $|\mathcal{N}(F_a)| = 2n - 1$.
7. $\mathcal{N}(E_{b,c}) = \{A \in Z(M_2(\mathbb{F}))\colon E_{b,c}A^t = 0\}$. Note that $E_{b,c}E_a \neq 0$ giving $(E_a)^t = F^a \notin \mathcal{N}(E_{b,c})$. Also, $E_{b,c}E^a \neq 0$ implies $(E^a)^t = F_a \notin \mathcal{N}(E_{b,c})$. Also, $E_{b,c}N \neq 0$ and $E_{b,c}M \neq 0$. Therefore $M, N \notin \mathcal{N}(E_{b,c})$. Next, $E_{b,c}(N_k)^t = 0$ gives $b + ck = 0$ i.e., $k = -bc^{-1}$. Therefore $N_{-bc^{-1}} \in \mathcal{N}(E_{b,c})$. $E_{b,c}(E_a)^t = 0$ if and only if $ba + c = 0$, which yields $a = -cb^{-1}$. Now, $E_{b,c}(E^a)^t = 0$ if and only if $b + ac = 0$, which implies $a = -bc^{-1}$. Next, $E_{b,c}(E_{a_k,a_l})^t = 0$ if and only if $ba_k + a_lc = 0$ i.e., $a_l = -bc^{-1}a_k$. Therefore $\mathcal{N}(E_{b,c}) = \{E_{-cb^{-1}}, E^{-bc^{-1}}, E_{a_k,-a_kbc^{-1}}, kN_{-bc^{-1}}\colon a_k \in \mathbb{F} \setminus \{0,1\}, k \in \mathbb{F} \setminus \{0\}\}$. Hence $|\mathcal{N}(E_{b,c})| = 2n - 1$.

Thus, if $p \equiv 3 \pmod 4$, then $IdN(R)$ is a $2n - 1$ regular graph. □

The converse of the above result is also true. We prove the converse of the above result in Theorem 3.9. In the following result, we prove that if $p \equiv 3 \pmod 4$, then $IdN(R)$ is disconnected and the components of $IdN(R)$ are complete bipartite graphs.

Proposition 3.4. *Let $R = M_2(\mathbb{F})$ and $p \equiv 3$ (mod 4). Then $IdN(R)$ is a disjoint union of $\dfrac{n+1}{2}$ copies of $K_{2n-1,2n-1}$.*

Proof. Suppose $p \equiv 3 \pmod 4$, hence $a^2 \neq -1$ for any $a \in \mathbb{F}$.
Let $X_1 = \{E_0, E^0, F_a, F^a, aM, aN\}$ and $Y_1 = \{E_b, E^b, E_{a,b}, kN_b\colon a, b, k \in \mathbb{F} \setminus \{0\}, a \neq 1\}$. From Figures 1, 2, and 3, observe that no vertex in X_1 is connected to any vertex in Y_1. Hence X_1 and Y_1 form two disconnected components of $IdN(R)$. Additionally, the vertices in X_1 form a complete bipartite graph with the partition of vertices $\{E^0, F_a, aN\}$ and $\{E_0, F^a, aM\}$, where $a \in \mathbb{F} \setminus \{0\}$ as shown in Figure 2. Consider a partition of the vertex set Y_1 as $U_1 = \{E_a, E^{a^{-1}}, E_{b,ba^{-1}}, E^a, E_{a^{-1}}, E_{b,ba}, kN_a, kN_{a^{-1}}\}$ and $V_1 = \{E^{-a}, E_{-a^{-1}}, E_{b,-ba}, E_{-a}, E^{-a^{-1}}, E_{b,-ba^{-1}}, kN_{-a}, kN_{-a^{-1}}\}$. The neighborhoods of the elements in U_1 and V_1 are given below:

1. $\mathcal{N}(E_a) = \{E^{-a}, E_{-a^{-1}}, E_{b,-ba}, kN_{-a}\} = \mathcal{N}(E^{a^{-1}}) = \mathcal{N}(E_{b,ba^{-1}}) = \mathcal{N}(kN_{a^{-1}})$.
2. $\mathcal{N}(E^{-a}) = \{E_a, E^{a^{-1}}, E_{b,ba^{-1}}, kN_{a^-1}\} = \mathcal{N}(E_{-a^{-1}}) = \mathcal{N}(E_{b,-ba}) = \mathcal{N}(kN_{-a})$.
3. $\mathcal{N}(E^a) = \{E_{-a}, E^{-a^{-1}}, E_{b,-ba^{-1}}, kN_{-a^-1}\} = \mathcal{N}(E_{a^{-1}}) = \mathcal{N}(E_{b,ba}) = \mathcal{N}(kN_a)$.
4. $\mathcal{N}(E_{-a}) = \{E^a, E_{a^{-1}}, E_{b,ba}, kN_a\} = \mathcal{N}(E^{-a^{-1}}) = \mathcal{N}(E_{b,-ba^{-1}}) = \mathcal{N}(kN_{-a^{-1}})$.

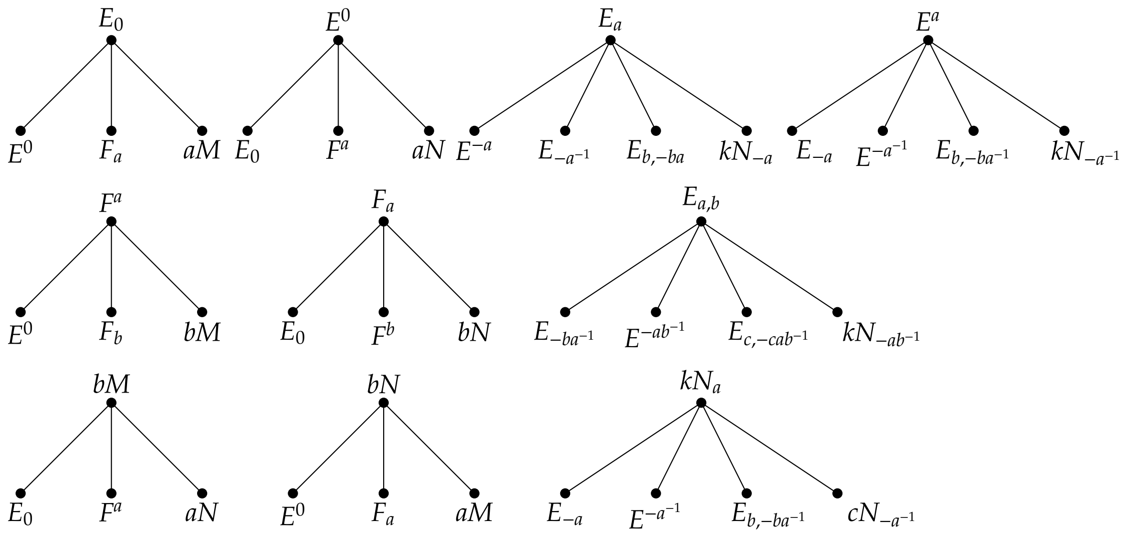


Figure 1: Neighborhoods of vertices in $IdN(R)$

Clearly, $\{U_1, V_1\}$ forms a partition of Y_1 such that no vertex in U_1 is adjacent to any other vertex in U_1 and no vertex in V_1 is adjacent to any other vertex in V_1 (see Figure 3). Therefore, the component Y_1 is a bipartite graph. Hence $IdN(R)$ is a disconnected graph having bipartite components.

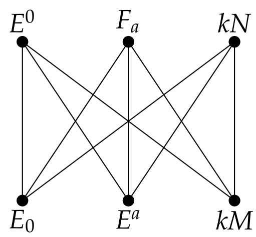


Figure 2: The component X_1

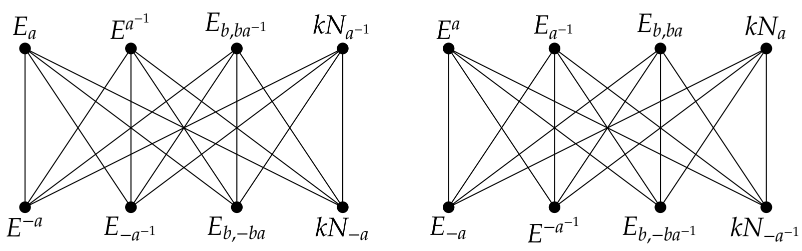


Figure 3: The component Y_1

By Remark 2.3, the number of non-trivial idempotent elements in $IdN(R)$ is $n(n+1)$. Also, the number of nonzero nilpotent elements is n^2-1 . Therefore $|IdN(R)| = n(n+1)+n^2-1 = 2n^2+n-1 = (n+1)(2n-1)$. Since each component of $IdN(R)$ is regular bipartite with valency $2n-1$. Each component of $IdN(R)$ is $K_{2n-1,2n-1}$. Therefore, each component of $IdN(R)$ contains $4n-2$ vertices. Hence the number of connected components in $IdN(R)$ is $\frac{(n+1)(2n-1)}{4n-2} = \frac{n+1}{2}$. Thus, $IdN(R)$ is a disjoint union of $\frac{n+1}{2}$ copies of $K_{2n-1,2n-1}$. □

Recall the following results from spectral graph theory [6]. Let $\lambda^{(s)}$ denote the eigenvalue λ with multiplicity s. Let G be a simple undirected regular graph with valency k. Then:
(i) k is the largest eigenvalue of the adjacency matrix of G.
(ii) The multiplicity of k as an eigenvalue equals the number of connected components of G.
(iii) The adjacency spectrum of K_n is the multiset: $\{(n-1)^{(1)}, (-1)^{(n-1)}\}$.
(iv) The adjacency spectrum of $K_{n,n}$ is the multiset: $\{n^{(1)}, -n^{(1)}, (0)^{(2n-2)}\}$.
(v) The Laplacian spectrum of K_n is the multiset: $\{0^{(1)}, (n)^{(n-1)}\}$.
(vi) The Laplacian spectrum of $K_{n,n}$ is the multiset: $\{0^{(1)}, (n)^{(2n-2)}, (2n)^{(1)}\}$.
We now apply this to determine the spectrum and the energy of $IdN(R)$.

Proposition 3.5. *Let $R = M_2(\mathbb{F})$ and $p \equiv 3 \pmod 4$. Then the adjacency spectrum of $IdN(R)$ is the multiset*

$$\left\{(2n-1)^{(\frac{n+1}{2})}, (-(2n-1))^{(\frac{n+1}{2})}, (0)^{(2(n^2-1))}\right\}, \tag{1}$$

whereas the Laplacian spectrum of $IdN(R)$ is the multiset

$$\left\{0^{(\frac{n+1}{2})}, (2n-1)^{(2(n^2-1))}, (2(2n-1))^{(\frac{n+1}{2})}\right\}. \tag{2}$$

Proof. By Proposition 3.4, $IdN(R)$ has $\frac{n+1}{2}$ connected components, and each component is $K_{2n-1,2n-1}$. Therefore $2n-1$ is the largest eigenvalue of $IdN(R)$, and its multiplicity is $\frac{n+1}{2}$. The eigenvalues of $K_{2n-1,2n-1}$ are given by: $(2n-1)^{(1)}, -(2n-1)^{(1)}, 0^{(4n-4)}$. Thus the adjacency spectrum of $IdN(R)$ is the multiset (1).

The Laplacian eigenvalues of $K_{2n-1,2n-1}$ are $0^{(1)}, (2n-1)^{4n-4}, (2(2n-1))^{(1)}$. Hence the Laplacian spectrum of $IdN(R)$ is the multiset (2). □

Recall that if the graph G is regular, then $LE(G) = E(G)$ ([11], Lemma 1). The following corollary is an immediate consequence of the above proposition.

Corollary 3.6. *Let $R = M_2(\mathbb{F})$ and $p \equiv 3 \pmod 4$. Both the energy and Laplacian energy of $IdN(R)$ are equal to $(2n-1)(n+1)$.*

Proof. By the above proposition, the adjacency spectrum of $IdN(R)$ is the multiset (1), and thus the energy of $IdN(R)$ is given by:

$$E(IdN(R)) = \frac{n+1}{2}|2n-1| + \frac{n+1}{2}|-(2n-1)| = (2n-1)(n+1).$$

By Lemma 3.3, the graph $IdN(R)$ is regular. Therefore the Laplacian energy is same as the energy of $IdN(R)$, i.e., $LE(IdN(R)) = (2n-1)(n+1)$. □

As an application consider the following example.

Example 3.7. *Let $R = M_2(\mathbb{Z}_3)$ be a ring with transpose involution. The idempotent and nilpotent elements aM, aN, aN_j in $M_2(\mathbb{Z}_3)$ are given below:*
$E_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, E^0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_1 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}, E^1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, E^2 = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, F^1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, F^2 = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix},$

$F_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, F_2 = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, E_{22} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, E_{21} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, M = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, N_1 = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, N_2 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix},$
$2M = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, 2N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 2N_1 = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, 2N_2 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}.$

The neighborhoods of the vertices are given below:
$\mathcal{N}(E_0) = \{E^0, F_1, F_2, M, 2M\}, \mathcal{N}(E^0) = \{E_0, F^1, F^2, N, 2N\}, \mathcal{N}(E_1) = \{E_2, E^2, E_{21}, N_2, 2N_2\}, \mathcal{N}(E_2) = \{E_1, E^1, E_{22}, N_1, 2N_1\},$
$\mathcal{N}(E^1) = \{E_2, E^2, E_{21}, N_2, 2N_2\}, \mathcal{N}(E^2) = \{E_1, E^1, E_{22}, N_1, 2N_1\}, \mathcal{N}(F^1) = \{E^0, F_1, F_2, M, 2M\}, \mathcal{N}(F^2) = \{E^0, F_1, F_2, M, 2M\},$
$\mathcal{N}(F_1) = \{E_0, F^1, F^2, N, 2N\}, \mathcal{N}(F_2) = \{E_0, F^1, F^2, N, 2N\}, \mathcal{N}(E_{22}) = \{E_2, E^2, E_{21}, N_2, 2N_2\}, \mathcal{N}(E_{21}) = \{E_1, E^1, E_{22}, N_1, 2N_1\},$
$\mathcal{N}(M) = \{E_0, F^1, F^2, N, 2N\}, \mathcal{N}(N) = \{E^0, F_1, F_2, M, 2M\}, \mathcal{N}(N_1) = \{E_2, E^2, E_{21}, N_2, 2N_2\}, \mathcal{N}(N_2) = \{E_1, E^1, E_{22}, N_1, 2N_1\}.$
The graph $IdN(R)$ is depicted in Figure 4.

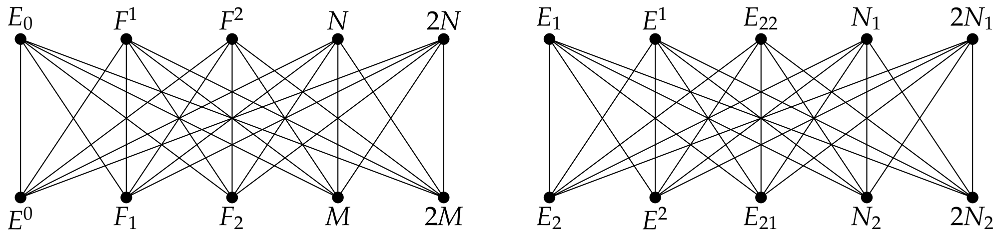


Figure 4: $IdN(M_2(\mathbb{Z}_3))$.

Here, $n = 3$, and thus the graph $IdN(R)$ is $(2n-1) = 5$-regular, with 2 complete bipartite components. The eigenvalues of each component are $-5^{(1)}, 0^{(8)}, 5^{(1)}$. The adjacency spectrum of $IdN(R)$ is the multiset $\{5^{(2)}, -5^{(2)}, 0^{(16)}\}$. The Laplacian spectrum of $IdN(R)$ is $\{0^{(2)}, 5^{(16)}, 10^{(2)}\}$. The energy and Laplacian energy are $E(IdN(R)) = LE(IdN(R)) = 20$.

If $p \not\equiv 3 \pmod 4$, then the following result demonstrates that $IdN(R)$ is not regular.

Lemma 3.8. *Let $R = M_2(\mathbb{F})$ and $p \not\equiv 3 \pmod 4$. Then $IdN(R)$ is not a regular graph.*

Proof. Suppose $p \not\equiv 3 \pmod 4$. Then there exists an element $a \in \mathbb{F}$ with $a^2 = -1$. The neighborhood of E_0 is $\mathcal{N}(E_0) = \left\{E^0, F_a, aM : a \in \mathbb{F} \setminus \{0\}\right\}$, which means that the size of $\mathcal{N}(E_0)$ is $2n-1$. For $a \in \mathbb{F} \setminus \{0\}$, the neighborhood of E_a is given by: $\mathcal{N}(E_a) = \{A \in M_2(\mathbb{F}) : E_a A^t = 0 \text{ i.e., } A(E_a)^t = AF^a = 0\}$. If $E_{b,c} \in \mathcal{N}(E_a)$, then $E_{b,c}F^a = 0$. This implies
$\begin{pmatrix} b & c \\ b(1-b)c^{-1} & 1-b \end{pmatrix} \begin{pmatrix} 0 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, which yields $ba + c = 0$, giving $c = -ba$. Additionally, note that $E^{-a}F^a = 0$ and $E_{-a^{-1}}F^a = 0$. Therefore $E_{-a^{-1}} \in \mathcal{N}(E_a)$ if and only if $E_{-a^{-1}} \neq E_a$ i.e., if and only if $a^2 \neq -1$ in $\mathbb{F}$. Thus, if $a^2 = -1$ in $\mathbb{F}$, then $E_{-a^{-1}} \notin \mathcal{N}(E_a)$. Therefore $\mathcal{N}(E_a) = \{E^{-a}, E_{b,-ba}, kN_a : b, k \in \mathbb{F} \setminus \{0,1\}, b \neq 1\}$, for $a^2 = -1$. Hence for $a^2 = -1$, $|\mathcal{N}(E_a)| = 2n-2$. Since the degree of the vertex E_0 is $2n-1$, but the degree of the vertex E_a is $2n-2$ for $a^2 = -1$, it follows that $IdN(R)$ is not a regular graph. □

The following result gives a necessary and sufficient condition for the regularity of $IdN(R)$.

Theorem 3.9. *Let $R = M_2(\mathbb{F})$. Then $IdN(R)$ is regular if and only if $p \equiv 3 \pmod 4$.*

Proof. The proof follows by Lemmas 3.3 and 3.8. □

The neighborhoods of the elements are given in Lemmas 3.3 and 3.8. The next result characterizes the structure of $IdN(R)$ for $p \not\equiv 3 \pmod 4$.

Proposition 3.10. *Let $R = M_2(\mathbb{F})$ and $p \not\equiv 3 \pmod 4$. Then:*

1. *The components of $IdN(R)$ corresponding to the vertices E^a and E_a are complete bipartite graphs $K_{2n-1,2n-1}$ for $a^2 \neq -1$.*
2. *The components of $IdN(R)$ corresponding to E^a and E_a are complete graphs K_{2n-1} for $a^2 = -1$.*

Proof. Since $p \not\equiv 3 \pmod 4$, there exists an element $a \in \mathbb{F}$ with $a^2 = -1$.
Case (1): Let $a \in \mathbb{F}\backslash\{0\}$, with $a^2 \neq -1$.
(i) Consider the neighborhoods of the vertices adjacent to E^a.
$\mathcal{N}(E^a) = \{E_{-a}, E^{-a^{-1}}, E_{b,-ba^{-1}}, kN_{-a^{-1}} : b, k \in \mathbb{F} \setminus \{0\}, b \neq 1\} = \mathcal{N}(E_{a^{-1}}) = \mathcal{N}(E_{b,ba}) = \mathcal{N}(kN_a)$. Note that $|\mathcal{N}(E^a)| = 2n-1$. And $\mathcal{N}(E^{-a^{-1}}) = \{E_{a^{-1}}, E^a, E_{b,ba}, kN_a : b, k \in \mathbb{F} \setminus \{0\}, b \neq 1\} = \mathcal{N}(E_{-a}) = \mathcal{N}(E_{b,-ba^{-1}}) = \mathcal{N}(kN_{-a^{-1}})$. Observe that $|\mathcal{N}(E^{-a^{-1}})| = 2n-1$.
Let $C = \{E_{b,ba} : b \in \mathbb{F} \setminus \{0,1\}\}$ and $D = \{E_{b,-ba^{-1}} : b \in \mathbb{F} \setminus \{0,1\}\}$. Note that $E_{b,ba}E^t_{c,ca} \neq 0$, and $E_{b,ba}E^t_{c,-ca^{-1}} = 0$. Every vertex in C is adjacent to every vertex in D, and no vertex within C or D are adjacent. Therefore the component corresponding to E^a forms a complete bipartite graph $K_{2n-1,2n-1}$ in $IdN(R)$ as shown in Figure 5 (a).

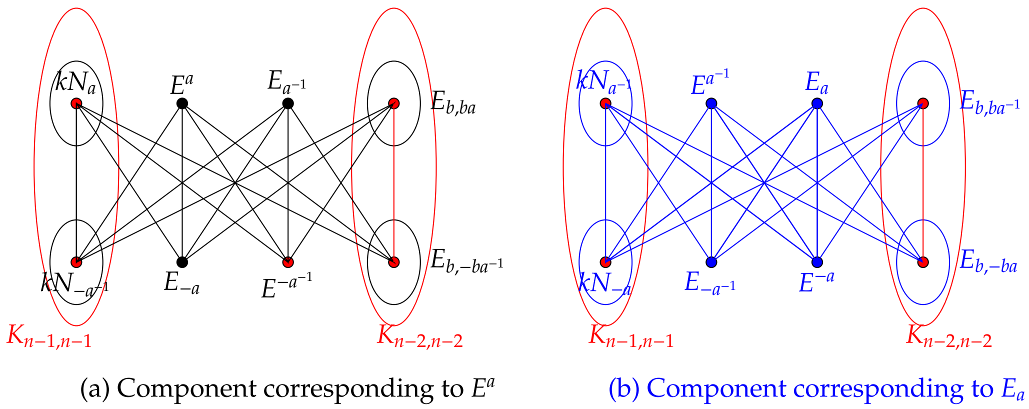
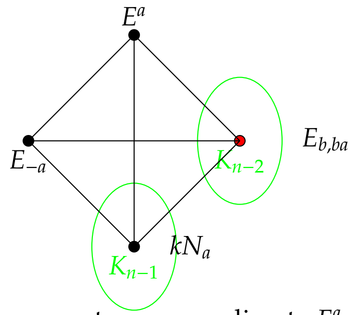
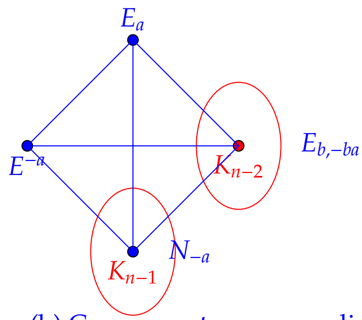


Figure 5: The components of $IdN(R)$ corresponding to the vertices E^a and E_a for $a^2 \neq -1$.

(ii) Next, consider the neighborhoods of the elements in component of $IdN(R)$ corresponding to E_a.
$\mathcal{N}(E_a) = \{E^{-a}, E_{-a^{-1}}, kN_{-a}, E_{b,-ba} : b, k \in \mathbb{F} \setminus \{0\}, b \neq 1\} = \mathcal{N}(E^{a^{-1}}) = \mathcal{N}(E_{b,ba^{-1}}) = \mathcal{N}(kN_{a^{-1}})$. Observe that $|\mathcal{N}(E_a)| = 2n-1$. And $\mathcal{N}(E_{-a^{-1}}) = \{E^{a^{-1}}, E_a, kN_{a^{-1}}, E_{b,ba^{-1}} : b, k \in \mathbb{F} \setminus \{0\}, b \neq 1\} = \mathcal{N}(E^{-a}) = \mathcal{N}(E_{b,-ba}) = \mathcal{N}(kN_{-a})$. Observe that $|\mathcal{N}(E_{-a^{-1}})| = 2n-1$.
Let $C_1 = \{E_{b,-ba} : b \in \mathbb{F} \setminus \{0,1\}\}$ and $D_1 = \{E_{b,ba^{-1}} : b \in \mathbb{F} \setminus \{0,1\}\}$. Note that $E_{b,-ba}E^t_{c,-ca} \neq 0$, and $E_{b,-ba}E^t_{c,ca^{-1}} = 0$. Every vertex in C_1 is adjacent to every vertex in D_1, and no vertex within C_1 or within D_1 are adjacent. Thus, the component corresponding to E_a forms a complete bipartite graph $K_{2n-1,2n-1}$ as shown in Figure 5 (b).
Case (2): Let $a \in \mathbb{F}\backslash\{0\}$, with $a^2 = -1$.
(i) The neighborhood of E^a is: $\mathcal{N}(E^a) = \{E_{-a}, E_{b,ba}, kN_a : b, k \in \mathbb{F} \setminus \{0\}, b \neq 1\}$. Hence $|\mathcal{N}(E^a)| = 2n-2$. Next, consider the neighborhoods of the elements adjacent to E^a. $\mathcal{N}(E_{-a}) = \{E^a, E_{b,ba}, kN_a : b, k \in \mathbb{F} \setminus \{0\}, b \neq 1\}$.
$\mathcal{N}(E_{b,ba}) = \{E^a, E_{-a}, E_{c,ca}, kN_a : c, k \in \mathbb{F} \setminus \{0\}, c \neq 1\}$. Since $a^2 = -1$, $-1/a = a$. This gives $N_a = N_{-1/a} = (N_a)^t$. Therefore $(kN_a)(bN_a)^t = 0$, which yields $bN_a \in \mathcal{N}(kN_a)$, for each $b \neq k$ i.e., the vertices kN_a are mutually adjacent. Therefore $\mathcal{N}(kN_a) = \{E^a, E_{-a}, E_{b,ba}, cN_a : b \in \mathbb{F} \setminus \{0,1\}, c \in \mathbb{F} \setminus \{0\}, c \neq k\}$. Hence $|\mathcal{N}(kN_a)| = 2n-2$. For each vertex v adjacent to E_a, the neighborhoods $\mathcal{N}(x)$ demonstrate that all vertices are mutually connected. Therefore, the component corresponding to E^a forms a complete graph K_{2n-1} (see Figure 6 (a)).



(a) Component corresponding to E^a



(b) Component corresponding to E_a

Figure 6: The components of $IdN(R)$ corresponding to the vertices E^a and E_a for $a^2 = -1$.

(ii) $\mathcal{N}(E_a) = \{E^{-a}, E_{b,-ba}, kN_{-a} : b, k \in \mathbb{F} \setminus \{0\}, b \neq 1\}$. Hence $|\mathcal{N}(E_a)| = 2n - 2$.
Consider the neighborhoods of the elements adjacent to E_a.
$\mathcal{N}(E^{-a}) = \{E_a, E_{b,-ba}, kN_{-a} : b, k \in \mathbb{F} \setminus \{0\}, b \neq 1\}$.
$\mathcal{N}(E_{b,-ba}) = \{E_a, E^{-a}, E_{c,-ca}, kN_{-a} : c, k \in \mathbb{F} \setminus \{0\}, c \neq b, c \neq 1\}$.
$\mathcal{N}(kN_{-a}) = \{E_a, E^{-a}, E_{b,-ba}, cN_a : b \in \mathbb{F} \setminus \{0, 1\}, c \in \mathbb{F} \setminus \{0\}, c \neq k\}$. Similar to part (i) above the component corresponding to the vertex E_a forms complete graph K_{2n-1} (see Figure 6 (b)). □

If $p \equiv 1 \pmod 4$, the following result characterizes the graph $IdN(R)$.

Proposition 3.11. *Let $R = M_2(\mathbb{F})$ and $p \equiv 1 \pmod 4$. Then $IdN(R)$ is a disjoint union of 2 copies of K_{2n-1} and $\dfrac{n-1}{2}$ copies of $K_{2n-1,2n-1}$.*

Proof. The total number of vertices in $IdN(R)$ is $(n+1)(2n-1)$. According to Proposition 3.10, the components of $IdN(R)$ are either bipartite or complete graphs. The components E_a and E^a corresponding to an element $a^2 = -1$ are complete graphs K_{2n-1}. Also, note that the components associated with $a \in \mathbb{F}$ and $-a \in \mathbb{F}$ are the same for $a^2 = -1$. By Remark 3.2, there are only 2 elements in $\mathbb{F}$ with $a^2 = -1$. Therefore there are exactly two copies of K_{2n-1}. The remaining components are regular bipartite graphs $K_{2n-1,2n-1}$. The total number of vertices in bipartite components is $(n+1)(2n-1) - 2(2n-1) = (2n-1)(n-1)$. Therefore the number of bipartite components in $IdN(R)$ is $\dfrac{(2n-1)(n-1)}{2(2n-1)} = \dfrac{n-1}{2}$. Hence $IdN(R)$ is a disjoint union of 2 copies of K_{2n-1} and $\dfrac{n-1}{2}$ copies of $K_{2n-1,2n-1}$. □

If $p \equiv 1 \pmod 4$, the following result determines the spectrum of $IdN(R)$.

Proposition 3.12. *Let $R = M_2(\mathbb{F})$ and $p \equiv 1 \pmod 4$. Then the adjacency spectrum of $IdN(R)$ is*

$$\left\{(2n-2)^{(2)}, (-1)^{(2(2n-2))}, (2n-1)^{(\frac{n-1}{2})}, (-(2n-1))^{(\frac{n-1}{2})}, (0)^{(2(n-1)^2)}\right\}, \tag{3}$$

whereas the Laplacian spectrum of $IdN(R)$ is

$$\left\{0^{(\frac{n+3}{2})}, (2n-1)^{(2(n^2-1))}, (2(2n-1))^{(\frac{n-1}{2})}\right\}. \tag{4}$$

Proof. The adjacency eigenvalues corresponding to K_{2n-1} are $2n-2$ with multiplicity 1, and -1 with multiplicity $2n-2$. Thus, the adjacency eigenvalues corresponding to these two copies of K_{2n-1} are $(2n-2)^{(2)}$ and $(-1)^{(2(2n-2))}$. The adjacency eigenvalues of $K_{2n-1,2n-1}$ are $(2n-1)^{(1)}, (-(2n-1))^{(1)}, (0)^{(2(2n-2))}$. Therefore the eigenvalues of $\frac{n-1}{2}$ copies of $K_{2n-1,2n-1}$ are $(2n-1)^{(\frac{n-1}{2})}, (-(2n-1))^{(\frac{n-1}{2})}, (0)^{(2(n-1)^2)}$. Hence the adjacency spectrum of $IdN(R)$ is the multiset (3).

The Laplacian eigenvalues of K_{2n-1} are $0^{(1)}, (2n-1)^{(2n-2)}$. Therefore, the Laplacian eigenvalues corresponding with two components of the complete graphs K_{2n-1} are $0^{(2)}, (2n-1)^{(2(2n-2))}$. The Laplacian eigenvalues corresponding with $K_{2n-1,2n-1}$ are $0^{(1)}, (2n-1)^{(4n-4)}, (2(2n-1))^{(1)}$. Since there are $\frac{n-1}{2}$ copies of $K_{2n-1,2n-1}$, the Laplacian eigenvalues corresponding to these components are $0^{(\frac{n-1}{2})}, (2n-1)^{(2(n-1)^2)}, (2(2n-1))^{(\frac{n-1}{2})}$. Thus, the Laplacian spectrum of $IdN(R)$ is the multiset (4). □

Recall that if G is a graph with N vertices and M edges, and $\mu_1, \mu_2, \ldots, \mu_N$ are the Laplacian eigenvalues of G, then $LE(G) = \sum_{i=1}^{N} \left|\mu_i - \frac{2M}{N}\right|$. In the following result, we find the energy of $IdN(R)$.

Corollary 3.13. *Let $R = M_2(\mathbb{F})$ and $p \equiv 1 \ (mod\ 4)$. Then the energy is $E(IdN(R)) = (n-1)(2n+7)$, and the Laplacian energy is $LE(IdN(R)) = \frac{(n-1)(n+3)(2n+3)}{n+1}$.*

Proof. The adjacency spectrum of $IdN(R)$ is the multiset (3). Hence The energy is given by:
$E(IdN(R)) = 2|2n-2| + 2(2n-2)|-1| + (\frac{n-1}{2})|2n-1| + (\frac{n-1}{2})|-(2n-1)| = (n-1)(2n+7)$.
Total number of vertices in $IdN(R)$ is $N = (n+1)(2n-1)$. Since there are two copies of K_{2n-1} and $\frac{n-1}{2}$ copies of $K_{2n-1,2n-1}$ in $IdN(R)$. Total number of edges in $IdN(R)$ is given by:
$M = 2\left(\frac{(2n-1)(2n-2)}{2}\right) + \left(\frac{n-1}{2}\right)(2n-1)^2 = \frac{(n-1)(2n-1)(2n+3)}{2}$.
The Laplacian spectrum of $IdN(R)$ is the multiset (4). Hence the Laplacian energy is given by:

$$\begin{aligned}
LE(G) &= \sum_{i=1}^{N} \left|\mu_i - \frac{2M}{N}\right| \\
&= \sum_{i=1}^{\frac{n+3}{2}} \left|0 - \frac{(n-1)(2n-1)(2n+3)}{(n+1)(2n-1)}\right| + \sum_{i=1}^{2(n^2-1)} \left|(2n-1) - \frac{(n-1)(2n-1)(2n+3)}{(n+1)(2n-1)}\right| \\
&+ \sum_{i=1}^{\frac{n-1}{2}} \left|2(2n-1) - \frac{(n-1)(2n-1)(2n+3)}{(n+1)(2n-1)}\right| \\
&= \sum_{i=1}^{\frac{n+3}{2}} \frac{(n-1)(2n+3)}{n+1} + \sum_{i=1}^{2(n^2-1)} \frac{2}{n+1} + \sum_{i=1}^{\frac{n-1}{2}} \frac{2n^2+n+1}{n+1} \\
&= \left(\frac{n+3}{2}\right)\left(\frac{(n-1)(2n+3)}{n+1}\right) + 2(n^2-1)\left(\frac{2}{n+1}\right) + \left(\frac{n-1}{2}\right)\left(\frac{2n^2+n+1}{n+1}\right) \\
&= \frac{(n-1)(n+3)(2n+3)}{n+1}.
\end{aligned}$$

□

For $p = 2$. The following result gives the structure of $IdN(R)$.

Proposition 3.14. *Let $R = M_2(\mathbb{F})$ and $p = 2$. Then $IdN(R)$ is a disjoint union of K_{2n-1} and $\frac{n}{2}$ copies of $K_{2n-1,2n-1}$.*

Proof. The total number of vertices in $IdN(R)$ is $(n+1)(2n-1)$. According to Proposition 3.4, the components of $IdN(R)$ are either bipartite or complete graphs. In this case, when $p = 2$, there is only one element $a \in \mathbb{F}$ such that $a^2 = -1$ (see Remark 3.2). Therefore there is exactly one component K_{2n-1}. The total number of vertices in bipartite components is $(n+1)(2n-1) - (2n-1) = n(2n-1)$. Therefore the number of bipartite components in $IdN(R)$ is $\frac{n(2n-1)}{2n-1} = \frac{n}{2}$. Hence $IdN(R)$ is a disjoint union of one copy of K_{2n-1} and $\frac{n}{2}$ copies of $K_{2n-1,2n-1}$. □

Let $p = 2$. In the following result we determine the spectrum of $IdN(R)$.

Proposition 3.15. *Let $R = M_2(\mathbb{F})$ and $p = 2$. Then the adjacency spectrum of $IdN(R)$ is given by the multiset*

$$\left\{(2n-2)^{(1)}, (-1)^{(2n-2)}, (2n-1)^{(\frac{n}{2})}, (-(2n-1))^{(\frac{n}{2})}, (0)^{(2n(n-1))}\right\}, \tag{5}$$

whereas the Laplacian spectrum is the multiset

$$\left\{0^{(\frac{n+2}{2})}, (2n-1)^{(2n^2-2)}, (2(2n-1))^{(\frac{n}{2})}\right\}. \tag{6}$$

Proof. The adjacency eigenvalues of K_{2n-1} are $2n-2$ with multiplicity 1, and -1 with multiplicity $2n-2$. The adjacency eigenvalues of $K_{2n-1,2n-1}$ are $(2n-1)^{(1)}, (-(2n-1))^{(1)}, 0^{(4n-4)}$. Therefore, the adjacency eigenvalues of $\frac{n}{2}$ copies of $K_{2n-1,2n-1}$ are $(2n-1)^{(\frac{n}{2})}, (-(2n-1))^{(\frac{n}{2})}, (0)^{(2n(n-1))}$. Hence the adjacency spectrum of $IdN(R)$ is the multiset (5).

The Laplacian eigenvalues corresponding to K_{2n-1} are $0^{(1)}, (2n-1)^{(2n-2)}$. The Laplacian eigenvalues corresponding to $K_{2n-1,2n-1}$ are $0^{(1)}, (2n-1)^{(4n-4)}, (2(2n-1))^{(1)}$. Therefore, the Laplacian eigenvalues corresponding to $\frac{n}{2}$ copies of $K_{2n-1,2n-1}$ are $0^{(\frac{n}{2})}, (2n-1)^{(2n(n-1))}, (2(2n-1))^{(\frac{n}{2})}$. Thus, the Laplacian spectrum of $IdN(R)$ is the multiset (6). □

The following corollary follows directly from the above result.

Corollary 3.16. *Let $R = M_2(\mathbb{F})$ and $p = 2$. Then the energy of $IdN(R)$ is $E(IdN(R)) = 2n^2 + 3n - 4$ whereas the Laplacian energy is*

$$LE(IdN(R)) = \frac{(n+2)(2n^2+n-2)}{n+1}.$$

Proof. The adjacency spectrum of $IdN(R)$ is the multiset (5). Hence the energy of $IdN(R)$ is given by:

$$E(IdN(R)) = 1|2n-2| + (2n-2)|-1| + \frac{n}{2}|2n-1| + \frac{n}{2}|-(2n-1)| = 2n^2 + 3n - 4.$$

Let N and M be the number of vertices and edges of $IdN(R)$. Then $N = (n+1)(2n-1)$. Since there is 1 copy of K_{2n-1} and $\frac{n}{2}$ copies of $K_{2n-1,2n-1}$, we have $M = \frac{(2n-1)(2n-2)}{2} + \frac{n}{2}(2n-1)^2 = \frac{(2n-1)(2n^2+n-2)}{2}$. Thus

$$\frac{2M}{N} = \frac{(2n-1)(2n^2+n-2)}{(n+1)(2n-1)} = \frac{2n^2+n-2}{n+1}.$$

The Laplacian spectrum of $IdN(R)$ is the multiset (6) and then the Laplacian energy of $IdN(R)$ is given by:

$$\begin{aligned}
LE(G) &= \sum_{i=1}^{N}\left|\mu_i - \frac{2M}{N}\right| \\
&= \sum_{i=1}^{\frac{n+2}{2}}\left|0 - \frac{2n^2+n-2}{n+1}\right| + \sum_{i=1}^{2(n^2-1)}\left|2n-1-\frac{2n^2+n-2}{n+1}\right| + \sum_{i=1}^{\frac{n}{2}}\left|2(2n-1) - \frac{2n^2+n-2}{n+1}\right| \\
&= \sum_{i=1}^{\frac{n+2}{2}}\frac{2n^2+n-2}{n+1} + \sum_{i=1}^{2(n^2-1)}\frac{2}{n+1} + \sum_{i=1}^{\frac{n}{2}}\frac{n(2n+1)}{n+1} \\
&= \left(\frac{n+2}{2}\right)\left(\frac{2n^2+n-2}{n+1}\right) + 2(n^2-1)\left(\frac{2}{n+1}\right) + \left(\frac{n}{2}\right)\left(\frac{n(2n+1)}{n+1}\right) \\
&= \frac{2n^3+5n^2-4}{n+1} = \frac{(n+2)(2n^2+n-2)}{n+1}.
\end{aligned}$$

□

Remark 3.17. *Let $R = M_2(\mathbb{F})$. Since $IdN(R)$ contains $K_{3,3}$ as a subgraph, $IdN(R)$ is not planar.*

We conclude this section by proving Beck's conjecture for $IdN(R)$.

Proposition 3.18. *Let $R = M_2(\mathbb{F})$. Then the chromatic number and clique number of $IdN(R)$ are:*

$$\omega(IdN(R)) = \chi(IdN(R)) = \begin{cases} 2, & \text{if } p \equiv 3 \ (mod\ 4) \\ 2n-1, & \text{if } p \not\equiv 3 \ (mod\ 4). \end{cases}$$

Proof. Suppose that $p \equiv 3 \pmod 4$. By Proposition 3.4, each component of $IdN(R)$ forms a bipartite graph. The chromatic number and clique number of a bipartite graph are both 2. Hence $\omega(IdN(R)) = \chi(IdN(R)) = 2$. Suppose $p \not\equiv 3 \pmod 4$, then there exists $a \in \mathbb{F}$ with $a^2 = -1$. By Proposition 3.10, the components of $IdN(R)$ include complete graphs K_{2n-1} and complete bipartite graphs $K_{2n-1,2n-1}$. The largest clique in $IdN(R)$ is K_{2n-1}, so $\omega(IdN(R)) = 2n-1$. The graph K_{2n-1} requires $2n-1$ colors since it is a complete graph, and we need one color for each vertex. For the bipartite components $K_{2n-1,2n-1}$, any two of the $2n-1$ available colors can be used to color the vertices, so they do not affect the total chromatic number. Thus $\chi(IdN(R)) = 2n-1$. Therefore, in both cases, we have $\omega(IdN(R)) = \chi(IdN(R))$, and the values are given by the above cases depending on whether $p \equiv 3 \pmod 4$ or not. Hence

$$\omega(IdN(R)) = \chi(IdN(R)) = \begin{cases} 2, & \text{if } p \equiv 3 \pmod 4 \\ 2n-1, & \text{if } p \not\equiv 3 \pmod 4. \end{cases}$$

□

References

[1] S. Akbari, M. Habibi, A. Majidinya, and R. Manaviyat, On the idempotent graph of a ring, Journal of Algebra and Its Applications 12(6) (2013) #1350003.

[2] M. N. Authman, M. Q. Husam and S. H. Nazar, The domination number of idempotent divisor graphs of commutative rings, Palestine Journal of Mathematics 11(4) (2022) 300–306.

[3] I. Beck, Coloring of commutative rings, Journal of Algebra 116 (1988) 208–226.

[4] D. S. Djordjević, M. Z. Kolundzija, S. Radosavljevic and D. Mosic, A survey on 2×2 operator matrices and their applications. Zb. Rad.(Beogr.) 20(28) (2022) 7–81.

[5] K. Cvetko-Vah and D. Dolzan, Indecomposability graphs of rings, Bulletin of the Australian Mathematical Society 77 (2008) 151–159.

[6] D. Cvetković, M. Doob and H. Sachs, Spectra of Graphs –Theory and Application, Academic Press, New York, (1980).

[7] C. Godsil and G. Royle, Algebraic Graph Theory, Springer, New York, (2001).

[8] I. Gutman, The energy of a graph: Old and new results, in: A. Betten, A. Kohnert, R. Laue, A. Wassermann (Eds.), Algebraic Combinatorics and Applications, Springer, Berlin, (2001), pp. 196–211.

[9] I. Gutman and B. Furtula, The total π-electron energy saga, Croatica Chemica Acta 90 (2017), 359–368.

[10] I. Gutman and B. Furtula, Energies of Graphs – Survey, Census, Bibliography, Center for Scientific Research, Kragujevac, (2019).

[11] I. Gutman and B. Zhou, Laplacian energy of a graph, Linear Algebra and Its Applications 414(1) (2006) 29–37.

[12] F. Harary, Graph Theory, CRC Press, Boca Raton, (1969).

[13] A. Khairnar and B. N.Waphare, Zero-divisor graphs of Laurent polynomials and Laurent power series, in: S. Rizvi, A. Ali, V. Filippis (Eds.), Algebra and Its Applications, Springer, Singapore (2016) 345–349.

[14] A. Khairnar and B. N.Waphare, Generalized projections in $\mathbb{Z}_n$, in: AKCE International Journal of Graphs and Combinatorics 16(1) (2019) 1-7.

[15] N. Kumbhar, A. Khairnar and B. N. Waphare, Strong zero-divisor graph of rings with involution, Asian-European Journal of Mathematics 16(10) (2023) #2350179.

[16] X. Li, Y. Shi and I. Gutman, Graph Energy, Springer, New York, (2012).

[17] K. Masalkar, A. Khairnar, A. Lande and L. Kadam, On spectrum of the zero-divisor graph of matrix ring, arXiv:2312.09934v1 (2023).

[18] N. Mihajlović, and D. Djordjević, Perturbation results and forward order law for the Moore–Penrose inverse in rings with involution. Georgian Mathematical Journal, 29(3) (2022) 425–439.

[19] A. Patil and P. Momale, Idempotent graphs, weak perfectness, and zero-divisor graphs, Soft computing 25(15) (2021) 10083–10088.

[20] A. Patil, P. S. Momale and C. M. Jadhav, On the idempotent graph of matrix ring, arXiv:2402.11256v1 [math.CO] (2024).

[21] A. Patil and B. Waphare, The zero-divisor graph of a ring with involution. Journal of Algebra and its Applications 17(03) (2018) #1850050.

[22] H. S. Ramane, Energy of graphs, in: M. Pal, S. Samanta, A. Pal (Eds.), Handbook of Research on Advanced Applications of Graph Theory in Modern Society, IGI Global, Hershey, (2020), pp. 267–296.

[23] P. Sharma and S. Dutta, On Idempotent graph of rings, Palestine Journal of Mathematics 12(1) (2023) 883–891.